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Approach to multi-attribute decision-making problems based on neutrality aggregation operators of T-spherical fuzzy information

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Abstract

In the process of decision-making, uncertain information is always challenging to deal with. T-spherical fuzzy set (TSFS) operates vagueness of data by analysing three independent functions, namely membership, non-membership, and abstinence function. The TSFS provides us robust scheme with parameter $q \geq 1$ to handle the countless opportunities. Hence, this set proves its superiority over the existing picture fuzzy set (PFS) and spherical fuzzy set (SFS). Now a day, decision-makers usually assign impartial values throughout the assessment. This manuscript demonstrates some new operational laws by fusing the neutral characteristics of the degrees of membership and using the probability sum (PS) function. Meanwhile, we determine several aggregation operators (AOs) including weighted averaging neutral, ordered weighed neutral, and hybrid averaging neutral AOs to aggregate the data under T-spherical fuzzy (TSF) environment. As it came to the notice that weighted neutral averaging aggregation operators of the Pythagorean fuzzy set (PyFS), single-valued neutrosophic fuzzy set (SVNFS), and q-rung orthopair fuzzy set (q-ROFS) have some restrictions during the decision-making problems. So, to overcome this, we introduce a new multi-attribute group decision-making method (MAGDM) based on proposed AOs. Lastly, we provide various numerical instances to explain the method and exhibit its supremacy. Furthermore, a comparative analysis is conducted to compare the potential of proposed AOs with some other existing methods.

Keywords T-spherical fuzzy set · Neutrality operations · Aggregation operators · Multi-attribute group decision-making

Mathematics Subject Classification 47S40 · 62C86 · 90B50 · 03F55

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1 Introduction

MAGDM is an activity that is carried out to obtain a favourable alternative among a list of alternatives. During this procedure, we aim to attain an optimal solution using fuzzy information. It is fact that crisp numbers are always not sufficient to evaluate the alternatives; sometimes uncertain information is required to handle such situations. Zadeh (1965) introduced the abstraction of fuzzy sets (FSs), which presumes the membership degree (MD) to illustrate the data. Following on, Atanassov (1986) presented another term associated with the MD known as the non-membership degree (NMD) to the science of FSs. As a result, intuitionistic fuzzy set (IFS) came into existence. Atanassov IFS cannot manage the information of MD ($\dot{\check{s}}$) and NMD ($ḑ$) outside this linear inequality $\dot{\check{s}} + ḑ \leq 1$, where the selection of $\dot{\check{s}}$ and $ḑ$ must be within [0, 1]. Yager (2013a) extensively stiffen this concept by initiating the PyFS so, the aforementioned inequality converted into quadratic inequality $\dot{\check{s}}^2 + ḑ^2 \leq 1$. In the form of q-ROFS Yager (2016) further relaxed the condition by raising the power of inequality to q th power where q is an integer with $q \geq 1$. Clearly, q-ROFS generates a larger set of fuzzy data with IFS and PyFS are its special cases. Fuzzy structures discussed till now, contain information in the shape of duplets. These fuzzy duplets make us feel sometimes not sufficient enough to portray human judgment. Some real-life scenarios, like voting, require four membership functions instead of two. To resolve such issues Cuong (2015) made his contributions and opened new horizons of research for scholars. Picture FS (PFS) depicts four dimensions of human opinion favour, disfavour, abstinence, and degree of refusal. PFS has certain limitations as scholars had faced earlier in the case of IFS. Mahmood et al. (2019) broke this hindrance by introducing a spherical FS (SFS) and TSFS. Among the all defined fuzzy structures, TSFS provides the maximum degree of freedom and the decision-maker can assign any value to membership functions from anywhere in the interval [0, 1].

The vagueness of data has assembled a substantial analysis in MADM where the best possible alternatives are chosen. Accumulation of data is the main device to resolve decision-making problems. Inevitably AOs have caught the attention of researchers, so Garg (2017a, 2016a) made contributions by suggesting interactive aggregation operators to deal with MADM problems under different fuzzy environments. Some useful investigations (Wang and Liu 2011; Ali et al. 2021; Garg 2016b, 2017b) have been conducted using Einstein t-norm operations under different fuzzy frameworks. Several pieces of research (Ashraf et al. 2022; Liu and Wang 2018; Sahu et al. 2021; Limboo and Dutta 2022; Kazimieras Zavadskas et al. 2020; Karamaşa et al. 2021; Yager 2013b) have played their part to remove the obscurities during the process of MAGDM or MADM under IFSs, PyFSs, and q-ROFSs. Besides it, some researchers made valuable contributions to enhancing the scope of MADM problems under different fuzzy environments (Wang and Triantaphyllou 2008; Arora and Garg 2019, 2018; Xu and Hu 2010; Xu 1988, 2005; Viriyasitavat 2016; Kaur and Garg 2018). Recently, Senapati (2022) introduced Aczel-Alsina operations to support the decision-making schemes for PFS. Furthermore, Senapati and Chen (2021, 2022), Senapati et al. (2021) proposed some useful AOs under interval-valued intuitionistic fuzzy set (IVIFS) and interval-valued Pythagorean fuzzy set (IVPyFS). To discuss the DMPs more vigorously Garg (2017c) presented the weighted average and geometric operators using the PFS environment whilst Wang et al. (2017) illustrated some novel geometric operators. Later on Mahmood et al. (2019) discussed operator to the SFS whilst Ullah et al. (2019) extended the idea of TSFS. Munir et al. (2021) intensified the studies by presenting some interactive geometric operators under the TSF environment. Some authors discussed few crucial decision-making issues for IVTSFSs and bipolar-valued hesitant fuzzy sets (Hussain et al. 2022; Akram et al. 2022; Ullah et al. 2018a; Khan et al. 2017) Further, these operators have been applied to MADM problems. Garg et al. (2021) examined power aggregation operators. Furthermore, Ullah et al. (2020a, b, 2018b) evaluated the functioning of rescue robots by utilizing Hamacher aggregation operators, some similarity measures for TSFS, and some averaging aggregation operators and their applications in multi-attribute decision-making problems.

The above-mentioned already existing studies, under the TSF framework, are vigorously useful to resolve the ambiguities during the procedure of decision-making, but most of them do not involve neutral character to discuss the neutral values assigned by decision-makers. Although He et al. (2015) and Garg (2020a, b), Garg and Chen (2020) has discussed and utilized the neutral character in many MADM problems under IFSs, PyFSs, SVNSs, and q-ROFSs. Recently, Javed et al. (2022) extended this novel idea to PFS and proposed useful AOs to evaluate the DM problems. Furthermore, Garg et al. (2022) investigated neutrality and a novel DM scheme for complex q-rung orthopair fuzzy sets. But all aforementioned fuzzy frameworks have certain limitations regarding range, loss of useful information, and application in many real-life problems. Henceforth, in this manuscript, we have established the notion of T-spherical fuzzy neutral aggregation (NA) operators that could handle the information in an adaptable manner from anywhere in the interval [0, 1] without any restriction. Here we established some innovative operational laws for TSFSs by combining the probability sum (PS) function and the proportional distribution rules of MD, AD, NMD, and degree of refusal. Having regard to this point, we intend to establish some new weighted averaging aggregation operators for TSFSs and consequently a MAGDM to figure out the decision-making problems. Thus, the goals of this manuscript are:

1. To show the data of decision-makers by utilizing TSFSs environment.
2. To suggest some novel neutral averaging AOs for TSFNs.
3. To lay out the MAGDM procedure and explain with several numerical examples to evaluate the study.
4. To demonstrate the importance and superiority of suggested neutral aggregation operators over several existing aggregation operators with the help of practical examples.

In this paper, we intend to discuss TSF weighted, ordered, and hybrid neutral averaging AOs represented by TSFWNA, TSFOWNA, and TSFHNA respectively, and develop their designating attributes. Further, we propose a new MAGDM procedure to handle the DM problems fusing with suggested AOs. Lastly, the superiority of the MAGDM procedure is exhibited via evident examples with the existing methods (Ullah et al. 2020a, b, 2021).

The rest of the manuscript is organized as follows. Section 2 contains some preliminaries associated with TSFS. Section 3 discusses some novel neutral operational laws and their properties for TSFNs. In Sect.4, TSFWNA, TSFOWNA, and TSFHNA operators are defined and their important properties are discussed. In Sect. 5, a new MAGDM plan is presented. In Sect. 6, some examples are furnished to verify the proposed method and a comparative survey is summarized. In Sect. 7, concrete results are given.

2 Preliminaries

In this section, we recollect some fundamental definitions related to TSFS are revised; we recall some basic but necessary definitions. Throughout the paper, X denotes a non-empty set, and $\dot{\check{s}}$, ị, ḑ, and r denote MD, NMD, abstinence degree (AD), and refusal degree (RD) respectively. The concept of TSFS was proposed by Mahmood et al. (2019) by narrating the fuzzy data using MD, NMD, AD, and RD.

Definition 1 (Ullah et al. 2018b) A TSFS based on an MD denoted by $\dot{\dot{s}}$, NMD ḑ, AD ị, and RD r, is of the from

$$\beta = \left\{ \begin{array}{c} \left(x, \left(\dot{\dot{s}}, \text{ị}, \text{ḑ}\right)\right) : \dot{\dot{s}} : X \to [0,1], \text{ị} : X \to [0,1] \quad and \quad \text{ḑ} : X \to [0,1] \forall x \in X \\ 0 \leq (\dot{\dot{s}}^q(x) + \text{ị}^q(x) + \text{ḑ}^q(x) \leq 1, q \in \mathbb{Z}^+ \\ r(x) = \sqrt[q]{1 - (\dot{\dot{s}}^q(x) + \text{ị}^q(x) + \text{ḑ}^q(x))} \end{array} \right\} \tag{1}$$

The triplet $\left(\dot{\dot{s}}(x), \text{ị}(x), \text{ḑ}(x)\right)$ is known as a TSF number (TSFN).

Definition 2 (Mahmood et al. (2019; Ullah et al. 2018b) For two TSFNs $\beta_1 = (\dot{\dot{s}}_1, \text{ị}_1, \text{ḑ}_1)$ and $\beta_2 = (\dot{\dot{s}}_2, \text{ị}_2, \text{ḑ}_2)$ and a real $\lambda > 0$, then characteristic axioms are defined as

(1) $\beta_1^C = \left(\text{ḑ}_1, \text{ị}_1, \dot{\dot{s}}_1\right)$,

(2) $\beta_1 \subseteq \beta_2$ if $\dot{\dot{s}}_1 \leq \dot{\dot{s}}_2$, $\text{ị}_1 \geq \text{ị}_2$ and $\text{ḑ}_1 \geq \text{ḑ}_2$,

(3) $\beta_1 = \beta_2$ if $\beta_1 \subseteq \beta_2$ and $\beta_2 \subseteq \beta_1$,

(4) $\beta_1 \oplus \beta_2 = \left(\sqrt[q]{\dot{\dot{s}}_1^q + \dot{\dot{s}}_2^q - \dot{\dot{s}}_1^q \dot{\dot{s}}_2^q}, \text{ị}_1\text{ị}_2, \text{ḑ}_1\text{ḑ}_2\right)$,

(5) $\beta_1 \otimes \beta_2 = \left(\dot{\dot{s}}_1\dot{\dot{s}}_{2,}, \text{ị}_1\text{ị}_2, \sqrt[q]{\text{ḑ}_1^q + \text{ḑ}_2^q - \text{ḑ}_1^q\text{ḑ}_2^q}\right)$,

(6) $\lambda\beta_1 = \left(\sqrt[q]{1 - \left(1 - \dot{\dot{s}}_1^q\right)^\lambda}, \text{ị}_1^\lambda, \text{ḑ}_1^\lambda\right)$,

(7) $\beta_1{}^\lambda = \left(\dot{\dot{s}}_1^\lambda, \text{ị}_1^\lambda, \sqrt[q]{1 - \left(1 - \dot{\dot{s}}_1^q\right)^\lambda}\right)$.

Definition 3 (Mahmood et al. (2019) For a TSFN $\beta = \left(\dot{\dot{s}}(x), \text{ị}(x), \text{ḑ}(x)\right)$, the score function and accuracy function for TSFNs are defined as, respectively.

$$SC(\beta) = \dot{\dot{s}}^q - \text{ḑ}^q, \quad \text{and} \quad SC(\beta) \in [-1, 1] \tag{2}$$

$$AC(\beta) = \dot{\dot{s}}^q + \text{ị}^q + \text{ḑ}^q, \quad \text{and} \quad AC(\beta) \in [0, 1] \tag{3}$$

Definition 4 (Mahmood et al. (2019) Let β_1 and β_2 be two TSFNs, $SC(\beta_i)$ is "score function" and $AC(\beta_i)$ is "accuracy function" of β_i, then $\beta_1 > \beta_2$ where the notation $>$ stands for "preferred to" if either $SC(\beta_1) > SC(\beta_2)$ or $SC(\beta_1) = SC(\beta_2)$ and $AC(\beta_1) > AC(\beta_2)$ holds.

For a collection of TSFNs $\beta_j = \left(\dot{\dot{s}}_j, \text{ị}_j, \text{ḑ}_j\right)$, $j = 1, 2, ..., n$, the weighted averaging AOs are formulated as under.

Definition 5 (Ullah et al. 2020b) Let $\beta_j = \left(\dot{\dot{s}}_j, \text{ị}_j, \text{ḑ}_j\right)$, $j = 1, 2, ..., n$ be a group of TSFNs and $\omega = (\omega_1, \omega_2, ..., \omega_n)$ be the weight vector of β_j with$\sum_{j=1}^n \omega_j = 1$, then a TSF weighted and ordered weighted averaging, represented by TSFWA and TSFOWA operators are determined as

$$\mathrm{TSFWA}(\beta_1, \beta_2, ..., \beta_n) = \left(\sqrt[q]{1 - \prod_{j=1}^{n} \left(1 - \dot{s}_j^q\right)^{\omega_j}}, \prod_{j=1}^{n} \left(\text{ị}_j\right)^{\omega_j}, \prod_{j=1}^{n} \left(\text{ḑ}_j\right)^{\omega_j} \right), \tag{4}$$

$$\mathrm{TSFOWA}(\beta_1, \beta_2, ..., \beta_n) = \left(\sqrt[q]{1 - \prod_{j=1}^{n} \left(1 - \dot{s}_{\sigma(j)}^q\right)^{\omega_j}}, \prod_{j=1}^{n} \left(\text{ị}_{\sigma(j)}\right)^{\omega_j}, \prod_{j=1}^{n} \left(\text{ḑ}_{\sigma(j)}\right)^{\omega_j} \right), \tag{5}$$

where σ is a permutation map of $(1, 2, ..., n)$ such that $\beta_{\sigma(j-1)} \geq \beta_{\sigma(j)}$ for all $j = 2, 3, ..., n$.

3 New operational laws on TSFNs

This section establishes some novel operational laws known as neutrality operations and a novel score function is defined to rank the TSFNs.

3.1 A new score function

The given score function in Definition 3 cannot obtain the accurate ranking of all TSFNs. For instance, $\beta_1 = (0.7, 0.7, 0.7)$ and $\beta_2 = (0.8.0.8, 0.8)$**,** we get $SC(\beta_1) = SC(\beta_2) = 0$ by utilizing Definition 3. To handle such issues under TSFSs, a novel score function is introduced for TSFS.

Definition 6 For a TSFN $\beta = \left(\dot{s}, \text{ị}, \text{ḑ}\right)$, a new score function is defined as.

$$\mathcal{S}(\beta) = \frac{e^{\dot{s}^q - \text{ị}^q - \text{ḑ}^q}}{1 + r^q}, \tag{6}$$

where r denotes the degree of refusal and is calculated as $r = \sqrt[q]{1 - \dot{s}^q - \text{ị}^q - \text{ḑ}^q}$.

Theorem 1 *For a TSFN $\beta = \left(\dot{s}, \text{ị}, \text{ḑ}\right)$,the score function $\mathcal{S}$increases monotonically with respect to $\dot{s}$and decreases with respect to* ị *and* ḑ.

Proof Differentiating the suggested score function partially with respect to $\dot{s}$ provides $\frac{\partial \mathcal{S}}{\partial \dot{s}} = \frac{(2+r^q)q\dot{s}^{q-1}e^{\dot{s}^q - \text{ị}^q - \text{ḑ}^q}}{\left(2-\dot{s}^q-\text{ị}^q-\text{ḑ}^q\right)^2} \geq 0$ while with respect to ị and ḑ gives $\frac{\partial \mathcal{S}}{\partial \text{ị}} = \frac{-r^q q \text{ị}^{q-1} e^{\dot{s}^q - \text{ị}^q - \text{ḑ}^q}}{\left(2-\dot{s}^q-\text{ị}^q-\text{ḑ}^q\right)^2} \leq 0$ and $\frac{\partial \mathcal{S}}{\partial \text{ḑ}} = \frac{-r^q q \text{ḑ}^{q-1} e^{\dot{s}^q - \text{ị}^q - \text{ḑ}^q}}{\left(2-\dot{s}^q-\text{ị}^q-\text{ḑ}^q\right)^2} \leq 0$ hence result proved.

Theorem 2 *For a TSFN β,a new score function $\mathcal{S}$satisfies.*

(1) $e^{-1} \leq \mathcal{S} \leq e$,
(2) $\mathcal{S}(\beta) = e$ iff $\beta = (1, 0, 0)$,
(3) $\mathcal{S}(\beta) = e^{-1}$ iff $\beta = (0, 1, 0)$ or $\beta = (0, 0, 1)$.

