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ORIGINAL ARTICLE

Approaches for multicriteria decision-making based on the hesitant fuzzy best–worst method

Jian Li[1] · Li-li Niu[2] · Qiongxia Chen[1] · Zhong-xing Wang[3]

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Abstract
Hesitant fuzzy preference relations (HFPRs) have been widely applied in multicriteria decision-making (MCDM) for their ability to efficiently express hesitant information. To address the situation where HFPRs are necessary, this paper develops several decision-making models integrating HFPRs with the best worst method (BWM). First, consistency measures from the perspectives of additive/multiplicative consistent hesitant fuzzy best worst preference relations (HFBWPRs) are introduced. Second, several decision-making models are developed in view of the proposed additive/multiplicatively consistent HFBWPRs. The main characteristic of the constructed models is that they consider all the values included in the HFBWPRs and consider the same and different compromise limit constraints. Third, an absolute programming model is developed to obtain the decision-makers' objective weights utilizing the information of optimal priority weight vectors and provides the calculation of decision-makers' comprehensive weights. Finally, a framework of the MCDM procedure based on hesitant fuzzy BWM is introduced, and an illustrative example in conjunction with comparative analysis is provided to demonstrate the feasibility and efficiency of the proposed models.

Keywords Multicriteria decision-making · Hesitant fuzzy best worst preference relations · Additive consistency · Multiplicative consistency · Best–worst method

Introduction

In multicriteria decision-making (MCDM) problems, we need to choose the best alternative/alternatives according to several determined criteria from a set of alternatives [1–3]. Different approaches have been developed from different perspectives, where preference relations (PRs) are one of the commonly used technologies. Their principle is to rank alternatives in view of the priority weight vector obtained from pairwise comparisons of the alternatives [4]. According to the evaluation scale of the pairwise comparisons, PRs can be divided into fuzzy PRs (FPRs) and multiplicative PRs (MPRs). The former uses the [0, 1] scale, and the latter expresses the comparison with [1/9, 9]. In the process of developing PRs, consistency analysis is necessary to avoid contradictory ranking. Tanino [5] developed two consistency concepts for FPRs, namely, additive consistent FPRs and multiplicative consistent FPRs. The former indicates additive transitivity, and the latter shows multiplicative transitivity. For MPRs, Saaty [6] defined the concept of multiplicatively consistent MPRs, which indicates the multiplicative transitivity among three related comparisons. Since then, MCDM approaches based on two types of PRs have been proposed [7–12].

Noticeably, FPRs and MPRs only employ an exact numeric value to denote the membership degree of pairwise comparisons, which limits the applications because hesitant information exists extensively in MCDM problems. To address this issue, Torra and Narukawa [13] employed several values in [0, 1] to denote the pairwise comparisons and developed the concept of hesitant fuzzy set (HFS). The primary advantage of HFS is that several values can be used to represent decision makers' hesitant

✉ Qiongxia Chen
qxchen@nnnu.edu.cn

Li-li Niu
nllgxu@163.com

1 School of Logistics Management and Engineering, Nanning Normal University, Nanning 530001, People's Republic of China

2 Guangxi University Xingjian College of Science and Liberal Arts, Nanning 530005, People's Republic of China

3 School of Mathematics and Information Science, Guangxi University, Nanning 530004, People's Republic of China

information; it can also overcome the shortcoming of fuzzy sets, which use only one value. Since the concept of HFS has been proposed, a large number of studies focusing on HFS have been developed [14–16]. Later, Xia and Xu [17] introduced the concept of hesitant fuzzy preference relations (HFPRs). Following the original work of Xia and Xu [17], many MCDM approaches based on HFPRs have been developed. For example, MCDM approaches are based on additive consistent HFPRs [18–20], MCDM approaches are based on multiplicatively consistent HFPRs [21–23], and MCDM approaches are based on multiplicatively consistent hesitant MPRs (HMPRs) [24, 25]. According to the principle of using the number of elements, including hesitant fuzzy elements (HFEs), these approaches can be classified into four categories [21]. A concise literature review of these approaches is presented as follows and summarized in Table 1.

(1) Consider only one FPR derived from HFPRs [26, 27]. This method is also named optimistic consistency; that is, a reduced FPR with the highest consistency degree is derived from HFPRs. The optimistic consistency method can reflect the highest consistency degree of HFPRs, but it cannot reflect the hesitancy of decision-makers. It leads to substantial information loss. (2) Based on ordered FPRs derived from normalized HFPRs [28, 29]. This method also named normalized consistency. The normalized consistency requires that any two HFEs have an equal number of elements; if two HFEs have an unequal number of elements, a normalized process is needed. In the review of the previous work [30], the shorter HFE needs to add some values until two HFEs have the same number of elements in the normalized process. Therefore, the normalized consistency method may distort the original information provided by decision-makers. (3) Based on all possible FPRs, including HFPRs [18, 31]. This method defines the concept of consistent HFPRs as too restricted. It is difficult for decision-makers to provide such pairwise comparisons in the actual decision-making process. (4) Based on the derived FPRs for each value in HFEs [32, 33]. The main feature of this method is that it considers all the evaluation information and neither adds values to HFEs nor removes values from HFEs. Compared with (3), this method only used some possible FPRs, including HFPRs.

Rezaei [34] developed a novel MCDM method named the best–worst method (BWM), which can be taken as an enhancement of the traditional analytic hierarchy process (AHP). With the BWM, it can remedy the drawbacks of AHP in terms of numerous comparisons and low consistency [35]; thus, it is much easier to use. At the same time, the weights derived from the BWM are more reliable than the AHP, as it only needs to provide the best and worst vectors [36, 37]. Due to these advantages, the BWM has attracted wide attention from scholars [38–43]. For example, Ming et al. [44] managed patient satisfaction in a blood-collection room integrated with BWM and probabilistic linguistic gained and lost dominance score method. Karimi et al. [45] introduced a fully fuzzy BWM with a triangular fuzzy number. Chen and Ming [46] developed a smart product service module integrated with a BWM and data envelopment analysis. Liang et al. [47] established the thresholds for the consistency ratios. Mohammadi and Rezaei [48] introduced the Bayesian BWM for group decision-making problems. In addition, an overview of the BWM can be found in [49].

The concept of HFPRs has been introduced, and several scholars have studied some MCDM methods based on BWM. However, there are still some important issues that need to be further studied. (1) The concept of additive/multiplicatively consistent HFPRs. As HFPRs, additive/multiplicatively consistent HFPRs develop in considering one FPR derived from HFPRs, which may lead to information loss [50]; develops in considering ordered FPRs derived from normalized HFPRs may distort the preference information [50, 51]; and develops in considering all possible FPRs in HFPRs seems too restrictive [31]. (2) The different expertise levels of decision-makers and the complexity of MCDM problems lead to the appearance of uncertainty in decision-making processes. In these cases, uncertain techniques, such as fuzzy numbers, interval numbers and triangular fuzzy numbers, were integrated with the BWM. Unfortunately, few scholars have studied integrating BWM with HFEs. (3) In the review of the previous work related to BWM, the scholars study the maximum absolute differences are minimized problem are transferred to two list of same compromise limit constraints. Considering that decision-makers with different constraints may have different compromise limits, it is necessary to study the maximum absolute differences when the minimized problem is transferred to two lists of different compromise limit constraints.

