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Approach to Equilibrium of Glauber Dynamics in the One Phase Region

I. The Attractive Case

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Abstract: Various finite volume mixing conditions in classical statistical mechanics are reviewed and critically analyzed. In particular some *finite size conditions* are discussed, together with their implications for the Gibbs measures and for the approach to equilibrium of Glauber dynamics in *arbitrarily large* volumes. It is shown that Dobrushin-Shlosman's theory of *complete analyticity* and its dynamical counterpart due to Stroock and Zegarlinski, cannot be applied, in general, to the whole one phase region since it requires mixing properties for regions of *arbitrary* shape. An alternative approach, based on previous ideas of Olivieri, and Picco, is developed, which allows to establish results on rapid approach to equilibrium deeply inside the one phase region. In particular, in the ferromagnetic case, we considerably improve some previous results by Holley and Aizenman and Holley. Our results are optimal in the sense that, for example, they show for the first time fast convergence of the dynamics *for any temperature* above the critical one for the d-dimensional Ising model with or without an external field. In part II we extensively consider the general case (not necessarily attractive) and we develop a new method, based on renormalizations group ideas and on an assumption of strong mixing in a finite cube, to prove hypercontractivity of the Markov semigroup of the Glauber dynamics.

0. Introduction

Recently many efforts have been devoted, with increasing interest, to analyze the precise connections between i) mixing properties of Gibbs measures for lattice spin systems (typically expressed in terms of rapid decay of the truncated correlations), and ii) the (properly defined) speed of approach towards equiilibrium of some associated spin flib Glauber type dynamics. We have in mind, in particular, the basic paper by Holley [H2], and the subsequent works by Aizenman and Holley [AH] and Stroock and Zegarlinski [SZ], where such connections were established, first for ferromagnetic

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Ising type models [H2, AH] and then for very general discrete or continuous spin systems [SZ].

The goal of those papers was to show, under very general hypotheses on the flip rates, that a Glauber dynamics for values of thermodynamical parameters (e.g. temperature and magnetic field) which do not give rise to a phase transition, must have a rapid (typically exponentially fast in the sup-norm) approach to equilibrium.

In all these works the mixing properties of the Gibbs state were expressed in terms of some finite volume condition similar if not equal (see [SZ]) to the famous Dobrushin and Shlosman *complete analyticity* conditions [DS2, DS3]. If such conditions hold then the Glauber dynamics approaches its invariant Gibbs measure μ exponentially fast either in the $L^2(d\mu)$ sense or in the L^∞ norm in any finite or infinite volume and with rates uniformly bounded in the volume and in the boundary conditions. In the standard Ising model the various conditions are easily verified at high temperature or large external magnetic field.

The first result referring to the region of parameters really close to the line of first order phase transition was proved a few years ago by the authors of the present paper in collaboration with Scoppola [MOS] while working on metastability for the dynamical Ising model. By purely dynamical arguemnts we established the rapid approach to equilibrium for the standard 2D stochastic (= dynamical) Ising model for any value of the magnetic field h, provided that the temperature T was low enough (depending on h!). In the arguments of proof in [MOS] a crucial role was played by the results of Neves and Schonmann [NS] on the metastable behaviour of the 2D Ising model in finite volume. In order to extend the resul to 3D in the same region of the phase diagram and since detailed results on metastability were (and still are) not available in 3D, we tried to see whether one of the various finite volume mixing conditions of the above mentioned papers could be satisfied by our model. To our surprise only very weak results could be deduced in our case since we could not verify the majority of these conditions; moreover, thanks also to some simple examples by Schonmann that are described in Sect. 2, we realized that, in general, any Dobrushin-Shlosman type of finite volume mixing condition is probably bound to fail near to a first order phase transition. The main reason for this surprising result is that, in such conditions, one is required to control the Gibbs state in a finite collection of sets of the lattice some of which with a ratio surface/volume of order 1 (e.g. a layer with thickness equal to one in 3D). Clearly, close to a first order phase transition line, one should expect to be able to establish mixing properties of the Gibbs state only in sufficiently regular regions (for example such that their surface/volume ratio tends to zero as the volume tends to ∞).

A second important observation on the approach to equilibrium for Glauber dynamics that comes out of the present work is the following one: as in equilibrium statistical mechanics where sometimes one is able to prove rapid decay of the *infinite volume* correlation functions but *not* of the *finite volume* ones, with bounds uniform in the volume and in the boundary conditions, also for the dynamics one has to carefully distinguish between *infinite volume* results and *finite volume* ones (with bounds uniform in the volume and in the boundary conditons). The reason is that even if there exists a unique infinite volume Gibbs state with exponentially decaying correlation functions, it may happen that, in arbitrarily large but finite volume (e.g. a big cube), a sort of long-range order close to the boundary occurs with a consequent decay of the correlation functions non-uniform in the location of the observables; such a non-uniformity must give rise to a global slowing down of the dynamics in the whole volume (see the remark at the end of Sect. 4).

Such a "boundary phase transition" is known to occur in 3D for the so-called Czech models [Sh]; even for the Ising model at low temperature and very small (depending on the temperature) magnetic field a "layering phase transition" is expected to take place (Basuev phenomenon [D1]). However it is reasonable to conjecture that the above phenomenon should never appear in 2D since, in this case, the boundary is one-dimensional and, in this case, phase transitions for a short range interaction can never take place. This is exactly what we prove in a paper in preparation done in collaboration with Schonmann [MoSh].

As a consequence of the above discussion, if one is willing to prove rapid convergence to equilibrium for the Glauber dynamics in the whole one-phase region, one should try to envisage a method that works directly for the infinite volume dynamics without any assumptions on the finite volume one. In [AH] first and later on in [SZ] the exponential convergence to equilibrium, directly in the infinite volume in the $L^2(d\mu)$, is proved without requiring anything on the finite volume dynamics. However in order to get a stronger result in which the $L^2(d\mu)$ norm is replaced by the L^∞ one, all the existing method had to assume a uniform lower bound on the gap of the generator of the dynamics in a finite region uniformly in the size, in the shape of the region as well as in the boundary conditions.

By the above discussion this seems too strong a requirement in order to cover the whole one phase region.

In this paper we make what we believe is an important step toward the solution of the above problems at least for discrete, finite range spin systems.

In the attractive (= ferromagnetic) case we show that rapid approach to equilibrium in the *infinite volume* in the *uniform norm* is equivalent to exponentially weak dependence on the boundary conditions for the magnetization at the origin. In the Ising model such a condition is expected to be true in the whole phase diagram outside the coexistence line but we are able to verify it only for:

i) $T > T_c$ and any uniform magnetic field h,
ii) for T small enough and any $h \neq 0$.

In order to obtain stronger results, namely rapid approach to equilibrium in *finite volume* in the uniform norm with bounds uniform in the volume and in the boundary conditions, we make a mixing assumption (rapid decay of two point truncated correlations) on the Gibbs state on a given large enough cube Λ_0 (thus we have no arbitrariness on the shape). Such a finite volume mixing condition was introduced some years ago by Olivieri [O] and Olivieri and Picco [OP] in order to derive, by renormalization group methods (decimation) and cluster expansion, analyticity and decay of truncated correlations of the Gibbs state in infinite and finite volumes, provided that the latter are in some sense a "multiple" of Λ_0. Although their results may appear weaker than those of Dobrushin and Shlosman, since not all possible geometric shapes are covered, they are certainly suited to deal with systems close (but not too close because e.g. of the Checks models) to a first order phase transition. We show that under the hypothesis of [O] and [OP] we can get the rapid approach to equilibrium, both in the $L^2(d\mu)$ and in the L^∞ sense, for volumes that are a "multiple" of Λ_0. Again we verify that for the Ising model our condition holds for high temperature or low temperature and arbitrarily small (not vanishing) magnetic field h but with $h/T \gg 1$. This in particular covers the case of metastability in the 3D case.

In this paper we only give the proofs in the attractive case; the general case requires proving the hypercontractivity of the Markov semigroup generated by the dynamics, which in turn is equivalent to proving the existence of a finite Logarithmic Sobolev constant for the Gibbs state. This was provided for the first time by Stroock and Zegarlinski [SZ] under a Dobrushin-Shloshman complete analyticity assumption; in our case we found a different approach, based on renormalization group methods. This new method is the argument of a forthcoming paper [MO].

The present paper is organized as follows:

In Sect. 1 we define the models and the Glauber dynamics.

In Sect. 2 we critically review the existing finite volume mixing conditions together with their implications both for the equilibrium problem and for the approach to equilibrium of the dynamics.

In Sect. 3 for the attractive case (not necessarily reversible with respect to a Gibbs measure) we prove rapid convergence to equilibrium in the infinite volume under a weak dependence on the boundary conditions of the magnetization in the origin.

In Sect. 4 we establish finite volume results.

In Sect. 5 we discuss the implications of our results for the stochastic Ising model.

Yau and ShengLin Lu [SY], starting from mixing properties of the Gibbs measure, proved a very interesting lower bound on the spectral gap of the generator of the Kawasaki dynamics. Their method, based on the strong mixing condition $SM(\Lambda, C, \gamma)$ for *all* cubes Λ, allows them to treat also the Glauber dynamic.

1. General Definitions and Notation

In this section we define the statistical mechanics models and the associated Glauber dynamics that we want to examine.

We will consider lattice spin systems. We start giving a list of basic definitions.

– Configuration space of a single spin: finite set $S = \{1, \ldots, N\}$, $N \in \mathbf{N}$.

– Configuration space in a subset $\Lambda \subset \mathbf{Z^d}$.

$$\Omega_\Lambda = S^\Lambda ,$$

thus an element σ_Λ in Ω_Λ is a function

$$\sigma_\Lambda : \Lambda \to S .$$

– Configuration space in the whole $\mathbf{Z^d}$:

$$\Omega = S^{\mathbf{Z^d}}$$

thus an element σ in Ω is a function

$$\sigma : \mathbf{Z^d} \to S .$$

– $\sigma_x \equiv \sigma(x)$ is called value of the spin at the site $x \in \Lambda$ in the configuration σ.

– By $|X|$ we denote the cardinality of

$$X \subset\subset \mathbf{Z^d}$$

(we write $X \subset\subset \mathbf{Z^d}$ to express that X is a *finite* subset of $\mathbf{Z^d}$).

– Potential $U =$ family of functions indexed by finite sets in $\mathbf{Z^d}$

$$U = \{U_X, X \subset\subset \mathbf{Z^d}\}$$

where, for every finite X.

$$U_X : \Omega_X \to R .$$

On the potential U we will always assume the following hypotheses:

H1. *Finite range:* $\exists r > 0 : U_x \equiv 0$ *if* diam $X > r$ *(we use Euclidean distance).*

H2. *Translation invariance*

$$\forall X \subset\subset \mathbf{Z^d} \qquad \forall k \in \mathbf{Z^d} \qquad U_{X+k} = U_X .$$

– Given $\Lambda \subset \mathbf{Z^d}$ and $\tau \in \Omega_{\Lambda^c}(\Lambda^c = \mathbf{Z^d} \backslash \Lambda)$, for every $\sigma \in \Omega$ we denote by $\sigma\tau$ the configuration obtained from σ by substituting τ to σ outside Λ:

$$\begin{aligned} (\sigma\tau)_x &= \sigma_x , \qquad \forall x \in \Lambda , \\ (\sigma\tau)_x &= \tau_x , \qquad \forall x \in \Lambda^c . \end{aligned} \tag{1.1}$$

– Given a set $\Lambda \subset\subset \mathbf{Z^d}$, a *boundary condition*, (b.c.), is a configuration

$$\tau \in \Omega_{\Lambda^c} .$$

– Given $\Lambda \subset\subset \mathbf{Z^d}$ the *energy* associated to a configuration $\sigma \in \Omega_\Lambda$ when the boundary condition outside Λ is $\tau \in \Omega_{\Lambda^c}$ given by:

$$H_\Lambda^\tau(\sigma) = H_\Lambda(\sigma \mid \tau) = \sum_{X : X \cap \partial\Lambda \neq \emptyset} U_X((\sigma\tau)_X) , \tag{1.2}$$

because of the hypothesis H1, $H_\Lambda^\tau(\sigma)$ depends only on τ_x for x in $\partial_r^+\Lambda$:

$$\partial_r^+\Lambda = \{x \notin \Lambda : \text{dist}(x, \Lambda) \le r\} . \tag{1.3}$$

– The *Gibbs measure* in Λ with b.c. $\tau \in \Omega_{\Lambda^c}$ and inverse temperature $\beta > 0$ is

$$\mu_\Lambda^\tau(\sigma) = \frac{\exp(-\beta H_\Lambda^\tau(\sigma))}{Z_\Lambda^\tau} . \tag{1.4}$$

The normalization factor, called *partition function* is given by

$$Z_\Lambda^\tau = \sum_{\sigma \in \Omega_\Lambda} \exp(-\beta H_\Lambda^\tau(\sigma)) . \tag{1.5}$$

If there exists a unique limiting Gibbs measure for $\Lambda \to \mathbf{Z^d}$, independent on τ, it will be denoted by μ.

– The *variation distance* between two probability measures P, Q on a finite set Y is:

$$\text{Var}(P, Q) = \tfrac{1}{2} \sum_{y \in Y} |P(y) - Q(y)| = \sup_{X \subset Y} |P(X) - Q(X)| . \tag{1.6}$$

– Given a metric $\varrho(\cdot\,,\cdot)$ on a finite space Y (a much more general framework can also be considered) the *Kantorovich-Rubinstein-Ornstein-Vasserstein distance with respect to* ϱ between two probability measures μ_1, μ_2 on Y, that we denote by $\text{KROV}_\varrho(\mu_1, \mu_2)$, is defined as

$$\text{KROV}_\varrho(\mu_1, \mu_2) = \inf_{\mu \in K(\mu_1, \mu_2)} \sum_{y, y' \in Y} \varrho(y, y')\, \mu(y, y') , \tag{1.7}$$

where $K(\mu_1, \mu_2)$ is the set of joint representation of μ_1, μ_2 namely the set of measures on the cartesian product $Y \times Y$ whose marginals with respect to the factors are, respectively, given by μ_1, μ_2. Namely we have $\forall B \subset Y$:

$$\mu(B \times Y) = \sum_{y \in B\, y' \in Y} \mu(y, y') = \mu_1(B),$$
$$\mu(Y \times B) = \sum_{y \in Y\, y' \in B} \mu(y, y') = \mu_2(B).$$

For the particular case

$$\varrho(y, y') = 1 \quad \text{iff } y \neq y' \text{ and } 0 \text{ otherwise}, \tag{1.8}$$

$\mathrm{KROV}_\varrho(\cdot\,,\cdot)$ coincides with the variation distance $\mathrm{Var}(\cdot\,,\cdot)$.

– Given a measure μ_Λ on Ω_Λ we call *relativization* of μ_Λ to Ω_Δ with $\Delta \subset \Lambda$, the measure $\mu_{\Lambda,\Delta}$ on Ω_Δ given by

$$\mu_{\Lambda,\Delta}(\sigma_\Delta) = \sum_{\sigma_{\Lambda \setminus \Delta}} \mu_\Lambda(\sigma_{\Lambda\setminus\Delta}, \sigma_\Delta). \tag{1.9}$$

Next we define the stochastic *jump* dynamics, given by a continuous time Markov process on $\Omega = S^{\mathbf{Z^d}}$, that will be studied in the sequel. Discrete time versions can also be considered.

Given $\Lambda \subset\subset \mathbf{Z^d}$ let

$$D(\Lambda) = \{f : \Omega \to R : f(\eta) = f(\sigma) \text{ if } \eta_x = \sigma_x\ \forall x \in \Lambda\}$$

be the set of *cylindrical functions* with support Λ. The set

$$D = \bigcup_\Lambda D(\Lambda)$$

is the set of cylindrical functions and by $C(\Omega)$ we denote the set of all continuous functions on $\Omega = \prod_x S_x$ with respect to the product topology of discrete topologies on S_x.

The dynamics is defined by means of its *generator* L which is given, for $f \in D$, by:

$$Lf(\sigma) = \sum_{x,a} c_x(\sigma, a)\,(f(\sigma^{a,x}) - f(\sigma)), \tag{1.10}$$

where $\sigma^{a,x}$ is the configuration obtained from σ by setting the spin at x equal to the value α and the non-negative quantities $c_x(\sigma, a)$ are called "jump rates."

The general hypotheses on the jump rates, that we shall always assume, are the following ones:

H3. *Finite range r. This means that if $\eta(y) = \sigma(y)\, \forall x, y : |y - x| \leq r$, then $c_x(\sigma, a) = c_x(\eta, a)$.*

H4. *Translation invariance. That is if $\eta(y) = \sigma(y + x)\, \forall y$, then $c_x(\sigma, a) = c_x(\eta, a)$.*

H5. *Positivity. There exists a positive constant k such that $\inf_{\sigma,x,a} c_x(\sigma, a) \geq k > 0$.*

For reasons that will be clear in the sequel it will be important for us to consider also the Markov processes associated to the above described jump rates in a *finite* *volume* Λ with boundary condition τ outside Λ. By this we mean the dynamics on Ω_Λ generated by L^τ_Λ defined as before starting from the jump rates

$$c_x^{\tau,\Lambda}(\sigma, a) \equiv c_x(\sigma\tau, a)\,,$$

where, given $\tau \in \Omega_{\Lambda^c}$, $\sigma \in \Omega_\Lambda$ and $\sigma\tau$ has been defined in (1.1).