This result is trivially true, so omitted.

To verify the utility of the suggested score function we consider TSFNs $\beta_1 = (0.7, 0.7, 0.7)$ and $\beta_2 = (0.6, 0.6, 0.6)$ for ranking purposes. By applying existing and suggested functions it is observed that the newly proposed score function is superior to the existing ones. The following result justifies our claim.

Theorem 3 *For two TSFNs $\beta_1$and β_2,if $SC(\beta_1) = SC(\beta_2)$, $AC(\beta_1) > AC(\beta_2)$then $\mathcal{S}(\beta_1) > S(\beta_2)$.Further, if $SC(\beta_1) = SC(\beta_2)$,$AC(\beta_1) = AC(\beta_2)$then $\mathcal{S}(\beta_1) = \mathcal{S}(\beta_2)$.*

Proof Let $\beta_1 = \left(\dot{s}_1, \mathrm{i}_1, \mathrm{d}_1\right)$ and $\beta_2 = \left(\dot{s}_2.\mathrm{i}_2, \mathrm{d}_2\right)$ be two TSFNs, then by Definition 4, if $SC(\beta_1) = SC(\beta_2)$,$AC(\beta_1) > AC(\beta_2)$ we have $\dot{s}_1^q - \mathrm{d}_1^q = \dot{s}_2^q - \mathrm{d}_2^q$, $\dot{s}_1^q + \mathrm{i}_1^q + \mathrm{d}_1^q > \dot{s}_2^q + \mathrm{i}_2^q + \mathrm{d}_2^q$. Therefore, we can get $\mathcal{S}(\beta_1) = \frac{e^{\dot{s}_1^q - \mathrm{i}_1^q - \mathrm{d}_1^q}}{2 - \left(\dot{s}_1^q + \mathrm{i}_1^q + \mathrm{d}_1^q\right)} \geq \frac{e^{\dot{s}_2^q - \mathrm{i}_2^q - \mathrm{d}_2^q}}{2 - \left(\dot{s}_2^q + \mathrm{i}_2^q + \mathrm{d}_2^q\right)} = \mathcal{S}(\beta_2)$.

3.2 Geometric meaning of PS and rules of TSFNs

For two TSFNs $\beta_1 = \left(\dot{s}_1, \mathrm{i}_1, \mathrm{d}_1\right)$ and $\beta_2 = \left(\dot{s}_2.\mathrm{i}_2, \mathrm{d}_2\right)$, the pictorial representation of the probability sum (PS) is depicted in Fig. 1. Here, $\dot{s}_1$ and $\dot{s}_2$ denote the MDs of β_1 and β_2 while $\mathrm{i}_1, \mathrm{i}_2$, and $\mathrm{d}_1, \mathrm{d}_2$ denote the ADs and NMDs respectively. Then clearly $\sqrt[q]{\dot{s}_1^q + \mathrm{i}_1^q + \mathrm{d}_1^q}$ and $\sqrt[q]{\dot{s}_2^q + \mathrm{i}_2^q + \mathrm{d}_2^q}$ are two events that do not influence the probability of each other, under the TSF environment. Express $\sqrt[q]{\dot{s}_\varepsilon^q + \mathrm{i}_\varepsilon^q + \mathrm{d}_\varepsilon^q}$ be the PS of occurring at least one events of $\sqrt[q]{\dot{s}_1^q + \mathrm{i}_1^q + \mathrm{d}_1^q}$ and $\sqrt[q]{\dot{s}_2^q + \mathrm{i}_2^q + \mathrm{d}_2^q}$; thus,

$$\sqrt[q]{\dot{s}_\varepsilon^q + \mathrm{i}_\varepsilon^q + \mathrm{d}_\varepsilon^q} = PS\left(\sqrt[q]{\dot{s}_1^q + \mathrm{i}_1^q + \mathrm{d}_1^q}, \sqrt[q]{\dot{s}_2^q + \mathrm{i}_2^q + \mathrm{d}_2^q}\right) \tag{7}$$

It is quite obvious that the total sum of the TSFNs β_1 and β_2 is concluded as under:

(1) Accumulate the MD, NMD, AD, and refusal degree of the TSFNs by the algebraic sum operations $T(u, v) = \sqrt[q]{u^q + v^q}$, then we get $\sqrt[q]{\dot{s}_1^q + \dot{s}_2^q}$,$\sqrt[q]{\mathrm{i}_1^q + \mathrm{i}_2^q}$,$\sqrt[q]{\mathrm{d}_1^q + \mathrm{d}_2^q}$ and $\sqrt[q]{r_1^q + r_2^q}$. as shown in Fig. 2. It is absolutely clear that their sum of q th exponent is greater than 1 which is no more a TSFN.

(2) To compute the total sum under TSF environment we choose a mutual interaction coefficient $\sqrt[q]{\frac{1 - r_1^q r_2^q}{\dot{s}_1^q + \dot{s}_2^q + \mathrm{i}_1^q + \mathrm{i}_2^q + \mathrm{d}_1^q + \mathrm{d}_2^q}}$ to $\sqrt[q]{\dot{s}_1^q + \dot{s}_2^q}$, $\sqrt[q]{\mathrm{i}_1^q + \mathrm{i}_2^q}$ and $\sqrt[q]{\mathrm{d}_1^q + \mathrm{d}_2^q}$, which are considered

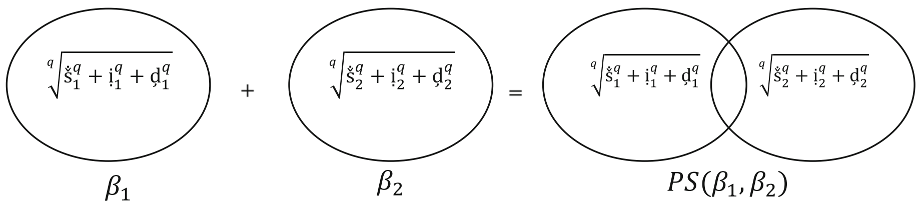

Fig. 1 Geometrical description of PS function

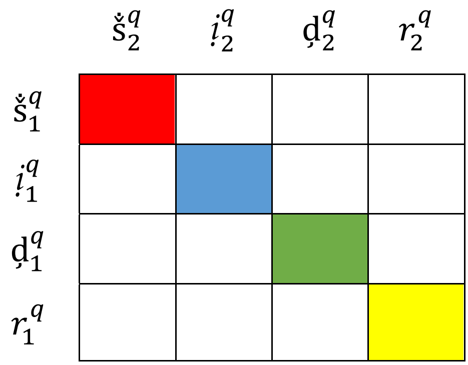

Fig. 2 Pictorial representation of suggested operations

as MD, AD, and NMD with the sum of q th exponent sum is less than 1. Similarly, the degree of refusal is $r_1 r_2$. So,

$$\dot{s}_{\varepsilon} = \sqrt[q]{\frac{\dot{s}_1^q + \dot{s}_2^q}{\dot{s}_1^q + \dot{s}_2^q + \mathrm{i}_1^q + \mathrm{i}_2^q + \mathrm{d}_1^q + \mathrm{d}_2^q}\left(1 - r_1^q r_2^q\right)},$$

$$\mathrm{i}_{\varepsilon} = \sqrt[q]{\frac{\mathrm{i}_1^q + \mathrm{i}_2^q}{\dot{s}_1^q + \dot{s}_2^q + \mathrm{i}_1^q + \mathrm{i}_2^q + \mathrm{d}_1^q + \mathrm{d}_2^q}\left(1 - r_1^q r_2^q\right)},$$

$$\mathrm{d}_{\varepsilon} = \sqrt[q]{\frac{\mathrm{d}_1^q + \mathrm{d}_2^q}{\dot{s}_1^q + \dot{s}_2^q + \mathrm{i}_1^q + \mathrm{i}_2^q + \mathrm{d}_1^q + \mathrm{d}_2^q}\left(1 - r_1^q r_2^q\right)}.$$

Before we illustrate some novel operational laws for TSFNs, it is necessary to mention that Eq. (7) preserves characteristics like commutatively, associatively, and boundedness. Furthermore, such a procedure provides us advantages in the case when the MD, AD, and NMD have the same values.

3.3 Neutral operational laws

Definition 7 Consider $\beta_1 = \left(\dot{s}_1, \mathrm{i}_1, \mathrm{d}_1\right)$ and $\beta_2 = \left(\dot{s}_2.\mathrm{i}_2, \mathrm{d}_2\right)$ be two TSFNs. The neutrality operation of β_1 and β_2 is defined as

$$\beta_1 \ominus \beta_2 = \begin{pmatrix} \sqrt[q]{\frac{MCS^q(\beta_1,\beta_2)}{MCS^q(\beta_1,\beta_2)+NCS^q(\beta_1,\beta_2)+ACS^q(\beta_1,\beta_2)}}.PS\left(\sqrt[q]{\dot{s}_1^q+\mathrm{i}_1^q+\mathrm{d}_1^q},\sqrt[q]{\dot{s}_2^q+\mathrm{i}_2^q+\mathrm{d}_2^q}\right), \\ \sqrt[q]{\frac{ACS^q(\beta_1,\beta_2)}{MCS^q(\beta_1,\beta_2)+NCS^q(\beta_1,\beta_2)+ACS^q(\beta_1,\beta_2)}}.PS\left(\sqrt[q]{\dot{s}_1^q+\mathrm{i}_1^q+\mathrm{d}_1^q},\sqrt[q]{\dot{s}_2^q+\mathrm{i}_2^q+\mathrm{d}_2^q}\right), \\ \sqrt[q]{\frac{NCS^q(\beta_1,\beta_2)}{MCS^q(\beta_1,\beta_2)+NCS^q(\beta_1,\beta_2)+ACS^q(\beta_1,\beta_2)}}.PS\left(\sqrt[q]{\dot{s}_1^q+\mathrm{i}_1^q+\mathrm{d}_1^q},\sqrt[q]{\dot{s}_2^q+\mathrm{i}_2^q+\mathrm{d}_2^q}\right) \end{pmatrix} \quad (8)$$

where MCS $(\beta_1, \beta_2) = \sqrt[q]{\dot{s}_1^q + \dot{s}_2^q}$, NCS $(\beta_1, \beta_2) = \sqrt[q]{\mathrm{d}_1^q + \mathrm{d}_2^q}$ and ACS $(\beta_1, \beta_2) = \sqrt[q]{\mathrm{i}_1^q + \mathrm{i}_2^q}$ denotes the MD, NMD, and AD coefficient sum of β_1 and $\beta_{2,}$ respectively.

For TSFN $\beta_1 = \left(\dot{s}_1, \mathrm{i}_1, \mathrm{d}_1\right)$ and real $\tau > 1$, we have

$$PS\left(\tau\sqrt[q]{\dot{s}_1^q + \mathrm{i}_1^q + \mathrm{d}_1^q}\right) = PS\left(\sqrt[q]{\dot{s}_1^q + \mathrm{i}_1^q + \mathrm{d}_1^q}, PS\left((\tau - 1)\sqrt[q]{\dot{s}_1^q + \mathrm{i}_1^q + \mathrm{d}_1^q}\right)\right) \quad (9)$$

and, therefore, we stated necessary propositions.

Proposition 1 *For a TSFN* $\beta_1 = \left(\dot{s}_1, \underset{\cdot}{i}_1, d_1\right)$*and a real* $\tau > 0$,*we obtain.*

$$\mathrm{PS}\left(\tau\sqrt[q]{\dot{s}_1^q + \underset{\cdot}{i}_1^q + d_1^q}\right) = \sqrt[q]{1 - \left(r_1^q\right)^\tau}. \tag{10}$$

The proof can be seen in the Appendix.

Take a collection of "n" TSFNs $\beta_j = \left(\dot{s}_j, \underset{\cdot}{i}_j, d_j\right)$, $j = 1, 2, ..., n$, such that $MCS\left(\beta_j\right) = \dot{s}_j$ and$MCS^q(\beta_1, ...\beta_n) = MCS^q(\beta_1, ...\beta_{n-1}) + \dot{s}_n^q$. Thus, we have$MCS^q(\beta_1, ...\beta_n) = \sum_{j=1}^n \dot{s}_j^q$. Likewise, we get $ACS^q(\beta_1, ...\beta_n) = \sum_{j=1}^n \underset{\cdot}{i}_j^q$ and$NCS^q(\beta_1, ...\beta_n) = \sum_{j=1}^n d_j^q$.

Proposition 2 *For TSFN* $\beta_1 = \left(\dot{s}_1, \underset{\cdot}{i}_1, d_1\right)$*and a real number* $\tau > 0$,*we have* $MCS(\tau\beta_1) = \sqrt[q]{\tau}MCS(\beta_1)$, $NCS(\tau\beta_1) = \sqrt[q]{\tau}NCS(\beta_1)$,*and* $ACS(\tau\beta_1) = \sqrt[q]{\tau}ACS(\beta_1)$.

Proof If we take $\beta_1 = \beta_2$ in MCS then $MCS(2\beta_1) = MCS(\beta_1, \beta_1) = \sqrt[q]{\dot{s}_1^q + \dot{s}_1^q} = \sqrt[q]{2}MCS(\beta_1)$. Using mathematical induction, we see that $MCS(\tau\beta_1) = \sqrt[q]{\tau}MCS(\beta_1)$, $NCS(\tau\beta_1) = \sqrt[q]{\tau}NCS(\beta_1)$, and $ACS(\tau\beta_1) = \sqrt[q]{\tau}ACS(\beta_1)$.

Definition 8 For TSFN $\beta_1 = \left(\dot{s}_1, \underset{\cdot}{i}_1, d_1\right)$ and a real number $\tau \geq 0$, the "scalar neutrality operation" is defined as.

$$\tau.\beta_1 = \begin{pmatrix} \sqrt[q]{\dfrac{MCS^q(\tau.\beta_1)}{MCS^q(\tau.\beta_1) + NCS^q(\tau.\beta_1) + ACS^q(\tau.\beta_1)}}.PS\left(\tau.\sqrt[q]{\dot{s}_1^q + \underset{\cdot}{i}_1^q + d_1^q}\right), \\ \sqrt[q]{\dfrac{ACS^q(\tau.\beta_1)}{MCS^q(\tau.\beta_1) + NCS^q(\tau.\beta_1) + ACS^q(\tau.\beta_1)}}.PS\left(\tau.\sqrt[q]{\dot{s}_1^q + \underset{\cdot}{i}_1^q + d_1^q}\right), \\ \sqrt[q]{\dfrac{NCS^q(\beta_1, \beta_2)}{MCS^q(\tau.\beta_1) + NCS^q(\tau.\beta_1) + ACS^q(\tau.\beta_1)}}.PS\left(\tau.\sqrt[q]{\dot{s}_1^q + \underset{\cdot}{i}_1^q + d_1^q}\right) \end{pmatrix} \tag{11}$$

where $(\tau\beta_1) = \sqrt[q]{\tau\dot{s}_1}$,$NCS(\tau\beta_1) = \sqrt[q]{\tau}d_1$ and $ACS(\tau\beta_1) = \sqrt[q]{\tau}\underset{\cdot}{i}_1$.

To prove the suggested operation defined in Eq. (8) and Eq. (11) gives procedural fairness, we represent them in a particular proposition.

Proposition 3 *For two TSFNs* $\beta_1 = \left(\dot{s}_1, \underset{\cdot}{i}_1, d_1\right)$*and* $\beta_2 = \left(\dot{s}_2.\underset{\cdot}{i}_2, d_2\right)$. *If* $\dot{s}_1 = \underset{\cdot}{i}_1 = d_1$*and* $\dot{s}_2 = \underset{\cdot}{i}_2 = d_2$ *then* $\dot{s}_{\beta_1 \ominus \beta_2} = d_{\beta_1 \ominus \beta_2} = \underset{\cdot}{i}_{\beta_1 \ominus \beta_2}$.