To eliminate the abovementioned defects, consider the advantage of HFS in showing the evaluation information and the advantage of BWM in solving MCDM problems. It is necessary to propose a new hesitant fuzzy BWM for MCDM. In this study, consistency measures from the perspectives of

Table 1 A summary of different consistent HFPRs

The category of different consistent HFPRs	Main characteristic	Representative literature
Optimistic consistency	Only considers one FPR derived from HFPRs	[26, 27]
Normalized consistency	Based on ordered FPRs derived from normalized HFPRs	[28, 29]
Average consistency	Based on all possible FPRs including in HFPRs	[18, 31]
Partial average consistency	Based on the derived FPRs for each value in HFEs	[32, 33]

additive/multiplicative consistent hesitant fuzzy best worst preference relations (HFBWPRs) are defined. Several decision-making models are developed in view of the proposed additive consistency and multiplicative consistency measures. The primary contributions of this study are summarized as follows:

1. Consistency measures from the perspectives of additive/multiplicatively consistent HFBWPRs are introduced, which integrate the advantages of HFPRs and BWM.
2. Several decision-making models are developed in view of the proposed additive/multiplicatively consistent HFBWPRs. The main characteristic of the constructed models is that they consider all the values included in the HFBWPRs and consider the same and different compromise limit constraints.
3. An absolute programming model is developed to obtain the decision-makers' objective weights utilizing the optimal priority weight vector information, and the calculation of decision-makers' comprehensive weights is provided.

The remainder of the paper is organized as follows. In "Preliminaries", some basic knowledge of FPRs, HFS, HFPRs and the BWM is introduced. In "Hesitant fuzzy BWM", the concepts of additive/multiplicatively consistent HFBWPRs are presented, and several decision-making models are developed in view of the proposed additive consistency and multiplicative consistency measures. In "A framework of MCDM procedure based on hesitant fuzzy BWM", an absolute programming model is developed to obtain the decision-makers' objective weights, and a procedure for MCDM problems with hesitant fuzzy BWM is given. In "Illustrative example", the proposed methods are illustrated by an example, and a comparative analysis is provided. Finally, conclusions are presented in "Conclusion".

Preliminaries

In this section, some basic knowledge of FPRs, HFS, HFPRs and the BWM is introduced.

FPRs

Let $X = \{x_1, x_2, \ldots, x_n\}$ denote a finite set of alternatives, where x_i represents the *ith* alternate. Orlovsky [52] developed the concept of FPRs.

Definition 1 [52]. An FPR on a set of alternatives X is represented by a matrix $H = (r_{ij})_{n \times n} \subset X \times X$, where r_{ij} is interpreted as the degree to which alternative x_i is preferred to x_j. Furthermore, r_{ij} should satisfy the following conditions: $r_{ij} + r_{ji} = 1, r_{ii} = 0.5$ for all $i, j \in N$.

To measure the rationality of FPRs provided by decision-makers, the concepts of additive consistent and multiplicative consistent FPRs were introduced.

Definition 2 [5]. Let $H = (r_{ij})_{n \times n}$ be an FPR, and $W = (w_1, w_2, \ldots, w_n)$ be the priority weight vector derived from R, where $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. For all $i, j \in N$, the FPR is additive consistency if $r_{ij} = \frac{1}{2}(w_i - w_j) + 0.5$ and FPR is multiplicative consistency if $r_{ij} = \frac{w_i}{w_i + w_j}$.

HFS and HFPRs

To express the hesitant information, Torra and Narukawa [13] introduced an effective tool which named HFS.

Definition 3 [13]. Let X be a fixed set. Accordingly, a HFS E on X is defined in terms of a function $h_E(x)$ that when applied to X returns a finite subset of [0, 1].

To be easily understood, Xia and Xu [53] utilized the following mathematical symbol to express the HFS:$E = \{< x, h_E(x) > | x \in X\}$. Where $h_E(x)$ is a set of values in [0, 1] representing the possible membership degrees of the element x in X to E, and $h_E(x)$ is named HFE and denoted as $h = \{\gamma^s | s = 1, 2, \ldots, \#h\}$, $\#h$ is the number of elements including in h.

With the effective of HFE, Xia and Xu [17] proposed the concept of HFPRs. However, the need for sequence relationships of the elements including in HFPRs, this leads to some complexity in actual application. To address this issue, Xu et al. [54] developed a new definition of HFPRs that does not need to arrange the elements in descending or ascending sequence.

Definition 4 [54]. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a fixed set, HFPRs on X is represented by a matrix $R = (h_{ij})_{n \times n} \subset X \times X$, where $h_{ij} = \left\{\gamma_{ij}^s \middle| s = 1, 2, \ldots, \#h_{ij}\right\}$ is a HFE indicating the possible preference degrees of alternative x_i is preferred to alternative x_j. For all $i, j \in N$, h_{ij} should satisfy: $\gamma_{ij}^s + \gamma_{ji}^{\#h_{ij} - s + 1} = 1, \gamma_{ii} = 0.5, \#h_{ij} = \#h_{ji}$, where γ_{ij}^s refers to the sth element in h_{ij}.

Integrating into Tanino [5]'s additive consistency and the concept of HFPRs, Xu et al. [55] developed the concept of additive consistent HFPRs.

Definition 5 [55]. Let R be the same as those given in Definition 4. If R satisfies the following condition: $\frac{1}{2}(w_i - w_j) + 0.5 = \gamma_{ij}^1$ *or* γ_{ij}^2 or … or $\gamma_{ij}^{\#h_{ij}}$. Then R is called additive consistent HFPR, where $W = (w_1, w_2, \ldots, w_n)$ is the priority weight vector derived from R.

Similarly, integrating into Tanino [5]'s multiplicative consistency and the concept of HFPRs, Zhu and Xu [56] developed the concept of multiplicative consistent HFPRs.

Definition 6 [56]. Let R be the same as those given in Definition 4. If R satisfies the following condition: $\frac{w_i}{w_i+w_j} = \gamma_{ij}^1 \; or \; \gamma_{ij}^2 \; or \; \cdots \; or \; \gamma_{ij}^{\#h_{ij}}$, then R is called multiplicative consistent HFPR.

BWM

BWM is a frequently used MCDM technique since it was developed by Rezaei [34]. Later, Li et al. [10] extended this method to the evaluation scale as a number from 0.1 to 1 to address FPRs. It mainly includes the following steps:

Step 1: Define the decision criteria.

The decision criteria are defined on the basis of alternatives' characteristics and denoted as $\{c_1, c_2, \ldots, c_n\}$.

Step 2: Identify the best and worst criteria.

How to identify the best and worst criteria is puzzled by decision-makers when using the BWM. Some scholars developed it with the out-degree and in-degree of the node [57] and belief degree [58], while more scholars suggested that with respect to decision-makers' professional judgment [59].

Step 3: Determine the priority of the best criterion over each of the other criteria.

When the best criteria are identified, the decision-makers determine the priority of the best criterion over each of the other criteria as a number from 0.1 to 1 and denote it as $A_{Bj} = (a_{B1}, a_{B2}, \cdots, a_{Bn})$. The value a_{Bj} is the priority of the best criterion over the jth criterion and $a_{BB} = 0.5$, where the evaluation result is an FPR.

Step 4: Determine the priority of each criterion over the worst criteria.