It is well known (see [L]) that under the above conditions $L(L^\tau_\Lambda)$ generates a unique positive contraction semigroup on the space $C(\Omega)\,(C(\Omega_\Lambda))$ that will be denoted by T_t or $T_t^{\Lambda,\tau}$.

Sometimes we will use the more probabilistic notation $E_\sigma f(\sigma_t)$ for $T_t f(\sigma)$, where σ_t denotes the Markov process generated by L at time t and $E_\sigma(\cdot)$ denotes expectation starting from the configuration σ.

It is also easy to see, using positivity, that in finite volume there exists a unique invariant measure that will be denoted by ν^τ_Λ.

In this paper we will mostly consider attractive dynamics. *Attractivity* is an important property enjoyed by some interesting spin dynamics and it can be formulated as follows:

H6. *Attractivity: If* $\sigma(x) \geq \eta(x)$ *for all* x *then:*
If $a \leq \eta(x)$, *then* $\sum\limits_{b\leq a} c_x(\sigma, b) \leq \sum\limits_{b\leq a} c_x(\eta, b)$.
If $a \geq \sigma(x)$, *then* $\sum\limits_{b\geq a} c_x(\sigma, b) \geq \sum\limits_{b\geq a} c_x(\eta, b)$.

It is easy to show (see [L]) that attractivity is equivalent to the following condition on the semigroup T_t: if in the space of spin configurations we introduce the partial order $\sigma \leq \eta$ iff $\sigma_x \leq \eta_x$ for all x, then the Markov semigroup T_t leaves invariant the set of increasing (decreasing) functions w.r.t. the above partial order.

Another important class of spin dynamics on Ω, generally called Glauber dynamics, are those which are *reversible* with respect to an a priori given Gibbs measure μ (in finite or infinite volume). We will say that the generator L (1.10) is *reversible* with respect to a Gibbs measure μ corresponding to a Hamiltonian $H(\sigma)$ iff:

$$\begin{aligned}&\exp\Big(-\beta \sum_{X\ni x} U_X(\sigma_X)\Big)\, c_x(\sigma, a)\\ &\quad = \exp\Big(-\beta \sum_{X\ni x} U_X((\sigma_X^{x,a}))\Big)\, c_x(\sigma^{x,a}, \sigma_x)\ \forall x \in \Lambda\,.\end{aligned} \tag{1.11}$$

A similar equation holds in finite volume Λ with boundary conditions τ, provided that we replace in (1.11) σ with the configuration $\sigma\tau$. It is immediate to check that in finite volume (1.11) implies that the unique invariant measure of the dynamics coincides with the Gibbs measure μ^τ_Λ. This important fact holds also in infinite volume provided that there exists a unique Gibbs measure in the thermodynamic limit. In the sequel such kind of dynamics will be referred to as Glauber dynamics.

– Finally we recall the definition of *stochastic Ising models* that will be analyzed in Sect. 4. They are stochastic processes on Ω, reversible with respect to the Gibbs measure of an Ising-like spin system ($S = \{-1, 1\}$).

To introduce them it is enough to define the class of their Hamiltonians. They will be of the form given in (1.2) with

$$U_X = -J_X \prod_{x\in X} \sigma_x \tag{1.12}$$

and $J_X \in R$.

– We say that an Ising spin system is *ferromagnetic* if the local field at the origin

$$h(\sigma) = \sum_{X;0\in X} J_X \prod_{x\in X\setminus\{0\}} \sigma_x \tag{1.13}$$

is an increasing function of the spins σ_x, $x \neq 0$.

The condition to be ferromagnetic is easily seen to be implied by the following more usual condition on the interaction J_X (see e.g. [FKG]) which ensures the validity of the F.K.G. inequalities for the Gibbs state.

Let us introduce the lattice gas variables $\varrho_x = \frac{1+\sigma_x}{2}$ and write the Hamiltonian $H(\sigma)$ as

$$\bar{H}(\varrho) = - \sum_{X\cap\Lambda\neq\emptyset} \Phi_X \prod_{x\in X} \varrho_x \,. \tag{1.14}$$

If the new potential Φ_X is non-negative for any set X consisting of more than one point then the system is ferromagnetic.

It is easy to check that in the case of only two body interaction the system is ferromagnetic iff $J_{(x,y)} \geq 0$.

2. Critical Analysis of Finite Volume Mixing Conditions

In this section we will critically review the existing notions of mixing for *finite volume* measures and their implications for the Gibbs state in the thermodynamic limit as well as for the rate of convergence to equilibrium of an associated Glauber dynamics.

We will distinguish between *strong* and *weak* finite volume mixing conditions. Both notions can be expressed as weak dependence, inside, Λ, say in $x \in \Lambda$, on the value of a conditioning spin, say in $y \in \partial_r^+ \Lambda$. We have strong mixing if the influence of what happens in x decays with the distance $|x-y|$ of x from y whereas we speak of weak mixing when the influence decays with the distance of x from the boundary $\partial\Lambda$ and not from y.

Strong Mixing. Strong mixing properties of measures are naturally expressed in terms of truncated expectation.

A mixing condition of *strong type* for a measure μ_Λ on Ω_Λ is a relation of the form:

For every pair of *cylindrical functions* fg with supports $S_f, S_g \subset \Lambda$ there exists a constant $C_{f,g}$ such that

$$|\mu_\Lambda(f,g)| \equiv |\mu_\Lambda(fg) - \mu_\Lambda(f)\mu_\Lambda(g)| \leq C_{f,g} \exp(-\gamma \operatorname{dist}(S_f, S_g)) \tag{2.1}$$

for some $\gamma > 0$.

For example $C_{f,g}$ can be given by

$$C_{f,g} = C\|f\|\,\|g\|\,|S_f|\,|S_g|$$

with

$$\|f\| = \sup_\sigma |f(\sigma)|$$

and C independent of f, g.

A mixing condition in the present form is not particularly meaningful; the exponential function in the r.h.s. of Eq. 10 is just a way of parametrizing the dependence between the variables.

Of course (2.1) would become interesting if it was true for arbitrarily large volumes Λ with γ independent of Λ, f, g.

For instance, Λ could be a generic element of a van Hove sequence in $\mathbf{Z^d}$. In this last case a condition like the one given in (2.1), uniform in Λ, is of course at least as strong as the corresponding infinite volume analogue.

– We say that a *strong mixing* condition in the sense of *truncated* expectations holds for the measure μ_Λ on Ω_Λ, with constants D, C, γ if for every cylindrical functions f, g with S_f, $S_g \subset \Lambda$, diam S_f, diam $S_g \leq D$,

$$|\mu_\Lambda(f, g)| \leq C\|f\|\,\|g\|\, e^{-\gamma\, \mathrm{dist}(S_f, S_g)}\,, \tag{2.2}$$

and we denote it by $SMT(\Lambda, D, C, \gamma)$.

– We simply say that a Gibbs measure μ_Λ on Ω_Λ satisfies a *strong mixing* conditon with constants C, γ if for every subset $\Delta \subset \Lambda$:

$$\sup_{\tau, \tau^{(y)} \in \Omega_{\Lambda^c}} \mathrm{Var}(\mu^\tau_{\Lambda,\Delta}, \mu^{\tau^{(y)}}_{\Lambda,\Delta}) \leq C e^{-\gamma\, \mathrm{dist}(\Delta, y)}\,, \tag{2.3}$$

where $\tau^{(y)}_x = \tau_x$ for $x \neq y$.

We denote this condition by $SM(\Lambda, C, \gamma)$.

It is easy to prove (see [SZ], proof of Eq. 3.4) that $SMT(\Lambda, r, C, \gamma)$ (r is the range of the interaction) implies that there exists $C' > 0$ such that $SM(\Lambda, C', \gamma)$ holds (notice that Λ and γ are unchanged).

As it has been initially discussed by Dobrushin and Pecherski [DP], and more extensively by Dobrushin and Shlosman [DS2, DS3], the assumption that $SM(\Lambda, C, \gamma)$ holds *for all (finite or infinite)* volumes Λ with uniform constants C, γ, is equivalent to many other conditions of mixing as, for instance, SMT and analyticity properties of the thermodynamical functions and correlation functions always *for all* volumes.

To partially clarify this point we give the main result of [DP].

Theorem 2.1 (Dobrushin-Pecherski [DP]). *If for some metric ϱ on S, every $\Lambda \subset \mathbf{Z^d}$, $\Delta \subset \Lambda$, $\tau, T' \in \Omega_{\Delta^c}$,*

$$\mathrm{KROV}_\varrho(\mu^\tau_{\Lambda,\Delta}, \mu^{\tau'}_{\Lambda,\Delta}) \leq \sum_{x \in \Delta} \sum_{y \in \partial^+_r \Lambda} \varphi(|x - y|) \cdot \varrho(\tau_y, \tau'_y) \tag{2.4}$$

with $\lim_{t \to \infty} \varphi(t) t^\alpha \to 0$, $\alpha > 2d$, *then it follows that there exist* $C > 0$, $\gamma > 0$ *such that* $SM(\Lambda, C, \gamma)$ *holds for every* Λ.

Dobrushin and Shlosman called *complete analytical interactions* the class of potentials whose Gibbs measures in *any finite or infinite volume* satisfies $SM(\Lambda, C, \gamma)$ and proved a result (stronger than the above quoted Theorem DP) of equivalence of $SM(\Lambda, C, \gamma), \forall \Lambda$ to some fifteen other mixing or analyticity conditions always considering all (finite or infinite) volumes with arbitrary size and shape. In their theory the arbitariness of the volumes involved seems to play a crucial role (see [DS2, DS3]).

An important concept introduced by Dobrushin and Shlosman in [DS3] is the one of *constructive condition.* Namely in suitable circumstances supposing only that a condition like $SM(\Lambda, C, \gamma)$ is true for a suitable *finite* family of regions Λ is sufficient to guarantee that the same condition holds *for all* (finite or infinite) Λ that is it implies *complete analyticity*.

More generally, it is natural to introduce the notion of *effectiveness*:

– Given two families Γ, Γ' of subsets of $\mathbf{Z^d}$ a strong mixing condition $SM(\,.\,, C, \gamma)$ is called (Γ, Γ')-*effective* if, supposing that $SM(\Lambda, C, \gamma)$ holds for any Λ in the class Γ, we have that there exist C', γ' such that $SM(\Lambda', C', \gamma')$ holds for every Λ' in Γ'.

Of course the interesting cases correspond to a *finite* family Γ and an *infinite* Γ' (finite size conditon for exponential decay of truncated correlations on arbitrarily large volumes).

Weak Mixing. We want now to give an interesting notion of *weak mixing*.

– We say that a Gibbs measure μ_Λ^τ satisfies a weak mixing condition with constants C, γ if for every subset $\Delta \subset \Lambda$

$$\sup_{\tau,\tau' \in \Omega_{\Lambda^c}} \operatorname{Var}(\mu_{\Lambda,\Delta}^{\tau}, \mu_{\Lambda,\Delta}^{\tau'}) \le C \sum_{x \in \Delta, y \in \partial_r^+ \Lambda} \exp(-\gamma |x-y|)\,. \tag{2.5}$$

We denote this condition by $WM(\Lambda, C, \gamma)$.

Condition (2.5) implies:

$$\sup_{\tau,\tau' \in \Omega_{\Lambda^c}} \operatorname{Var}(\mu_{\Lambda,x}^{\tau}, \mu_{\Lambda,x}^{\tau'}) \le C' \sum_{y \in \partial_r^+ \Lambda} \exp(-\gamma' |x-y|)\,. \tag{2.6}$$

$\forall x \in \Lambda$ for suitable $C' > 0$, $\gamma' > 0$.

It is easy to see that in the attractive (ferromagnetic) case (2.5) and (2.6) are equivalent. A similar weak mixing condition, that we call $WM_\varrho(\Lambda, C, \gamma)$, is given in the following way: Suppose $\varrho(\cdot\,,\cdot)$ is a metric on the single spin space S.

Given $\Lambda \subset\subset \mathbf{Z^d}$, let $\varrho_\Lambda(\cdot\,,\cdot)$ be the metric on Ω_Λ given by

$$\varrho_\Lambda(\sigma_\Lambda, \sigma'_\Lambda) = \sum_{x \in \Lambda} \varrho(\sigma_x, \sigma'_x)\,. \tag{2.7}$$

We say that $WM_\varrho(\Lambda, C, \gamma)$ holds if $\forall \Delta \subset \Lambda$

$$\sup_{\tau,\tau' \in \Omega_{\Lambda^c}} \operatorname{KROV}_{\varrho_\Lambda}(\mu_{\Lambda,\Delta}^{\tau}, \mu_{\Lambda,\Delta}^{\tau'}) \le \sum_{x \in \partial_r^+ \Lambda} \exp(-\gamma \operatorname{dist}(x, \Delta))\,. \tag{2.8}$$

It is immediate to see that, when ϱ is given by (1.8) $WM_\varrho(\Lambda, C, \gamma)$ implies the validity of the bound given by (2.5) with *the same* constant γ (but with a different C) so that the validity of $WM_\varrho(\Lambda, C, \gamma)$ implies, in that case, that: $\exists C' : WM(\Lambda, C', \gamma)$ is satisfied. It is immediate to see that $SM(\Lambda, C, \gamma)$ implies $WM(\Lambda, C, \gamma)$. The converse is not true. There exist potentials, the so-called Czech potentials, (see [DS1, Sh]) which satisfy $WM(\Lambda, C, \gamma)$ but do not satisfy $SM(\Lambda, C, \gamma)$, uniformly on Λ for any $C > 0$, $\gamma > 0$.

These models, that in Dobrushin-Shlosman's language are not completely analytical, exhibit a sort of boundary phase transition even though the phase in the bulk is unique.

It is expected that also for the standard Ising model for $d \ge 3$ at very low temperature and for special values of the magnetic field (depending on the temperature) some "layering phase transition" involving long range order along the boundary takes place. This analysis is due to Basuev [DS1].

Nothing similar is expected in $d = 2$ since, in that case, the boundary is one-dimensional [MOSh].

In the sequel, while analyzing the concept of complete analyticity in the Dobrushin-Shlosman's sense we will exhibit some counterexamples, involving "pathological" shapes, violating complete analyticity, namely the validity of $SM(\Lambda, C, \gamma)$ *for every* Λ. These models, however, satisfy as we will see, some weaker form of strong mixing involving only sufficiently regular shapes (see below).

We can say that the way in which the Czech models or the Ising model in the Basuev situation violate complete analyticity is more "intrinsic" and it is related to a real phase transition that, however, is not detected inside the bulk.

In a paper in preparation, [MOSh], the authors of the present paper, in collaboration with Schonmann, analyze the relations between strong and weak mixing conditions and show that in two dimensions, given $C > 0$, $\gamma > 0$ if $WM(\Lambda, C, \gamma)$ holds for a sufficiently large square then $SM(\Lambda, C', \gamma')$ for some $C' > 0$, $\gamma' > 0$ holds *for all* squares.

We want to notice, at this point, that $WM(\Lambda, C, \gamma)$ implies not only uniqueness of limiting Gibbs measure but also decay of infinite volume correlations (see [DS1]). However, the example of Czech models shows that the notion of exponential decay of finite volume correlations (uniformly in the volume) namely, for instance, the validity of $SM(\Lambda, C, \gamma)$ for some fixed $C > 0$, $\gamma > 0$ and *for any cube* Λ is *strictly stronger* than the corresponding infinite volume property.

Finally we remark that the above definitions can be extended to the case of non-Gibbsian measures for which there is natural notion of imposing boundary conditions outside Λ (see Sect. 3).

Review of Known Results: the Gibbs State. Let us now review some of the known results concerning finite size conditions and mixing properties of Gibbs measures. We begin with a result by Dobrushin and Shlosman concerning uniqueness of infinite volume Gibbs measures.

This result generalizes previous results by Dobrushin based on a "one point condition" on Gibbs conditional distribution (see [D2]).

First we need a definition.

– Given a metric ϱ on the single spin space S we say that condition $DSU_\varrho(\Lambda_0, \delta)$ is satisfied if:

there exists a finite set $\Lambda_0 \subset\subset \mathbf{Z^d}$, a $\delta > 0$ such that: $\forall \tau, \tau' \in \Omega^c_{\Lambda_0}$ with $\tau'_x = \tau_x \forall x \neq y$ and $\forall y \in \partial^+_r \Lambda_0$ there is a number α_y such that:

$$\sup_{\tau,\tau'} \mathrm{KROV}_{\varrho_{\Lambda_0}}(\mu^\tau_{\Lambda_0}, \mu^{\tau'}_{\Lambda_0}) \leq \alpha_y \varrho(\tau_y, \tau'_y), \tag{2.9}$$

where

$$\sum_{y \in \partial^+_r \Lambda_0} \alpha_y \leq \sigma |\Lambda_0| . \tag{2.10}$$

– We simply say that $DSU(\Lambda_0, \delta)$ is satisfied if (2.9), (2.10) hold with ϱ given by Eq. (1.8). We observe that, for this choice of ϱ, in the ferromagnetic case we can substitute, in (2.9) KROV with Var.