Proof If $\dot{s}_1 = \underset{\cdot}{i}_1 = d_1$ and $\dot{s}_2 = \underset{\cdot}{i}_2 = d_2$ then using the neutrality operation of β_1 and β_2, we have

$\frac{\ddot{s}_{\beta_1\ominus\beta_2}}{\underset{.}{i}_{\beta_1\ominus\beta_2}} = \sqrt[q]{\frac{MCS^q(\beta_1,\beta_2)}{ACS^q(\beta_1,\beta_2)}} = \sqrt[q]{\frac{\ddot{s}_1^q+\ddot{s}_2^q}{\underset{.}{i}_1^q+\underset{.}{i}_2^q}} = 1$ and $\frac{\ddot{s}_{\beta_1\ominus\beta_2}}{\underset{,}{d}_{\beta_1\ominus\beta_2}} = \sqrt[q]{\frac{MCS^q(\beta_1,\beta_2)}{NCS^q(\beta_1,\beta_2)}} = \sqrt[q]{\frac{\ddot{s}_1^q+\ddot{s}_2^q}{\underset{,}{d}_1^q+\underset{,}{d}_2^q}} = 1.$
Hence, proved

$$\ddot{s}_{\beta_1\ominus\beta_2} = \underset{,}{d}_{\beta_1\ominus\beta_2} = \underset{.}{i}_{\beta_1\ominus\beta_2}..$$

Additionally, it is quite clear from Definition 2, we can treat Eq. (8) as a "neutrality operation"" and likewise Eq. (11) as a "scalar neutrality operation". Now, Eq. (8) can be rearranged by substituting the values of MCS, NCS, ACS, and PS function

$$\beta_1 \ominus \beta_2 = \left(\begin{array}{c} \sqrt[q]{\frac{\ddot{s}_1^q + \ddot{s}_2^q}{\ddot{s}_1^q + \ddot{s}_2^q + \underset{.}{i}_1^q + \underset{.}{i}_2^q + \underset{,}{d}_1^q + \underset{,}{d}_2^q}\left(1 - r_1^q r_2^q\right)}, \\ \sqrt[q]{\frac{\underset{.}{i}_1^q + \underset{.}{i}_2^q}{\ddot{s}_1^q + \ddot{s}_2^q + \underset{.}{i}_1^q + \underset{.}{i}_2^q + \underset{,}{d}_1^q + \underset{,}{d}_2^q}\left(1 - r_1^q r_2^q\right)}, \\ \sqrt[q]{\frac{\underset{,}{d}_1^q + \underset{,}{d}_2^q}{\ddot{s}_1^q + \ddot{s}_2^q + \underset{.}{i}_1^q + \underset{.}{i}_2^q + \underset{,}{d}_1^q + \underset{,}{d}_2^q}\left(1 - r_1^q r_2^q\right)} \end{array} \right). \tag{12}$$

Following, we illustrate the origin of Definition 8. As Eq. (8) gives us

$$\beta_1 \ominus \beta_1 = 2\beta_1$$

$$= \left(\begin{array}{c} \sqrt[q]{\frac{MCS^q(\beta_1,\beta_1)}{MCS^q(\beta_1,\beta_1) + NCS^q(\beta_1,\beta_1) + ACS^q(\beta_1,\beta_1)}.PS\left(\sqrt[q]{\ddot{s}_1^q + \underset{.}{i}_1^q + \underset{,}{d}_1^q}, \sqrt[q]{\ddot{s}_1^q + \underset{.}{i}_1^q + \underset{,}{d}_1^q}\right)}, \\ \sqrt[q]{\frac{ACS^q(\beta_1,\beta_1)}{MCS^q(\beta_1,\beta_1) + NCS^q(\beta_1,\beta_1) + ACS^q(\beta_1,\beta_1)}.PS\left(\sqrt[q]{\ddot{s}_1^q + \underset{.}{i}_1^q + \underset{,}{d}_1^q}, \sqrt[q]{\ddot{s}_1^q + \underset{.}{i}_1^q + \underset{,}{d}_1^q}\right)}, \\ \sqrt[q]{\frac{NCS^q(\beta_1,\beta_1)}{MCS^q(\beta_1,\beta_1) + NCS^q(\beta_1,\beta_1) + ACS^q(\beta_1,\beta_1)}.PS\left(\sqrt[q]{\ddot{s}_1^q + \underset{.}{i}_1^q + \underset{,}{d}_1^q}, \sqrt[q]{\ddot{s}_1^q + \underset{.}{i}_1^q + \underset{,}{d}_1^q}\right)} \end{array} \right)$$

$$= \left(\sqrt[q]{\frac{\ddot{s}_1^q}{\ddot{s}_1^q + \underset{.}{i}_1^q + \underset{,}{d}_1^q}\left(1 - \left(r_1^q\right)^2\right)}, \sqrt[q]{\frac{\underset{.}{i}_1^q}{\ddot{s}_1^q + \underset{.}{i}_1^q + \underset{,}{d}_1^q}\left(1 - \left(r_1^q\right)^2\right)}, \sqrt[q]{\frac{\underset{,}{d}_1^q}{\ddot{s}_1^q + \underset{.}{i}_1^q + \underset{,}{d}_1^q}\left(1 - \left(r_1^q\right)^2\right)} \right)$$

Similarly,

$$3\beta_1 = \left(\sqrt[q]{\frac{\ddot{s}_1^q}{\ddot{s}_1^q + \underset{.}{i}_1^q + \underset{,}{d}_1^q}\left(1 - \left(r_1^q\right)^3\right)}, \sqrt[q]{\frac{\underset{.}{i}_1^q}{\ddot{s}_1^q + \underset{.}{i}_1^q + \underset{,}{d}_1^q}\left(1 - \left(r_1^q\right)^3\right)}, \sqrt[q]{\frac{\underset{,}{d}_1^q}{\ddot{s}_1^q + \underset{.}{i}_1^q + \underset{,}{d}_1^q}\left(1 - \left(r_1^q\right)^3\right)} \right)$$

For any $\tau > 0$, we have

$$\tau\beta_1 = s\left(\sqrt[q]{\frac{\ddot{s}_1^q}{\ddot{s}_1^q + \underset{.}{i}_1^q + \underset{,}{d}_1^q}\left(1 - \left(r_1^q\right)^\tau\right)}, \sqrt[q]{\frac{\underset{.}{i}_1^q}{\ddot{s}_1^q + \underset{.}{i}_1^q + \underset{,}{d}_1^q}\left(1 - \left(r_1^q\right)^\tau\right)}, \sqrt[q]{\frac{\underset{,}{d}_1^q}{\ddot{s}_1^q + \underset{.}{i}_1^q + \underset{,}{d}_1^q}\left(1 - \left(r_1^q\right)^\tau\right)} \right)$$

Proposition 4 *The MCS, NCS, and ACS for two TSFNs $\beta_1$and $\beta_2$have the following characteristics.*

(1) $MCS(\beta_1, \beta_2) = MCS(\beta_2, \beta_1)$,
(2) $MCS(\tau(\beta_1, \beta_2)) = MCS\left(\tau\beta_1, \tau\beta_2\right)$,
(3) $MCS\left(\tau_1\beta_1, \tau_2\beta_1\right) = MCS((\tau_1 + \tau_{21})\beta_1)$,

(4) $NCS(\beta_1,\beta_2) = NCS(\beta_2,\beta_1)$,
(5) $NCS(\tau(\beta_1,\beta_2)) = NCS\big(\tau\beta_1,\tau\beta_2\big)$,
(6) $NCS\big(\tau_1\beta_1,\tau_2\beta_1\big) = NCS((\tau_1+\tau_{21})\beta_1)$,
(7) $ACS(\beta_1,\beta_2) = ACS(\beta_2,\beta_1)$,
(8) $ACS(\tau(\beta_1,\beta_2)) = ACS\big(\tau\beta_1,\tau\beta_2\big)$,
(9) $ACS\big(\tau_1\beta_1,\tau_2\beta_1\big) = ACS((\tau_1+\tau_{21})\beta_1)$.

Proof For TSFNs $\beta_1 = \left(\dot{\check{s}}_1, \dot{\mathrm{i}}_1, \mathrm{d}_1\right)$ and $\beta_2 = \left(\dot{\check{s}}_2.\dot{\mathrm{i}}_2, \mathrm{d}_2\right)$, we have $MCS(\beta_1,\beta_2) = \sqrt[q]{\dot{\check{s}}_1^q + \dot{\check{s}}_2^q}, NCS(\beta_1,\beta_2) = \sqrt[q]{\mathrm{d}_1^q + \mathrm{d}_2^q}$ and $ACS(\beta_1,\beta_2) = \sqrt[q]{\dot{\mathrm{i}}_1^q + \dot{\mathrm{i}}_2^q}, MCS(\tau\beta_1) = \sqrt[q]{\tau}\dot{\check{s}}_1$, $NCS(\tau\beta_1) = \sqrt[q]{\tau}\mathrm{d}_1$, and $ACS(\tau\beta_1) = \sqrt[q]{\tau}\dot{\mathrm{i}}_1$. Thus, $MCS\big(\tau\beta_1,\tau\beta_2\big) = \sqrt[q]{\left(\sqrt[q]{\tau}\dot{\check{s}}_1\right)^q + \left(\sqrt[q]{\tau}\dot{\check{s}}_2\right)^q} = \sqrt[q]{\tau}\sqrt[q]{\dot{\check{s}}_1^q + \dot{\check{s}}_2^q} = \sqrt[q]{\tau}MCS(\beta_1,\beta_2) = MCS(\tau(\beta_1,\beta_2))$. The remaining properties have similar proofs.

3.4 Properties of the proposed operations

In this section, we illustrate some properties of the proposed operational laws as under.

Theorem 4 *If* $\beta_1 = \left(\dot{\check{s}}_1, \dot{\mathrm{i}}_1, \mathrm{d}_1\right)$ *and* $\beta_2 = \left(\dot{\check{s}}_2.\dot{\mathrm{i}}_2, \mathrm{d}_2\right)$ *be TSFNs. Then operations of* β_1 *and* β_2, $\beta_1 \ominus \beta_2$ *and* $\tau\beta_1$ *are also TSFNs for* $\tau > 0$.

Proof For TSFNs $\beta_1 = \left(\dot{\check{s}}_1, \dot{\mathrm{i}}_1, \mathrm{d}_1\right)$ and $\beta_2 = \left(\dot{\check{s}}_2.\dot{\mathrm{i}}_2, \mathrm{d}_2\right)$, we have $\dot{\check{s}}_1, \dot{\mathrm{i}}_1, \mathrm{d}_1, \dot{\check{s}}_2.\dot{\mathrm{i}}_2, \mathrm{d}_2 \in [0,1]$, $r_1^q = 1 - \dot{\check{s}}_1^q - \dot{\mathrm{i}}_1^q - \mathrm{d}_1^q \in [0,1]$ and $r_2^q = 1 - \dot{\check{s}}_2^q - \dot{\mathrm{i}}_2^q - \mathrm{d}_2^q \in [0,1]$. Let $\beta_1 \ominus \beta_2 = \left(\dot{\check{s}}_{\beta_1\ominus\beta_2}, \dot{\mathrm{i}}_{\beta_1\ominus\beta_2}, \mathrm{d}_{\beta_1\ominus\beta_2}\right)$.

To show $\beta_1 \ominus \beta_2 \in$ TSFN, it is sufficient to show that $\dot{\check{s}}_{\beta_1\ominus\beta_2}, \dot{\mathrm{i}}_{\beta_1\ominus\beta_2}, \mathrm{d}_{\beta_1\ominus\beta_2} \in [0,1]$ and $\dot{\check{s}}^q_{\beta_1\ominus\beta_2} + \dot{\mathrm{i}}^q_{\beta_1\ominus\beta_2} + \mathrm{d}^q_{\beta_1\ominus\beta_2} \le 1$. Since $\dot{\check{s}}_1^q + \dot{\mathrm{i}}_1^q + \mathrm{d}_1^q \le 1, \dot{\check{s}}_2^q + \dot{\mathrm{i}}_2^q + \mathrm{d}_2^q \le 1$ and hence $\frac{\dot{\check{s}}_1^q + \dot{\check{s}}_2^q}{\dot{\check{s}}_1^q + \dot{\check{s}}_2^q + \dot{\mathrm{i}}_1^q + \dot{\mathrm{i}}_2^q + \mathrm{d}_1^q + \mathrm{d}_2^q} \le 1$, provided $\dot{\check{s}}_1^q + \dot{\check{s}}_2^q + \dot{\mathrm{i}}_1^q + \dot{\mathrm{i}}_2^q + \mathrm{d}_1^q + \mathrm{d}_2^q \ne 0$. Also,$r_1, r_2 \in [0,1]$ which implies that $1 - r_1^q r_2^q \in [0,1]$. Hence, we can obtain $\dot{\check{s}}_{\beta_1\ominus\beta_2}, \dot{\mathrm{i}}_{\beta_1\ominus\beta_2}, \mathrm{d}_{\beta_1\ominus\beta_2} \in [0,1]$.

Moreover, $\dot{\check{s}}^q_{\beta_1\ominus\beta_2} + \dot{\mathrm{i}}^q_{\beta_1\ominus\beta_2} + \mathrm{d}^q_{\beta_1\ominus\beta_2} = 1 - r_1^q r_2^q \in [0,1]$. Hence $\beta_1 \ominus \beta_2$ is TSFN. Analogously,$\tau\beta_1$ is also TSFN.

Theorem 5 *Let* $\beta_1 = \left(\dot{\check{s}}_1, \dot{\mathrm{i}}_1, \mathrm{d}_1\right)$ *and* $\beta_2 = \left(\dot{\check{s}}_2.\dot{\mathrm{i}}_2, \mathrm{d}_2\right)$ *be two TSFNs and* $\tau, \tau_1, \tau_2 \ge 0$ *be real numbers. Then*

(1) $\beta_1 \ominus \beta_2 = \beta_2 \ominus \beta_1$
(2) $\tau(\beta_1 \ominus \beta_2) = \left(\tau\beta_1 \ominus \tau\beta_2\right)$
(3) $\tau_1\beta_1 \ominus \tau_2\beta_1 = (\tau_1+\tau_2)\beta_1$

Proof

(1) It follows from Eq. (12).
(2) Eq. (8) gives

$$\tau\beta_1 \ominus \tau\beta_2 = \begin{pmatrix} \sqrt[q]{\dfrac{MCS^q(\tau\beta_1,\tau\beta_2)}{MCS^q(\tau\beta_1,\tau\beta_2)+NCS^q\tau\beta_1,\tau\beta_2 + ACS^q(\tau\beta_1,\tau\beta_2)}}.PS\left(PS\left(\tau\sqrt[q]{\dot{s}_1^q+{\underset{\cdot}{\mathrm{i}}}_1^q+{\underset{,}{\mathrm{d}}}_1^q}\right), PS\left(\tau\sqrt[q]{\dot{s}_2^q+{\underset{\cdot}{\mathrm{i}}}_2^q+{\underset{,}{\mathrm{d}}}_2^q}\right)\right), \\ \sqrt[q]{\dfrac{NCS^q(\tau\beta_1,\tau\beta_2)}{MCS^q(\tau\beta_1,\tau\beta_2)+NCS^q(\tau\beta_1,\tau\beta_2) + ACS^q(\tau\beta_1,\tau\beta_2)}}.PS\left(PS\left(\tau\sqrt[q]{\dot{s}_1^q+{\underset{\cdot}{\mathrm{i}}}_1^q+{\underset{,}{\mathrm{d}}}_1^q}\right), PS\left(\tau\sqrt[q]{\dot{s}_2^q+{\underset{\cdot}{\mathrm{i}}}_2^q+{\underset{,}{\mathrm{d}}}_2^q}\right)\right), \\ \sqrt[q]{\dfrac{ACS^q(\tau\beta_1,\tau\beta_2)}{MCS^q(\tau\beta_1,\tau\beta_2)+NCS^q(\tau\beta_1,\tau\beta_2) + ACS^q(\tau\beta_1,\tau\beta_2)}}.PS\left(PS\left(\tau\sqrt[q]{\dot{s}_1^q+{\underset{\cdot}{\mathrm{i}}}_1^q+{\underset{,}{\mathrm{d}}}_1^q}\right), PS\left(\tau\sqrt[q]{\dot{s}_2^q+{\underset{\cdot}{\mathrm{i}}}_2^q+{\underset{,}{\mathrm{d}}}_2^q}\right)\right) \end{pmatrix},$$

$$= \begin{pmatrix} \sqrt[q]{\dfrac{\tau\left(\dot{s}_1^q+\dot{s}_2^q\right)}{\tau\left(\dot{s}_1^q+\dot{s}_2^q\right)+\tau\left({\underset{\cdot}{\mathrm{i}}}_1^q+{\underset{\cdot}{\mathrm{i}}}_2^q\right)+\tau\left({\underset{,}{\mathrm{d}}}_1^q+{\underset{,}{\mathrm{d}}}_2^q\right)}}PS\left(\sqrt[q]{1-\left(r_1^q\right)^\tau}, \sqrt[q]{1-\left(r_2^q\right)^\tau}\right), \\ \sqrt[q]{\dfrac{\tau\left({\underset{\cdot}{\mathrm{i}}}_1^q+{\underset{\cdot}{\mathrm{i}}}_2^q\right)}{\tau\left(\dot{s}_1^q+\dot{s}_2^q\right)+\tau\left({\underset{\cdot}{\mathrm{i}}}_1^q+{\underset{\cdot}{\mathrm{i}}}_2^q\right)+\tau\left({\underset{,}{\mathrm{d}}}_1^q+{\underset{,}{\mathrm{d}}}_2^q\right)}}PS\left(\sqrt[q]{1-\left(r_1^q\right)^\tau}, \sqrt[q]{1-\left(r_2^q\right)^\tau}\right), \\ \sqrt[q]{\dfrac{\tau\left({\underset{,}{\mathrm{d}}}_1^q+{\underset{,}{\mathrm{d}}}_2^q\right)}{\tau\left(\dot{s}_1^q+\dot{s}_2^q\right)+\tau\left({\underset{\cdot}{\mathrm{i}}}_1^q+{\underset{\cdot}{\mathrm{i}}}_2^q\right)+\tau\left({\underset{,}{\mathrm{d}}}_1^q+{\underset{,}{\mathrm{d}}}_2^q\right)}}PS\left(\sqrt[q]{1-\left(r_1^q\right)^\tau}, \sqrt[q]{1-\left(r_2^q\right)^\tau}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt[q]{\dfrac{\left(\dot{s}_1^q+\dot{s}_2^q\right)}{\left(\dot{s}_1^q+\dot{s}_2^q\right)+\left({\underset{\cdot}{\mathrm{i}}}_1^q+{\underset{\cdot}{\mathrm{i}}}_2^q\right)+\left({\underset{,}{\mathrm{d}}}_1^q+{\underset{,}{\mathrm{d}}}_2^q\right)}}\left(1-\left(r_1^q\right)^\tau\left(r_2^q\right)^\tau\right), \\ \sqrt[q]{\dfrac{\left({\underset{\cdot}{\mathrm{i}}}_1^q+{\underset{\cdot}{\mathrm{i}}}_2^q\right)}{\left(\dot{s}_1^q+\dot{s}_2^q\right)+\left({\underset{\cdot}{\mathrm{i}}}_1^q+{\underset{\cdot}{\mathrm{i}}}_2^q\right)+\left({\underset{,}{\mathrm{d}}}_1^q+{\underset{,}{\mathrm{d}}}_2^q\right)}}\left(1-\left(r_1^q\right)^\tau\left(r_2^q\right)^\tau\right), \\ \sqrt[q]{\dfrac{\left({\underset{,}{\mathrm{d}}}_1^q+{\underset{,}{\mathrm{d}}}_2^q\right)}{\left(\dot{s}_1^q+\dot{s}_2^q\right)+\left({\underset{\cdot}{\mathrm{i}}}_1^q+{\underset{\cdot}{\mathrm{i}}}_2^q\right)+\left({\underset{,}{\mathrm{d}}}_1^q+{\underset{,}{\mathrm{d}}}_2^q\right)}}\left(1-\left(r_1^q\right)^\tau\left(r_2^q\right)^\tau\right) \end{pmatrix}$$