Similarly, when the worst criteria are identified, the decision-makers determine the priority of each criterion over the worst criteria as a number from 0.1 to 1 and denote it as $A_{jW} = (a_{1W}, a_{2W}, \ldots, a_{nW})$. The value a_{jW} is the priority of the jth criterion over the worst criteria and $a_{WW} = 0.5$, where the evaluation result is an FPR.

Step 5: Calculate the optimal weights of the criteria.

To obtain the optimal weight of each criterion, there are two cases, including Case 1: suppose the FPRs have additive consistency. We form the pairs $\frac{1}{2}(w_B - w_j) + 0.5 - a_{Bj}$ and $\frac{1}{2}(w_j - w_W) + 0.5 - a_{jW}$ and then try to minimize the maximum of $\left|\frac{1}{2}(w_B - w_j) + 0.5 - a_{Bj}\right|$ and $\left|\frac{1}{2}(w_j - w_W) + 0.5 - a_{jW}\right|$ for each j. Based on the theory of maximum–minimum, the optimal weight model is constructed as follows:

$$\min \; z = \varsigma$$
$$\text{s.t.} \begin{cases} \left|\frac{1}{2}(w_B - w_j) + 0.5 - a_{Bj}\right| \le \varsigma, & \forall j \\ \left|\frac{1}{2}(w_j - w_W) + 0.5 - a_{jW}\right| \le \varsigma, & \forall j \\ \sum_{j=1}^{n} w_j = 1 \\ w_j \ge 0 \end{cases} \tag{1}$$

Case 2: Suppose the FPRs have multiplicative consistency. We form the pairs $\frac{w_B}{w_B+w_j} - a_{Bj}$ and $\frac{w_j}{w_j+w_W} - a_{jW}$ and then try to minimize the maximum of $\left|\frac{w_B}{w_B+w_j} - a_{Bj}\right|$ and $\left|\frac{w_j}{w_j+w_W} - a_{jW}\right|$ for each j. Similarly, to Eq. (1), the optimal weight model is constructed as follows:

$$\min \; z = \varsigma$$
$$\text{s.t.} \begin{cases} \left|\frac{w_B}{w_B+w_j} - a_{Bj}\right| \le \varsigma, & \forall j \\ \left|\frac{w_j}{w_j+w_W} - a_{jW}\right| \le \varsigma, & \forall j \\ \sum_{j=1}^{n} w_j = 1 \\ w_j \ge 0 \end{cases} \tag{2}$$

Step 6: Calculate the consistency ratio of FPRs.

Since the optimal weights of each criterion are derived from different models, there are also two cases to calculate the consistency ratio.

Case 1: We derive the optimal solution ς_A by solving model (1), and combining the consistency index δ_A presented in Table 2, the consistency ratio of FPRs is calculated as follows: $\mathrm{CR}_A = \frac{\varsigma_A}{\delta_A}$.

Case 2: Similarly, we derive the optimal solution ς_M by solving model (2), and combining the consistency index δ_M presented in Table 3, the consistency ratio of FPRs is calculated as follows: $\mathrm{CR}_M = \frac{\varsigma_M}{\delta_M}$.

Step 7: Improve the consistency of FPRs.

When a desired consistency level is not achieved, the consistency of FPRs can be improved by modifying some values, including in the FPRs. Two issues need to be considered in this process: the first issue is how to determine the threshold of consistency, and

Table 2 Consistency index for BWM with additive consistent FPRs

a_{BW}	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
δ_A	0.13	0.10	0.07	0.03	0.00	0.03	0.07	0.10	0.13	0.17

Table 3 Consistency index for BWM with multiplicatively consistent FPRs

a_{BW}	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
δ_M	0.0633	0.0746	0.0599	0.0325	0.0000	0.0325	0.0599	0.0746	0.0633	0.0000

the second issue is how to modify the values to improve the consistency. For the first one, some scholars developed it with the Monte Carlo simulation method [60], while more scholars suggested that decision-makers determine it with respect to the background of practical decision problems [10]. For the last one, several scholars designed different algorithms based on different decision environments, such as intuitionistic fuzziness [57], hesitant fuzzy linguistics [61], and probabilistic hesitant fuzziness [10].

Remark 1 The consistency index for the fuzzy BWM presented in Tables 2 and 3 includes only the numbers ranging from 0.1 to 1, and other numbers, including those in [0,1], can be determined according to Eq. (13) if BWM with additive consistent FPR or Eq. (21) if the BWM has a multiplicatively consistent FPR, which is developed in [10].

Hesitant fuzzy BWM

In this section, we first introduce the concepts of additive/multiplicatively consistent HFBWPRs and then develop several programming models for deriving priority weight vectors from the proposed HFBWPRs, which include two cases. That is, programming models consider the same and different compromise limit constraints.

Additive and multiplicative consistent HFBWPRs

To further consider Definitions 5 and 6, the concepts of additive and multiplicatively consistent HFPRs are defined on the basis of relationships between the formula consisting of priority weights and the values included in HFEs. However, the relationships present in Definitions 5 and 6 only consider the relationships between one priority weight formula and all the values included in HFPRs but cannot reflect the hesitancy of decision-makers. It is reasonable that every value included in HFPRs has a relationship to one priority weight formula. That is, the additive and multiplicatively consistent HFPRs are in accordance with the derived FPRs with respect to each fixed value. Integrating HFPRs into the idea of BWM, new concepts for additive and multiplicatively consistent HFBWPRs are developed as follows.

Definition 7 Let R be the same as those given in Definition 4. Then, R is called additive consistent HFBWPRs if the elements of best and worst including in R satisfy the following conditions:

$$\begin{cases} \frac{1}{2}\left(w_B^k - w_j^k\right) + 0.5 = \sum_{s=1}^{\#h_{Bj}} \alpha_{Bj}^s \gamma_{Bj}^s, & k = 1,2,\ldots,\prod_{j=1}^n \#h_{Bj} \\ \frac{1}{2}\left(w_j^k - w_W^k\right) + 0.5 = \sum_{s=1}^{\#h_{jW}} \beta_{jW}^s \gamma_{jW}^s, & k = 1,2,\ldots,\prod_{j=1}^n \#h_{jW}, \end{cases} \tag{3}$$

for all $j = 1,2,\ldots,n$, where w_j^k are the priority weights such that $w_j^k \geq 0, j = 1,2,\ldots,n, \sum_{\substack{j=1 \\ j\neq B}}^n w_j^k + w_B^k = 1$, for all $k = 1,2,\ldots,\prod_{j=1}^n \#h_{Bj}$ or $\sum_{\substack{j=1 \\ j\neq W}}^n w_j^k + w_W^k = 1$, for all $k = 1,2,\ldots,\prod_{j=1}^n \#h_{jW}$, the above two equations hold at least one of them. In addition, α_{Bj}^s, $s = 1,2,\ldots,\#h_{Bj}$ and β_{jW}^s, $s = 1,2,\ldots,\#h_{jW}$ are two lists of 0–1 indicator variables that satisfy $\sum_{s=1}^{\#h_{Bj}} \alpha_{Bj}^s = 1$ and $\sum_{s=1}^{\#h_{jW}} \beta_{jW}^s = 1$.

In a similar way, the concept of multiplicatively consistent HFBWPRs is developed as follows.