Theorem 2.2 (Dobrushin-Shloshman [DS3]). *Let $DSU_\varrho(\Lambda_0, \delta)$ be satisfied for some ϱ, Λ_0 and $\delta < 1$; then $\exists C > 0, \gamma > 0$ such that condition $WM_\varrho(\Lambda, C, \gamma)$ holds for every Λ.*

Notice that the result of the above theorem is valid *for every Λ* but obviously it loses interest when Λ is such that any point of Λ is near to some point of $\partial\Lambda$ (one can say that, in this case, the boundary "penetrates" inside the bulk). Examples of Λ's with this kind of shapes will be analyzed later on. One can apply Theorem 2.2 to, say van Hove sequences of regions Λ.

Remark. Theorem 2.2 implies, in particular, the uniqueness of infinite volume Gibbs measure. Then (2.9), (2.10) provide an example of finite size conditon: one supposes true some properties of *finite volume* Gibbs measure and deduces properties for *infinite volume* distributions.

Remark. One can see that $SM(\Lambda, C, \gamma)$ for *every Λ* implies $DSU(\Lambda_0, \delta)$ with $\delta < 1$ for a sufficiently large Λ_0 (depending on C, γ). It can be shown for the above mentioned Czech models that even though they satisfy $WM(\Lambda, C, \gamma)$ for all cubes Λ, they violate $DSU(\Lambda_0, \delta)$ with $\delta < 1$ for any cube Λ_0.

Theorem 2.3 (Dobrushin-Shloshman [DS3]). *There exists a function $L = L(C, \gamma)$ such that $SM(\cdot, C, \gamma)$ is (Γ, Γ')-effective with Γ given by the set of all subsets of a cube of edge $L(C, \gamma)$ and $\Gamma' \equiv$ the set of all (finite or infinite) subsets Λ of $\mathbf{Z^d}$.*

Remark. The above theorem requires to verify a strong mixing condition for regions of *arbitrary* shape with given maximal diameter and insures the validity of the strong mixing *for any volume* (finite or infinite). One can ask oneself whether or not it is reasonable to expect the validity of the above notion of complete analyticity in the Dobrushin-Shlosman's sense for the Ising model either in the whole pure phase region or at least when Basuev phenomena are excluded: for example for any given positive magnetic field, for all sufficiently large inverse temperature β.

The simplest example where no phase transition of any kind takes place which, however, violates complete analyticity (in its strong from) is simply given by the usual 3D Ising model with coupling constant $J = 1$ β larger than 2D critical value $\beta_c^{(2)}$ and $h = 2$.

Consider a (horizontal) squared layer of size L namely a parallelepiped ($\equiv$ *box*) Λ with dimensions $L, L, 1$ in the directions 1, 2, 3, respectively. Suppose to introduce -1 boundary conditions on the sites contiguous to Λ from direction 3 (namely the sites belonging to the $L \times L$ squared layer adjacent to Λ from above and below). The effective field inside Λ is zero and since $\beta > \beta_c^{(2)}$ the spins inside Λ are very sensible to the value of the conditioning external spins belonging to the same horizontal layer as Λ.

Certainly for these values of thermodynamical parameter both strong and weak mixing conditons are violated for these flat regions. However, as we will see later on and as it is very reasonable, one can prove strong mixing for every $h > 0$ and β sufficient large *for any (arbitrarily large) cube* and even *for a very wide class of "sufficiently fat"* regions.

Another even more interesting example has been found by Schonmann [S].

Consider a 2D ferromagnetic Ising model with nearest neighbours and next nearest neighbours interactions whose hamiltonian in the finite region Λ, with open b.c. (no interaction with the exterior), is given by

$$H = -J \sum_{x,y \in \Lambda : |x-y|=1} \sigma_x \sigma_y - K \sum_{x,y \in \Lambda : |x-y|=\sqrt{2}} \sigma_x \sigma_y - h \sum_{x \in \Lambda} \sigma_x ,$$

where $J = K = 1$; $h = 4$.

Consider the partition of $\mathbf{Z}^2$ into two sublattices $E_o \equiv \mathbf{Z}^2_{\sqrt{2},\mathrm{o}}$, $E_e \equiv \mathbf{Z}^2_{\sqrt{2},\mathrm{e}}$ of spacing $\sqrt{2}$ and directions at 45 degrees with respect to the original lattice directions

$$E_o = \{x \equiv (x_1, x_2) \in \mathbf{Z}^2 : x_1 + x_2 = \text{odd}\} ,$$
$$E_e = \{x \equiv (x_1, x_2) \in \mathbf{Z}^2 : x_1 + x_2 = \text{even}\} .$$

Consider the square Λ with (oblique) edges parallel to the axes of E_e, E_o *contained in* E_e and containing $(2L+1)^2$ points:

$$\Lambda = \{x \equiv (x_1, x_2) \in \mathbf{Z}^2 : x_1 + x_2 = \text{even}, -L \le x_1 + x_2 \le L, -L \le x_1 - x_2 \le L\} .$$

The set of sites in $\mathbf{Z}^2$ exterior to Λ but conditioning Λ, namely $\partial^+_{\sqrt{2}}\Lambda$, is given by

$$\partial^+_{\sqrt{2}}\Lambda = \partial^+_o \cup \partial^+_e , \qquad \partial^+_{o,e}\Lambda = \partial^+_{\sqrt{2}}\Lambda \cap E_{o,e} .$$

Notice that ∂^+_o "penetrates in the bulk" of Λ, whereas ∂^+_e contains just the sites of E_e adjacent (at distance $\sqrt{2}$) from the exterior to Λ.

Consider any boundary condition τ with -1 in $\partial^+_o : \tau_{\partial^+_o} = -1$.

In this way we reduce ourselves to a usual nearest neighbour Ising model in a oblique square with zero effective field and boundary condition simply given by $\tau_{\partial^+_e}$.

If β is large enough our system will be, for every L, sensitive to the boundary condition $\tau_{\partial^+_e}$ (first order phase transition) and then strong mixing condition for this particular sequence of regions Λ will certainly fail.

Other interesting examples violating, for some pathological shapes, DS complete analyticity (without exhibiting any real phase transition) are provided by van Enter, Fernandez and Sokal in the framework of their critical analysis of renormalization group transformations [EFS].

We want to stress that for these counterexamples to the DS complete analyticity it is essential to have chosen "strange" (pathological) shapes.

Again one can see that in the above examples strong mixing $SM(\Lambda, C, \gamma)$ holds true for some $C > 0$, $\gamma > 0$ *for every regular* (without holes) box in $\mathbf{Z}^2$.

In the context of studying properties of approach to equilibrium of Glauber dynamics several authors: Holley [H2], Aizenman and Holley [AH], Stroock and Zegarlinski [SZ] have considered relations between finite size conditions and different types of mixing conditions as those considered by Dobrushin and Shlosman or similar ones.

We want to quote first a result by Holley: one among many other results contained in the basic paper [H2]. Holley consideres Ising spin systems enclosed in a particular kind of regions: the boxes where:

$\Lambda \subset\subset \mathbf{Z}^\mathbf{d}$ is a *box* of it is the cartesian product of d finite intervals in $\mathbf{Z}$.

Holley introduces a finite size condition referring to a cube Λ_0, that we call condition $H(\Lambda_0, \delta)$; it can be considered as a stronger version of $DSU(\Lambda_0, \delta)$ and it is given by: for every $x \in \Lambda_0$, $y \in \partial_r^+ \Lambda_0$, there exists $\bar{\alpha}_{x,y} > 0$ such that for every box $\Lambda \subset \Lambda_0$:

$$\sup_{\tau,\tau^{(y)} \in \Omega_{\Lambda^c}} \operatorname{Var}(\mu_\Lambda^\tau, \mu_\Lambda^{\tau^{(y)}}) \leq \sum_{x \in \Lambda} \bar{\alpha}_{x,y}$$

with

$$\sum_{x \in \Lambda_0, y \in \partial_r^+ \Lambda_0} \bar{\alpha}_{x,y} \leq \delta |\Lambda_0| \, .$$

Theorem 2.4 (Holley [H2]). *Consider a ferromagnetic Ising model. Then the existence of a cube Λ_0 such that $H(\Lambda_0, \delta)$ holds with $\delta < 1$ is equivalent to the existence of $C > 0$, $\gamma > 0$ such that $SM(\Lambda, C, \gamma)$ holds for every box Λ.*

A similar statement is contained in [AH]. A generalization of Theorem 4.2 is due to Stroock and Zegarlinski and it is based on a condition that we call $SZ(\Lambda_0, \delta)$. This condition refers to an arbitrary finite subset $\Lambda_0 \subset\subset \mathbf{Z^d}$; it is exactly like $H(\Lambda_0, \delta)$ with "for every box $\Lambda \subset \Lambda_0$" replaced by "for every $\Lambda \subset \Lambda_0$."

Theorem 2.5 (Stroock, Zegarlinski [SZ]). *In the general case (potentials satisfying hypotheses* H1, H2*) the existence of a region Λ_0 such that $SZ(\Lambda_0, \delta)$ holds with $\delta < 1$ is equivalent to the existence of $C > 0$, $\gamma > 0$ such that $SM(\Lambda, C, \gamma)$ holds for every set Λ.*

Remark. Condition $SZ(\Lambda_0, \delta)$, $\delta < 1$, is called in [SZ] condition $DSM(\Lambda_0)$ and it is erroneously attributed to Dobrushin-Shlosman. We want to notice, at this point, that also Theorem 1 in [AH] (and the same in [SZ]), even though it is attributed to Dobrushin-Shloshman, differs both in the hypothesis and in the thesis from the analogous Theorem 2.2 (see Theorem 3.1 of [DS1]). The difference in the hypothesis is the use of Var (in Theorem 1 of [AH]) instead of KROV (in Theorem 2.2). The difference in the thesis is in a prefactor, corresponding to the boundary of the volume, in front of the exponential (in Theorem 1 of [AH]) and absent in Theorem 2.2 We refer to [AH] and [DS1] for more details.

The proof of Theorems 2.4 and 2.5 uses dynamical arguments similar, in spirit, to the "surgery" methods of [DS1, DS2, DS3], which are based on subsequent local modifications of joint representations of Gibbs measures in a big volume Λ.

It provides a very simple way to deduce $SM(\Lambda, C, \gamma)$ for every Λ, starting from a finite size condition, $SZ(\Lambda_0, \delta)$, $\delta < 1$, that is easily seen to be implied by the validity of $SM(\Lambda, C, \gamma)$ for every $\Lambda \subset \Lambda_0$ for a sufficiently large cube Λ_0. Thus it provides an alternative proof of Theorem 2.3.

Holley's argument of proof takes into account all the translates $\Lambda_0(x)$ of the basic cube Λ_0 (of edge L) in Λ; here x is a vector in $\mathbf{Z^d}$ not necessarily of the form: $x = Ly$, $y \in \mathbf{Z^d}$. In this case sometimes it happens that $\Lambda_0(x) \cap \Lambda$ is not a cube but, rather, a box and this leads to the consideration of properties of a Gibbs measure in an arbitrary box subset of Λ_0.

Now we want to introduce a last condition that we call $K(\Lambda_0, \delta)$, very similar to $SZ(\Lambda_0, \delta)$ (and also to $H(\Lambda_0, \delta)$). It is exactly $SZ(\Lambda_0, \delta)$ with the substitution of Var with KROV:

– Condition $K(\Lambda_0, \delta)$: for $\Lambda \subset\subset \mathbf{Z^d}$ let ϱ_Λ be given by (2.7) where, for simplicity, we choose ϱ as in (1.8). Then for every $x \in \Lambda_0$, $y \in \partial_r^+ \Lambda_0$, there exists $\bar{\alpha}_{x,y} > 0$

such that for every $\Lambda \subset \Lambda_0$:

$$\sup_{\tau,\tau^{(y)} \in \Omega_{\Lambda^c}} \mathrm{KROV}_{\varrho_\Lambda}(m_\Lambda^\tau, \mu_\Lambda^{\tau^{(y)}}) \le \sum_{x\in\Lambda} \bar{\alpha}_{x,y}$$

with

$$\sum_{x\in\Lambda_0, y\in\partial_r^+\Lambda_0} \bar{\alpha}_{x,y} \le \delta|\Lambda_0|\,.$$

Then we have:

Theorem 2.5′. *In the general case (potentials satisfying hypotheses* H1, H2*) the existence of a region Λ_0 such that $K(\Lambda_0,\delta)$ holds with $\delta < 1$ is equivalent to the existence of $C > 0$, $\gamma > 0$ such that $SM(\Lambda, C, \gamma)$ holds for every set Λ.*

Theorem 2.5′ is very similar to Theorem 2.5 and it can also be considered as another proof of Theorem 2.3.

In Appendix 3 we give a proof of Theorem 2.5′ which hopefully will shed some light on why hypothesis $K(\Lambda_0,\delta)$ with $\delta < 1$, or the similar conditions $H(\Lambda_0,\delta)$ and $SZ(\Lambda_0,\delta)$, are so natural within the Dobrushin-Shlosman' approach to complete analyticity. We will show that Theorem 2.5′, and in a proper sense also Theorem 2.4 and 2.5, can be reduced to a corollary of Theorem 2.2: one finds in fact that $K(\Lambda,\delta)$ is the correct strengthening of $DSU(\Lambda,\delta)$ that is needed in order to show SM instead of WM. The price to pay, in this way, is to consider regions with arbitrary shape.

In the work of Olivieri [O] and Olivieri and Picco, [OP], an approach to the same problem, substantially different with respect to the one of Dobrushin and Shlosman, was developed; it uses a block decimation procedure and the theory of cluster expansion; it can be considered as the analogue, for a suitable class of regular domains, of the DS theory of complete analytical interacitons (that, we repeat, is intrinsically formulated in terms of arbitrary shapes).

Here (see Appendix 2) we propose a further simplification of the assumptions and statements of [O, OP]. In this formulation it is sufficient to assume Strong Mixing *only* for a suitable *cube* in order to ensure the same property for *any multiple* of this cube. Let us give the corresponding definitions.

Given the odd integer L let

$$Q_L(x) = \left\{ y \in \mathbf{Z^d};\, |x_i - y_i| \le \frac{L-1}{2},\, i = 1, \ldots, d \right\}$$

be the cube of edge L centered at x.

We say that Λ is a *multiple* of the cube $\Lambda_0 = Q_L(0)$ if it is a union of translated cubes $Q_L(x)$ with disjoint interior:

$$\Lambda = \bigcup_{y\in Y} Q_L(Ly)$$

for some $Y \subset \mathbf{Z^d}$.

Theorem 2.6 (Olivieri, Picco [O], [OP]). *In the general case (hypotheses* H1, H2 *satisfied) $\exists L = L(C,\gamma)$ such that $SM(\cdot, C, \gamma)$ is (Γ,Γ')-effective where Γ consists just in the cube $\Lambda_0 = Q_L(0)$ and Γ' is the class of all multiples Λ_0.*

A proof of the theorem in this form (a corollary of Propositions 2.5.1–2.5.4 of [OP]) can be found in Appendix 2. For an alternative dynamical proof see Sect. 4.

Remark. It is easy to see that Γ' can be extended to contain all properly defined "sufficiently fat" regions.

Remarks. The approach in [OP] makes use of a somehow complicated geometrical construction and of a suitable polymer expansion; it proves not only effectiveness but also analyticity properties (similar to the ones proved by DS in the case of their completely analytical interactions) by expressing any quantity of interest, referring to an arbitrary volume Λ multiple of Λ_0, in terms of a series expansion which is convergent by virtue of the assumed finite size condition on Λ_0. It is remarkable that the proof of the effectiveness alone can be given by avoiding this complicated approach and relying only on simple dynamical arguments.

The [OP] theory, by omitting the consideration of arbitrary regions (practically excluding *only* pathological shapes), can be successfully applied near to the coexisting line corresponding to a first order phase transition where the previous DS theory failed. In particular for the Schonmann's example one can immediately show complete analyticity in the above (weaker) sense (other examples will be discussed in Sect. 5).

Review of Known Results: the Dynamics. In what follows we define various different notions of exponential convergence to equilibrium for the stochastic spin dynamics defined in Sect. 1. As we have already explained in the introduction one has to carefully distinguish among the various notions if one wants to derive results in a region of the phase diagram very close to a phase transition line. In what follows we will assume that there exists a unique invariant measure μ.