$$
= \left(\begin{array}{l}
\sqrt[q]{\dfrac{\left(\dot{\check{s}}_1^q + \dot{\check{s}}_2^q\right)}{\left(\dot{\check{s}}_1^q + \dot{\check{s}}_2^q\right) + \left(\underset{\cdot}{\mathrm{i}}_1^q + \underset{\cdot}{\mathrm{i}}_2^q\right) + \left(\underset{,}{\mathrm{d}}_1^q + \underset{,}{\mathrm{d}}_2^q\right)}}\, PS\left(\tau\left(\sqrt[q]{1 - r_1^q r_2^q}\right)\right), \\
\sqrt[q]{\dfrac{\left(\underset{\cdot}{\mathrm{i}}_1^q + \underset{\cdot}{\mathrm{i}}_2^q\right)}{\left(\dot{\check{s}}_1^q + \dot{\check{s}}_2^q\right) + \left(\underset{\cdot}{\mathrm{i}}_1^q + \underset{\cdot}{\mathrm{i}}_2^q\right) + \left(\underset{,}{\mathrm{d}}_1^q + \underset{,}{\mathrm{d}}_2^q\right)}}\, PS\left(\tau\left(\sqrt[q]{1 - r_1^q r_2^q}\right)\right), \\
\sqrt[q]{\dfrac{\left(\underset{,}{\mathrm{d}}_1^q + \underset{,}{\mathrm{d}}_2^q\right)}{\left(\dot{\check{s}}_1^q + \dot{\check{s}}_2^q\right) + \left(\underset{\cdot}{\mathrm{i}}_1^q + \underset{\cdot}{\mathrm{i}}_2^q\right) + \left(\underset{,}{\mathrm{d}}_1^q + \underset{,}{\mathrm{d}}_2^q\right)}}\, PS\left(\tau\left(\sqrt[q]{1 - r_1^q r_2^q}\right)\right)
\end{array} \right)
$$

$$
= \tau(\beta_1 \ominus \beta_2)
$$

(3) For $\tau_1, \tau_2 > 0$,

$$
\tau_1\beta_1 = \left(\sqrt[q]{\frac{\dot{\check{s}}_1^q}{\dot{\check{s}}_1^q + \underset{\cdot}{\mathrm{i}}_1^q + \underset{,}{\mathrm{d}}_1^q}\left(1 - \left(r_1^q\right)^{\tau_1}\right)}, \sqrt[q]{\frac{\underset{\cdot}{\mathrm{i}}_1^q}{\dot{\check{s}}_1^q + \underset{\cdot}{\mathrm{i}}_1^q + \underset{,}{\mathrm{d}}_1^q}\left(1 - \left(r_1^q\right)^{\tau_1}\right)}, \sqrt[q]{\frac{\underset{,}{\mathrm{d}}_1^q}{\dot{\check{s}}_1^q + \underset{\cdot}{\mathrm{i}}_1^q + \underset{,}{\mathrm{d}}_1^q}\left(1 - \left(r_1^q\right)^{\tau_1}\right)} \right)
$$

and

$$
\tau_2\beta_1 = \left(\sqrt[q]{\frac{\dot{\check{s}}_1^q}{\dot{\check{s}}_1^q + \underset{\cdot}{\mathrm{i}}_1^q + \underset{,}{\mathrm{d}}_1^q}\left(1 - \left(r_1^q\right)^{\tau_2}\right)}, \sqrt[q]{\frac{\underset{\cdot}{\mathrm{i}}_1^q}{\dot{\check{s}}_1^q + \underset{\cdot}{\mathrm{i}}_1^q + \underset{,}{\mathrm{d}}_1^q}\left(1 - \left(r_1^q\right)^{\tau_2}\right)}, \sqrt[q]{\frac{\underset{,}{\mathrm{d}}_1^q}{\dot{\check{s}}_1^q + \underset{\cdot}{\mathrm{i}}_1^q + \underset{,}{\mathrm{d}}_1^q}\left(1 - \left(r_1^q\right)^{\tau_2}\right)} \right)
$$

Hence, using Eq. (8), we obtain

$$
\tau_1\beta_1 \ominus \tau_2\beta_1 = \left(\begin{array}{l}
\sqrt[q]{\dfrac{MCS^q(\tau_1\beta_1, \tau_2\beta_1)}{MCS^q(\tau_1\beta_1, \tau_2\beta_1) + NCS^q(\tau_1\beta_1, \tau_2\beta_1) + ACS^q(\tau_1\beta_1, \tau_2\beta_1)}} . PS\left(PS\left(\tau_1\sqrt[q]{\dot{\check{s}}_1^q + \underset{\cdot}{\mathrm{i}}_1^q + \underset{,}{\mathrm{d}}_1^q}\right), PS\left(\tau_2\sqrt[q]{\dot{\check{s}}_2^q + \underset{\cdot}{\mathrm{i}}_2^q + \underset{,}{\mathrm{d}}_2^q}\right)\right), \\
\sqrt[q]{\dfrac{ACS^q(\tau_1\beta_1, \tau_2\beta_1)}{MCS^q(\tau_1\beta_1, \tau_2\beta_1) + NCS^q(\tau_1\beta_1, \tau_2\beta_1) + ACS^q(\tau_1\beta_1, \tau_2\beta_1)}} . PS\left(PS\left(\tau_1\sqrt[q]{\dot{\check{s}}_1^q + \underset{\cdot}{\mathrm{i}}_1^q + \underset{,}{\mathrm{d}}_1^q}\right), PS\left(\tau_2\sqrt[q]{\dot{\check{s}}_2^q + \underset{\cdot}{\mathrm{i}}_2^q + \underset{,}{\mathrm{d}}_2^q}\right)\right), \\
\sqrt[q]{\dfrac{NCS^q(\tau_1\beta_1, \tau_2\beta_1)}{MCS^q(\tau_1\beta_1, \tau_2\beta_1) + NCS^q(\tau_1\beta_1, \tau_2\beta_1) + ACS^q(\tau_1\beta_1, \tau_2\beta_1)}} . PS\left(PS\left(\tau_1\sqrt[q]{\dot{\check{s}}_1^q + \underset{\cdot}{\mathrm{i}}_1^q + \underset{,}{\mathrm{d}}_1^q}\right), PS\left(\tau_2\sqrt[q]{\dot{\check{s}}_2^q + \underset{\cdot}{\mathrm{i}}_2^q + \underset{,}{\mathrm{d}}_2^q}\right)\right)
\end{array} \right),
$$

$$
= \begin{pmatrix}
\sqrt[q]{\frac{(\tau_1+\tau_2)\dot{s}_1^q}{(\tau_1+\tau_2)\dot{s}_1^q+(\tau_1+\tau_2)\underset{.}{\mathrm{i}}_1^q+(\tau_1+\tau_2)\underset{,}{\mathrm{d}}_1^q}}.PS\left(\sqrt[q]{1-\left(r_1^q\right)^{\tau_1}}, \sqrt[q]{1-\left(r_1^q\right)^{\tau_2}}\right), \\
\sqrt[q]{\frac{(\tau_1+\tau_2)\underset{.}{\mathrm{i}}_1^q}{(\tau_1+\tau_2)\dot{s}_1^q+(\tau_1+\tau_2)\underset{.}{\mathrm{i}}_1^q+(\tau_1+\tau_2)\underset{,}{\mathrm{d}}_1^q}}.PS\left(\sqrt[q]{1-\left(r_1^q\right)^{\tau_1}}, \sqrt[q]{1-\left(r_1^q\right)^{\tau_2}}\right), \\
\sqrt[q]{\frac{(\tau_1+\tau_2)\underset{,}{\mathrm{d}}_1^q}{(\tau_1+\tau_2)\dot{s}_1^q+(\tau_1+\tau_2)\underset{.}{\mathrm{i}}_1^q+(\tau_1+\tau_2)\underset{,}{\mathrm{d}}_1^q}}.PS\left(\sqrt[q]{1-\left(r_1^q\right)^{\tau_1}}, \sqrt[q]{1-\left(r_1^q\right)^{\tau_2}}\right)
\end{pmatrix}
$$

$$
= \left(\sqrt[q]{\frac{\dot{s}_1^q}{\dot{s}_1^q+\underset{.}{\mathrm{i}}_1^q+\underset{,}{\mathrm{d}}_1^q}\left(1-\left(r_1^q\right)^{\tau_1+\tau_2}\right)}, \sqrt[q]{\frac{\underset{.}{\mathrm{i}}_1^q}{\dot{s}_1^q+\underset{.}{\mathrm{i}}_1^q+\underset{,}{\mathrm{d}}_1^q}\left(1-\left(r_1^q\right)^{\tau_1+\tau_2}\right)}, \sqrt[q]{\frac{\underset{,}{\mathrm{d}}_1^q}{\dot{s}_1^q+\underset{.}{\mathrm{i}}_1^q+\underset{,}{\mathrm{d}}_1^q}\left(1-\left(r_1^q\right)^{\tau_1+\tau_2}\right)}\right)
$$

$$
= (\tau_1+\tau_2)\beta_1.
$$

4 Neutral aggregation operators for TSFNs

This section illustrates TSFWNA, TSFOWNA, and TSFHNA operators of TSFNs.

For this purpose, consider Ω to be the collection of TSFNs.

Definition 9 Let β_i be a collection of "n" TSFNs. The TSFWNA is a function defined on $(\beta_1, \beta_2, ..., \beta_n)$ by

$$
TSFWNA(\beta_1, \beta_2, ..., \beta_n) = \ominus_{i=1}^n \omega_i \beta_i \tag{13}
$$

where $\omega_i \geq 0$ is the weight vector of β_i. and $\sum_{i=1}^n \omega_i = 1$

Theorem 6 *The aggregation of "n"TSFNs $\beta_i = \left(\dot{s}_i, \underset{.}{\mathrm{i}}_i, \underset{,}{\mathrm{d}}_i\right)$using Definition 9 is also a TSFN, where.*

$$
TSFWNA(\beta_1, \beta_2, ..., \beta_n) = \begin{pmatrix}
\sqrt[q]{\frac{\sum_{i=1}^n \omega_i \dot{s}_i^q}{\sum_{i=1}^n \omega_i\left(\dot{s}_i^q+\underset{.}{\mathrm{i}}_i^q+\underset{,}{\mathrm{d}}_i^q\right)}.\left(1-\prod_{i=1}^n\left(r_i^q\right)^{\omega_i}\right)}, \\
\sqrt[q]{\frac{\sum_{i=1}^n \omega_i \underset{.}{\mathrm{i}}_i^q}{\sum_{i=1}^n \omega_i\left(\dot{s}_i^q+\underset{.}{\mathrm{i}}_i^q+\underset{,}{\mathrm{d}}_i^q\right)}.\left(1-\prod_{i=1}^n\left(r_i^q\right)^{\omega_i}\right)}, \\
\sqrt[q]{\frac{\sum_{i=1}^n \omega_i \underset{,}{\mathrm{d}}_i^q}{\sum_{i=1}^n \omega_i\left(\dot{s}_i^q+\underset{.}{\mathrm{i}}_i^q+\underset{,}{\mathrm{d}}_i^q\right)}.\left(1-\prod_{i=1}^n\left(r_i^q\right)^{\omega_i}\right)}
\end{pmatrix}. \tag{14}
$$

The proof is provided in Appendix.

Moreover, from TSFWNA operator, a few necessary characteristics are given as under.

Theorem 7 (Idempotency):*Let $\beta_i = \left(\dot{s}_i, \underset{.}{\mathrm{i}}_i, \underset{,}{\mathrm{d}}_i\right)(i = 1, 2, 3..., n)$be a set of TSFNs. If $\beta_i = \left(\dot{s}_0, \underset{.}{\mathrm{i}}_0, \underset{,}{\mathrm{d}}_0\right)$for all i,then $TSFWNA(\beta_1, \beta_2, ..., \beta_n) = \left(\dot{s}_0, \underset{.}{\mathrm{i}}_0, \underset{,}{\mathrm{d}}_0\right)$.*

The proof is provided in Appendix.

Theorem 8 (Boundedness):*For a set of "n"TSFNs β_i, we have*

(1) $min\left\{\dot{s}_i^q + \underset{\cdot}{\mathrm{i}}_i^q + \underset{\varsigma}{\mathrm{d}}_i^q\right\} \leq \dot{s}_p^q + \underset{\cdot}{\mathrm{i}}_p^q + \underset{\varsigma}{\mathrm{d}}_p^q \leq max\left\{\dot{s}_i^q + \underset{\cdot}{\mathrm{i}}_i^q + \underset{\varsigma}{\mathrm{d}}_i^q\right\}$,

(2) $\frac{min\left\{\dot{s}_i^q+\underset{\cdot}{\mathrm{i}}_i^q+\underset{\varsigma}{\mathrm{d}}_i^q\right\}.min\left\{\dot{s}_i^q\right\}}{max\left\{\dot{s}_i^q+\underset{\cdot}{\mathrm{i}}_i^q+\underset{\varsigma}{\mathrm{d}}_i^q\right\}} \leq \dot{s}_p^q \leq \frac{max\left\{\dot{s}_i^q+\underset{\cdot}{\mathrm{i}}_i^q+\underset{\varsigma}{\mathrm{d}}_i^q\right\}.max\left\{\dot{s}_i^q\right\}}{min\left\{\dot{s}_i^q+\underset{\cdot}{\mathrm{i}}_i^q+\underset{\varsigma}{\mathrm{d}}_i^q\right\}}$,

(3) $\frac{min\left\{\dot{s}_i^q+\underset{\cdot}{\mathrm{i}}_i^q+\underset{\varsigma}{\mathrm{d}}_i^q\right\}.min\left\{\underset{\cdot}{\mathrm{i}}_i^q\right\}}{max\left\{\dot{s}_i^q+\underset{\cdot}{\mathrm{i}}_i^q+\underset{\varsigma}{\mathrm{d}}_i^q\right\}} \leq \underset{\cdot}{\mathrm{i}}_p^q \leq \frac{max\left\{\dot{s}_i^q+\underset{\cdot}{\mathrm{i}}_i^q+\underset{\varsigma}{\mathrm{d}}_i^q\right\}.max\left\{\underset{\cdot}{\mathrm{i}}_i^q\right\}}{min\left\{\dot{s}_i^q+\underset{\cdot}{\mathrm{i}}_i^q+\underset{\varsigma}{\mathrm{d}}_i^q\right\}}$,

(4) $\frac{min\left\{\dot{s}_i^q+\underset{\cdot}{\mathrm{i}}_i^q+\underset{\varsigma}{\mathrm{d}}_i^q\right\}.min\left\{\underset{\varsigma}{\mathrm{d}}_i^q\right\}}{max\left\{\dot{s}_i^q+\underset{\cdot}{\mathrm{i}}_i^q+\underset{\varsigma}{\mathrm{d}}_i^q\right\}} \leq \underset{\varsigma}{\mathrm{d}}_p^q \leq \frac{max\left\{\dot{s}_i^q+\underset{\cdot}{\mathrm{i}}_i^q+\underset{\varsigma}{\mathrm{d}}_i^q\right\}.max\left\{\underset{\varsigma}{\mathrm{d}}_i^q\right\}}{min\left\{\dot{s}_i^q+\underset{\cdot}{\mathrm{i}}_i^q+\underset{\varsigma}{\mathrm{d}}_i^q\right\}}$,

where TSFWNA$(\beta_1, \beta_2, ..., \beta_n) = \left(\dot{s}_p, \underset{\cdot}{\mathrm{i}}_p, \underset{\varsigma}{\mathrm{d}}_p\right)$. *It is known as Boundedness.*

The proof can be seen in Appendix.