Definition 8 Let R be the same as those given in Definition 4. Then, R is called multiplicatively consistent HFBWPRs if the elements of best and worst, including in R, satisfy the following conditions:

$$\begin{cases} \frac{w_B^k}{w_B^k + w_j^k} = \prod_{s=1}^{\#h_{Bj}} \left(\gamma_{Bj}^s\right)^{\alpha_{Bj}^s}, & k = 1,2,\ldots,\prod_{j=1}^n \#h_{Bj} \\ \frac{w_j^k}{w_j^k + w_W^k} = \prod_{s=1}^{\#h_{jW}} \left(\gamma_{jW}^s\right)^{\beta_{jW}^s}, & k = 1,2,\ldots,\prod_{j=1}^n \#h_{jW}, \end{cases} \tag{4}$$

for all $j = 1,2,\ldots,n$. The meanings of symbols w_j^k, α_{Bj}^s and β_{jW}^s are the same as those given in Eq. (3).

Remark 2 It can be easily found that the additive and multiplicatively consistent HFBWPRs present in Definitions 7 and 8 only consider the elements of best and worst; that is, the elements including in the Bth row and Wth column are considered. Other elements, including in R, do not require satisfying Eq. (3) or Eq. (4).

Example 1 Let R be an HFPR obtained from pairwise comparisons of four criteria, namely, x_1, x_2, x_3 and x_4, where

$$R = \begin{bmatrix} \{0.5\} & \{0.3\} & \{0.5, 0.6, 0.7\} & \{0.5\} \\ \{0.7\} & \{0.5\} & \{0.7\} & \{0.6, 0.7\} \\ \{0.3, 0.4, 0.5\} & \{0.3\} & \{0.5\} & \{0.8\} \\ \{0.5\} & \{0.3, 0.4\} & \{0.2\} & \{0.5\} \end{bmatrix}.$$

Suppose x_3 is the best criterion and x_4 is the worst criterion.
According to Definition 7, if

$$\begin{cases} \frac{1}{2}\left(w_3^1 - w_1^1\right) + 0.5 = 0.3\alpha_{31}^1 + 0.4\alpha_{31}^2 + 0.5\alpha_{31}^3; \quad \frac{1}{2}\left(w_3^2 - w_1^2\right) + 0.5 = 0.3\alpha_{31}^1 + 0.4\alpha_{31}^2 + 0.5\alpha_{31}^3 \\ \frac{1}{2}\left(w_3^3 - w_1^3\right) + 0.5 = 0.3\alpha_{31}^1 + 0.4\alpha_{31}^2 + 0.5\alpha_{31}^3; \quad \frac{1}{2}\left(w_3^1 - w_2^1\right) + 0.5 = 0.3; \quad \frac{1}{2}\left(w_3^1 - w_4^1\right) + 0.5 = 0.8 \\ \frac{1}{2}\left(w_1^1 - w_4^1\right) + 0.5 = 0.5; \quad \frac{1}{2}\left(w_2^1 - w_4^1\right) + 0.5 = 0.6\beta_{24}^1 + 0.7\beta_{24}^2; \quad \alpha_{31}^1 + \alpha_{31}^2 + \alpha_{31}^3 = 1; \quad \beta_{24}^1 + \beta_{24}^2 = 1 \end{cases}$$

holds, then R is called additive consistent HFBWPRs.

Deriving priority weight vectors from HFBWPRs with the same compromise limit constraint

Consistency of PRs is related to rationality. By comparison, inconsistent PRs often lead to misleading solutions. Therefore, developing approaches to obtain the expected consistency level is necessary. However, only a few scholars have focused on optimization-based methods to obtain the expected consistent HFBWPRs at present. Therefore, in this section, several mathematical programming models are proposed to obtain acceptable consistent HFBWPRs that consider the minimized deviation from the target of the goal. There are two cases, namely, deriving priority weight vectors from HFBWPRs based on additive consistency and multiplicative consistency.

Case 1; Deriving priority weight vectors from additive consistent HFBWPRs

According to the definition of additive consistent HFBWPRs, we obtain $\frac{1}{2}\left(w_B^k - w_j^k\right) + 0.5 = \sum_{s=1}^{\#h_{Bj}} \alpha_{Bj}^s \gamma_{Bj}^s$ and $\frac{1}{2}\left(w_j^k - w_W^k\right) + 0.5 = \sum_{s=1}^{\#h_{jW}} \beta_{jW}^s \gamma_{jW}^s$. The priority weights of complete additive consistent HFBWPRs can be derived by solving a list of equations $\frac{1}{2}\left(w_B^k - w_j^k\right) + 0.5 = \sum_{s=1}^{\#h_{Bj}} \alpha_{Bj}^s \gamma_{Bj}^s$, $j = 1, 2, \ldots, n$, $k = 1, 2, \ldots, \prod_{j=1}^n \#h_{Bj}$ and $\frac{1}{2}\left(w_j^k - w_W^k\right) + 0.5 = \sum_{s=1}^{\#h_{jW}} \beta_{jW}^s \gamma_{jW}^s$, $j = 1, 2, \cdots, n$, $k = 1, 2, \ldots, \prod_{j=1}^n \#h_{jW}$. However, the abovementioned equations do not constantly hold in general given a deviation between $\frac{1}{2}\left(w_B^k - w_j^k\right) + 0.5$ and $\sum_{s=1}^{\#h_{Bj}} \alpha_{Bj}^s \gamma_{Bj}^s$ for each pair of possible value (B, j_0) with $j_0 = 1, 2, \ldots, n$ for each $s = 1, 2, \ldots, \#h_{Bj}$. Similarly, there is also a deviation between $\frac{1}{2}\left(w_j^k - w_W^k\right) + 0.5$ and $\sum_{s=1}^{\#h_{jW}} \beta_{jW}^s \gamma_{jW}^s$ for each pair of possible value $\left(j_0, W\right)$ with $j_0 = 1, 2, \ldots, n$ for each $s = 1, 2, \ldots, \#h_{jW}$. Moreover, the more $\frac{1}{2}\left(w_B^k - w_j^k\right) + 0.5 - \sum_{s=1}^{\#h_{Bj}} \alpha_{Bj}^s \gamma_{Bj}^s$ and $\frac{1}{2}\left(w_j^k - w_W^k\right) + 0.5 - \sum_{s=1}^{\#h_{jW}} \beta_{jW}^s \gamma_{jW}^s$ approaches to 0, the more valid and reasonable the priority weights are.