1) Exponential convergence in L^2 for the infinite volume dynamics. We denote it by

$$EC\,, \qquad L^2(d\mu)\,, \qquad \mathbf{Z^d}\,.$$

It means that there exists $\gamma > 0$ such that $\forall f \in L^2(d\mu)$:

$$\|T_t f - \mu(f)\|_{L^2(\mu)} \le \|f - \mu(f)\|_{L^2(\mu)} e^{-\gamma t}\,.$$

2) Uniform (L^∞) exponential convergence for infinite volume dynamics, denoted by

$$UEC\,, \qquad \mathbf{Z^d}\,.$$

It means:

$$\exists \gamma > 0 : \forall f \in D\, \exists C_f > 0 : \|T_t f - \mu(f)\|_u \le C_f e^{-\gamma T}\,,$$

namely

$$\sup_\sigma |E_\sigma f(\sigma_t) - \mu(f)| \le C_f e^{-\gamma t}\,.$$

3) Exponential convergence in L^2 for finite volume dynamics in Λ uniformly in $\Lambda \in \Gamma$ and in the b.c. τ namely:

$$\exists \gamma > 0 : \forall \Lambda \in \Gamma, \forall \tau \in \Omega^c_\Lambda, \forall f \in L^2(d\mu^\tau_\Lambda) : \|T^{\Lambda,\tau}_t f - \mu^\tau_\Lambda(f)\|_{L^2(\mu^\tau_\Lambda)} \le \|f - \mu^\tau_\Lambda(f)\|_{L^2(\mu^\tau_\Lambda)} e^{-\gamma t}\,.$$

We denote it by

$$EC\,, \qquad L^2(\mu^\tau_\Lambda)\,\forall \Lambda \in \Gamma\,.$$

4) Uniform exponential convergence for finite volume dynamics in Λ uniformly in Λ varying in a class Γ and in the b.c. τ; namely:

$$\exists \gamma > 0 : \forall f \in D(\Lambda)\, \exists C_f > 0 : \sup_{\tau \in \Omega_{\Lambda^c}} \|T^{\Lambda,\tau}_t f - \mu^\tau_\Lambda(f)\|_u \le C_f e^{-\gamma t}\,.$$

We denote it by

$$UEC\,, \quad \forall \Lambda \in \Gamma\,.$$

Many authors and in particular Holley investigated the relationship between the above (and other) notions of convergence; on the other hand, for the case of dynamics reversible with respect to Gibbs measures like Stochastic Ising Models, they studied the relations between the speed of approach to equilibrium and mixing properties of invariant Gibbs measure.

In particular the problem of deducing exponential approach to equilibrium (in the different above senses) from *finite size* condition involving properties of finite volume Gibbs measure has been recently the object of many studies.

In this context we mention the following theorems:

Theorem 2.7 (Holley [H2]). *In the attractive case, suppose that there exists a cube Λ_0 such that $H(\Lambda_0, \delta)$, with $\delta < 1$, holds; then $UEC, \mathbf{Z^d}$ holds; moreover $EC, L^2(\mu_\Lambda^\tau)$ holds for every box Λ.*

Notice that, as previously remarked, the hypotheses of Theorem 2.7 do not apply to situations (like the previously discussed 3-D Ising system with $h = 2J$) for which, however, the thesis is certainly expected to be true provided that we replace *for every box Λ* with *for every cube Λ*.

Theorem 2.8 (Aizenman, Holley [AH]). *In the general, not necessarily attractive, case if there is a cube Λ_0 such that $DSU(\Lambda_0, \delta)$ is satisfied with $\delta < 1$, then $EC, L^2(d\mu), \mathbf{Z^d}$ holds.*

Finally we want to quote the following theorem, due to Stroock and Zegarlinski, obtained in the framework of the theory making use of logarithmic Sobolev inequalities.

Theorem 2.9 (Stroock, Zegarlinski [SZ]). *In the general case the following statements are equivalent:*
 i) *There exists a finite region Λ_0 such that $SZ(\Lambda_0, \delta)$ holds with $\delta < 1$.*
 ii) *UEC for every Λ holds.*
 iii) *EC, $L^2(d\mu_\Lambda^\tau)$ for every Λ holds.*

Notice that, by Theorem 2.5, points i), ii), iii) of Theorem 2.9 are also equivalent to the existence of $C > 0$, $\gamma > 0$ such that $SM(\Lambda, C, \gamma)$ holds *for every set Λ*.

Following the previously developed critical analysis it is reasonable to try to prove a theorem being the analogue of Theorem 2.9 for some class of sufficiently regular regions. In particular, giving up with the consideration of *every* shape, one would like to substitute point i) of Theorem 2.9 with a finite size condition referring *only* to a cube (for example $SM(Q_L, C, \gamma)$ for L chosen sufficiently large in terms of (C, γ) and, moreover, to substitute "for every set" in Theorem 2.5 and in ii), iii) of Theorem 2.9 with: "for every multiple of Q_L."

Finally, in analogy with the case of equilibrium statistical mechanics, it is reasonable to expect that $UEC\,\forall \Lambda$ or even $UEC\,\forall \Lambda \in \Gamma$ with $\Gamma \equiv$ class of regular domains (for example van Hove) is a strictly stronger notion than $UEC\,\mathbf{Z^d}$. Thus it is conceivable to look for some theorem stronger than, for example, Theorem 2.7, and such that the statement: "validity of $UEC\,\mathbf{Z^d}$" follows *only* from some hyothesis *strictly weaker* than $H(\Lambda, \delta)$, $\delta < 1$: this hypothesis should not imply $EC\; L^2(d\mu_\Lambda^\tau)\;\forall$ box Λ otherwise $H(\Lambda, \delta)$, $\delta < 1$ would follow via Lemma 3.1 of [SZ].

In the present paper and in [MO] we develop the above sketched program by positively answering the above quoted open questions.

3. Exponential Convergence to Equilibrium Under a Weak Mixing Condition

In this section we state and prove the first one of our main results, namely that for an attractive spin system, not necessarily reversible with respect to a Gibbs measure, a *weak mixing* condition on the invariant measure of the dynamics implies exponential convergence in a *strong sense* for the dynamics in the whole lattice $\mathbf{Z}^{\mathbf{d}}$ to its equilibrium measure.

The infinitesimal generator L of our spin dynamics on Ω is given by (1.10) and the hypotheses on the jump rates are those discussed in Sect. 1: H3, H4, H5, H6, namely finite range, positivity, translation invariance and attractivity. However we *do not* require the detailed balance condition with respect to some Gibbs measure (1.11). For notation convenience we will denote by μ_Λ^+ and μ_Λ^- the invariant measures obtained using as boundary conditions the extreme configurations identically equal to the maximum value N and to the minimal value 1 of the spins, respectively; moreover any given realization of our Markov process on Ω at time t will always be denoted by σ_t independently of the starting point. The latter will always be specified in the expectation value of the observables over σ_t, e.g. $E_\xi f(\sigma_t)$ if the starting point was the configuration ξ. We will also denote the average of an arbitrary function f with respect to a measure ν on Ω or Ω_Λ by $\nu(f)$.

It is easy to check, using attractivity, that if f is an increasing cylindrical function with support inside the finite set Λ, then the following inequalities holds:

a) If $\tau \le \tau'$, then $T_t^{\Lambda,\tau}(f) \le T_t^{\Lambda,\tau'}(f)$.

b) If $\Lambda \subset \Lambda'$ and if $\tau(x) = N$ for all x in $\Lambda'\backslash\Lambda$, then $T_t^{\Lambda',\tau}(f) \le T_t^{\Lambda,\tau}(f)$.

c) $T_t^{\Lambda,+}f(\xi) \ge T_t f(\xi)$ if f is an increasing function.

Remark. Clearly by taking the limit as $t \to \infty$ analogous inequalities hold for the invariant measures. The invariant measure in finite volume is unique because of the positivity of the jump rates.

We formulate now a condition on the finite volume invariant measure which ensures the ergodicity of the infinite volume Markov process and the exponential convergence of its distribution at time t to the unique invariant measure as $t \to \infty$. Such a condition, in analogy to the weak mixing condition for Gibbs states, will also be called *weak mixing*.

We recall that $Q_L(x)$ is the cube of side L, L odd, centered at x; we will write Q_L for $Q_L(0)$.

Weak Mixing. There exist two positive constants C and ε such that for any integer L,

$$\mu_{Q_L}^+(\sigma(0)) - \mu_{Q_L}^-(\sigma(0)) \le C\exp(-\varepsilon L) . \tag{3.1}$$

Remark. One sees immediately that the above mixing condition implies that there exists a unique invariant measure for the Markov process on Ω that will be denoted by μ.

Our main result then reads as follows:

Theorem 3.1. *The following ar equivalent:*
i) *Weak mixing.*
ii) *There exists a positive constant* m *and for any cylindrical function* f *there exists a constant* C_f *such that:*

$$\sup_{\xi} |T_t(f)(\xi) - \mu(f)| \leq C_f \exp(-mt),$$

namely UEC, $\mathbf{Z^d}$ *holds.*

Proof. i) ⇒ ii). Let us define

$$\varrho(t) = E_t(\sigma(0)) - E_-(\sigma_t(0)), \tag{3.2}$$

where $E_+()$ and $E_-()$ denote the expectations over the Markov process starting from the configurations identically equal to N and to 1 respectively. It is easy to see that if $\varrho(t)$ decays exponentially fast to zero then the theorem follows. It is an important result by Holley [H2] (see also [AH] for a different derivation) that the exponential decay of $\varrho(t)$ follows once one is able to show that $\varrho(t)$ goes to zero faster than $\frac{1}{t^d}$. In order to prove such a weaker decay of $\varrho(t)$ the main new technical tool is a recursive inequality satisfied by $\varrho(t)$ that for convenience we state as a proposition:

Proposition 3.1. *Under the hypotheses of Theorem* 3.3 *there exist two finite positive constants* C *and* ε *such that for any integer* L*:*

$$\varrho(2t) \leq 2(L)^d \varrho(t)^2 + 2C \exp(-\varepsilon L).$$

Proof. We write $\varrho(2t)$as:

$$\varrho(2t) = \int d\mu(z) [E_+(\sigma_{2t}(0)) - E_z(\sigma_{2t}(0))] + \int d\mu(z) [E_z(\sigma_{2t}(0)) - E_-(\sigma_{2t}(0))], \tag{3.3}$$

and we show that each one of the two integrals is bounded by a half of the r.h.s. of the recursive inequality.

Because of the attractivity assumption the distribution of the process at time t starting from the "+" configuration is stochastically larger than the one starting from a generic configuration z. Therefore, using the results of [H1], there exists a joint representation $\nu_t^{+,z}$ of the two distributions $E_+(\cdot)$ and $E_z(\cdot)$ which is above the diagonal, i.e. $\nu_t^{+,z}((\xi,\eta){:}\,\xi \geq \eta) = 1$. In what follows ξ and η represent the evoluted at time t of the configurations + and z respectively. Let now χ_L be the characteristic function of the event that $\xi(j) = \eta(j)\ \forall j \in Q_L$. Then, using the Markov property, we can write:

$$\begin{aligned}\int d\mu(z) &[E_+(\sigma_{2t}(0)) - E_z(\sigma_{2t}(0))] \\ &= \int d\mu(z) \int d\nu_t^{+,z}(\xi,\eta) \chi_L [E_\xi(\sigma_t(0)) - E_\eta(\sigma_t(0))] \\ &\quad + \int d\mu(z) \int d\nu_t^{+,z}(\xi,\eta)(1-\chi_L)[E_\xi(\sigma_t(0)) - E_\eta(\sigma_t(0))].\end{aligned} \tag{3.4}$$

Again by using attractivity and translation invariance, the second term in the r.h.s. of (3.4) can be bounded by:

$$(L)^d\,\varrho(t)\int d\mu(z)\,\nu_t^{+,z}(\xi(0)\neq\eta(0))\leq (L)^d\,\varrho(t)^2\,. \tag{3.5}$$

If we now denote by τ the common projection in Q_L of the configurations ξ and η and we denote by $\hat{\chi}_{L,\tau}$ the characteristic function of the event:

$$\xi(j)=\eta(j)=\tau(j)\,\forall j\in Q_L\,,$$

then χ_L is equal to

$$\chi_L=\sum_{\tau\in\Omega_{Q_L}}\hat{\chi}_{L,\tau}\,,$$

and therefore the first term in the r.h.s. of (3.4) can be written as:

$$\int d\mu(z)\sum_{\tau\in\Omega_{Q_L}}\int d\nu_t^{+,z}(\xi,\eta)\,\hat{\chi}_{\lambda,\tau}[E_\xi(\sigma_t(0))-E_\eta(\sigma_t(0))]\,. \tag{3.6}$$

Attractivity allows us to bound the quantity $[E_\xi(\sigma_t(0))-E_\eta(\sigma_t(0))]$ by imposing extra "+" and "−" boundary conditions outside the cube Q_L. More precisely:

$$E_\xi(\sigma_t(0))-E_\eta(\sigma_t(0))\leq E_\xi^{Q_L,+}(\sigma_t(0))-E_\eta^{Q_L,-}(\sigma_t(0))\,, \tag{3.7}$$

where in general $E_\xi^{Q_L,\zeta}(\,)$ denotes the expectation over the process starting from the configuration ξ and evolving in the box Q_L with jump rates $c_x^{\zeta,\Lambda}(\xi,a)$. Thus (3.6) is bounded above by:

$$\begin{aligned}
&\int d\mu(z)\sum_{\tau\in\Omega_{Q_L}}\int d\nu_t^{+,t}(\xi,\eta)\,\hat{\chi}_{L,\tau}[E_\tau^{Q_L,+}(\sigma_t(0))-E_\tau^{Q_L,-}(\sigma_t(0))]\\
&\quad\leq\int d\mu(z)\sum_{\tau\in\Omega_{Q_L}}\nu_t^{+,z}(\eta(j))=\tau(j)\forall j\in Q_L)\,[E_\tau^{Q_L,+}(\sigma_t(0))-E_\tau^{Q_L,-}(\sigma_t(0))]\\
&\quad=\int d\mu(z)\,E_z(E_{z_t}^{Q_L,+}(\sigma_t(0)))-\int d\mu(z)\,E_z(E_{z_t}^{Q_L,-}(\sigma_t(0)))
\end{aligned} \tag{3.8}$$

where, by an abuse of notation, z_t is the value of the process at time t starting from the configuration z. Since $E_z^{Q_L,+}(\sigma_t(0))$ is increasing in z (because of attractivity) $E_z(E_{z_t}^{Q_L,+}(\sigma_t(0)))$ is smaller than $E_z^{Q_L,+}(E_{z_t}^{Q_L,+}(\sigma_t(0)))$. Analogously $E_z(E_{z_t}^{Q_L,-}(\sigma_t(0)))$ is larger than $E_z^{Q_L,-}(E_{z_t}^{Q_L,-}(\sigma_t(0)))$. Thus

$$\begin{aligned}
\int d\mu(z)\,E_z(E_{z_t}^{Q_L,+}(\sigma_t(0)))&\leq\int d\mu(z)\,E_z^{Q_L,+}(E_{z_t}^{Q_L,+}(\sigma_t(0)))\\
&\leq\int d\mu_{Q_L}^{+}(z)\,E_z^{Q_L,+}(E_{z_t}^{Q_L,+}(\sigma_t(0)))=\mu_{Q_L}^{+}(\xi(0))
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
\int d\mu(z)\,E_z(E_{z_t}^{Q_L,-}(\sigma_t(0)))&\geq\int d\mu(z)\,E_z^{Q_L,-}(E_{z_t}^{Q_L,-}(\sigma_t(0)))\\
&\leq\int d\mu_{Q_L}^{-}(z)\,E_z^{Q_L,-}(E_{z_t}^{Q_L,-}(\sigma_t(0)))=\mu_{Q_L}^{-}(\xi(0))\,.
\end{aligned} \tag{3.10}$$

In order to derive the last two equalities we used the fact that $\mu^+_{Q_L}$ is the invariant measure of the process in Q_L with "+" boundary conditions and analogously for $\mu^-_{Q_L}$. Thus the r.h.s. of (3.8) is bounded from above by:

$$|\mu^+_{Q_L}(\xi(0)) - \mu^-_{Q_L}(\sigma(0))| \le C\exp(-\varepsilon L) \tag{3.11}$$

because of the weak mixing assumption.

Exactly the same steps show that also the second term in the r.h.s. of (3.8) is bounded from above by (3.11). Thus combining together (3.8) and (3.5) we get the proposition.

The main idea at this stage is to use the recursive inequality as a tool to transform a very rough and weak decay in time of $\varrho(t)$ of the form:

$$\varrho(t) \le \exp(-\gamma(\log(t))^{1/d}) \tag{3.12}$$

into a much better decay of the form:

$$\varrho(t) \le \exp(-\exp(+\gamma(\log(t))^{1/d}))\,. \tag{3.13}$$

Once the above bound is established, then one has that $\varrho(t)$ decays for large times faster than the inverse of any power of t and therefore, thanks to Holley's theorem (see Theorem 0.1 of [H2]), $\varrho(t)$ has to decay exponentially fast.

Let us first prove the rough bound (3.12).

Proposition 3.2. *There exists a finite time t_0 and a positive constant γ such that:*

$$\varrho(t) \le \exp(-\gamma(\log(t))^{1/d})$$

for all t greater than t_0.