Theorem 9 (Monotonicity):*Let* $\beta_i = \left(\dot{s}_{\beta_i}, \underset{\cdot}{\mathrm{i}}_{\beta_i}, \underset{\varsigma}{\mathrm{d}}_{\beta_i}\right)$ *and* $\gamma_i = \left(\dot{s}_{\gamma_i}, \underset{\cdot}{\mathrm{i}}_{\gamma_i}, \underset{\varsigma}{\mathrm{d}}_{\gamma_i}\right)$ *be sets of "n"TSFNs. Then,*

(1) $\dot{s}_{p\beta}^q + \underset{\cdot}{\mathrm{i}}_{p\beta}^q + \underset{\varsigma}{\mathrm{d}}_{p\beta}^q \leq \dot{s}_{p\gamma}^q + \underset{\cdot}{\mathrm{i}}_{p\gamma}^q + \underset{\varsigma}{\mathrm{d}}_{p\gamma}^q$ if $\dot{s}_{\beta_i}^q + \underset{\cdot}{\mathrm{i}}_{\beta_i}^q + \underset{\varsigma}{\mathrm{d}}_{\beta_i}^q \leq \dot{s}_{\gamma_i}^q + \underset{\cdot}{\mathrm{i}}_{\gamma_i}^q + \underset{\varsigma}{\mathrm{d}}_{p\gamma}^q$.

(2) $\dot{s}_{p\beta} \leq \dot{s}_{p\gamma}, \underset{\cdot}{\mathrm{i}}_{p\beta} \geq \underset{\cdot}{\mathrm{i}}_{p\gamma}, \underset{\varsigma}{\mathrm{d}}_{p\beta} \geq \underset{\varsigma}{\mathrm{d}}_{p\gamma}$ *if* $\dot{s}_{\beta_i}^q + \underset{\cdot}{\mathrm{i}}_{\beta_i}^q + \underset{\varsigma}{\mathrm{d}}_{\beta_i}^q = \dot{s}_{\gamma_i}^q + \underset{\cdot}{\mathrm{i}}_{\gamma_i}^q + \underset{\varsigma}{\mathrm{d}}_{p\gamma}^q$ *and* $\dot{s}_{\beta_i} \leq \dot{s}_{\gamma_i}$.

(3) $TSFWNA(\beta_1, \beta_2, ..., \beta_n) \leq TSFWNA(\gamma_1, \gamma_2, ..., \gamma_n)$ *if* $\dot{s}_{\beta_i}^q + \underset{\cdot}{\mathrm{i}}_{\beta_i}^q + \underset{\varsigma}{\mathrm{d}}_{\beta_i}^q = \dot{s}_{\gamma_i}^q + \underset{\cdot}{\mathrm{i}}_{\gamma_i}^q + \underset{\varsigma}{\mathrm{d}}_{p\gamma}^q$ *and* $\dot{s}_{\beta_i} \leq \dot{s}_{\gamma_i}$.

The proof is given in Appendix.

Definition 10 Let $\beta_i = \left(\dot{s}_i, \underset{\cdot}{\mathrm{i}}_i, \underset{\varsigma}{\mathrm{d}}_i\right)(i = 1, 2, 3..., n)$ be a collection of TSFNs. Then TSFOWNA operator of β_i is defined by.

$$TSFOWNA(\beta_1, \beta_2, ..., \beta_n) = \ominus_{i=1}^n \omega_i \beta_{\sigma(i)} \tag{15}$$

where σ is the permutation map of $1, 2, 3..., n$ such that $\beta_{\sigma(i-1)} \geq \beta_{\sigma(i)}$ and $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ be associated weight with TSFOWNA operator, having $\omega_i > 0$ and $\sum_{i=1}^n \omega_i = 1$.

Theorem 10 *The aggregated value of TSFNs using Definition 10 is also a TSFN and given by.*

$$TSFOWNA(\beta_1, \beta_2, ..., \beta_n) = \begin{pmatrix} \sqrt[q]{\frac{\sum_{i=1}^n \omega_i \dot{s}_{\sigma(i)}^q}{\sum_{i=1}^n \omega_i \left(\dot{s}_{\sigma(i)}^q + \underset{\cdot}{\mathrm{i}}_{\sigma(i)}^q + \underset{\varsigma}{\mathrm{d}}_{\sigma(i)}^q\right)}.\left(1 - \prod_{i=1}^n \left(r_{\sigma(i)}^q\right)^{\omega_i}\right)}, \\ \sqrt[q]{\frac{\sum_{i=1}^n \omega_i \underset{\cdot}{\mathrm{i}}_{\sigma(i)}^q}{\sum_{i=1}^n \omega_i \left(\dot{s}_{\sigma(i)}^q + \underset{\cdot}{\mathrm{i}}_{\sigma(i)}^q + \underset{\varsigma}{\mathrm{d}}_{\sigma(i)}^q\right)}.\left(1 - \prod_{i=1}^n \left(r_{\sigma(i)}^q\right)^{\omega_i}\right)}, \\ \sqrt[q]{\frac{\sum_{i=1}^n \omega_i \underset{\varsigma}{\mathrm{d}}_{\sigma(i)}^q}{\sum_{i=1}^n \omega_i \left(\dot{s}_{\sigma(i)}^q + \underset{\cdot}{\mathrm{i}}_{\sigma(i)}^q + \underset{\varsigma}{\mathrm{d}}_{\sigma(i)}^q\right)}.\left(1 - \prod_{i=1}^n \left(r_{\sigma(i)}^q\right)^{\omega_i}\right)} \end{pmatrix}.$$

The proof is analogous to Theorem 6.

Definition 11 A TSFHNA: $\Omega^n \rightarrow \Omega$ is a map defined as.

$$TSFHNA(\beta_1, \beta_2, ..., \beta_n) = \ominus_{i=1}^{n} \psi_i \dot{\beta}_{\sigma(i)} \tag{16}$$

where $\dot{\beta}_{\sigma(i)}$ is the i^{th} largest of the weighted TSFNs $\dot{\beta}_i$ and $\dot{\beta}_i = n\omega_i \beta_i \forall i$. Further,$\psi_i > 0$; $\sum_{i=1}^{n} \psi_i = 1$ be weight vector given to the operator.

Theorem 11 *The aggregated values of "n"TSFNs* $\beta_i = \left(\dot{\dot{s}}_i, \dot{\mathrm{i}}_i, \d{\mathrm{d}}_i\right)$*using TSFHNA operator is also a TSFN and formulated by.*

$$TSFHNA(\beta_1, \beta_2, ..., \beta_n) = \begin{pmatrix} \sqrt[q]{\frac{\sum_{i=1}^{n} \psi_i \dot{\dot{s}}^q_{\sigma(i)}}{\sum_{i=1}^{n} \psi_i \left(\dot{\dot{s}}^q_{\sigma(i)} + \dot{\mathrm{i}}^q_{\sigma(i)} + \d{\mathrm{d}}^q_{\sigma(i)}\right)} \cdot \left(1 - \prod_{i=1}^{n} \left(\dot{r}^q_{\sigma(i)}\right)^{\psi_i}\right)}, \\ \sqrt[q]{\frac{\sum_{i=1}^{n} \psi_i \dot{\mathrm{i}}^q_{\sigma(i)}}{\sum_{i=1}^{n} \psi_i \left(\dot{\dot{s}}^q_{\sigma(i)} + \dot{\mathrm{i}}^q_{\sigma(i)} + \d{\mathrm{d}}^q_{\sigma(i)}\right)} \cdot \left(1 - \prod_{i=1}^{n} \left(\dot{r}^q_{\sigma(i)}\right)^{\psi_i}\right)}, \\ \sqrt[q]{\frac{\sum_{i=1}^{n} \psi_i \d{\mathrm{d}}^q_{\sigma(i)}}{\sum_{i=1}^{n} \psi_i \left(\dot{\dot{s}}^q_{\sigma(i)} + \dot{\mathrm{i}}^q_{\sigma(i)} + \d{\mathrm{d}}^q_{\sigma(i)}\right)} \cdot \left(1 - \prod_{i=1}^{n} \left(\dot{r}^q_{\sigma(i)}\right)^{\psi_i}\right)} \end{pmatrix} \tag{17}$$

This Theorem has the same proof as Theorem 6.

Remark 1. The TSFOWNA and TSFHNA operators satisfy all the characteristics which are satisfied by the TSFWNA aggregation operator; therefore, their proofs are identical to them so excluded.

5 A MAGDM algorithm based on proposed operators

In this section, an innovative MAGDM approach is established for designating the possible option to choose the best one, using TSFSs environment. Consider a set of 'm" available choices $\mathcal{C} = \{\mathcal{C}_1, \mathcal{C}_2, ...\mathcal{C}_m\}$ which are assessed under 'n' attributes $\mathcal{H} = \{\mathcal{H}_1, \mathcal{H}_2, \ldots \mathcal{H}_{\boldsymbol{n}}\}$ with weight vector $\omega > 0$ such that$\sum_{j=1}^{n} \omega_j = 1$. The set of attributes is divided into two disjoint sets, namely the cost type attributes and benefit type. Consider a team of experts $E = \left\{E^{(1)}, E^{(2)}, ...E^{(l)}\right\}$ which evaluates the given alternatives with weight vector $w_k > 0$ such that$\sum_{k=1}^{l} w_k = 1$. The team of experts will rate the each attribute under TSFSs environment as $\delta_{ij}^{(k)} = \left(\dot{\dot{s}}_{ij}^{(k)}, \dot{\mathrm{i}}_{ij}^{(k)}, \d{\mathrm{d}}_{ij}^{(k)}\right)$ with $0 \le \dot{\dot{s}}_{ij}^{(k)}, \dot{\mathrm{i}}_{ij}^{(k)}, \d{\mathrm{d}}_{ij}^{(k)} \le 1$ such that $0 \le \dot{\dot{s}}_{ij}^{(k)} + \dot{\mathrm{i}}_{ij}^{(k)} + \d{\mathrm{d}}_{ij}^{(k)} \le 1$ for $i = 1, 2 \ldots, m; j = 1, 2 \ldots n; k = 1, 2, \ldots l$. A decision matrix under TSFS is provided as

$$\mathcal{M}^k = \begin{pmatrix} \delta_{11}^{(k)} & \delta_{12}^{(k)} & \cdots & \delta_{1n}^{(k)} \\ \delta_{21}^{(k)} & \delta_{22}^{(k)} & \cdots & \delta_{2n}^{(k)} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{m1}^{(k)} & \delta_{m2}^{(k)} & \cdots & \delta_{mn}^{(k)} \end{pmatrix}.$$

The following steps will explain the novel approach in detail:

Step 1: First, each decision-maker will provide a decision matrix $\mathcal{M}$ in the form of TSFNs.

Step 2: In this step, proposed AOs that is TSFWNA or TSFOWNA or TSFHNA will be utilized to aggregate all the decision-makers' preferences $\delta_{ij}^{(k)}, k = 1, 2, \ldots, l$ into $\delta_{ij} = \left(\dot{\check{s}}_{ij}, \underset{.}{\mathrm{i}}_{ij}, \underset{,}{\mathrm{d}}_{ij}\right)$.

Step 3: If the set of attributes $\mathcal{H}$ contains two types of attributes, named as the cost attributes (F_1) and the benefit attributes ($F\ _2$), thus we will normalize δ_{ij} into r_{ij}.

$$r_{ij} = \begin{cases} \left(\underset{,}{\mathrm{d}}_{ij}, \underset{.}{\mathrm{i}}_{ij}, \dot{\check{s}}_{ij}\right), & for \quad F_1\, criteria \\ \left(\dot{\check{s}}_{ij}, \underset{.}{\mathrm{i}}_{ij}, \underset{,}{\mathrm{d}}_{ij}\right), & for \quad F_2\, criteria \end{cases}. \tag{18}$$

Step 4: Following optimization model is used to obtain the weight vector of attributes in the case if weight information is partially known.

$$\begin{aligned} max\, f &= \sum_{i=1}^{m} \sum_{j=1}^{n} \omega_{ij} \mathcal{S}_{ij}, \\ s.t. \sum_{j=1}^{n} \omega_j &= 1; \omega_j \geq 0; \omega \in \mathcal{G}, \end{aligned} \tag{19}$$

where $\mathcal{S}_{ij}$ represents the sum of scores of each alternative $\mathcal{C}_i$. After some calculations above model gives $\omega = (\omega_1, \omega_2, ...\omega_n)$.

Step 5: Using suggested TSF averaging AOs, compute the comprehensive values $r_i = \left(\dot{\check{s}}_i, \underset{.}{\mathrm{i}}_i, \underset{,}{\mathrm{d}}_i\right)$ of alternatives $\mathcal{C}_i (i = 1, 2, ..., m)$.

Step 6: Compute the final ranking orders of $r_i = \left(\dot{\check{s}}_i, \underset{.}{\mathrm{i}}_i, \underset{,}{\mathrm{d}}_i\right), i = 1, 2, ..., m$ by utilizing Eq. (20).

$$\mathcal{S}(r_i) = \frac{e^{\dot{\check{s}}_i{}^q - \underset{.}{\mathrm{i}}_i{}^q - \underset{,}{\mathrm{d}}_i{}^q}}{2 - \dot{\check{s}}_i{}^q - \underset{.}{\mathrm{i}}_i{}^q - \underset{,}{\mathrm{d}}_i{}^q}. \tag{20}$$

Step 7: The provided alternatives $\mathcal{C}_i (i = 1, 2, ..., m)$ are categorized using Definition 4 and selecting the optimal one(s).

6 Numerical examples

The proposed MAGDM approach is tested and validated by applying on a practical example from the investment sector. Moreover, in this section, we will study some examples from a comparison point of view.

6.1 Illustration of proposed MAGDM

Example 1 Investment is always an uncertain event, especially in the present circumstances. Immediately after the COVID-19 pandemic, economic trends are down and irregular. A local group of investors is interested to invest in the electronics manufacturing company among the four most emerging companies. They decided to hire three decision-makers $E^{(1)}$, $E^{(2)}$ and $E^{(3)}$ which are evaluating the four companies: $\mathcal{C}_1$ ("a mobile company"), $\mathcal{C}_2$ ("a LED company"), $\mathcal{C}_3$ ("a ceiling fan company") and $\mathcal{C}_4$("an AC company") with four attributes: $\mathcal{H}_1$("legal entity"), $\mathcal{H}_2$ ("perpetual succession"), $\mathcal{H}_3$("the economic benefit") and $\mathcal{H}_4$("the

Table 1 The decision matrix where "$\mathcal{C}_i$" denote alternatives and "$\mathcal{H}_i$" denote the attributes

Experts		$\mathcal{H}_1$	$\mathcal{H}_2$	$\mathcal{H}_3$	$\mathcal{H}_4$
$\boldsymbol{E}^{(1)}$	$\mathcal{C}_1$	(0.5, 0.01, 0.4)	(0.6, 0.21, 0.2)	(0.5, 0.16, 0.3)	(0.7, 0.31, 0.7)
	$\mathcal{C}_2$	(0.4, 0.01, 0.3)	(0.5, 0.17, 0.3)	(0.7, 0.22, 0.1)	(0.6, 0.32, 0.2)
	$\mathcal{C}_3$	(0.2, 0.04, 0.6)	(0.4, 0.19, 0.1)	(0.3, 0.23, 0.2)	(0.4, 0.21, 0.4)
	$\mathcal{C}_4$	(0.4, 0.4, 0.4)	(0.5, 0.13, 0.4)	(0.4, 0.29, 0.3)	(0.5, 0.12, 0.3)
$\boldsymbol{E}^{(2)}$	$\mathcal{C}_1$	(0.5, 0.12, 0.3)	(0.7, 0.22, 0.2)	(0.4, 0.23, 0.6)	(0.6, 0.16, 0.4)
	$\mathcal{C}_2$	(0.7, 0.15, 0.2)	(0.4, 0.25, 0.5)	(0.8, 0.12, 0.4)	(0.5, 0.22, 0.3)
	$\mathcal{C}_3$	(0.3, 0.01, 0.4)	(0.5, 0.19, 0.6)	(0.7, 0.13, 0.3)	(0.3, 0.23, 0.5)
	$\mathcal{C}_4$	(0.5, 0.4, 0.4)	(0.6, 0.17, 0.3)	(0.5, 0.27, 0.2)	(0.4, 0.14, 0.2)
$\boldsymbol{E}^{(3)}$	$\mathcal{C}_1$	(0.7, 0.21, 0.2)	(0.5, 0.21, 0.3)	(0.4, 0.25, 0.5)	(0.8, 0.23, 0.2)
	$\mathcal{C}_2$	(0.5, 0.15, 0.3)	(0.6, 0.25, 0.5)	(0.8, 0.15, 0.2)	(0.6, 0.32, 0.3)
	$\mathcal{C}_3$	(0.2, 0.21, 0.6)	(0.3, 0.19, 0.4)	(0.7, 0.17, 0.3)	(0.5, 0.24, 0.4)
	$\mathcal{C}_4$	(0.4, 0.4, 0.4)	(0.4, 0.31, 0.3)	(0.5, 0.31, 0.4)	(0.7, 0.31, 0.2)

Table 2 Aggregated values of experts by TSFWNA operator

	$\mathcal{H}_1$	$\mathcal{H}_2$	$\mathcal{H}_3$	$\mathcal{H}_4$
$\mathcal{C}_1$	(0.6011, 0.1618, 0.3211)	(0.5992, 0.2138, 0.2514)	(0.4415, 0.2211, 0.4855)	(0.7362, 0.2581, 0.5274)
$\mathcal{C}_2$	(0.5477, 0.1316, 0.2848)	(0.5306, 0.2298, 0.4526)	(0.7721, 0.1776, 0.2709)	(0.5792, 0.3014, 0.2735)
$\mathcal{C}_3$	(0.2338, 0.1732, 0.5787)	(0.4084, 0.1935, 0.4388)	(0.6266, 0.1928, 0.2782)	(0.4298, 0.2280, 0.4299)
$\mathcal{C}_4$	(0.4, 0.4, 0.4)	(0.4993, 0.2412, 0.3429)	(0.4709, 0.2946, 0.3342)	(0.5915, 0.2403, 0.2488)

business growth ability"). Suppose that $w = (0.35, 0.4, 0.25)$ is the weight vector of the decision-makers and their assessment matrices $\mathcal{M}^1, \mathcal{M}^2$, and $\mathcal{M}^3$ under TSFNs, where $q = 3$ are provided in Table 1. The objective of this study is to choose a suitable firm for financing. In the following, we will discuss MAGDM process step by step:

Step 1: Table 1 is consist of the rating values provided by experts.