In this regard, motivated by the idea developed in Rezaei [34], the priority weight vectors are obtained by minimizing the maximum absolute differences $\left|\frac{1}{2}\left(w_B^k - w_j^k\right) + 0.5 - \sum_{s=1}^{\#h_{Bj}} \alpha_{Bj}^s \gamma_{Bj}^s\right|$ and $\left|\frac{1}{2}\left(w_j^k - w_W^k\right) + 0.5 - \sum_{s=1}^{\#h_{jW}} \beta_{jW}^s \gamma_{jW}^s\right|$. Thus, the following programming models can be constructed:

$$\min \quad z^k = \varsigma$$
$$\text{s.t.} \begin{cases} \left|\frac{1}{2}\left(w_B^k - w_j^k\right) + 0.5 - \sum_{s=1}^{\#h_{Bj}} \alpha_{Bj}^s \gamma_{Bj}^s\right| \le \varsigma \\ \left|\frac{1}{2}\left(w_j^k - w_W^k\right) + 0.5 - \sum_{s=1}^{\#h_{jW}} \beta_{jW}^s \gamma_{jW}^s\right| \le \varsigma \\ \alpha_{Bj_0}^s \left(\sum_{\substack{j=1 \\ j \ne B}}^n w_j^k + w_B^k - 1\right) = 0, & \forall\, k = 1, 2, \ldots, \prod_{j=1}^n \#h_{Bj} \\ \beta_{j_0 W}^s \left(\sum_{\substack{j=1 \\ j \ne W}}^n w_j^k + w_W^k - 1\right) = 0, & \forall\, k = 1, 2, \ldots, \prod_{j=1}^n \#h_{jW} \\ \left(\alpha_{Bj_0}^s = 1\right) \vee \left(\beta_{j_0 W}^s = 1\right), & j_0 = 1, 2, \ldots, n \\ \sum_{s=1}^{\#h_{Bj}} \alpha_{Bj}^s = 1, \quad \sum_{s=1}^{\#h_{jW}} \beta_{jW}^s = 1 \\ \alpha_{Bj}^s = 0 \vee 1, & s = 1, 2, \ldots, \#h_{Bj} \\ \beta_{jW}^s = 0 \vee 1, & s = 1, 2, \ldots, \#h_{jW} \\ w_j^k \ge 0, & \varsigma \ge 0 \\ j = 1, 2, \ldots, n \end{cases} \tag{5}$$

In Eq. (5), the first and second constraints are derived from the theory of maximum–minimum; the third constraint is hold when $\alpha_{Bj_0}^s = 1$, which ensure that all the values including in the HFBWPRs are considered. Similarly, when $\alpha_{j_0 W}^s = 1$, the fourth constraint is hold. The fifth constraint ensures that at least one of the third and fourth constraints holds. In addition, the sixth to eighth constraints indicates that α_{Bj}^s and β_{jW}^s are two lists of 0–1 indicator variables. Solving Eq. (5), a list of priority weight vectors w^k, $k = 1, 2, \ldots, t$, where $t = \prod_{j=1}^n \#h_{Bj} + \prod_{j=1}^n \#h_{jW}$ can be derived. Since w^k can be viewed as the possible priority weight vector of R.

Remark 3 Solving Eq. (5), a list of priority weight vectors w^k, $k = 1, 2, \ldots, t$, are derived. However, in most cases, we will obtain more than two identical priority weight vectors.

Based on the ideas of Zhang et al. [31] and Wu et al. [62]. The distance between w^k and R is developed to select the best priority weight vector of R.

Definition 9 Let $R = \left(h_{ij}\right)_{n \times n} \subset X \times X$ be an HFBWPR, and $w^k = \left(w_1^k, w_2^k, \cdots, w_n^k\right)$, $k = 1, 2, \cdots, t$ be a list of priority weight vectors derived from Eq. (5). Then, the distance between w^k and R is developed as follows:

$$d_1\left(w^k,R\right)=\frac{1}{2(n-1)}\left(\sum_{j=1}^{n}\left|\frac{1}{2}\left(w_B^k-w_j^k\right)+0.5-\sum_{s=1}^{\#h_{Bj}}\alpha_{Bj}^s\gamma_{Bj}^s\right|+\sum_{j=1}^{n}\left|\frac{1}{2}\left(w_j^k-w_W^k\right)+0.5-\sum_{s=1}^{\#h_{jW}}\beta_{jW}^s\gamma_{jW}^s\right|\right). \quad (6)$$

It can be easily found that the distance $d_1\left(w^k,R\right)$ reflects the sum of the average of the absolute deviation between $\frac{1}{2}\left(w_B^k-w_j^k\right)+0.5$ and $\sum_{s=1}^{\#h_{Bj}}\alpha_{Bj}^s\gamma_{Bj}^s$ for the elements including in the B th row and the absolute deviation between $\frac{1}{2}\left(w_j^k-w_W^k\right)+0.5$ and $\sum_{s=1}^{\#h_{jW}}\beta_{jW}^s\gamma_{jW}^s$ for the elements including in the Wth column. The number of above absolute deviations is $2(n-1)$, and the coefficient presented in Eq. (6) ensures the value range in interval [0, 1]. It is natural that the optimal priority weight vector minimizes the deviation $d_1\left(w^k,R\right)$. As a consequence, the priority weight vector of R is developed as follows.

Definition 10 Let R and w^k be the same as those given in Definition 9. Then, the priority weight vector of R is developed as follows:

$$w^{k*}=\left(w_1^{k*},w_2^{k*},\dots w_n^{k*}\right)=\arg\min_{w^k}d_1\left(w^k,R\right), \quad (7)$$

where the symbol arg represents the correspondence of the priority weight vector w^k in the minimum distance $d_1\left(w^k,R\right)$, which is obtained from Eq. (6).

Remark 4 There may be more than one priority weight vector, including in $\min_{w^k}d_1\left(w^k,R\right)$; that is, sometimes the solution of Eq. (7) is not unique. In this case, the priority weight vector R is developed as:

$$w^{k*}=\left(w_1^{k*},w_2^{k*},\dots,w_n^{k*}\right)=\left(\frac{1}{l_0}\sum_{k=1}^{l_0}w_1^k,\frac{1}{l_0}\sum_{k=1}^{l_0}w_2^k,\dots,\frac{1}{l_0}\sum_{k=1}^{l_0}w_n^k\right), \quad (8)$$

where w_i^k, $i=1,2,\cdots,n$, and it indicates that the number of priority weight vectors included in $\min_{w^k}d_1\left(w^k,R\right)$ is l_0. It can be easily found that the priority weight vectors are derived from the average value of w_i^k, $k=1,2,\cdots,l_0$.

Case 2. Deriving priority weight vectors from multiplicatively consistent HFBWPRs

Similar to the idea of additive consistent HFBWPRs presented in case 1, the following programming models can be constructed when we consider multiplicatively consistent HFBWPRs.

$$\min \quad z^k=\varsigma$$
$$\text{s.t.}\begin{cases}\left|\frac{w_B^k}{w_B^k+w_j^k}-\prod_{s=1}^{\#h_{Bj}}\left(\gamma_{Bj}^s\right)^{\alpha_{Bj}^s}\right|\le\varsigma\\ \left|\frac{w_j^k}{w_j^k+w_W^k}-\prod_{s=1}^{\#h_{jW}}\left(\gamma_{jW}^s\right)^{\beta_{jW}^s}\right|\le\varsigma\\ \alpha_{Bj_0}^s\left(\sum_{j=1,j\ne B}^{n}w_j^k+w_B^k-1\right)=0, & \forall\ k=1,2,\dots,\prod_{j=1}^{n}\#h_{Bj}\\ \beta_{j_0W}^s\left(\sum_{j=1,j\ne W}^{n}w_j^k+w_W^k-1\right)=0, & \forall\ k=1,2,\dots,\prod_{j=1}^{n}\#h_{jW}\\ \left(\alpha_{Bj_0}^s=1\right)\vee\left(\beta_{j_0W}^s=1\right), & j_0=1,2,\dots,n\\ \sum_{s=1}^{\#h_{Bj}}\alpha_{Bj}^s=1,\quad \sum_{s=1}^{\#h_{jW}}\beta_{jW}^s=1\\ \alpha_{Bj}^s=0\vee1, & s=1,2,\dots,\#h_{Bj}\\ \beta_{jW}^s=0\vee1, & s=1,2,\dots,\#h_{jW}\\ w_j^k\ge0, & \varsigma\ge0\\ j=1,2,\dots,n\end{cases}. \quad (9)$$

The difference between Eqs. (5) and (9) is that we use multiplicatively consistent HFBWPRs in the first and second constraints of Eq. (9) to replace additive consistent HFBWPRs in the first and second constraints of Eq. (5), the rest of the constraints are exactly the same. The meaning of symbols presents in Eq. (9) is the same as those given in Eq. (5). Similar to additive consistent HFBWPRs, the distance between w^k and R is developed to select the best priority weight vector R.