Proof. Using the attractivity of the dynamics, we have that for any cube Q_L:

$$\varrho(t) \le E^{Q_L,+}_{+}(\sigma_t(0)) - E^{Q_L,-}_{-}(\sigma_t(0))\,. \tag{3.14}$$

By adding and subtracting $\mu^+_{Q_L}(\xi(0)) + \mu^-_{Q_L}(\xi(0))$ the r.h.s. of (3.14) becomes equal to:

$$\begin{aligned} &E^{Q_L,+}_{+}(\sigma_t(0)) - \mu^+_{Q_L}(\xi(0)) + \mu^-_{Q_L}(\xi(0)) \\ &\quad - E^{Q_L,-}_{-}(\sigma_t(0)) + \mu^+_{Q_L}(\xi(0)) - \mu^-_{Q_L}(\xi(0))\,. \end{aligned} \tag{3.15}$$

The *weak mixing* condition implies that the third term in (3.,15) is bounded from above by

$$C\exp(-\varepsilon L)\,. \tag{3.16}$$

The estimate of the first and of the second term is identical and one gets:

$$E^{Q_L,+}_{+}(\sigma_t(0)) - \mu^+_{Q_L}(\xi(0)) \le C\exp(-\varepsilon L) \tag{3.17}$$

provided that

$$t \ge \exp(c_0 L^d)\,, \tag{3.18}$$

where c_0 is a suitable positive constant depending only on the jump rates. The above one is a very poor estimate which uses only the fact that the jump rates are uniformly positive. This fact implies that starting from an arbitrary pair of configurations ξ and η and coupling them together with e.g. the basic coupling (see [L]) there is a positive probability, at most exponentially small in the volume of Q_L, that at time $t = 1$ they have become identical in the cube Q_L. This fact immediately implies the above rough bound on the first and second term of (3.15).

We now choose the side L of the cube as $L = L(t) = 2\left[\left\{\frac{1}{4c_0}\log(t)\right\}^{1/d}\right]$. With this choice we have that $t \geq \exp(c_0|\Lambda|)$ and thus:

$$\varrho(t) \leq 3C\exp(-\varepsilon L(t)). \tag{3.19}$$

The proposition is proved.

We now use Proposition 3.1 to transform the weak decay of $\varrho(t)$ given by (3.19) into a fast decay. The key point is the following lemma:

Lemma 3.1. *Let $R(t)$ be a positive increasing function of t tending to plus infinity as $t \to \infty$ and such that for some $B < 2$: $R(2t) \leq BR(t)$ for all sufficiently large times t. Then there exists a finite time t_0 and a positive constant A such that if for some time $t_1 \geq t_0$ it happens that*

$$\varrho(t_1) \leq AR(t_1)^{-d},$$

then there exists a time $t_2 \geq t_1$ such that

$$\varrho(t_2) \leq \exp\left(-\frac{\varepsilon}{4}R(t_2)\right)$$

where ε is the constant appearing in Proposition 3.1.

Proof. Let us choose t_0 be so large that for all $t \geq t_0$ the following conditions are satisfied:

i) $[12R(t)]^d 2C\exp(-\varepsilon R(t)) \leq \exp\left(-\frac{\varepsilon}{2}R(t)\right)$;
ii) $R(t) > 1$;
iii) (3.19) holds;
iv) $R(2t) \leq BR(t)$,

where C appears in Proposition 3.1.

We then set $x(t) = (6R(t))^d\varrho(t)$. Then, using Proposition 3.1, the assumption on $R(t)$ and the definiton of t_0, we have:

$$x(2t) \leq x(t)^2 + \exp\left(-\frac{\varepsilon}{2}R(t)\right). \tag{3.20}$$

Let us now take the constant A of the lemma equal to $\frac{1}{3(6^d)}$. Then by hypothesis there exists a time $t_1 \geq t_0$ such that $x(t_1) \leq \frac{1}{3}$. Let $x_n = x(2^n t_1)$; we will show that, by assuming

$$x(2^n t_1) \geq \exp\left(-\frac{\varepsilon}{4}R(2^n t_1)\right)\ \forall n \tag{3.21}$$

we would get a contradiction. For, from (3.20) we get

$$x_{n+1} \leq 2x_n^2 \tag{3.22}$$

which implies

$$x_n \leq \tfrac{1}{2}(2x_0)^{2^n} \leq \left(\tfrac{2}{3}\right)^{2^n}. \tag{3.23}$$

On the other hand the assumption $R(2t) \leq BR(t)$ with B less than 2 implies that

$$\exp\left(-\frac{\varepsilon}{4}R(2^n t_1)\right) \geq \exp\left(-\frac{\varepsilon}{4}B^n R(t_1)\right), \tag{3.24}$$

which clearly contradicts (3.21). Thus there exists n_0 such that

$$x(2^{n_0}t_1) \le \exp\left(-\frac{\varepsilon}{4} R(2^{n_0}t_1)\right).$$

We then take $t_2 = 2^{n_0}t_1$. This lemma is proved.

We can finally conclude the proof of the theorem. Let

$$R(t) = \exp\left(\frac{\gamma}{2d} (\log(t))^{1/d}\right).$$

Clearly $R(t)$ satisfy the hypotheses of Lemma 3.1. Moreover, using Proposition 3.2, for all $t_1 \ge t_0$ and sufficiently large:

$$\varrho(t_1) \le AR(t_1)^{-d},$$

where A and t_0 are the constants appearing in Lemma 3.1. Thus, thanks to the lemma, there exists a time $t_2 \ge t_1$ such that:

$$\varrho(t_2) \le \exp\left(-\frac{\varepsilon}{4} R(t_2)\right). \tag{3.25}$$

Since t_1 can be taken arbitrarily large, the above bound implies that for any finite N there exists an arbitrarily large time T such that:

$$\varrho(T) \le \frac{1}{T^N}.$$

Thanks to Holley's theorem this implies that $\varrho(t)$ decays exponentially fast in time.

i) $\Leftarrow$ ii) This was proved years ago by Holley and Stroock [HS] for the stochastic Ising model. For completeness we give the proof also in the more general case of non-reversible spin dynamics. Clearly ii) implies that the infinite volume dynamics is ergodic with a unique invariant measure μ. Thus we write:

$$\mu^+_{Q_L}(\xi(0)) - \mu^-_{Q_L}(\xi(0)) = \mu^+_{Q_L}(\xi(0)) - \mu(\xi(0)) + \mu(\xi(0)) - \mu^-_{Q_L}(\xi(0)). \tag{3.26}$$

Let us estimate $\mu^+_{Q_L}(\xi(0)) - \mu(\xi(0))$. By adding and subtracting $E_+(\sigma_t(0))$ and using the exponential convergence to equilibrium together with attractivity, we get:

$$0 \le \mu^+_{Q_L}(\xi(0)) - \mu(\xi(0)) \le C\exp(-\gamma t) + \mu^+_{Q_L}(\xi(0)) - E_+(\sigma_t(0)). \tag{3.27}$$

We now choose the time t as $t = \delta L$. Since the jump rates are finite range it easily follows (see e.g. [H2], Lemma 1.1) that if δ is small enough one has:

$$E^{\Lambda,+}_+(\sigma_t(0)) - E_+(\sigma_t(0)) \le \exp(-L).$$

Thus the r.h.s. of (3.27) can be bounded by:

$$\begin{aligned} &C\exp(-\gamma\delta L) + \mu^+_{Q_L}(\xi(0)) - E^{\Lambda,+}_+(\sigma_t(0)) + \exp(-L) \\ &\le C\exp(-\gamma\delta L) + \exp(-L), \end{aligned} \tag{3.28}$$

since by attractivity $\mu^+_{Q_L}(\xi(0)) - E^{\Lambda,+}_+(\sigma_t(0))$ is negative. In conclusion we have shown that $\mu^+_{Q_L}(\xi(0)) - \mu(\xi(0))$ is smaller than $C\exp(-\gamma\delta L) + \exp(-L)$.

The same argument applies also to the other term in the r.h.s. of (3.26) $\nu(\xi(0)) - \mu^-_{Q_L}(\xi(0))$.

The theorem is proved.

4. Exponential Convergence in Finite Volumes: The Stochastic Ising Model

We prove in this section the exponential convergence to equilibrium in finite volumes with rates that are estimated uniformly in the volume for the stochastic Ising model under a strong mixing condition on the Gibbs state.

The Hamiltonian H^{τ}_{Λ} of our spin system satisfies hypotheses **H1** and **H2** of Sect. 1, but for simplicity we assume that the spins can take only the two values $+1$ or -1. If there exists a unique Gibbs state in the infinite volume limit $\Lambda \to \mathbf{Z^d}$ independent of the boundary conditions τ then it will be simply denoted by μ.

Later on, in order to simplify some of the proofs, we will make the assumption that the Hamiltonian is ferromagnetic or attractive (see Sect. 1); we emphasize, however, that all the results of this section can also be proved without the assumption of ferromagnetism by using the logarithmic Sobolev inequalities (see [MO]).

The stochastic dynamics that will be the object of study in this section will be one of the Glauber dynamics associated to the hamiltonian H^{τ}_{Λ} (see (1.2), (1.12)). We will need to analyze the stochastic Ising model in finite volume Λ with boundary conditions τ as well as in the whole lattice $\mathbf{Z^d}$. Both cases are defined through their jump rates. In order to simplify the exposition and the computations we decided to take from the beginning precise form for our jump rates:

$$c_x(\sigma, a) = \mu^{\sigma}_{\{x\}}(\eta(x) = a) = \frac{1}{1 + \exp\Big(- 2\beta a \sum\limits_{X; x \in X} J_X \prod\limits_{y \in X \setminus \{x\}} \sigma_y \Big)} . \tag{4.1}$$

where it is understood that if we are in a finite volume Λ the configuration σ agrees with the boundary configuration τ outside Λ. This choice corresponds to what is known as the *heat bath* dynamics.

Remark. In the finite volume case our expression for the jump rates makes sure that the Markov process generated by the jump rates on $\{-1, 1\}^{\Lambda}$ is reversible with respect to the Gibbs state μ^{τ}_{Λ}. This means that the generator of the process L becomes a non-positive selfadjoint operator in the Hilbert space $L^2(\Omega_{\Lambda}, d\mu^{\tau}_{\Lambda})$ and that μ^{τ}_{Λ} is the unique invariant measure of the process. This important fact holds also in the infinite volume limit if the Gibbs state is unique (see [L]). Moreover if the interaction is ferromagnetic then automatically the above defined jump rates become attractive in the sense made precise in Sect. 1.

For the reader's convenience we recall now our finite volume strong mixing condition (see Sect. 1) that in the sequel we will refer to as $SM(L_0, C, \gamma)$:

Let Λ_0 be the cube of side $2L_0 + 1$ with sides parallel to the coordinate axes and let for any $V \subset \Lambda_0 \mu^{\tau}_{\Lambda_0, V}$ be the relativization of $\mu^{\sigma}_{\Lambda_0}$ to the set V. Then for any y outside Λ_0 and any V in Λ_0 we must have:

$$\mathrm{Var}(\mu^{\tau}_{\Lambda_0, V}, \mu^{\tau^{(y)}}_{\Lambda_0, V}) \le C \exp(-\gamma\, \mathrm{dist}(y, V))\, \forall \tau \in \{-1, +1\}^{\Lambda_0^C} ,$$

Remark. It is easy to check that the above condition implies that if for two given configurations τ and τ' we denote by $V_{\tau,\tau'}$ the set $\{x \notin \Lambda_0; \tau(x) \neq \tau'(x)\}$ and by Q the maximal subset of Λ_0 which is at distance greater than $L_0^{1/2}$ from $V_{\tau,\tau'}$, then we have:

$$\mathrm{Var}(\mu^{\tau}_{\Lambda_0, Q}, \mu^{\tau'}_{\Lambda_0, Q}) \le c(r) \frac{1}{L_0^{d+2}} , \tag{4.2}$$

where $c(r)$ is a numerical constant depending only on the range r.

Obviously since our condition has to hold only in a definite geometric shape, in our case a cube, contrary to what is assumed by Aizenman and Holley or Zegarlinski and Stroock, we will prove our results only in volumes that are multiples of the elementary volume Λ_0 (see the definition before Theorem 2.6). As already discussed in the introduction this has to be the case if we want to apply our condition to a system at low temperature near a first order phase transition for which it can be proved (see Sect. 1) that the Dobrushin-Shlosman complete analyticity fails.

Let us now state our main results.

– In what follows we will call L_0-compatible any subset of the lattice $\mathbf{Z^d}$ which is the union of translates of the cube Λ_0 such that their vertices lay on the rescaled lattice $(2L_0+1)\mathbf{Z^d}$, and we will denote by gap(L^τ_Λ) the lowest positive eigenvalue of $-L^\tau_\Lambda$ in $L^2(\Omega_\Lambda, d\mu^\tau_\Lambda)$.

Remark. For simplicity the next three results are stated only for volumes Λ that are L_0-compatible. It is however relatively easy to check that once they hold for this rather restricted class of volumes, then they hold also for the larger class of sets Λ such that for any x in Λ it is possible to find a suitable translated $\Lambda_0(y)$ of the cube $\Lambda_0(y)$ entirely contained in Λ and such that $\mathrm{dist}(x, \partial\Lambda \backslash \partial\Lambda \cap \partial\Lambda_0(y)) \geq \frac{L_0}{2}$.

The next results says that $SM(L_0, C, \gamma)$ implies exponential convergence to equilibrium in any L_0-compatible finite volume in the L^2-norm.

Theorem 4.1. *Given C and γ there exists a positive constant $\bar{L}$ depending on C, γ, d and the range of the interaction such that if $SM(L_0, C, \gamma)$ holds with $L_0 \geq \bar{L}$, then there exists a positive constant m_0 such that for any L_0-compatible set Λ and for any function f in $L^2(d\mu^\tau_\Lambda)$:*

$$\|T^{\Lambda,\tau}_t(f) - \mu^\tau_\Lambda(f)\|_{L^2(d\mu^\tau_\Lambda)} \leq \|f - \mu^\tau_\Lambda(f)\|_{L^2(d\mu^\tau_\Lambda)} \exp(-m_0 t),$$

where $T^{\Lambda,\tau}_t$ denotes the Markov semigroup of the process evolving in Λ with boundary conditons τ.

Proof of Theorem 4.1. Let us fix an L_0-compatible set Λ and a boundary configuration τ and let $\{Q_i\}$ be a covering of the set Λ with the following two properties:

i) Each element of the covering is a cube of side $2L_0 + 1$ with sides parallel to the coordinate axes.

ii) If two different cubes Q_i and Q_j overlap then necessarily each one of them is translated by L_0, along at least one coordinate axis, of the other.

It is very easy to check that for any L_0-compatible set Λ such a covering always exists.

Next we introduce a new dynamics (Gibbs sampling) on $\{-1,+1\}^\Lambda$ by defining its generator L_Q as:

$$L_Q f(\sigma) = \sum_{\eta, i} c_{Q_i}(\sigma, \eta)\,(f(\eta) - f(\sigma)), \tag{4.3}$$

where the new jump rates $c_{Q_i}(\sigma, \eta)$ are a generalization of those of the heat bath dynamics and are given by:

$$c_{Q_i}(\sigma, \eta) = \mu^\sigma_{Q_i}(\eta) \tag{4.4}$$

if η agrees with σ outside the cube Q_i and zero otherwise. It is understood that outside Λ the configurations σ and η agree with τ.

Remark. The above version of Gibbs sampling is different from the one employed by Holley [H2], Aizenman and Holley [AH] and Stroock and Zegarlinski [SZ]. In these previous works the updating was as follows: each site x is chosen in $\mathbf{Z}^\mathbf{d}$ with rate one and then the configuraton in $\Lambda_0(x) \cap \Lambda$ is put equal to η with probability $\mu^\sigma_{\Lambda_0(x)\cap\Lambda}(\eta)$, where $\Lambda_0(x)$ is the cube of side $2L_0 + 1$ centered at x. This dynamics has, however, sometimes the inconvenience to update regions that are not squares Λ_0 but rather boxes (= intersection between two cubes) on which, contrary to what happen for cubes Λ_0, we have no control at all and for which our mixing condition may very well fail!

Warning: Within the present proof of Theorem 4.1, by an abuse of notation, we will continue to use the notation σ_t to denote the evolution at time t of the configuration σ according to the above defined Gibbs sampling, instead of the perhaps more precise notation $\sigma_t^{\{Q_i\}}$.

It is rather simple to show that the above Gibbs sampling is still reversible with respect to the Gibbs state in Λ with boundary conditions τ; more important: one easily proves (see Lemma 2.3 of [SZ]) that if gap(L_Q) and gap(L) denote the gap in the spectrum of the generators L_Q and L respectively, then there exists a positive constant c independent of Λ and τ such that:

$$\text{gap}(L) \geq \exp(-cL_0^d)\,\text{gap}(L_Q)\,. \tag{4.5}$$

Thus in order to prove the theorem we need only to estimate from below gap(L_Q) uniformly in Λ and τ.

For this purpose we adopt a scheme very similar to the one already used in Sect. 3 even if we are working with a very general, not necessarily ferromagnetic, system.