Step 2: The initial information provided by experts is aggregated by TSFWNA operator with $w = (0.35, 0.25, 0.4)$ as the corresponding weight vector for decision-makers.

Step 3: There is no need for normalization because all the attributes provided in this study are benefit types (Table 2).

Step 4: We presume that the partial weight knowledge of attribute's importance is suggested by the experts is $\mathcal{G} = \{0.2 \leq \omega_1 \leq 0.3, 0.25 \leq \omega_2 \leq 0.35, 0.15 \leq \omega_3 \leq 0.4, 0.2 \leq \omega_4 \leq 0.35, \omega_1 + \omega_3 \leq 2\omega_2, \omega_1 + 2\omega_4 \leq \omega_3\}$, we exhibit the optimization model using Eq. (19) which provides $\omega = (0.2.0.25, 0.3, 0.25)$.

Step 5: By applying the TSFWNA operator and weights ω, we get r_i with $q = 3$ as $r_1 = (0.4711, 0.2669, 0.4426)$,$r_2 = (0.5099, 0.2206, 0.3960)$, $r_3 = (0.6227, 0.2326, 0.3567)$ and $r_4 = (0.6007, 0.2640, 0.4042)$.

Step 6: The final ranking orders are computed as $\mathcal{S}(r_1) = 0.5581$,$\mathcal{S}(r_2) = 0.5916$,$\mathcal{S}(r_3) = 0.7065$ and $\mathcal{S}(r_4) = 0.6719$.

Table 3 Ranking results for different values of "q"

Values for $\boldsymbol{q}$	Score values				Ranking
	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	
$\boldsymbol{q} = 3$	0.5581	0.5916	0.7065	0.6719	$\mathcal{C}_3 > \mathcal{C}_4 > \mathcal{C}_2 > \mathcal{C}_1$
$\boldsymbol{q} = 4$	0.5335	0.5492	0.6372	0.6042	$\mathcal{C}_3 > \mathcal{C}_4 > \mathcal{C}_2 > \mathcal{C}_1$
$\boldsymbol{q} = 5$	0.5207	0.5276	0.5947	0.5669	$\mathcal{C}_3 > \mathcal{C}_4 > \mathcal{C}_2 > \mathcal{C}_1$
$\boldsymbol{q} = 6$	0.5134	0.5160	0.5672	0.5446	$\mathcal{C}_3 > \mathcal{C}_4 > \mathcal{C}_2 > \mathcal{C}_1$
$\boldsymbol{q} = 20$	0.500083	0.500036	0.501583	0.50079	$\mathcal{C}_3 > \mathcal{C}_4 > \mathcal{C}_2 > \mathcal{C}_1$

Step 7: As $\mathcal{S}(r_3) > \mathcal{S}(r_4) > S(r_2) > S(r_1)$, therefore the preferences are ranked as $\mathcal{C}_3 > \mathcal{C}_4 > \mathcal{C}_2 > \mathcal{C}_1$. So $\mathcal{C}_3$ is the optimal selection.

6.2 Impact of "q" and different AOs on the results

To review the impact of constant variable q on the ranking pattern, we summarized several values of q, respectively, in Step 2 and Step 5 of the proposed method. It is evident from the values in Table 3 that q has its significance to determine the preference order. We notice that the preference order is unchanged for different values of q that is $\mathcal{C}_3 > \mathcal{C}_4 > \mathcal{C}_2 > \mathcal{C}_1$ further, $\mathcal{C}_3$ remains the best possible choice for different values of q. As we increase the values of perimeter q, values of scores keep on decreasing and approach 0.5, meanwhile, the ranking pattern remains unchanged. This variation of perimeter q depicts that higher powers of q do not have much significance on the final results. Alteration of AOs under Step 5 and Step 2 exhibits stable behaviour final ranking and $\mathcal{C}_3$ remains the optimal choice but the significance of the components during the steps alters. For example, if the opinion of the

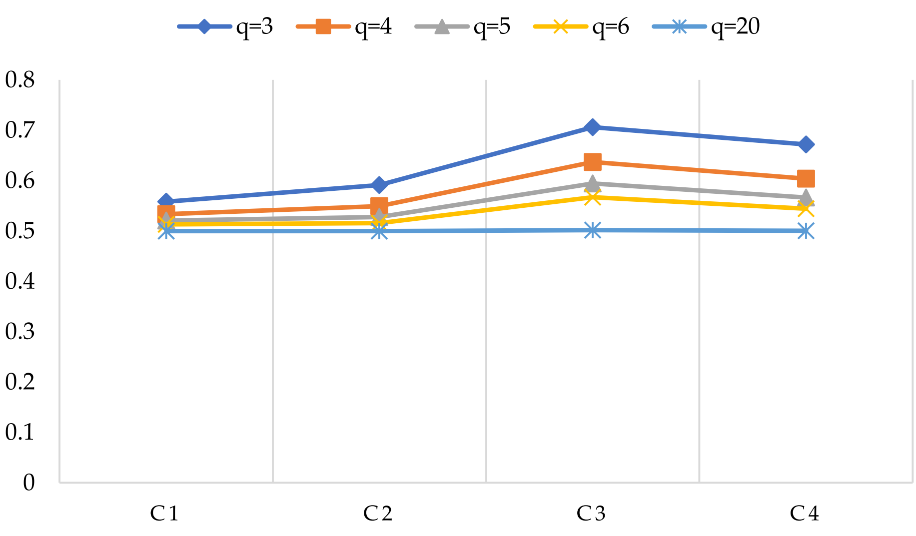

Fig. 3 Geometric interpretation of different values of "q"

Table 4 Impact of different AOs and ranking patterns

Operators used		Score values				Ranking
In Step 2	In Step 5	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	
TSFWNA	TSFWNA	0.5581	0.5916	0.7065	0.6719	$\mathcal{C}_3 > \mathcal{C}_4 > \mathcal{C}_2 > \mathcal{C}_1$
	TSFOWNA	0.5594	0.5946	0.6857	0.6787	$\mathcal{C}_3 > \mathcal{C}_4 > \mathcal{C}_2 > \mathcal{C}_1$
	TSFHNA	0.5565	0.5899	0.6729	0.6375	$\mathcal{C}_3 > \mathcal{C}_4 > \mathcal{C}_2 > \mathcal{C}_1$
TSFOWNA	TSFWNA	0.5647	0.5971	0.6627	0.6597	$\mathcal{C}_3 > \mathcal{C}_4 > \mathcal{C}_2 > \mathcal{C}_1$
	TSFOWNA	0.5618	0.5922	0.6979	0.6532	$\mathcal{C}_3 > \mathcal{C}_4 > \mathcal{C}_2 > \mathcal{C}_1$
	TSFHNA	0.5594	0.5902	0.6650	0.6235	$\mathcal{C}_3 > \mathcal{C}_4 > \mathcal{C}_2 > \mathcal{C}_1$
TSFHNA	TSFHNA	0.5594	0.5755	0.6658	0.6506	$\mathcal{C}_3 > \mathcal{C}_4 > \mathcal{C}_2 > \mathcal{C}_1$
	TSFWNA	0.5574	0.5758	0.6977	0.6466	$\mathcal{C}_3 > \mathcal{C}_4 > \mathcal{C}_2 > \mathcal{C}_1$
	TSFOWNA	0.5594	0.5755	0.6658	0.6506	$\mathcal{C}_3 > \mathcal{C}_4 > \mathcal{C}_2 > \mathcal{C}_1$

experts is concentrated more, then one can use TSFOWNA in Step 2 and TSFWNA in Step 5. Likewise, other pairs of AOs can also be described (Fig. 3, Table 4).

6.3 Advantages

In this subsection, we show the advantages of working in TSF environment with the help of an example. We have discussed earlier that the AD and RD do play a significant role in the description of the information under uncertainty.

Example 2 Reconsider the information from Example 1 without taking the abstinence degree into account. Table 5 consists of information from decision-makers in the form of duplets. Understandably all the given information is in the form of q-ROFNs for $q = 3$.

Table 5 Decision matrix obtained after excluding the AD from the decision matrix of Table 1

Experts		$\mathcal{H}_1$	$\mathcal{H}_2$	$\mathcal{H}_3$	$\mathcal{H}_4$
$\boldsymbol{E}^{(1)}$	$\mathcal{C}_1$	(0.5, 0.4)	(0.6, 0.2)	(0.5, 0.3)	(0.7, 0.7)
	$\mathcal{C}_2$	(0.4, 0.3)	(0.5, 0.3)	(0.7, 0.1)	(0.6, 0.2)
	$\mathcal{C}_3$	(0.2, 0.6)	(0.4, 0.1)	(0.3, 0.2)	(0.4, 0.4)
	$\mathcal{C}_4$	(0.4, 0.4)	(0.5, 0.4)	(0.4, 0.3)	(0.5, 0.3)
$\boldsymbol{E}^{(2)}$	$\mathcal{C}_1$	(0.5, 0.3)	(0.7, 0.2)	(0.4, 0.6)	(0.6, 0.4)
	$\mathcal{C}_2$	(0.7, 0.2)	(0.4, 0.5)	(0.8, 0.4)	(0.5, 0.3)
	$\mathcal{C}_3$	(0.3, 0.4)	(0.5, 0.6)	(0.7, 0.3)	(0.3, 0.5)
	$\mathcal{C}_4$	(0.5, 0.4)	(0.6, 0.3)	(0.5, 0.2)	(0.4, 0.2)
$\boldsymbol{E}^{(3)}$	$\mathcal{C}_1$	(0.7, 0.2)	(0.5, 0.3)	(0.4, 0.5)	(0.8, 0.2)
	$\mathcal{C}_2$	(0.5, 0.3)	(0.6, 0.5)	(0.8, 0.2)	(0.6, 0.3)
	$\mathcal{C}_3$	(0.2, 0.6)	(0.3, 0.4)	(0.7, 0.3)	(0.5, 0.4)
	$\mathcal{C}_4$	(0.4, 0.4)	(0.4, 0.3)	(0.5, 0.4)	(0.7, 0.2)

Now, by taking the q-ROFWNA operator (Garg and Chen 2020) we accumulate the data, and the results are summed up as $\mathcal{S}(r_1) = 0.5726, \mathcal{S}(r_2) = 0.5616, \mathcal{S}(r_3) = 0.6733$ and $\mathcal{S}(r_4) = 0.7130$. Finally, we obtained ranking order of alternatives as $\mathcal{C}_4 > \mathcal{C}_3 > \mathcal{C}_2 > \mathcal{C}_1$ and consequently acquiring the best possible one is $\mathcal{C}_4$. The ranking pattern obtained in this way is different from the previously obtained results using TSFWNA, TSFOWNG, and TSFHNA operators. This difference in the final ranking patterns depicts the loss of some important information by dropping the abstinence degree. Hence, it is an advantage of the proposed series of AOs that human opinion can be modelled more appropriately using TSFWNA, TSFOWNG, and TSFHNA operators.

6.4 Comparative analysis

In this section, we aim to compare the results obtained using the neutrality AOs of TSFSs with the results obtained using other AOs from the literature. We also showed that the AOs of other fuzzy layouts cannot evaluate the information presented in the form of TSFNs.

Example 3 Using the data from Example 1, here we will calculate the aggregation value and validate the proposed operation with the help of different tools which already exist in the literature. It is quite obvious from the literature survey that already proposed neutrality aggregation operators of, PyFSs, q-ROFSs, and SVNFNs (Garg 2020a, b; Garg and Chen 2020) are unable to aggregate the data with the triplet given in Table 1. Here we apply the AOs suggested by Ullah et al. (2020a, b, 2021) to the information given in Table 1 to conduct a comparative study. Table 6 shows different results drawn from different aggregation operators. The ranking patterns obtained using the different existing AOs are recorded in Table 6. Results obtained using proposed TSFWNA, TSFOWNA, and TSFHNA AOs are consistent as these are alike to the ranking patterns calculated by Ullah et al. (2020a, b, 2021). Furthermore, it must be noted that several other existing AOs discussed in this manuscript are unable to deal

Table 6 Ranking patterns using existing and proposed methods

Aggregation operators	Ref.	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	Ranking
Hamacher AOs	Ullah et al. (2020a)	0.4557	0.5431	0.6527	0.6675	$\mathcal{C}_4 > \mathcal{C}_3 > \mathcal{C}_2 > \mathcal{C}_1$
		0.5293	0.7318	0.7717	0.7945	$\mathcal{C}_4 > \mathcal{C}_3 > \mathcal{C}_2 > \mathcal{C}_1$
Arithmetic AOs	Ullah et al. (2020b)	0.4100	0.4960	0.6193	0.6090	$\mathcal{C}_3 > \mathcal{C}_4 > \mathcal{C}_2 > \mathcal{C}_1$
		0.3815	0.4833	0.5565	0.5787	$\mathcal{C}_4 > \mathcal{C}_3 > \mathcal{C}_2 > \mathcal{C}_1$
Dombi AOs	Ullah et al. (2021)	0.4157	0.4990	0.6526	0.6295	$\mathcal{C}_3 > \mathcal{C}_4 > \mathcal{C}_2 > \mathcal{C}_1$
		0.3491	0.4726	0.5268	0.5661	$\mathcal{C}_4 > \mathcal{C}_3 > \mathcal{C}_2 > \mathcal{C}_1$
Neutrality AOs of TSFSs	This paper	0.5581	0.5916	0.7065	0.6719	$\mathcal{C}_3 > \mathcal{C}_4 > \mathcal{C}_2 > \mathcal{C}_1$
	This paper	0.5594	0.5946	0.6857	0.6787	$\mathcal{C}_3 > \mathcal{C}_4 > \mathcal{C}_2 > \mathcal{C}_1$
	This paper	0.5565	0.5899	0.6729	0.6375	$\mathcal{C}_3 > \mathcal{C}_4 > \mathcal{C}_2 > \mathcal{C}_1$
Neutrality AOs of qROFSs	Khan et al. (2017)	Unable to compute				Unable to specify
Neutrality AOs of PyFSs	Akram et al. (2022)					
Neutrality AOs of IFSs	Hussain et al. (2022)					

with the neutral attitude of the decision-makers. In this way, proposed AOs are a much more improved version of Ullah et al. (2020a, b, 2021).

7 Conclusion

In this paper, we have investigated a new scheme to solve the DM problems more efficiently. As TSFS provides us with a wider range of fuzzy information as compared to PFS and SFS. This structure broadens the scope of human opinion and makes decision-making more relevant to the real-world scenario. During the decision-making activity, neutral attitude of the experts plays a significant role. In this manuscript, we have considered the neutral character of attitude to draw a fair conclusion. For this purpose, we have established some novel neutral operational laws under TSFS. It is observed from the proposed operations, that when a decision-maker assigns equivalent degrees during the evaluation process then their aggregated degrees remain equivalent. Additionally, by combining them with the eminent averaging operator, we define some new weighted AOs containing TSFWNA, TSFOWNA, and TSFHNA operators. Finally, based on these AOs, a MAGDM scheme is illustrated, and some real-life examples are considered to make its comparison with the several existing schemes. In the future, we aim to extend our work to the environment of IVTSFSs.

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Data availability statement The manuscript has no associated data.

Declarations

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

Appendix

Proof of Preposition 1

Proof Results are proven by utilizing the principle of mathematical induction (PMI) on τ. The following steps are executed.

Step 1: For $\tau = 2$ and using Eq. (9), we have

$$\begin{aligned} PS\left(\tau \sqrt[q]{\dot{\check{s}}_1^q + \d{i}_1^q + \c{d}_1^q}\right) &= PS\left(\sqrt[q]{\dot{\check{s}}_1^q + \d{i}_1^q + \c{d}_1^q}, PS\left(\sqrt[q]{\dot{\check{s}}_1^q + \d{i}_1^q + \c{d}_1^q}\right)\right) \\ &= PS\left(\sqrt[q]{\dot{\check{s}}_1^q + \d{i}_1^q + \c{d}_1^q}, \sqrt[q]{1 - r_1^q}\right) \\ &= \sqrt[q]{1 - \left(1 - \dot{\check{s}}_1^q - \d{i}_1^q - \c{d}_1^q\right)\left(1 - 1 + r_1^q\right)} \\ &= \sqrt[q]{1 - \left(r_1^q\right)^2}. \end{aligned}$$

Thus, the result is valid for $\tau = 2$.
Step 2: Consider Eq. (10) is true for $\tau = n$, then for $\tau = n + 1$,

$$
\begin{aligned}
& PS\left((n+1)\sqrt[q]{\dot{s}_1^q + \text{i}_1^q + \text{d}_1^q}\right) \\
& = PS\left(\sqrt[q]{\dot{s}_1^q + \text{i}_1^q + \text{d}_1^q},\ PS\left(n\sqrt[q]{\dot{s}_1^q + \text{i}_1^q + \text{d}_1^q}\right)\right) \\
& = PS\left(\sqrt[q]{\dot{s}_1^q + \text{i}_1^q + \text{d}_1^q},\ \sqrt[q]{1-\left(r_1^q\right)^n}\right) \\
& = \sqrt[q]{1-\left(1-\dot{s}_1^q - \text{i}_1^q - \text{d}_1^q\right)\left(1-1+\left(r_1^q\right)^n\right)} \\
& = \sqrt[q]{1-\left(r_1^q\right)^{n+1}}.
\end{aligned}
$$

Hence, by PMI, Eq. (10) holds for all values of τ.