Definition 11 Let $R=\left(h_{ij}\right)_{n\times n}\subset X\times X$ be an HFBWPR, and let $w^k=\left(w_1^k,w_2^k,\cdots,w_n^k\right)$, $k=1,2,\cdots,t$ be a list of priority weight vectors derived from Eq. (9). Then, the distance between w^k and R is developed as follows.

$$d_2\left(w^k,R\right)=\frac{1}{2(n-1)}\left(\sum_{j=1}^{n}\left|\frac{w_B^k}{w_B^k+w_j^k}-\prod_{s=1}^{\#h_{Bj}}\left(\gamma_{Bj}^s\right)^{\alpha_{Bj}^s}\right|+\sum_{j=1}^{n}\left|\frac{w_j^k}{w_j^k+w_W^k}-\prod_{s=1}^{\#h_{jW}}\left(\gamma_{jW}^s\right)^{\beta_{jW}^s}\right|\right). \quad (10)$$

The difference between Eqs. (6) and (10) is that we use multiplicatively consistent HFBWPRs in the first and second sections of Eq. (10) to replace additively consistent HFBWPRs in the first and second sections of Eq. (6). The meaning of symbols presents in Eq. (10) is the same as those given in Eq. (6). Similar to additive consistent HFBWPRs, the priority weight vector of R is developed as follows.

Definition 12 Let R and w^k be the same as those given in Definition 11. Then, the priority weight vector of R is developed as follows:

$$w^{k*} = \left(w_1^{k*}, w_2^{k*}, \ldots, w_n^{k*}\right) = \arg\min_{w^k} d_2\left(w^k, R\right). \tag{11}$$

Remark 5 There may be more than one priority weight vector, including in $\min_{w^k} d_2\left(w^k, R\right)$. In this case, the priority weight vector R is developed according to Eq. (8).

Deriving priority weight vectors from HFBWPRs with different compromise limit constraints

In the above programming models, the constraints $\left|\frac{1}{2}\left(w_B^k - w_j^k\right) + 0.5 - \sum_{s=1}^{\#h_{Bj}} \alpha_{Bj}^s \gamma_{Bj}^s\right| \leq \varsigma$ and $\left|\frac{1}{2}\left(w_j^k - w_W^k\right) + 0.5 - \sum_{s=1}^{\#h_{jW}} \beta_{jW}^s \gamma_{jW}^s\right| \leq \varsigma$ present in Eq. (5) or $\left|\frac{w_B^k}{w_B^k + w_j^k} - \prod_{s=1}^{\#h_{Bj}} \left(\gamma_{Bj}^s\right)^{\alpha_{Bj}^s}\right| \leq \varsigma$ and $\left|\frac{w_j^k}{w_j^k + w_W^k} - \prod_{s=1}^{\#h_{jW}} \left(\gamma_{jW}^s\right)^{\beta_{jW}^s}\right| \leq \varsigma$ present in Eq. (9) with the same compromise limit ς for all j, $j = 1, 2, \ldots, n$. This is unreasonable in some circumstances because different HFEs, including HFBWPRs, have different numbers of elements; in other words, different numbers of elements may express different hesitant degrees of HFEs. It is more reasonable that for different HFEs corresponding to different compromise limits ς_j for different j, $j = 1, 2, \cdots, n$. There are also two cases, that is, deriving priority weight vectors from HFBWPRs based on additive consistency and multiplicative consistency.

Case 1: Deriving priority weight vectors from additive consistent HFBWPRs

In view of the above analysis, the maximum absolute differences are minimized, and the problem $\min\max\left\{\left|\frac{1}{2}\left(w_B^k - w_j^k\right) + 0.5 - \sum_{s=1}^{\#h_{Bj}} \alpha_{Bj}^s \gamma_{Bj}^s\right|, \left|\frac{1}{2}\left(w_j^k - w_W^k\right) + 0.5 - \sum_{s=1}^{\#h_{jW}} \beta_{jW}^s \gamma_{jW}^s\right|\right\}$ can be transferred to two lists of different compromise limit constraints; that is, $\left|\frac{1}{2}\left(w_B^k - w_j^k\right) + 0.5 - \sum_{s=1}^{\#h_{Bj}} \alpha_{Bj}^s \gamma_{Bj}^s\right| \leq \varsigma_j$ and $\left|\frac{1}{2}\left(w_j^k - w_W^k\right) + 0.5 - \sum_{s=1}^{\#h_{jW}} \beta_{jW}^s \gamma_{jW}^s\right| \leq \varsigma_j$ for different j and $j = 1, 2, \ldots, n$. Thus, the following programming models are constructed:

$$\min \quad z^k = \sum_{j=1}^{n} \varsigma_j$$
$$\text{s.t.} \begin{cases} \left|\frac{1}{2}\left(w_B^k - w_j^k\right) + 0.5 - \sum\limits_{s=1}^{\#h_{Bj}} \alpha_{Bj}^s \gamma_{Bj}^s\right| \leq \varsigma_j & \\ \left|\frac{1}{2}\left(w_j^k - w_W^k\right) + 0.5 - \sum\limits_{s=1}^{\#h_{jW}} \beta_{jW}^s \gamma_{jW}^s\right| \leq \varsigma_j & \\ \alpha_{Bj_0}^s\left(\sum\limits_{\substack{j=1 \\ j\neq B}}^{n} w_j^k + w_B^k - 1\right) = 0, & \forall\, k = 1, 2, \ldots, \prod\limits_{j=1}^{n} \#h_{Bj} \\ \beta_{j_0W}^s\left(\sum\limits_{\substack{j=1 \\ j\neq W}}^{n} w_j^k + w_W^k - 1\right) = 0, & \forall\, k = 1, 2, \ldots, \prod\limits_{j=1}^{n} \#h_{jW} \\ \left(\alpha_{Bj_0}^s = 1\right) \vee \left(\beta_{j_0W}^s = 1\right), & j_0 = 1, 2, \ldots, n \\ \sum\limits_{s=1}^{\#h_{Bj}} \alpha_{Bj}^s = 1, & \sum\limits_{s=1}^{\#h_{jW}} \beta_{jW}^s = 1 \\ \alpha_{Bj}^s = 0 \vee 1, & s = 1, 2, \ldots, \#h_{Bj} \\ \beta_{jW}^s = 0 \vee 1, & s = 1, 2, \ldots, \#h_{jW} \\ w_j^k \geq 0, & \varsigma \geq 0 \\ j = 1, 2, \ldots, n & \end{cases} \tag{12}$$

The difference between Eqs. (5) and (12) is that we use ς_j in the first and second constraints of Eq. (12) to replace ς in the first and second constraints of Eq. (5), and use $\sum_{j=1}^{n} \varsigma_j$ in the objective function of Eq. (12) to replace ς in the objective function of Eq. (5), the rest of the constraints are exactly the same. The meaning of symbols presents in Eq. (12) is the same as those given in Eq. (5).