Given two initial configurations σ and σ' we couple their dynamics by defining the generator $\tilde{L}_Q$ of the coupled process as:

$$\tilde{L}_Q f(\sigma,\sigma') = \sum_{\eta,\eta',i} \tilde{c}_{Q_i}(\sigma,\sigma',\eta,\eta')\,(f(\eta,\eta') - f(\sigma,\sigma'))\,, \tag{4.6}$$

where the jump rates $\tilde{c}_{Q_i}(\sigma,\sigma',\eta,\eta')$ are given by:

$$\tilde{c}_{Q_i}(\sigma,\sigma',\eta,\eta') = \mu^{\sigma,\sigma'}_{Q_i}(\eta,\eta') \tag{4.7}$$

if the pair η,η' agrees with the pair σ,σ' outside the cube Q_i and zero otherwise. Here the measure $\mu^{\sigma,\sigma'}_{Q_i}$ is an element of the set K of the joint representations of the two Gibbs states $\mu^\sigma_{Q_i}$ and $\mu^{\sigma'}_{Q_i}$ and it is such that it realizes the minimum

$$\text{Var}(\mu^\sigma_{Q_i,\tilde{Q}_i}, \mu^{\sigma'}_{Q_i,\tilde{Q}_i}) = \min_{\nu\in K} \sum_{\eta,\eta'\in\Omega_{Q_i}} \nu(\eta,\eta')\,\varrho_{\tilde{Q}_i}(\eta,\eta')\,.$$

In the above formula

$$\varrho_{\tilde{Q}_i}(\eta,\eta') = 1$$

if $\eta(x) \neq \eta'(x)$ for some x in $\tilde{Q}_i$ and zero otherwise and $\tilde{Q}_i$ is the maximal subset of Q_i which is at distance greater than $L_0^{1/2}$ from the set $V = \{x \notin Q_i; \sigma(x) \neq \sigma'(x)\}$.

Remark. It is well known that in the attractive case the joint representation $\mu^{\sigma,\sigma'}_{Q_i}$ is above the diagonal, i.e. $\mu^{\sigma,\sigma'}_{Q_i}(\eta \leq \eta') = 0$ if $\eta \geq \eta'$.

A concrete way to realize the coupled process is to attach an exponential clock of parameter one to each cube Q_i; then when a clock rings, say at time t and at the cube Q_i, one updates the pair σ_t, σ'_t inside Q_i to the pair η, η' with probability $\mu_{Q_i}^{\sigma,\sigma'}(\eta,\eta')$.

Once the coupling has been established we define the quantity $\varrho_\Lambda^\tau(t)$ as:

$$\varrho_\Lambda^\tau(t) = \sup_{\sigma,\eta x \in \Lambda} P(\sigma_t(x) \neq \sigma'_t(x)) . \tag{4.8}$$

It is elementary to verify that if $\varrho_\Lambda^\tau(t)$ decays exponentially with a rate m_Q bounded away from zero, uniformly in the volume Λ and in the boundary conditions τ, then $\text{gap}(L_Q) \geq m_Q$.

In order to prove the exponential decay of $\varrho_\Lambda^\tau(t)$ we would like, at this point, to apply to $\varrho_\Lambda^\tau(t)$ the usual Holley's criterion: if there exists a large enough finite time t_0 such that $\varrho_\Lambda^\tau(t_0) \ll \frac{1}{t_0^d}$ then $\varrho_\Lambda^\tau(t)$ decays exponentially fast. The idea then is to verify the existence of the basic time scale t_0 by just using our $SM(L_0, C, \gamma)$ condition. In fact the above described coupling is such that after the updating at time t of, say, the cube Q_i, the probability to see a difference between η and η' at a site x in Q_i at a distance greater than $L_0^{1/2}$ from the set $V = \{x \notin Q_i; \sigma_t(x) \neq \sigma'_t(x)\}$ is smaller than $\frac{1}{L_0^{d+2}} \ll 1$ uniformly in the configurations σ_t and σ'_t, provided L_0 is large enough. Thus, under this coupling, two arbitrary configurations σ, σ' should become equal everywhere in Λ in a short time and, in some sense, the Gibbs sampling behaves as a high temperature, almost independent, stochastic Ising model.

In order to implement this program we first prove the Holley's recursive inequality (see [H2]) for $\varrho_\Lambda^\tau(t)$:

$$\varrho_\Lambda^\tau(2t) \leq (C(2L_0 + 1 + r)t + 1)^d \, \varrho_\Lambda^\tau(t)^2 + \exp(-\gamma t) \tag{4.9}$$

for suitable positive constants C and γ independent of t, Λ and L_0.

To prove (4.9), let x, σ and σ' be fixed, let $A(x)$ be the box of side $C(2L_0 + 1 + r)t + 1$ centered at the site x, where C is a constant to be fixed later and let $\chi_{t,x}$ be the characteristic function of the event, for the coupled process $\{\sigma_t, \sigma'_t\}$ that $\sigma_t(j) = \sigma'_t(j) \, \forall j \in A(x) \cap \Lambda$. Then we can write:

$$\begin{aligned} P(\sigma_{2t}(x) \neq \sigma'_{2t}(x)) \leq\; & E\chi(\sigma_{2t}(x) \neq \sigma'_{2t}(x))\,(1 - \chi_{(t,x)})) \\ & + E\chi(\sigma_{2t}(x) \neq \sigma'_{2t}(x))\chi_{(t,x)} . \end{aligned} \tag{4.10}$$

The first term in the r.h.s. of (4.10), using the Markov property and the definition of $\varrho(t)$, is bounded from above by:

$$(C(2L_0 + 1 + r)t + 1)^d \, \varrho_\Lambda^\tau(t)^2 . \tag{4.11}$$

In order to bound the second term we observe that the Gibbs sampling has "finite speed of propagation of information" since one single updating can influence spins in a region with diameter not larger than $r + 2L_0 + 1$. It is then easy to check (see e.g. Lemma 1.1 of [SZ]) that if the constant C is taken large enough independently of L_0 and t then there exists another constant γ, e.g. larger than one for C large enough, such that the second term is bounded by:

$$\exp(-\gamma t) .$$

Using now (4.9) it follows immediately from Lemma 2.4 of [H2] that there exist two numerical constants δ and $\bar{t}$ depending on the constant C, d and r such that, if for some time $t_0 \geq \bar{t}$,

$$(C(2L_0 + 1 + r)t_0 + 1)^d \varrho_\Lambda^\tau(t_0) \leq \delta\,, \tag{4.12}$$

then there exists a positive constant $m_Q(t_0, \delta)$ such that

$$\varrho_\Lambda^\tau(t) \leq \exp(-m_Q t)\, \forall t \geq t_0\,. \tag{4.13}$$

We finally verify the existence of such a time t_0 uniformly in Λ and in the boundary conditions τ. Let $x \in \Lambda, \sigma, \sigma'$ and t_0 be fixed, let Q_i be a cube such that $x \in Q_i$ with $\operatorname{dist}(x, \partial Q_i \backslash (\partial\Lambda \cap \partial Q_i)) \geq \frac{L_0}{2}$, and let $Q_{i_1}, \ldots, Q_{i_n}$ be the other elements of the covering which intersect Q_i. The number n is clearly dependent on the geometry of Λ but can be bounded by a constant $n(d)$ dependent only on the dimension d. Let also $\nu(Q_i, t)$ be the number of ringings of the exponential clock of parameter one attached to the cube Q_i within time t and analogously for the other cubes $Q_{i_1}, \ldots, Q_{i_n}$. Then for any integer N small than $\frac{L_0^{1/2}}{10}$ we estimate $\varrho_\Lambda^\tau(t_0)$ by:

$$\begin{aligned}\varrho_\Lambda^\tau(t_0) \leq\ & P(\nu(Q_i, t_0) = 0) \\ & + \sum_{j=1}^n P(\nu(Q_{i_j}, t_0) \geq N) + P(\nu(Q_i, t_0) \geq N) \\ & + \sum_{j=1}^n \sum_{k=1}^N P \text{ (the } k^{\text{th}} \text{ updating of the cube } Q_{i_j} \text{ was "bad")} \\ & + \sum_{k=1}^N P \text{ (the } k^{\text{th}} \text{ updating of the cube } Q_i \text{ was "bad")}\,,\end{aligned} \tag{4.14}$$

where an updating $\{\sigma, \sigma' \to \eta, \eta'\}$ of a cube Q_{i_j} is "bad" if $\eta(x) \neq \eta'(x)$ for some x in Q_{i_j} with $\operatorname{dist}(x, \{y \in Q_{i_j}^c; \sigma_j \neq \sigma_j'\} \geq L_0^{1/2})$.

Let us in fact assume that within time t_0 the cube Q_i has been updated at least once and that all the updatings within time t_0 of the cubes $Q_i, Q_{i_1}, \ldots, Q_{i_n}$ have been "good" and not more than N. Then, for L_0 large enough, since right after the last update of the cube Q_i, say at time t, there is no difference in the two configurations σ_t and σ_t' in a cube $\bar{Q}_i \subset Q_i$ of side $\frac{L_0}{4}$ containing x with $\operatorname{dist}(x, \partial\bar{Q}_i \backslash (\partial\Lambda \cap \partial\bar{Q})) \geq \frac{L_0}{8}$, and since a "good" updating of one of the neighbor cubes Q_{i_j} can only bring a difference in the two configurations σ_t and σ_t' inside $\bar{Q}_i$ at a distance from $\partial\bar{Q}_i \backslash (\partial\Lambda \cap \partial\bar{Q}_i)$ smaller than $L_0^{1/2}$, after k updatings between times t and t_0 of the cubes $Q_{i_1}, \ldots, Q_{i_n}$, we have that the two configurations σ and σ' are still equal for all $y \in Q_i$ at distance from x less than or equal to $\frac{L_0}{8} - kL_0^{1/2}$. Thus if $k \leq N \leq \frac{L_0^{1/2}}{10}$ at the final time t_0 one still has $\sigma_{t_0}'(x) = \sigma_{t_0}(x)$.

The first term in (4.14) is equal to $\exp(-t_0)$. The second and third term can also be bounded by $\exp(-t_0)$ if $N = at_0$ for a large enough but $at_0 \le \frac{L_0^{1/2}}{10}$. Finally the sum of the fourth and fifth termis bounded by:

$$(n(d)+1)N \sup_{\sigma,\sigma'} \mathrm{Var}(\mu^{\sigma}_{\Lambda_0,Q}, \mu^{\sigma'}_{\Lambda_0,Q}) \le c(r)\,(n(d)+1)N\,\frac{1}{L_0^{d+2}}\,,$$

where Q is the maximal subset of Λ_0 which is at distance from $V_{\sigma,\sigma'} = \{x \notin \Lambda_0; \sigma(x) \neq \sigma'(x)\}$ greater than $L_0^{1/2}$. Here $c(r)$ is a numerical constant depending only on the range r.

Thus, by choosing, for example, $t_0 = \frac{L_0^{\frac{1}{d+2}}}{10a}$, $N = at_0$ with a large enough, we have that the quantity $(C(2L_0+1+r)t_0+1)^d\,\varrho^{\tau}_{\Lambda}(t_0)$ is bounded above by:

$$\begin{aligned}&(C(2L_0+1+r)t_0+1)^d\,\varrho^{\tau}_{\Lambda}(t_0)\\ &\le (C(2L_0+1+r)t_0+1)^d\left\{2\exp(-t_0)+c(r)\,(n(d)+1)\,at_0\,\frac{1}{L_0^{d+2}}\right\} \le \delta\,,\end{aligned}$$

provided that L_0 is large enough (depending of C, r, δ). The theorem is proved.

Warning: From now on we go back to the usual Glauber (i.e. single spin) dynamics.

The first result that we derive from the above theorem is Theorem 2.6, namely the (Γ,Γ')-effectiveness of $SM(\cdot, C, \gamma)$ with Γ consisting only of the cube Λ_{L_0} and Γ' the family of all L_0-compatible sets of $\mathbf{Z^d}$ provided that L_0 is large enough. The proof is based on the following nice result due to Stroock and Zegarlinski:

Proposition 4.1 (see Lemma 3.1 of [SZ]). *Let us assume that* $\mathrm{gap}(L^{\tau}_{\Lambda}) \ge m > 0$ *uniformly in* Λ *and* τ. *Then there exist positive constants* C' *and* γ' *independent of* Λ *such that for any subset* V *of* Λ *any site* k *outside* Λ *and any function* f *with support in* V*:*

$$\sup_{\tau} |\mu^{\tau^{(k)}}_{\Lambda}(f) - \mu^{\tau}_{\Lambda}(f)| \le C\exp(-\gamma\,\mathrm{dist}(V,k))\left\{|||f||| \wedge \sup_{\sigma}|f(\sigma)|\right\},$$

where $|||f||| = \sum_x \|\nabla_x f\|$ *and* $\|\nabla_k f\| = \sup_p |f(\sigma^{(k)}) - f(\sigma)|$.

Remark. Actually in Lemma 3.1 in [SZ] the dependence of the estimate on f was only through the seminorm $|||f|||$. That may be not so convenient if f depends on a large number of spins (e.g. f is the characteristic function of the event that all the spins in Λ at distance from k larger than L are $+1$) since one may introduce a spurious volume factor. A little effort shows however that the dependence on f can be improved to that of the proposition.

Clearly the above result proves the theorem since the variation distance between the relativization of the Gibbs states in V with boundary conditions outside Λ given by τ and $\tau^{(k)}$ respectively, is equal to:

$$\sup_{A \subset \Omega_{Q(L)}} |\mu^{\tau^{(k)}}_{\Lambda}(A) - \mu^{\tau}_{\Lambda}(A)|\,.$$

The last result strengthens the result given in Theorem 4.1:

Theorem 4.2. *There exists a positive constant $\bar{L}$ depending only on the range of the interaction and on the dimension d such that if $SM(L_0, C, \gamma)$ holds with $L_0 \geq \bar{L}$ then there exists a positive constant m such that for any L_0-compatible set Λ and for any function f on $\{-1, +1\}^\Lambda$:*

$$\sup_\sigma |T_t^{\Lambda,\tau}(f)(\sigma) - \mu(f)| \leq |||f||| \exp(-mt),$$

where $T_t^{\Lambda,\tau}$ denotes the Markov semigroup of the process evolving in Λ with boundary conditions τ.

Proof. In this paper we prove the theorem only in the attractive case. The proof for the general case can be found in [MO].

We proceed as in the proof of Theorem 3.1. We define similarly to (4.8) but for the Glauber dynamics:

$$\varrho_\Lambda^\tau(t) = \sup_{x\in\Lambda} E_+^{\Lambda_l,\tau}(\sigma_t(x)) - E_-^{\Lambda_L,\tau}(\sigma_t(x)).$$

As in Sect. 2 we need only to show that $\varrho_\Lambda^\tau(t)$ decays exponentially to zero with a rate independent of the volume Λ and of the boundary conditions τ. One easily checks that also the finite volume definition of $\varrho_\Lambda^\tau(t)$ obeys Holley's alternative: there exists a positive constant δ_0 independent of the boundary conditions τ such that if there exists a sufficiently large finite time t_0 such that:

$$t_o^d \varrho_\Lambda^\tau(t_0) \leq \delta_0, \tag{4.15}$$

then there exists a finite constant m depending on t_0 and δ_0 such that:

$$\varrho_\Lambda^\tau(t) \leq \exp(-mt) \quad \forall t \geq t_0. \tag{4.16}$$

Thus, in order to prove the theorem, we need only to show that there exists a time t_0, independent of the volume Λ and of the boundary condition τ, such that the above condition holds. In turn this will follow from a computation similar to that of Sect. 3 since we know from the previous theorem that the mass gap of the stochastic Ising model can be bounded from below uniformly in the boundary conditions and in the volume Λ provided that Λ is L_0-compatible.

More precisely let, for any $x \in \Lambda$, $\Lambda_N(x)$ be an L_0-compatible subset of Λ such that:

a) x is contained in $\Lambda_N(x)$,
b) $\text{dist}(x, \partial\Lambda_N(x)\backslash\partial\Lambda \cap \partial\Lambda_N(x)) \geq NL_0$.