Proof of Theorem 6

Proof For "n" TSFNs β_i and real $\omega_i > 0$, for the existence of Eq. 14 we apply PMI on "n" which is composed as:

Step 1: For $n = 1 \beta_i = \left(\dot{s}_i, \text{i}_i, \text{d}_i\right) \omega_i = 1$. Thus,

TSFWNA $(\beta_1) = \omega_1 \beta_1 = \left(\dot{s}_1, \text{i}_1, \text{d}_1\right)$

$$
= \left(\sqrt[q]{\frac{\omega_1 \dot{s}_1^q}{\omega_1\left(\dot{s}_1^q + \text{i}_1^q + \text{d}_1^q\right)}.\left(1-\left(r_1^q\right)^{\omega_1}\right)},\ \sqrt[q]{\frac{\omega_1 \text{i}_1^q}{\omega_1\left(\dot{s}_1^q + \text{i}_1^q + \text{d}_1^q\right)}.\left(1-\left(r_1^q\right)^{\omega_1}\right)},\ \sqrt[q]{\frac{\omega_1 \text{d}_1^q}{\omega_1\left(\dot{s}_1^q + \text{i}_1^q + \text{d}_1^q\right)}.\left(1-\left(r_1^q\right)^{\omega_1}\right)}\right)
$$

Thus, Eq. (14) is satisfied.
Step2: Suppose Eq. (14) is valid for $n = k$

$$
TSFWNA(\beta_1, \beta_2, ..., \beta_k) = \left(\begin{array}{c}
\sqrt[q]{\frac{\sum_{i=1}^k \omega_i \dot{s}_i^q}{\sum_{i=1}^k \omega_i\left(\dot{s}_i^q + \text{i}_i^q + \text{d}_i^q\right)}.\left(1-\prod_{i=1}^k \left(r_i^q\right)^{\omega_i}\right)}, \\
\sqrt[q]{\frac{\sum_{i=1}^k \omega_i \text{i}_i^q}{\sum_{i=1}^k \omega_i\left(\dot{s}_i^q + \text{i}_i^q + \text{d}_i^q\right)}.\left(1-\prod_{i=1}^k \left(r_i^q\right)^{\omega_i}\right)}, \\
\sqrt[q]{\frac{\sum_{i=1}^k \omega_i \text{d}_i^q}{\sum_{i=1}^k \omega_i\left(\dot{s}_i^q + \text{i}_i^q + \text{d}_i^q\right)}.\left(1-\prod_{i=1}^k \left(r_i^q\right)^{\omega_i}\right)}
\end{array}\right)
$$

Now for $n = k + 1$

$$
\begin{aligned}
& TSFWNA(\beta_1, \beta_2, ..., \beta_{k+1}) \\
& = TSFWNA(\beta_1, \beta_2, ..., \beta_k) \ominus (\omega_{k+1}\beta_{k+1})
\end{aligned}
$$

$$
= \left(\begin{array}{l}
\sqrt[q]{\dfrac{MCS^q(TSFWNA(\beta_1,\beta_2,...,\beta_k),\omega_{k+1}\beta_{k+1})}{\left(\begin{array}{l} MCS^q(TSFWNA(\beta_1,\beta_2,...,\beta_k),\omega_{k+1}\beta_{k+1}) \\ +NCS^q(TSFWNA(\beta_1,\beta_2,...,\beta_k),\omega_{k+1}\beta_{k+1}) \\ +ACS^q(TSFWNA(\beta_1,\beta_2,...,\beta_k),\omega_{k+1}\beta_{k+1}) \end{array}\right)}}.PS\left(\sqrt[q]{1-\prod_{i=1}^{k}\left(r_i^q\right)^{\omega_i}},\sqrt[q]{1-\left(r_{k+1}^q\right)^{\omega_{k+1}}}\right), \\
\sqrt[q]{\dfrac{NCS^q(TSFWNA(\beta_1,\beta_2,...,\beta_k),\omega_{k+1}\beta_{k+1})}{\left(\begin{array}{l} MCS^q(TSFWNA(\beta_1,\beta_2,...,\beta_k),\omega_{k+1}\beta_{k+1}) \\ +NCS^q(TSFWNA(\beta_1,\beta_2,...,\beta_k),\omega_{k+1}\beta_{k+1}) \\ +ACS^q(TSFWNA(\beta_1,\beta_2,...,\beta_k),\omega_{k+1}\beta_{k+1}) \end{array}\right)}}.PS\left(\sqrt[q]{1-\prod_{i=1}^{k}\left(r_i^q\right)^{\omega_i}},\sqrt[q]{1-\left(r_{k+1}^q\right)^{\omega_{k+1}}}\right), \\
\sqrt[q]{\dfrac{ACS^q(TSFWNA(\beta_1,\beta_2,...,\beta_k),\omega_{k+1}\beta_{k+1})}{\left(\begin{array}{l} MCS^q(TSFWNA(\beta_1,\beta_2,...,\beta_k),\omega_{k+1}\beta_{k+1}) \\ +NCS^q(TSFWNA(\beta_1,\beta_2,...,\beta_k),\omega_{k+1}\beta_{k+1}) \\ +ACS^q(TSFWNA(\beta_1,\beta_2,...,\beta_k),\omega_{k+1}\beta_{k+1}) \end{array}\right)}}.PS\left(\sqrt[q]{1-\prod_{i=1}^{k}\left(r_i^q\right)^{\omega_i}},\sqrt[q]{1-\left(r_{k+1}^q\right)^{\omega_{k+1}}}\right)
\end{array} \right)
$$

Using the expression for MCS, ACS, and NCS, we get

$$
\begin{aligned}
&MCS^q(TSFWNA(\beta_1,\beta_2,...,\beta_k),\omega_{k+1}\beta_{k+1}) \\
&\quad = MCS^q(TSFWNA(\beta_1,\beta_2,...,\beta_k)) + MCS^q(\omega_{k+1}\beta_{k+1}) \\
&\quad = \sum_{i=1}^{k}\omega_i \ddot{s}_i^q + \omega_{k+1}\ddot{s}_{k+1}^q = \sum_{i=1}^{k+1}\omega_i \ddot{s}_i^q .
\end{aligned}
$$

Similarly, we get,

$$
\begin{aligned}
&ACS^q\left(TSFWNA\left(\beta_1,\beta_2,...,\beta_k\right),\omega_{k+1}\beta_{k+1}\right) \\
&\quad = ACS^q\left(TSFWNA\left(\beta_1,\beta_2,...,\beta_k\right)\right) + ACS^q\left(\omega_{k+1}\beta_{k+1}\right)
\end{aligned}
$$

$$
= \sum_{i=1}^{k}\omega_i \underset{\cdot}{i}_i^q + \omega_{k+1}\underset{\cdot}{i}_{k+1}^q = \sum_{i=1}^{k+1}\omega_i \underset{\cdot}{i}_i^q
$$

$$
\begin{aligned}
&NCS^q(TSFWNA(\beta_1,\beta_2,...,\beta_k),\omega_{k+1}\beta_{k+1}) \\
&\quad = NCS^q(TSFWNA(\beta_1,\beta_2,...,\beta_k)) + NCS^q(\omega_{k+1}\beta_{k+1}) \\
&\quad = \sum_{i=1}^{k}\omega_i \underset{\cdot}{d}_i^q + \omega_{k+1}\underset{\cdot}{d}_{k+1}^q = \sum_{i=1}^{k+1}\omega_i \underset{\cdot}{d}_i^q ,
\end{aligned}
$$

also, by Definition of PS, we have.
$PS\left(\sqrt[q]{1-\prod_{i=1}^{k}\left(r_i^q\right)^{\omega_i}},\sqrt[q]{1-\left(r_{k+1}^q\right)^{\omega_{k+1}}}\right) = \sqrt[q]{1-\left(1-1+\prod_{i=1}^{k}\left(r_i^q\right)^{\omega_i}\right)\left(1-1+\left(r_{k+1}^q\right)^{\omega_{k+1}}\right)} = \sqrt[q]{1-\prod_{i=1}^{k+1}\left(r_i^q\right)^{\omega_i}}$. Thus,

$$TSFWNA(\beta_1, \beta_2, ..., \beta_{k+1}) = \begin{pmatrix} \sqrt[q]{\frac{\sum_{i=1}^{k+1}\omega_i \ddot{s}_i^q}{\sum_{i=1}^{k+1}\omega_i\left(\ddot{s}_i^q + \underset{.}{\mathrm{i}}_i^q + \underset{,}{\mathrm{d}}_i^q\right)}\cdot\left(1-\prod_{i=1}^{k+1}\left(r_i^q\right)^{\omega_i}\right)}, \\ \sqrt[q]{\frac{\sum_{i=1}^{k+1}\omega_i \underset{.}{\mathrm{i}}_i^q}{\sum_{i=1}^{k+1}\omega_i\left(\ddot{s}_i^q + \underset{.}{\mathrm{i}}_i^q + \underset{,}{\mathrm{d}}_i^q\right)}\cdot\left(1-\prod_{i=1}^{k+1}\left(r_i^q\right)^{\omega_i}\right)}, \\ \sqrt[q]{\frac{\sum_{i=1}^{k+1}\omega_i \underset{,}{\mathrm{d}}_i^q}{\sum_{i=1}^{k+1}\omega_i\left(\ddot{s}_i^q + \underset{.}{\mathrm{i}}_i^q + \underset{,}{\mathrm{d}}_i^q\right)}\cdot\left(1-\prod_{i=1}^{k+1}\left(r_i^q\right)^{\omega_i}\right)}, \end{pmatrix} \quad \text{i.e.,}$$

Eq. (14) satisfies for $n = k + 1$. Hence, using induction, Eq. (14) is true$\forall n$.

Proof of Theorem 7

Proof For "n" TSFNs $\beta_i = \left(\ddot{s}_i, \underset{.}{\mathrm{i}}_i, \underset{,}{\mathrm{d}}_i\right)$ and $\beta_0 = \left(\ddot{s}_0, \underset{.}{\mathrm{i}}_0, \underset{,}{\mathrm{d}}_0\right)$ such that $\beta_i = \beta_0$ we have $\ddot{s}_i = \ddot{s}_0$,$\underset{.}{\mathrm{i}}_{i=}\underset{.}{\mathrm{i}}_0$ and $\underset{,}{\mathrm{d}}_i = \underset{,}{\mathrm{d}}_0\ \forall i$, then by utilizing Eq. (14) having $\omega_i > 0$ and $\sum_{i=1}^{n}\omega_i = 1$, we yield.

$$TSFWNA(\beta_1, \beta_2, ..., \beta_n) = \begin{pmatrix} \sqrt[q]{\frac{\ddot{s}_0^q}{\left(\ddot{s}_0^q + \underset{.}{\mathrm{i}}_0^q + \underset{,}{\mathrm{d}}_0^q\right)}\cdot\left(1-\left(r_0^q\right)^{\sum_{i=1}^{n}\omega_i}\right)}, \\ \sqrt[q]{\frac{\underset{.}{\mathrm{i}}_0^q}{\left(\ddot{s}_0^q + \underset{.}{\mathrm{i}}_0^q + \underset{,}{\mathrm{d}}_0^q\right)}\cdot\left(1-\left(r_0^q\right)^{\sum_{i=1}^{n}\omega_i}\right)}, \\ \sqrt[q]{\frac{\underset{,}{\mathrm{d}}_0^q}{\left(\ddot{s}_0^q + \underset{.}{\mathrm{i}}_0^q + \underset{,}{\mathrm{d}}_0^q\right)}\cdot\left(1-\left(r_0^q\right)^{\sum_{i=1}^{n}\omega_i}\right)} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt[q]{\frac{\ddot{s}_0^q}{\left(\ddot{s}_0^q + \underset{.}{\mathrm{i}}_0^q + \underset{,}{\mathrm{d}}_0^q\right)}\cdot\left(\ddot{s}_0^q + \underset{.}{\mathrm{i}}_0^q + \underset{,}{\mathrm{d}}_0^q\right)}, \\ \sqrt[q]{\frac{\underset{.}{\mathrm{i}}_0^q}{\left(\ddot{s}_0^q + \underset{.}{\mathrm{i}}_0^q + \underset{,}{\mathrm{d}}_0^q\right)}\cdot\left(\ddot{s}_0^q + \underset{.}{\mathrm{i}}_0^q + \underset{,}{\mathrm{d}}_0^q\right)}, \\ \sqrt[q]{\frac{\underset{,}{\mathrm{d}}_0^q}{\left(\ddot{s}_0^q + \underset{.}{\mathrm{i}}_0^q + \underset{,}{\mathrm{d}}_0^q\right)}\cdot\left(\ddot{s}_0^q + \underset{.}{\mathrm{i}}_0^q + \underset{,}{\mathrm{d}}_0^q\right)} \end{pmatrix}$$

$$= \beta_0$$

Proof of Theorem 8

Proof For "n" TSFNs $\beta_i = \left(\ddot{s}_i, \underset{.}{\mathrm{i}}_i, \underset{,}{\mathrm{d}}_i\right)$, we have

(1) $$min\left\{\dot{s}_i^q+\underset{\cdot}{\mathrm{i}}_i^q+\underset{\varsigma}{\mathrm{d}}_i^q\right\}=1-\left(1-min\left\{\dot{s}_i^q+\underset{\cdot}{\mathrm{i}}_i^q+\underset{\varsigma}{\mathrm{d}}_i^q\right\}\right)^{\sum_{i=1}^n\omega_i}$$

$$=1-\prod_{i=1}^n\left(1-min\left\{\dot{s}_i^q+\underset{\cdot}{\mathrm{i}}_i^q+\underset{\varsigma}{\mathrm{d}}_i^q\right\}\right)^{\omega_i}\leq 1-\prod_{i=1}^n\left(1-\dot{s}_i^q-\underset{\cdot}{\mathrm{i}}_i^q-\underset{\varsigma}{\mathrm{d}}_i^q\right)^{\omega_i}$$

$$\leq 1-\prod_{i=1}^n\left(1-max\left\{\dot{s}_i^q+\underset{\cdot}{\mathrm{i}}_i^q+\underset{\varsigma}{\mathrm{d}}_i^q\right\}\right)^{\omega_i}=1-\left(1-max\left\{\dot{s}_i^q+\underset{\cdot}{\mathrm{i}}_i^q+\underset{\varsigma}{\mathrm{d}}_i^q\right\}\right)^{\sum_{i=1}^n\omega_i}$$

$$=max\left\{\dot{s}_i^q+\underset{\cdot}{\mathrm{i}}_i^q+\underset{\varsigma}{\mathrm{d}}_i^q\right\}.$$

Thus, we have

$$min\left\{\dot{s}_i^q+\underset{\cdot}{\mathrm{i}}_i^q+\underset{\varsigma}{\mathrm{d}}_i^q\right\}\leq 1-\prod_{i=1}^n\left(1-\dot{s}_i^q-\underset{\cdot}{\mathrm{i}}_i^q-\underset{\varsigma}{\mathrm{d}}_i^q\right)^{\omega_i}\leq max\left\{\dot{s}_i^q+\underset{\cdot}{\mathrm{i}}_i^q+\underset{\varsigma}{\mathrm{d}}_i^q\right\}.$$

Using theorem 6,

$$\dot{s}_p=\sqrt[q]{\frac{\sum_{i=1}^k\omega_i\dot{s}_i^q}{\sum_{i=1}^k\omega_i\left(\dot{s}_i^q+\underset{\cdot}{\mathrm{i}}_i^q+\underset{\varsigma}{\mathrm{d}}_i^q\right)}.\left(1-\prod_{i=1}^k\left(r_i^q\right)^{\omega_i}\right)},$$

$$\underset{\cdot}{\mathrm{i}}_p=\sqrt[q]{\frac{\sum_{i=1}^k\omega_i\underset{\cdot}{\mathrm{i}}_i^q}{\sum_{i=1}^k\omega_i\left(\dot{s}_i^q+\underset{\cdot}{\mathrm{i}}_i^q+\underset{\varsigma}{\mathrm{d}}_i^q\right)}.\left(1-\prod_{i=1}^k\left(r_i^q\right)^{\omega_i}\right)}.$$

Therefore,

$$\underset{\varsigma}{\mathrm{d}}_p=\sqrt[q]{\frac{\sum_{i=1}^k\omega_i\underset{\varsigma}{\mathrm{d}}_i^q}{\sum_{i=1}^k\omega_i\left(\dot{s}_i^q+\underset{\cdot}{\mathrm{i}}_i^q+\underset{\varsigma}{\mathrm{d}}_i^q\right)}.\left(1-\prod_{i=1}^k\left(r_i^q\right)^{\omega_i}\right)}$$

$$\dot{s}_p^q+\underset{\cdot}{\mathrm{i}}_p^q+\underset{\varsigma}{\mathrm{d}}_p^q=1-\prod_{i=1}^n\left(1-\dot{s}_i^q-\underset{\cdot}{\mathrm{i}}_i^q-\underset{\varsigma}{\mathrm{d}}_i^q\right)^{\omega_i}.$$