Case 2. Deriving priority weight vectors from multiplicatively consistent HFBWPRs

Similar to the idea of additive consistent HFBWPRs presented in case 1, the following programming models can be constructed when we consider multiplicatively consistent HFBWPRs.

$$\min \quad z^k = \sum_{j=1}^{n} \varsigma_j$$
$$\text{s.t.} \begin{cases} \left|\frac{w_B^k}{w_B^k + w_j^k} - \prod\limits_{s=1}^{\#h_{Bj}} \left(\gamma_{Bj}^s\right)^{\alpha_{Bj}^s}\right| \leq \varsigma_j & \\ \left|\frac{w_j^k}{w_j^k + w_W^k} - \prod\limits_{s=1}^{\#h_{jW}} \left(\gamma_{jW}^s\right)^{\beta_{jW}^s}\right| \leq \varsigma_j & \\ \alpha_{Bj_0}^s\left(\sum\limits_{\substack{j=1 \\ j\neq B}}^{n} w_j^k + w_B^k - 1\right) = 0, & \forall\, k = 1, 2, \ldots, \prod\limits_{j=1}^{n} \#h_{Bj} \\ \beta_{j_0W}^s\left(\sum\limits_{\substack{j=1 \\ j\neq W}}^{n} w_j^k + w_W^k - 1\right) = 0, & \forall\, k = 1, 2, \ldots, \prod\limits_{j=1}^{n} \#h_{jW} \\ \left(\alpha_{Bj_0}^s = 1\right) \vee \left(\beta_{j_0W}^s = 1\right), & j_0 = 1, 2, \ldots, n \\ \sum\limits_{s=1}^{\#h_{Bj}} \alpha_{Bj}^s = 1, & \sum\limits_{s=1}^{\#h_{jW}} \beta_{jW}^s = 1 \\ \alpha_{Bj}^s = 0 \vee 1, & s = 1, 2, \ldots, \#h_{Bj} \\ \beta_{jW}^s = 0 \vee 1, & s = 1, 2, \ldots, \#h_{jW} \\ w_j^k \geq 0, & \varsigma \geq 0 \\ j = 1, 2, \ldots, n & \end{cases} \tag{13}$$

The difference between Eqs. (9) and (13) is that we use ς_j in the first and second constraints of Eq. (13) to replace ς in the first and second constraints of Eq. (9), and use $\sum_{j=1}^{n} \varsigma_j$ in the objective function of Eq. (13) to replace ς in the objective function of Eq. (9), the rest of the constraints are exactly the same. The meaning of symbols presents in Eq. (13) is the same as those given in Eq. (9).

Remark 6 When solving Eqs. (12) and (13), a list of compromise limit constraint values ς_j and $j = 1, 2, \ldots, n$ are derived. In these cases, we use the value $\max\left\{ \varsigma_j \middle| j = 1, 2, \ldots, n \right\}$ to replace the value ς_A and ς_M present in step 6 of Sect. 2.3 when calculating the consistency ratio of FPRs.

A framework of MCDM procedure based on hesitant fuzzy BWM

In this section, including three subsections, the first subsection introduces hesitant fuzzy MCDM problems, the second subsection develops a method to derive the weights of decision-makers, and the last subsection introduces a framework of the MCDM procedure based on hesitant fuzzy BWM.

Hesitant fuzzy MCDM problems

Hesitant fuzzy MCDM problems involve m alternatives denoted as $A = \left\{a_1, a_2, \ldots, a_m\right\}$. Each alternative is assessed based on several feature criteria $C = \left\{c_1, c_2, \ldots, c_n\right\}$. Let $E = \left\{e_1, e_2, \ldots, e_z\right\}$ be a set of decision-makers and $\tau = \left(\tau_1, \tau_2, \ldots, \tau_z\right)$ be the decision-makers' weight vector. The evaluation of the alternative a_i, $i = 1, 2, \ldots, m$ with respect to the feature criterion is provided by the moderator, which is an HFE and denotes as $h = \{\gamma^s | s = 1, 2, \ldots, \#h\}$. We assume that the weights of criteria and decision-makers are completely unknown. Decision-makers are invited to determine the weights of the criteria. The evaluation of the criteria c_j, $j = 1, 2, \ldots, n$ with pairwise comparisons is provided by decision-maker e_k, $k = 1, 2, \ldots, z$, which are the HFPRs and denoted as $h_k = \left\{\gamma_k^s | s = 1, 2, \ldots, \#h\right\}$.

Derive the weights of the decision-makers

In this subsection, an optimization model is constructed to derive the objective weights of decision-makers with complete unknown information. Considering that decision-makers in the MCDM process typically construct from different knowledge backgrounds and have varied expertise in the domain area, each decision-maker has different judgment values, which influences the solution differently. Therefore, each decision-maker has a different importance weight when collecting the priority weights of the criteria. Given that the decision-maker whose judgment values are far away from the collect judgment values indicates that the judgment values he/she provides are the least reliable, the decision-maker should endow a smaller weight value. By comparison, the decision-maker whose judgment values are close to the collect judgment values indicates that the judgment values he/she provides are the most reliable, and the decision-maker should endow a larger weight value. On this basis, the optimization model is constructed as follows:

$$\min \quad z = \sum_{k=1}^{z} \sum_{i=1}^{n} \left| w_i^{k*} - \sum_{k=1}^{z} \tau_k^o w_i^{k*} \right|$$
$$\text{s.t.} \begin{cases} \sum_{k=1}^{z} \tau_k^o = 1 \\ 0 \le \tau_k^o \le 1 \\ k = 1, 2, \ldots, z. \end{cases} \tag{14}$$

In Eq. (14), w_i^{k*}, and $i = 1, 2, \ldots, n$ are the priority weight vectors of the criteria provided by decision-maker e_k. If we consider the maximum absolute differences to be a minimized problem with the same compromise limit constraints, then w_i^{k*} is obtained from Eqs. (5), (6) and (7) suppose HFBWPRs with additive consistency or obtained from Eqs. (9), (10) and (11) suppose HFBWPRs with multiplicative consistency. Moreover, if we consider that the maximum absolute differences are minimized problems with different compromise limit constraints, then w_i^{k*} is obtained from Eqs. (6), (7) and (12) suppose HFBWPRs with additive consistency or obtained from Eqs. (10), (11) and (13) suppose HFBWPRs with multiplicative consistency.

According to the knowledge of decision-makers, the subjective weights of decision-makers are given in advance as τ_k^s with $\tau_k^s \ge 0$ and $\sum_{k=1}^{z} \tau_k^s = 1$. Therefore, the comprehensive weight τ_k of the k th decision-maker is calculated as follows:

$$\tau_k = \theta \tau_k^o + (1 - \theta)\tau_k^s, \tag{15}$$

where $k = 1, 2, \ldots, z$ and the parameter θ $(0 \le \theta \le 1)$ trade-offs the subjective weights and objective weights of decision-makers. In general, we set $\theta = 0.5$.