Then, by attractivity, we can bound $\varrho_\Lambda^\tau(t)$ by:

$$E_+^{\Lambda_N(x),+}(\sigma_t(0)) - E_-^{\Lambda_N(x),-}(\sigma_t(0)), \tag{4.17}$$

where $E_+^{\Lambda_N(x),+}(\ )$ is the expectation over the process which starts from all pluses and evolves with $+$ boundary conditions on $\partial\Lambda_N(x)\backslash\partial\Lambda \cap \partial\Lambda_N(x)$ and the given τ-boundary conditions on $\partial\Lambda \cap \partial\Lambda_N(x)$ and analogously for $E_-^{\Lambda_N(x),-}(\ )$. Thus, as in Proposition 3.2, the r.h.s. of (4.17) is bounded from above by:

$$\begin{aligned} &E_+^{\Lambda_N(x),+}(\sigma_t(0)) - \mu_{\Lambda_N(x)}^+(\sigma(0)) + \mu_{\Lambda_N(x)}^-(\sigma(0)) \\ &\quad - E_-^{\Lambda_N(x),-}(\sigma_t(0)) + \mu_{\Lambda_N(x)}^+(\sigma(0)) - \mu_{\Lambda_N(x)}^-(\sigma(0)), \end{aligned} \tag{4.18}$$

where the Gibbs states $\mu^+_{\Lambda_N(x)}$ and $\mu^-_{\Lambda_N(x)}$ have $+$ and $-$ boundary conditions on $\partial\Lambda_N(x)\backslash\partial\Lambda\cap\partial\Lambda_N(x)$ and the given τ-boundary conditions on $\partial\Lambda\cap\partial\Lambda_N(x)$. Using Proposition 4.1 the third term is bounded from above by:

$$\exp(-\gamma N L_0)\,. \tag{4.19}$$

The estimate of the first and second term is the same and we get that each of them, e.g. the first one, is bounded from above by:

$$\exp(-\gamma N L_0) \tag{4.20}$$

provided that:

$$t \geq \frac{1}{\text{gap}(\Lambda_N(x),+)}\,[\log(\mu^+_{\Lambda_N(x)}(+)^{-1}) + \gamma N L_0) \tag{4.21}$$

(see for instance [Si]) where $\mu^-_{\Lambda_N(x)}(+)$ is the $\mu^-_{\Lambda_N(x)}$-measure of the configuration in $\Lambda_N(x)$ identically equal to plus one and gap$(\Lambda_N(x),+)$ is the gap in the spectrum of the (self-adjoint) generator of the stochastic Ising model in $\Lambda_N(x)$ with $+$ boundary conditions on $\partial\Lambda_N(x)\backslash\partial\Lambda\cap\partial\Lambda_N(x)$ and the given τ-boundary conditions on $\partial\Lambda\cap\partial\Lambda_N(x)$. Using the result of Proposition 4.1 we have that gap$(\Lambda_N(x),+)$ (and the same for gap$(\Lambda_N(x),-)$) is bounded below by m_0 uniformly in N. Therefore, since $\log(\mu^-_{\Lambda_N(x)}(+)) > -A(NL_0)^d$ for some constant A, if we take $NL_0 = c_0 t^{\frac{1}{d}}$ then, if c_0 is sufficiently small depending on A, we get that (4.21) is satisfied and therefore

$$\varrho^\tau_\Lambda(t) \leq 3\exp(-\gamma c_0 t^{\frac{1}{d}})\,.$$

Thus $\varrho^\tau_\Lambda(t)$ decays faster than $\frac{1}{t^d}$ uniformly in the volume Λ and the theorem follows.

Remark. One may wonder whether the rates m_0 and m of the exponential convergence to equilibrium in the L^2-sense and in the uniform norm are equal. The proof that we give of Theorem 4.3 in the attractive case, which basically rephrases in finite volumes the usual Holley's argument, does not allow us to derive any conclusion about this question. However if instead of Holley's argument one uses logarithmic Sobolev inequalities (see [MO]) then one can conclude that the two rates are actually the same.

Remark. One may wonder why we need in this section a condition like $SM(L_0, C, \gamma)$ which is much stronger than the weak mixing condition used in the previous section. A first simple answer to this question is the following: since exponential convergence to equilibrium in finite volume (in the L^2 or L^∞ sense) implies the exponential decay of truncated correlations in the given volume (see Proposition 4.1), certainly such convergence cannot take place for those systems, like the Czech models or the 3D Ising model at low temperature at very small magnetic fields (Bassuev phenomenon) in which truncated correlations do not decay exponentially fast uniformly in the locations of the two observables.

Another explanation which seems to be reasonable even in the attractive case, is the following: let us suppose that we have only a weak mixing property of the Gibbs state and let us consider the quantity $E^{\Lambda_L,\tau}_\sigma(f(\sigma_t))$, where f is an observable located well inside a box Λ_L of side L. If $t \leq \delta L$, where δ is a suitable small constant, then, because of the finite speed of propagation of information, $E^{\Lambda_L,\tau}_\sigma(f(\sigma_t))$

is exponentially close (in t) to its infinite volume version $E_\sigma(f(\sigma_t))$ which is indeed, because of Theorem 3.1, exponentially (in t) close to $\mu(f)$ which, in turn, because of weak mixing, is exponentially (in L) close to $\mu^\tau_{\Lambda_L}(f)$. Since $L \geq \frac{t}{\delta}$ it follows that for times t up to δL we have:

$$|E^{\Lambda_L,\tau}_\sigma(f(\sigma_t)) - \mu^\tau_{\Lambda_L}(f)| \leq C\exp(-mt)$$

for suitable C and m.

Let us now consider times t much larger than L and let us suppose that for these times the probability distribution of $\sigma_t(x)$ for x close to the boundary of Λ_L is not exponentially (in t) close to the invariant measure. That is not an unreasonable assumption if our system exhibits a kind of phase transition at the boundary as apparently does the 3D Ising model at low temperature for some very small magnetic fields (Basuev phenomenon). Let us now analyze the influence of this slow approach to equilibrium at the boundary on $E^{\Lambda_L,\tau}_\sigma(f(\sigma_t))$. Certainly, because of weak mixing, the effect will not be larger than a suitable negative exponential in L but we cannot exclude that it will be precisely of this order. If this is the case then, since $t \gg L$, the influence on $E^{\Lambda_L,\tau}_\sigma(f(\sigma_t))$ of the slow convergence to equilibrium at the boundary will be much larger than a negative exponential of the time t and thus, even in the bulk, we will have a convergence slower than exponential.

Finally, from a technical point of view we observe that in finite volume, as the reader can easily check, we cannot repeat the proof of Proposition 3.1 because for some site x the cube of side L and centered at x intersects the boundary of the cube. Thus we are forced to choose the strong hypothesis $(SM(L_0, C, \gamma))$.

5. Applications

In this section we discuss some applications of our results. In particular we prove the exponential convergence to equilibrium for the infinite volume stochastic Ising model for all temperatures above the critical one and for low temperature and non-zero external field.

The model that we will consider is the standard Ising model whose Hamiltonian in a finite volume Λ of the lattice $\mathbf{Z^d}$ with boundary conditions τ is given by:

$$H^\tau_\Lambda(\sigma) = -\tfrac{1}{2} \sum_{x,y\in\Lambda\,:\,|x-y|=1} \sigma(x)\sigma(y) - \tfrac{1}{2} \sum_{x\in\Lambda} \left[h + \sum_{y\notin\Lambda\,:\,|x-y|=1} \tau(y)\right]\sigma(x)\,.$$

The associate finite volume Gibbs state at inverse temperature β will be denoted by $\mu^{\tau,\beta,h}_\Lambda(\sigma)$. It is well known that if the dimension d is greater or equal than 2 there exists a critical value of β, denoted in the sequel by β_c, such that there exists a unique infinite volume Gibbs state $\mu^{\beta,h}$ iff $h \neq 0$ or $\beta < \beta_c$. Thus, if we consider the associated stochastic Ising model discussed in the previous section, then it will be an ergodic Markov process on $\{-1,+1\}^{\mathbf{Z^d}}$ with $\mu^{\beta,h}$ as unique invariant measure only for $h \neq 0$ or $\beta < \beta_c$. In the following theorem we will strengthen this result. Let us denote by $E^{\beta,h}_\sigma(f(\sigma_t))$ or by $E^{\Lambda,\tau,\beta,h}_\sigma(f(\sigma_t))$ the expected value at time t of the function f with respect to the distribution of the process evolving with external field h and inverse temperature β in the infinite lattice $\mathbf{Z^b}$ or in the finite set Λ with boundary conditions τ. Then we have:

Theorem 5.1. a) *Assume that $\beta < \beta_c$. Then there exists a positive constant m and for any cylindrical function f there exists a constant C_f such that:*

$$\sup_\sigma |E_\sigma^{\beta,h}(f(\sigma_t)) - \mu^{\beta,h}(f)| \leq C_f \exp(-mt).$$

b) *There exists a positive constant β_0 such that for any $\beta \geq \beta_0$ and $h > 0$ there exists a positive constant m and for cylindrical function f there exists a constant C_f such that:*

$$\sup_\sigma |E_\sigma^{\beta,h}(f(\sigma_t)) - \mu^{\beta,h}(f)| \leq C_f \exp(-mt).$$

c) *Given $h > 0$ there exist two positive constants $\beta_0(h)$ and $L_0(h)$ such that for any $\beta \geq \beta_0$ there exists a positive constant m such that for any $L_0(h)$-compatible set Λ and for any function f on $\{-1,+1\}\Lambda$:*

$$\sup_\sigma |E_\sigma^{\Lambda,\tau,\beta,h}(f(\sigma_t)) - \mu_\Lambda^{\tau,\beta,h}(f)| \leq |||f||| \exp(-mt).$$

Proof. a) Thanks to Theorem 3.1 we need only to verify our *weak mixing* condition (3.1). This in turn follows from part i) of Theorem 2 of a recent paper by Higuchi [Hi].

b) Also in this case we verify the *weak mixing* condition and for this purpose we use a result by Martyrosian [M]. In order to state his result we need some notation. A finite subset A of the cubic lattice is said to be connected if for any two sites x and y in A there is a sequence of nearest neighbor sites $x_0, x_1, \dots, x_n$ in A connecting x to y, i.e. $x_0 = x$ and $x_n = y$. The finite connected set A will be said to be simply-connected if its complement is connected. Given a connected set A, $\phi(A)$ will be the smallest simply-connected set containing A. Then we have:

Theorem 5.2 *(Martyrosian). There exists a positive constant β_0 such that for any $\beta \geq \beta_0$ and any $h > 0$ there exists a positive constant C such that for every L:*

$$\mu_{Q_L}^{-,\beta,h}(\sigma; \exists \text{ a connected set } A \text{ with } \sigma(x) = +1 \ \forall x \in A \text{ and } Q_{L-\log(L)} \subset \phi(A))$$

tends to one as $L \to \infty$.

Corollary 5.1. *There exists a positive constant β_0 such that for any $\beta \geq \beta_0$ and any $h > 0$ there exist positive constants C and ε such that for every L:*

$$\mu_{Q_L}^{+,\beta,h}(\sigma(0)) - \mu_{Q_L}^{-,\beta,h}(\sigma(0)) \leq C \exp(-\varepsilon L).$$

Proof. Using the method of Appendix 1, it is enough to show that

$$L^{d-1}[\mu_{Q_L}^{+,\beta,h}(\sigma(0)) - \mu_{Q_L}^{-,\beta,h}(\sigma(0))] \tag{5.1}$$

tends to zero as $L \to \infty$. For this purpose let us define

$$\begin{aligned} L_k &= [L/2 + kC\log(L)], \\ \Lambda_k &= Q_{L_k}, \end{aligned} \tag{5.2}$$

for $k = 1 \dots K = \left[\dfrac{L}{2C\log(L)}\right]$, where C is the constant appearing in Theorem 5.2 and $[\cdot]$ denotes the integer part. Let also, for any k, Ω_k be the event that in the annulus $\Lambda_k \backslash \Lambda_{k-1}$ there exists a connected set Γ_k such that:

i) $\sigma_x = +1 \ \forall x \in \Gamma_k$.

ii) The set $Q_L \backslash \Gamma_k$ splits into two disjoint connected sets A and B, with $0 \in A$.

Let finally $\hat{\Omega}$ be the union of the events Ω_k. It is easy to see, using F.K.G., that:

$$\mu_{Q_L}^{-,\beta,h}(\sigma(0) \mid \hat{\Omega}) \geq \mu_{Q_L}^{+,\beta,h}(\sigma(0)),$$

so that

$$L^{d-1}[\mu_{Q_L}^{+,\beta,h}(\sigma(0)) - \mu_{Q_L}^{-,\beta,h}(\sigma(0))] \leq 2L^{d-1}\mu_{Q_L}^{-,\beta,h}(\hat{\Omega}^c). \tag{5.3}$$

Thus we are left with the estimate of $\mu_{Q_L}^{-,\beta,h}(\hat{\Omega}^c)$.

If χ_k denotes the characteristic function of the event Ω_k^c, we can write:

$$\mu_{Q_L}^{-,\beta,h}(\hat{\Omega}^c) \leq \mu_{Q_L}^{-,\beta,h}\left(\prod_{k=2}^{K} \chi_k\right)\mu_{\Lambda_1}^{-,\beta,h}(\chi_1), \tag{5.4}$$

where we have used the D.L.R. equations and the fact that

$$\mu_{\Lambda_k}^{\tau,\beta,h}(\chi_k) \leq \mu_{\Lambda_k}^{-,\beta,h}(\chi_k).$$

for any τ.

By Theorem 5.2 we have that:

$$\mu_{\Lambda_1}^{-,\beta,h}(\chi_1) \leq 1/2, \tag{5.5}$$

provided that L is large enough.

Thus, if we iterate (5.4) K-times, we get that

$$\mu_{Q_L}^{-,\beta,h}(\hat{\Omega}^c) \leq 2^{-K},$$

which clearly proves the corollary since $K \geq L/2C\log(L) - 1$.

c) In this case we verify that for any $h > 0$ there exist two positive constants $\beta_0(h)$ and $L_0(h) \geq \bar{L}$, where $\bar{L}$ is the numerical constant appearing in Theorem 4.3, such that if $\beta \geq \beta_0$ then $SM(L_0, C, \gamma)$ mixing condition holds. Let us fix $h > 0$ and let us choose $2L_0(h) = \left\lceil \frac{A}{h} \right\rceil$. It is simple to verify that if the constant A is taken large enough (e.g. $A = 4$ in $d = 2$ and $A = 6$ for $d = 3$) then the configuration identically equal to $+1$ is the unique ground state configuration of the Hamiltonian $H_{\Lambda_{L_0(h)}}^{\tau}(\sigma)$ for any boundary condition τ. Thus if we estimate the variation distance appearing in our $SM(L_0, C, \gamma)$ mixing condition by:

$$2\sup_{\tau} \mu_{\Lambda}^{\tau,\beta,h}(\exists x \in \Lambda_{L_0(h)};\ \sigma(x) = -1), \tag{5.6}$$

then we can make the variation distance as small as we like by taking β large enough. Finally by taking the constant A large enough we can make the length scale $L_0(h)$ larger than the numerical constant $\bar{L}$ appearing in Theorem 4.3. It is important to stress here that $\bar{L}$ does not depend on the parameters β and h of the Hamiltonian.

The theorem is proved.

Appendix 1. A Simple Proof of Theorem 2.6 in the Attractive Case

We give a simple proof of the "effectiveness" of the $SM(L_0, C, \gamma)$ mixing condition in the case the interaction $J(X)$ is ferromagnetic.

Let Λ be an L_0-compatible set, let τ and $\tau^{(y)}$ be boundary configurations outside Λ, where $\tau^{(y)}$ is obtained from τ by flipping the spin at $y \in \Lambda^c$ and let x be a site of Λ. Without loss of generality we assume that $\tau^{(y)} \geq \tau$. Given C and γ we will prove that there exists a constant $\bar{L} \geq R$ such that if $SM(L_0, C, \gamma)$ holds for $L_0 \geq \bar{L}$, then there exist positive constants $C_0 \gamma'$ such that:

$$\mu_\Lambda^{\tau^{(y)}}(\sigma_x = +1) - \mu_\Lambda^{\tau}(\sigma_x = +1) \leq C_0 \exp(-\gamma' \operatorname{dist}(x, y)) . \tag{A1.1}$$

Clearly (A1.1) proves the theorem. Let in fact $\mu_\Lambda^{\tau^{(y)},\tau}(\sigma, \eta)$ be a joint representation of $\mu_\Lambda^{\tau^{(y)}}$ and μ_Λ^{τ} which is above the diagonal. Then we have:

$$\operatorname{Var}(\mu_{\Lambda,V}^{\tau}, \mu_{\Lambda,V}^{\tau^{(y)}}) \leq \sum_{x \in V} \mu_\Lambda^{\tau^{(y)}}(\sigma_x \neq \eta_x) . \tag{A1.2}$$

Using (A1.1) and the fact that $\mu_\Lambda^{\tau^{(y)},\tau}$ is above the diagonal, the r.h.s. of (A1.2) is bounded by $C' \exp(-\gamma' \operatorname{dist}(y, V))$ for some constant C'.