Hence, we get

$$min\left\{\dot{s}_i^q+\underset{\cdot}{\mathrm{i}}_i^q+\underset{\varsigma}{\mathrm{d}}_i^q\right\}\leq\dot{s}_p^q+\underset{\cdot}{\mathrm{i}}_p^q+\underset{\varsigma}{\mathrm{d}}_p^q\leq max\left\{\dot{s}_i^q+\underset{\cdot}{\mathrm{i}}_i^q+\underset{\varsigma}{\mathrm{d}}_i^q\right\}.$$

(2) Since, $\dot{s}_i\geq min\left\{\dot{s}_i\right\}$, so by expression $\dot{s}_p$ we have

$$\dot{s}_p^q\geq\frac{\sum_{i=1}^n\omega_i\left(min\left\{\dot{s}_i^q\right\}\right)}{\sum_{i=1}^n\omega_i\left(max\left\{\dot{s}_i^q+\underset{\cdot}{\mathrm{i}}_i^q+\underset{\varsigma}{\mathrm{d}}_i^q\right\}\right)}\left[1-\prod_{i=1}^n\left(1-min\left\{\dot{s}_i^q+\underset{\cdot}{\mathrm{i}}_i^q+\underset{\varsigma}{\mathrm{d}}_i^q\right\}\right)^{\omega_i}\right]$$

$$=\frac{min\left\{\dot{s}_i^q\right\}}{max\left\{\dot{s}_i^q+\underset{\cdot}{\mathrm{i}}_i^q+\underset{\varsigma}{\mathrm{d}}_i^q\right\}}\left[1-\left(1-min\left\{\dot{s}_i^q+\underset{\cdot}{\mathrm{i}}_i^q+\underset{\varsigma}{\mathrm{d}}_i^q\right\}\right)^{\sum_{i=1}^n\omega_i}\right]$$

$$=\frac{min\left\{\dot{s}_i^q+\underset{\cdot}{\mathrm{i}}_i^q+\underset{\varsigma}{\mathrm{d}}_i^q\right\}min\left\{\dot{s}_i^q\right\}}{max\left\{\dot{s}_i^q+\underset{\cdot}{\mathrm{i}}_i^q+\underset{\varsigma}{\mathrm{d}}_i^q\right\}}.$$

Moreover,

$$\begin{aligned}
\dot{s}_p^q &\leq \frac{\sum_{i=1}^{n}\omega_i\left(max\left\{\dot{s}_i^q\right\}\right)}{\sum_{i=1}^{n}\omega_i\left(min\left\{\dot{s}_i^q+\underset{\cdot}{\mathrm{i}}_i^q+\underset{\varsigma}{\mathrm{d}}_i^q\right\}\right)}\left[1-\prod_{i=1}^{n}\left(1-max\left\{\dot{s}_i^q+\underset{\cdot}{\mathrm{i}}_i^q+\underset{\varsigma}{\mathrm{d}}_i^q\right\}\right)^{\omega_i}\right]\\
&=\frac{max\left\{\dot{s}_i^q\right\}}{min\left\{\dot{s}_i^q+\underset{\cdot}{\mathrm{i}}_i^q+\underset{\varsigma}{\mathrm{d}}_i^q\right\}}\left[1-\left(1-max\left\{\dot{s}_i^q+\underset{\cdot}{\mathrm{i}}_i^q+\underset{\varsigma}{\mathrm{d}}_i^q\right\}\right)^{\sum_{i=1}^{n}\omega_i}\right]\\
&=\frac{max\left\{\dot{s}_i^q+\underset{\cdot}{\mathrm{i}}_i^q+\underset{\varsigma}{\mathrm{d}}_i^q\right\}max\left\{\dot{s}_i^q\right\}}{min\left\{\dot{s}_i^q+\underset{\cdot}{\mathrm{i}}_i^q+\underset{\varsigma}{\mathrm{d}}_i^q\right\}}
\end{aligned}$$

$$=\frac{min\left\{\dot{s}_i^q+\underset{\cdot}{\mathrm{i}}_i^q+\underset{\varsigma}{\mathrm{d}}_i^q\right\}min\left\{\dot{s}_i^q\right\}}{max\left\{\dot{s}_i^q+\underset{\cdot}{\mathrm{i}}_i^q+\underset{\varsigma}{\mathrm{d}}_i^q\right\}}\leq \mathrm{E}_p^q\leq\frac{max\left\{\dot{s}_i^q+\underset{\cdot}{\mathrm{i}}_i^q+\underset{\varsigma}{\mathrm{d}}_i^q\right\}.max\left\{\dot{s}_i^q\right\}}{min\left\{\dot{s}_i^q+\underset{\cdot}{\mathrm{i}}_i^q+\underset{\varsigma}{\mathrm{d}}_i^q\right\}}$$

(3) Similar to part (2).

Proof of Theorem 9

Proof For "n" TSFNs $\beta_1,\beta_2,\ldots,\beta_n$ and $\gamma_1,\gamma_2,\ldots,\gamma_n$ and by Theorem 6, we get TSFWNA $(\beta_1,\beta_2,\ldots,\beta_n)=\left(\dot{s}_{p_\beta},\underset{\cdot}{\mathrm{i}}_{p_\beta},\underset{\varsigma}{\mathrm{d}}_{p_\beta}\right)$ and TSFWNA $(\gamma_1,\gamma_2,\ldots,\gamma_n)=\left(\dot{s}_{p_\gamma},\underset{\cdot}{\mathrm{i}}_{p_\gamma},\underset{\varsigma}{\mathrm{d}}_{p_\gamma}\right)$

$$\begin{aligned}
\dot{s}_{p_\beta}^q &= \frac{\sum_{i=1}^{k}\omega_i\dot{s}_{\beta_i}^q}{\sum_{i=1}^{k}\omega_i\left(\dot{s}_{\beta_i}^q+\underset{\cdot}{\mathrm{i}}_{\beta_i}^q+\underset{\varsigma}{\mathrm{d}}_{\beta_i}^q\right)}\left[1-\prod_{i=1}^{n}\left(1-\dot{s}_{\beta_i}^q-\underset{\cdot}{\mathrm{i}}_{\beta_i}^q-\underset{\varsigma}{\mathrm{d}}_{\beta_i}^q\right)^{\omega_i}\right],\\
\underset{\cdot}{\mathrm{i}}_{p_\beta}^q &= \frac{\sum_{i=1}^{k}\omega_i\underset{\cdot}{\mathrm{i}}_{\beta_i}^q}{\sum_{i=1}^{k}\omega_i\left(\dot{s}_{\beta_i}^q+\underset{\cdot}{\mathrm{i}}_{\beta_i}^q+\underset{\varsigma}{\mathrm{d}}_{\beta_i}^q\right)}\left[1-\prod_{i=1}^{n}\left(1-\dot{s}_{\beta_i}^q-\underset{\cdot}{\mathrm{i}}_{\beta_i}^q-\underset{\varsigma}{\mathrm{d}}_{\beta_i}^q\right)^{\omega_i}\right],\\
\underset{\varsigma}{\mathrm{d}}_{p_\beta}^q &= \frac{\sum_{i=1}^{k}\omega_i\underset{\varsigma}{\mathrm{d}}_{\beta_i}^q}{\sum_{i=1}^{k}\omega_i\left(\dot{s}_{\beta_i}^q+\underset{\cdot}{\mathrm{i}}_{\beta_i}^q+\underset{\varsigma}{\mathrm{d}}_{\beta_i}^q\right)}\left[1-\prod_{i=1}^{n}\left(1-\dot{s}_{\beta_i}^q-\underset{\cdot}{\mathrm{i}}_{\beta_i}^q-\underset{\varsigma}{\mathrm{d}}_{\beta_i}^q\right)^{\omega_i}\right],\\
\dot{s}_{p_\gamma}^q &= \frac{\sum_{i=1}^{k}\omega_i\dot{s}_{\gamma_i}^q}{\sum_{i=1}^{k}\omega_i\left(\dot{s}_{\gamma_i}^q+\underset{\cdot}{\mathrm{i}}_{\gamma_i}^q+\underset{\varsigma}{\mathrm{d}}_{\gamma_i}^q\right)}\left[1-\prod_{i=1}^{n}\left(1-\dot{s}_{\gamma_i}^q-\underset{\cdot}{\mathrm{i}}_{\gamma_i}^q-\underset{\varsigma}{\mathrm{d}}_{\gamma_i}^q\right)^{\omega_i}\right],\\
\underset{\cdot}{\mathrm{i}}_{p_\gamma}^q &= \frac{\sum_{i=1}^{k}\omega_i\underset{\cdot}{\mathrm{i}}_{\gamma_i}^q}{\sum_{i=1}^{k}\omega_i\left(\dot{s}_{\gamma_i}^q+\underset{\cdot}{\mathrm{i}}_{\gamma_i}^q+\underset{\varsigma}{\mathrm{d}}_{\gamma_i}^q\right)}\left[1-\prod_{i=1}^{n}\left(1-\dot{s}_{\gamma_i}^q-\underset{\cdot}{\mathrm{i}}_{\gamma_i}^q-\underset{\varsigma}{\mathrm{d}}_{\gamma_i}^q\right)^{\omega_i}\right],\\
\underset{\varsigma}{\mathrm{d}}_{p_\gamma}^q &= \frac{\sum_{i=1}^{k}\omega_i\underset{\varsigma}{\mathrm{d}}_{\gamma_i}^q}{\sum_{i=1}^{k}\omega_i\left(\dot{s}_{\gamma_i}^q+\underset{\cdot}{\mathrm{i}}_{\gamma_i}^q+\underset{\varsigma}{\mathrm{d}}_{\gamma_i}^q\right)}\left[1-\prod_{i=1}^{n}\left(1-\dot{s}_{\gamma_i}^q-\underset{\cdot}{\mathrm{i}}_{\gamma_i}^q-\underset{\varsigma}{\mathrm{d}}_{\gamma_i}^q\right)^{\omega_i}\right].
\end{aligned}$$

By applying the above results, we get.

(1) If $\dot{s}_{\beta_i}^q+\underset{\cdot}{\mathrm{i}}_{\beta_i}^q+\underset{\varsigma}{\mathrm{d}}_{\beta_i}^q\leq\dot{s}_{\gamma_i}^q+\underset{\cdot}{\mathrm{i}}_{\gamma_i}^q+\underset{\varsigma}{\mathrm{d}}_{\gamma_i}^q$, then

$$\begin{aligned}
\dot{s}_{p_\beta}^q+\underset{\cdot}{\mathrm{i}}_{p_\beta}^q+\underset{\varsigma}{\mathrm{d}}_{p_\beta}^q &\leq 1-\prod_{i=1}^{n}\left(1-\dot{s}_{\beta_i}^q-\underset{\cdot}{\mathrm{i}}_{\beta_i}^q-\underset{\varsigma}{\mathrm{d}}_{\beta_i}^q\right)^{\omega_i}\\
&\leq 1-\prod_{i=1}^{n}\left(1-\dot{s}_{\gamma_i}^q-\underset{\cdot}{\mathrm{i}}_{\gamma_i}^q-\underset{\varsigma}{\mathrm{d}}_{\gamma_i}^q\right)^{\omega_i}=\dot{s}_{p_\gamma}^q+\underset{\cdot}{\mathrm{i}}_{p_\gamma}^q+\underset{\varsigma}{\mathrm{d}}_{p_\gamma}^q.
\end{aligned}$$

(2) If $\dddot{s}^q_{\beta_i} + \underset{\cdot}{\mathrm{i}}^q_{\beta_i} + \underset{,}{\mathrm{d}}^q_{\beta_i} = \dddot{s}^q_{\gamma_i} + \underset{\cdot}{\mathrm{i}}^q_{\gamma_i} + \underset{,}{\mathrm{d}}^q_{p_\gamma}$, and $\dddot{s}^q_{\beta_i} \le \dddot{s}^q_{\gamma_i}$, then

$$
\begin{aligned}
\dddot{s}^q_{p_\beta} &= \frac{\sum_{i=1}^{k} \omega_i \dddot{s}^q_{\beta_i}}{\sum_{i=1}^{k} \omega_i \left(\dddot{s}^q_{\beta_i} + \underset{\cdot}{\mathrm{i}}^q_{\beta_i} + \underset{,}{\mathrm{d}}^q_{\beta_i}\right)} \left[1 - \prod_{i=1}^{n} \left(1 - \dddot{s}^q_{\beta_i} - \underset{\cdot}{\mathrm{i}}^q_{\beta_i} - \underset{,}{\mathrm{d}}^q_{\beta_i}\right)^{\omega_i}\right] \\
&\le \frac{\sum_{i=1}^{k} \omega_i \dddot{s}^q_{\gamma_i}}{\sum_{i=1}^{k} \omega_i \left(\dddot{s}^q_{\gamma_i} + \underset{\cdot}{\mathrm{i}}^q_{\gamma_i} + \underset{,}{\mathrm{d}}^q_{\gamma_i}\right)} \left[1 - \prod_{i=1}^{n} \left(1 - \dddot{s}^q_{\gamma_i} - \underset{\cdot}{\mathrm{i}}^q_{\gamma_i} - \underset{,}{\mathrm{d}}^q_{\gamma_i}\right)^{\omega_i}\right] \\
&= \dddot{s}^q_{p_\gamma}
\end{aligned}
$$

and

$$
\begin{aligned}
\underset{\cdot}{\mathrm{i}}^q_{p_\beta} &= \frac{\sum_{i=1}^{k} \omega_i \underset{\cdot}{\mathrm{i}}^q_{\beta_i}}{\sum_{i=1}^{k} \omega_i \left(\dddot{s}^q_{\beta_i} + \underset{\cdot}{\mathrm{i}}^q_{\beta_i} + \underset{,}{\mathrm{d}}^q_{\beta_i}\right)} \left[1 - \prod_{i=1}^{n} \left(1 - \dddot{s}^q_{\beta_i} - \underset{\cdot}{\mathrm{i}}^q_{\beta_i} - \underset{,}{\mathrm{d}}^q_{\beta_i}\right)^{\omega_i}\right] \\
&\ge \frac{\sum_{i=1}^{k} \omega_i \underset{\cdot}{\mathrm{i}}^q_{\gamma_i}}{\sum_{i=1}^{k} \omega_i \left(\dddot{s}^q_{\gamma_i} + \underset{\cdot}{\mathrm{i}}^q_{\gamma_i} + \underset{,}{\mathrm{d}}^q_{\gamma_i}\right)} \left[1 - \prod_{i=1}^{n} \left(1 - \dddot{s}^q_{\gamma_i} - \underset{\cdot}{\mathrm{i}}^q_{\gamma_i} - \underset{,}{\mathrm{d}}^q_{\gamma_i}\right)^{\omega_i}\right] \\
&= \underset{\cdot}{\mathrm{i}}^q_{p_\gamma}
\end{aligned}
$$

and

$$
\begin{aligned}
\underset{,}{\mathrm{d}}^q_{p_\beta} &= \frac{\sum_{i=1}^{k} \omega_i \underset{,}{\mathrm{d}}^q_{\beta_i}}{\sum_{i=1}^{k} \omega_i \left(\dddot{s}^q_{\beta_i} + \underset{\cdot}{\mathrm{i}}^q_{\beta_i} + \underset{,}{\mathrm{d}}^q_{\beta_i}\right)} \left[1 - \prod_{i=1}^{n} \left(1 - \dddot{s}^q_{\beta_i} - \underset{\cdot}{\mathrm{i}}^q_{\beta_i} - \underset{,}{\mathrm{d}}^q_{\beta_i}\right)^{\omega_i}\right] \\
&\ge \frac{\sum_{i=1}^{k} \omega_i \underset{,}{\mathrm{d}}^q_{\gamma_i}}{\sum_{i=1}^{k} \omega_i \left(\dddot{s}^q_{\gamma_i} + \underset{\cdot}{\mathrm{i}}^q_{\gamma_i} + \underset{,}{\mathrm{d}}^q_{\gamma_i}\right)} \left[1 - \prod_{i=1}^{n} \left(1 - \dddot{s}^q_{\gamma_i} - \underset{\cdot}{\mathrm{i}}^q_{\gamma_i} - \underset{,}{\mathrm{d}}^q_{\gamma_i}\right)^{\omega_i}\right]. \\
&= \underset{,}{\mathrm{d}}^q_{p_\gamma}
\end{aligned}
$$

From part (2), we obtain $\dddot{s}^q_{p_\beta} \le \dddot{s}^q_{p_\gamma}$, $\underset{\cdot}{\mathrm{i}}^q_{p_\beta} \ge \underset{\cdot}{\mathrm{i}}^q_{p_\gamma}$ and $\underset{,}{\mathrm{d}}^q_{p_\beta} \ge \underset{,}{\mathrm{d}}^q_{p_\gamma}$ and hence by Eq. (2), we get.
$\dddot{s}^q_{p_\beta} - \underset{\cdot}{\mathrm{i}}^q_{p_\beta} - \underset{,}{\mathrm{d}}^q_{p_\beta} \le \dddot{s}^q_{p_\gamma} - \underset{\cdot}{\mathrm{i}}^q_{p_\gamma} - \underset{,}{\mathrm{d}}^q_{p_\gamma}$. So, by an order relation, we get

$$\mathrm{TSFWNA}(\beta_1, \beta_2, ..., \beta_n) \le \mathrm{TSFWNA}(\gamma_1, \gamma_2, ..., \gamma_n).$$

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