We now prove (A1.1). Let, for any Λ, $\sigma \geq \eta \in \Omega_{\Lambda^c} \mu_\Lambda^{\sigma,\eta}$ denote the joint representation of the Gibbs states in Λ with boundary conditions σ and η which is above the diagonal. Let also, for any x in Λ, Q_x be a cube of side $2L_0 + 1$ such that $x \in Q_x$ and $\operatorname{dist}(x, \partial Q_x \partial \Lambda \cap \partial Q_x) \geq \frac{L_0}{2}$. Clearly such a cube always exists. Let $\partial_r Q_x$ be the set of sites y in $\Lambda \backslash Q_x$ with $\operatorname{dist}(y, Q_x) \leq r$, where r is the range of the interaction. Then we can write:

$$\begin{aligned} \mu_\Lambda^{\tau^{(y)}}(\sigma_x = +1) - \mu_\Lambda^{\tau}(\sigma_x = +1) &= \sum_{\sigma,\sigma'} \mu_\Lambda^{\tau^{(y)}}(\sigma, \sigma') \mu_{Q_x}^{\sigma,\sigma'}(\eta_x \neq \eta'_x) \\ &\leq \sum_{\sigma,\sigma'} \mu_\Lambda^{\tau^{(y)},\tau}(\sigma, \sigma') \chi(\exists z \in \partial_r Q_x; \sigma_z \neq \sigma'_z) C \exp(-\gamma L_0/2) , \end{aligned} \tag{A1.3}$$

where we have used $SM(L_0, C, \gamma)$ in order to estimate $\mu_{Q_x}^{\sigma,\sigma'}(\eta_x \neq \eta'_x)$. It is at this point that attractivity becomes important. Since $\mu_\Lambda^{\tau^{(y)},\tau}$ is above the diagonal the term $\mu_\Lambda^{\tau^{(y)},\tau}(\sigma_z \neq \sigma'_z)$ is equal to $\mu_\Lambda^{\tau^{(y)}}(\sigma_z = +1) - \mu_\Lambda^{\tau}(\sigma_z = +1)$. Thus, if we denote with $F(x)$ the l.h.s. of (A1.3), we get:

$$F(x) \leq C \exp(-\gamma L_0/2) \sum_{z \in \partial_r Q_x} F(z) . \tag{A1.4}$$

Iteration of (A1.4) gives that $F(x)$ is bounded by the series:

$$F(x) \leq \sum_{n \geq \left[\frac{L}{L_0 + R}\right]} (C c(d, r) L_0^{d-1} \exp(-\gamma L_0/2))^n , \tag{A1.5}$$

where $c(d, r)$ is a numerical constant. Clearly (A1.5) gives the desired exponential bound for $F(x)$ provided that L_0 is large enough depending on r and the dimension d.

Appendix 2

In this appendix we want to prove Theorem 2.6. First we need some definitions.

Let $Q_{L,3L}(0)$ be the box:

$$Q_{L,3L}(0) = \left\{ x \in \mathbf{Z^d};\, |x_i| \le \frac{3L-1}{2},\, i = 1,\ldots,d-1,\, |x_d| \le \frac{L-1}{2} \right\}.$$

Let $\bar{\Gamma}$ be the set of all subsets of $Q_{L,3L}(0)$ which

i) are union of cubes $Q_L(x)$, $x = Ly$, $y \in \mathbf{Z^d}$,

ii) contain $Q_L(0)$ and,

iii) are symmetric with respect to all directions of the lattice.

Let us call "vertical" the d^{th} direction of the lattice and "horizontal" the hyperplane orthogonal to it.

For any $\Lambda \in \bar{\Gamma}$ consider pairs of sites k, k' in $\partial_r^+ \Lambda$ "adjacent" to opposite horizontal faces of Λ in the sense that there exist $x, x' \in \Lambda$, $x = (x_1, \ldots, x_d)$, $x' = (x'_1, \ldots, x'_d)$ with $|x-k| \le r$, $|x'-k'| \le r$, $|x_d| = |x'_d| = \dfrac{L-1}{2}$, $x_d = -x'_d$.

Let

$$\Delta_k = \{x \in \Lambda \colon \text{dist}(x,k) \le r\}\,,$$
$$\Delta_{k'} = \{x \in \Lambda \colon \text{dist}(x,k') \le r\}\,.$$

We also assume that if the horizontal distance between $\Delta_k, \Delta_{k'}$ is larger than one, then there exists an $x \in \Lambda$ such that the cube $Q_L(x)$ is such that $\Lambda \supset Q_L(x) \supset \Delta_k$, $\Delta_{k'} \subset \Lambda \backslash Q_L(x)$.

Following a simple argument already used by Stroock and Zegarlinski (see [SZ], proof of Eq. (3.4) we write, for $y \in \partial_r^+ \Lambda$:

$$\mu_\Lambda^\tau(f) - \mu_\Lambda^{\tau^{(y)}}(f) = \mu_\Lambda^\tau(f\psi_\Lambda^{(y)}) - \mu_\Lambda^\tau(f)\mu_\Lambda^\tau(\psi_\Lambda^{(y)}) \tag{A2.1}$$

with $\psi_\Lambda^{(y)}$ such that:

$$\mu_\Lambda^\tau(\psi_\Lambda^{(y)}) = 1$$

and

$$\|\psi_\Lambda^{(y)}\| \le \exp(4\|U\|)\,. \tag{A2.2}$$

Let us now state and prove a lemma.

Lemma A2.1. *In the general case (hypotheses $H1, H2$ satisfied) suppose that $SM(Q_L, C, \gamma)$ holds for some $C > 0, \gamma > 0, L > 8r, 2dr(L+r)^{d-1}\exp(-\gamma L/8) < 1$. Then, for any $\Lambda \in \bar{\Gamma}, k, k'$ as above, given any cylindrical function f, with support $S_f = \Delta_k$, we have:*

$$\sup_{\tau \in \Omega_{\Lambda^c}} \mu_\Lambda^\tau(f, g) \le \|f\| C' \exp(-\gamma' L) \tag{A2.3}$$

with

$$C' = C\exp(4\|U\|)\,, \qquad 4L\exp(-\gamma L/8) = \exp(-\gamma' L)\,,$$

where $g = \psi_\Lambda^{(k')}$, and

$$\|U\| = \sum_{X \subset\subset \mathbf{Z^d}, X \ni O} |U_X|\,.$$

Proof. For simplicity we shall only consider the case $d = 2$ where $\bar{\Gamma}$ contains only the square $Q_L(0)$ (for which (A2.1) is true by hypothesis with $C' = C$, $\gamma' = \gamma$) and the rectangle $\Lambda \equiv Q_{L,3L}(0)$ with edges parallel to the 1, 2 (horizontal and vertical) coordinate axes with length, respectively, $L_1 = 3L$, $L_2 = L$. The easy extension of the argument to the general, d-dimensional, case is left to the reader.

Considering Λ we distinguish two cases:
1) $\Delta_k, \Delta_{k'}$ have horizontal distance $\leq L/2$; namely:

$$\inf_{x \in \Delta_k, y \in \Delta_{k'}} |x_1 - y_1| \leq L/2 .$$

2) $\Delta_k, \Delta_{k'}$ have horizontal distance $> L/2$.

In the first case we observe that there exists $x \in \Lambda$ such that the square $Q_L(x)$, that for notation convenience we call V, is contained in Λ, contains both $\Delta_k, \Delta_{k'}$ and is such that $\text{dist}(\Delta_k, \partial V \cap \Lambda)$, $\text{dist}(\Delta_{k'}, \partial V \cap \Lambda) \geq L/8$.

We then have, $\forall \tau \in \Omega_{\Lambda^c}$:

$$\mu^\tau_\Lambda(fg) = \sum_{\omega \in \Omega_{\Lambda \setminus V}} \mu^\tau_{\Lambda, \Lambda\setminus V}(\omega) \mu^{\tau,\omega}_V(fg) \tag{A2.4}$$

from which we get:

$$\mu^\tau_\Lambda(f,g) \equiv \mu^\tau_\Lambda(fg) - \mu^\tau_\Lambda(f)\mu^\tau_\Lambda(g) = \varepsilon^\tau_\Lambda(f,g) + \bar{\varepsilon}^\tau_\Lambda(f,g) \tag{A2.5}$$

with

$$\varepsilon^\tau_\Lambda(f,g) = \sum_{\omega \in \Omega_{\Lambda\setminus V}} \mu^{\tau,\omega}_V(f,g)\mu^\tau_{\Lambda,\Lambda\setminus V}(\omega) , \tag{A2.6}$$

$$\bar{\varepsilon}^\tau_\Lambda(f,g) = \sum_{\omega,\omega' \in \Omega_{\Lambda\setminus V}} \mu^\tau_{\Lambda,\Lambda\setminus V}(\omega)\mu^\tau_{\Lambda,\Lambda\setminus V}(\omega')\mu^{\tau,\omega}_V(f)\,[\mu^{\tau,\omega}_V(g) - \mu^{\tau,\omega'}_V(g)] \tag{A2.7}$$

From $SM(Q_L, C, \gamma)$, (A2.6) we have immediately:

$$\varepsilon^\tau_\Lambda(f,g) \leq \|f\| C \exp(-\gamma L/8) . \tag{A2.8}$$

From $SM(Q_L, C, \gamma)$, (A2.7) and (A2.1) we get

$$\bar{\varepsilon}^\tau_\Lambda(f,g) \leq 4LrC\|f\|\,\|g\|\exp(-\gamma L/4) . \tag{A2.9}$$

Consider now the second case (horizontal distance of Δ_k, $\Delta_{k'} > L/2$). By hypothesis there exists an $x \in \Lambda$ such that the square $V \equiv Q_L(x)$ is such that $Q_{L,3L} \equiv \Lambda \supset V \supset \Delta_k$, $\Delta_{k'} \subset \Lambda \setminus V$ (suppose, for instance, that Δ_k, between $\Delta_k, \Delta_{k'}$ is the set at largest horizontal distance from the vertical edges of Λ).

We then have:

$$\mu^\tau_\Lambda(f,g) = \sum_{\omega,\omega' \in \Omega_{\Lambda\setminus V}} \mu^\tau_{\Lambda,\Lambda\setminus V}(\omega)\mu^\tau_{\Lambda,\Lambda\setminus V}(\omega')g(\omega)\,[\mu^{\tau,\omega}_V(f) - \mu^{\tau,\omega'}_V(f)] . \tag{A2.10}$$

From (A2.3), (A2.1) and (A2.10) we get

$$\mu^\tau_\Lambda(f,g) \leq 4LC\|f\|\,\|g\|\,\|\psi\|\exp(-\gamma L/8) . \tag{A2.11}$$

From (A2.5), (A2.8), (A2.9) and (A2.11) we get the lemma.

From Lemma A2.1 and Proposition 3.1, Eq.'s (3.9), (3.11) of [O] we get that there exists $L \equiv L(C, \gamma)$ such that Condition C_L of [OP] (see Eq. (1.8) there) holds. Then, from Propositions 2.5.1, 2.5.2, 2.5.3, 2.5.4 of [OP] Theorem 2.6 follows.

Appendix 3

In this final appendix we prove Theorem 2.5′. Our goal is to show that Theorem 2.6 can be viewed as a corollary of Theorem 2.2.

Let Λ be a subset of $\mathbf{Z^d}$, let τ be a boundary configuration outside Λ, i.e. $\tau \in \Omega_{L^c}$, let $y \in \partial_r^+ \Lambda$ and let $\Delta \subset \Lambda$. We want to estimate

$$\mathrm{Var}(\mu^{\tau}_{\Lambda,\Delta}, \mu^{\tau^{(y)}}_{\Lambda,\Delta}) \tag{A3.1}$$

by supposing true $K(\Lambda_0, \delta)$ for some finite set Λ_0 and $\delta < 1$.

For this purpose let for any $x \in \Delta$

$$l_x = \mathrm{dist}(x, y)\,,$$

and let

$$B = \bigcup_{x\in\Delta} \{z \in \mathbf{Z^d}; \mathrm{dist}(z,x) < l_x\} \cup \Lambda\,. \tag{A3.2}$$

Then by construction $\mathrm{dist}(\Delta, \partial_r^+ B) \geq \inf_{x\in\Delta} l_x = \mathrm{dist}(\Delta, y)$ and $y \in \partial_r^+ B$.

The idea at this point is to estimate (A3.1) by applying Theorem 2.2 to a suitable "Gibbs" measure ν_B^τ on Ω_B, whose specifications satisfy, thanks to $K(\Lambda_0, \delta)$, the condition $DSU(\Lambda_0, \delta)$ with $\delta < 1$. In order to define the new measure ν_B^τ, let us denote by ξ the restriction of the configuration τ to the set $B\backslash\Lambda$; by abuse of notation, the restriction of τ to $\mathbf{Z^d}\backslash B$ will also be called τ. If for every configuration $\sigma \in \Omega_B$ we denote by σ_A its restriction to $A \subset B$, then the measure ν_B^τ is given by:

$$\begin{aligned} \nu_B^\tau(\sigma) &= 0 \quad \text{if} \quad \sigma_{B\backslash\Lambda} \neq \xi\,, \\ \nu_B^\tau(\sigma) &= \mu_\Lambda^{\tau\xi}(\sigma_\Lambda) \quad \text{if} \quad \sigma_{B\backslash\Lambda} = \xi\,, \end{aligned} \tag{A3.3}$$

where $\tau\xi$ has been defined in (1.1).

Thus, by construction, (A3.1) can be written as:

$$\mathrm{Var}(\nu^{\tau}_{B,\Delta}, \nu^{\tau^{(y)}}_{B,\Delta})\,. \tag{A3.4}$$

It is easy to check that ν_B^τ is "Gibbsian" in the sense that it satisfies the DLR equations for the following local *specifications* q_V^ζ:

$$q_V^\zeta(\sigma_V) = \mu^\zeta_{V\cap\Lambda}(\sigma_{V\cap\Lambda}\mathbf{1}_{(\sigma_{V\cap(B\backslash\Lambda)}=\xi_{V\cap(B\backslash\Lambda)})}) \tag{A3.5}$$

where $\zeta \in \Omega_{C^c}$ and in general, for any set A

$$\begin{aligned} \mathbf{1}_{(\sigma_A=\eta_A)} &= 0 \quad \text{if} \quad \sigma_A \neq \eta_A\,, \\ \mathbf{1}_{(\sigma_A=\eta_A)} &= 1 \quad \text{if} \quad \sigma_A = \eta_A\,. \end{aligned}$$

We next show that $K(\Lambda_0, \delta)$, with $\delta < 1$, implies that the specifications q_V^ζ satisfy $DSU(\Lambda_0, \delta)$ with $\delta < 1$ *uniformly* in the location of the cube Λ_0 inside the set B.

Thus, let us choose $x \in B$ in such a way that $\Lambda_0 + x \subset B$ and let us compute:

$$\sup_{\zeta,\zeta^{(y')}} \mathrm{KROV}_\varrho(q^\zeta_{\Lambda_0+x}, q^{\zeta^{(y')}}_{\Lambda_0+x})\,, \tag{A3.6}$$

where ϱ is given by (1.8) and $y' \in \partial_r^+(\Lambda_0 + x)$. We have distinguish three different cases:

i) $\Lambda_0 + x \subset B \backslash \Lambda$; in this case (A3.6) is zero by construction.
ii) $\Lambda_0 + x \subset \Lambda$; in this case (A3.6) is equal to

$$\sup_{\zeta, \zeta^{y'}} \mathrm{KROV}_\varrho(\mu^\zeta_{\Lambda_0+X}, \mu^{\zeta^{(Y')}}_{\Lambda_0+X}), \tag{A3.7}$$

which, because of $K(\Lambda_0, \delta)$, is bounded from above by $\alpha_{y'}$ with

$$\sum_{y' \in \partial_r^+(\Lambda_0+x)} \alpha_{y'} \leq \delta |\Lambda_0| . \tag{A3.8}$$

iii) $\Lambda_0 + x$ intersects both Λ and $B \backslash \Lambda$; in this case let $V = (\Lambda_0 + x) \cap \Lambda$. Then (A3.6) becomes equal to

$$\sup_{\zeta, \zeta^{y'}} \mathrm{KROV}_\varrho(\mu^{\zeta\xi}_V, \mu^{\zeta^{(y')}\xi}_V), \tag{A3.9}$$

where $\zeta\xi$ is the configuration in V^c which coincides with ζ outside $\Lambda_0 + x$ and with ξ in $(\Lambda_0 + x) \cap (B \backslash \Lambda)$. Again because of $K(\Lambda_0, \delta)$, (A3.9) is bounded from above by $\alpha_{y'}$ with

$$\sum_{y' \in \partial_r^+(\Lambda_0+x)} \alpha_{y'} \leq \delta |\Lambda_0| . \tag{A3.10}$$

We stress that it is precisely in the third case iii) that one uses the full strength of $K(\Lambda_0, \delta)$ since the set V can be an arbitrary subset of Λ_0.

At this stage we can apply Theorem 2.2 to the measure ν_B^τ and estimate (A3.4) from above by:

$$\mathrm{Var}(\nu^\tau_{B,\Delta}, \nu^{\tau^{(y)}}_{B,\Delta}) \leq C \sum_{x \in \Delta, z \in \partial_r^+ B} \exp(-\gamma'|x - z|) \leq C' \exp(-\gamma'' \operatorname{dist}(\Delta, y)) \tag{A3.11}$$

for a suitable, positive constant γ''.

Remark. We notice that Theorem 2.2 has been stated only for translation invariant Gibbs measures and certainly the specifications in (A3.5) do not satisfy this requirement. However, as one can easily check in the original proof in [DS1], translation invariance becomes irrelevant provided that one is able to verify $DSU(\Lambda_0, \delta)$ with $\delta < 1$ *uniformly* in the location of the cube Λ_0 inside the set B.

The theorem is proved.

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