A framework of MCDM procedure based on hesitant fuzzy BWM

The proposed decision-making procedure is summarized in the following steps.

Step 1: Define the decision criteria and provide the evaluation values.

The decision criteria are defined on the basis of alternatives' characteristics, which are determined by the moderator, and the evaluation of the alternatives with respect to the decision criteria is also provided by the moderator, which is denoted as $R = \left(h_{ij}\right)_{m \times n}$. h_{ij} is an HFE indicating the evaluation value of alternative a_i under the criteria c_j.

Step 2: Identify the best and worst criteria.

According to the determination criteria, the decision-makers provide their pairwise comparison judgment matrices and denote them as $R_k = \left(h_{ij,k}\right)_{n\times n} \subset X \times X$, $k = 1, 2, \ldots, z$. $h_{ij,k} = \left\{ \gamma_{ij,k}^s \middle| \quad s = 1, 2, \ldots, \#h_{ij} \right\}$ is an HFPR indicating that the possible preference degrees of criteria c_i are preferred to criteria c_j. Then, the score function of criteria c_i is calculated as follows [53]:

$$s\left(h_{i,k}\right) = \frac{1}{\#h_{ij}} \sum_{j=1}^{n} \sum_{s=1}^{\#h_{ij}} \gamma_{ij,k}^s, \quad i = 1, 2, \ldots, n. \tag{16}$$

For decision-maker e_k, the best and worst criteria can be determined based on the score function; that is, the maximal value of $s\left(h_{i,k}\right)$ is the best criterion, and the minimum value is the worst criterion.

Step 3: Calculate the optimal weights of the criteria.

To obtain the optimal weight of each criterion, there are two cases, including.

Case 1: Suppose HFPRs with additive consistent HFBWPRs. If we consider the maximum absolute differences to be a minimized problem with the same compromise limit constraints, then the priority weight vectors of the criteria are obtained from Eqs. (5), (6) and (7); otherwise, if we consider the maximum absolute differences to be a minimized problem with different compromise limit constraints, the priority weight vectors of the criteria are obtained from Eqs. (6), (7) and (12).

Case 2: Suppose HFPRs with multiplicatively consistent HFBWPRs. If we consider the maximum absolute differences to be a minimized problem with the same compromise limit constraints, then the priority weight vectors of the criteria are obtained from Eqs. (9), (10) and (11); otherwise, if we consider the maximum absolute differences to be a minimized problem with different compromise limit constraints, the priority weight vectors of the criteria are obtained from Eqs. (10), (11) and (13).

Step 4: Calculate the consistency ratio.

Since the optimal weight of each criterion is derived from different models, there are also two cases to calculate the consistency ratio.

Case 1: Solving models (5) or (12), we derive the optimal solution ς_A and use the consistency index δ_A presented in Table 2. The consistency ratio is calculated according to $\mathrm{CR}_A = \frac{\varsigma_A}{\delta_A}$.

Case 2: Solving models (9) or (13), we derive the optimal solution ς_M and use the consistency index δ_M presented in Table 3. The consistency ratio is calculated according to $\mathrm{CR}_M = \frac{\varsigma_M}{\delta_M}$.

Step 5: Improve the consistency of FPRs.

Suppose the threshold of consistency is determined by the moderator. If a desired consistency level is not achieved, the consistency of FPRs can be improved by modifying some values, including in the FPRs. To save space, we only present the case in which the HFBWPRs with additive consistency and the maximum absolute differences are minimized problems with the same compromise limit constraint; other cases can be developed in a similar way.

In the improving process, the identification rule and direction rule are sequentially considered. First, identify the position that needs to be adjusted. In the first phase, we determine the maximum difference between $\left|\frac{1}{2}\left(w_B^k - w_j^k\right) + 0.5 - \sum_{s=1}^{\#h_{Bj}} \alpha_{Bj}^s \gamma_{Bj}^s\right|$ and ς, and $\left|\frac{1}{2}\left(w_j^k - w_W^k\right) + 0.5 - \sum_{s=1}^{\#h_{jW}} \beta_{jW}^s \gamma_{jW}^s\right|$ and ς. The maximum difference corresponds to the value γ_{Bj} or γ_{jW} that needs to be adjusted. In the second phase, determine the range that can be adjusted. If $\frac{1}{2}\left(w_B^k - w_j^k\right) + 0.5 - \sum_{s=1}^{\#h_{Bj}} \alpha_{Bj}^s \gamma_{Bj}^s + \varsigma = 0$ or $\frac{1}{2}\left(w_j^k - w_W^k\right) + 0.5 - \sum_{s=1}^{\#h_{jW}} \beta_{jW}^s \gamma_{jW}^s + \varsigma = 0$, the adjustment range is determined as $\gamma_{Bj} \in \left\{[0,1] \wedge \left[\gamma_{Bj}, \gamma_{Bj} + \varsigma\right]\right\}$ or $\gamma_{jW} \in \left\{[0,1] \wedge \left[\gamma_{jW}, \gamma_{jW} + \varsigma\right]\right\}$. If $\frac{1}{2}\left(w_B^k - w_j^k\right) + 0.5 - \sum_{s=1}^{\#h_{Bj}} \alpha_{Bj}^s \gamma_{Bj}^s - \varsigma = 0$ or $\frac{1}{2}\left(w_j^k - w_W^k\right) + 0.5 - \sum_{s=1}^{\#h_{jW}} \beta_{jW}^s \gamma_{jW}^s - \varsigma = 0$, the adjustment range is determined as $\gamma_{Bj} \in \left\{[0,1] \wedge \left[\gamma_{Bj} - \varsigma, \gamma_{Bj}\right]\right\}$ or $\gamma_{jW} \in \left\{[0,1] \wedge \left[\gamma_{jW} - \varsigma, \gamma_{jW}\right]\right\}$. For a better understanding, algorithm 1 for improving the consistency of FPRs is depicted in Fig. 1.

Step 6: Determine the weights of decision-makers.

First, the objective weights of the decision-makers are derived with respect to Eq. (14), and then the comprehensive weights of decision-makers are determined according to Eq. (15).

Step 7: Compute the collective optimal priority weights of the criteria.

The collective optimal priority weights of the criteria are determined by the following formula:

$$\lambda_i = \sum_{k=1}^{z} \tau_k w_i^{k*}, \quad i = 1, 2, \ldots, n, \tag{17}$$

where τ_k is the weight of decision-maker e_k, and w_i^{k*} is the optimal priority weight vector determined from Step 3.

Step 8: Calculate the collective evaluation values.

The collective evaluation values are calculated by the hesitant fuzzy weighted averaging (HFWA) operator [53]:

$$\mathrm{HFWA}\left(a_i\right) = \cup_{\gamma_{ij} \in h_{ij}} \left\{ 1 - \prod_{j=1}^{n} \left(1 - \gamma_{ij}\right)^{\lambda_j} \right\}, \quad i = 1, 2, \ldots, n, \tag{18}$$

where λ_j is the collective optimal priority weight vector determined from Eq. (17).

Step 9: Rank the alternatives.

The ranking order of all alternatives is obtained by the scores of collective evaluation values $\mathrm{HFWA}\left(a_i\right)$,

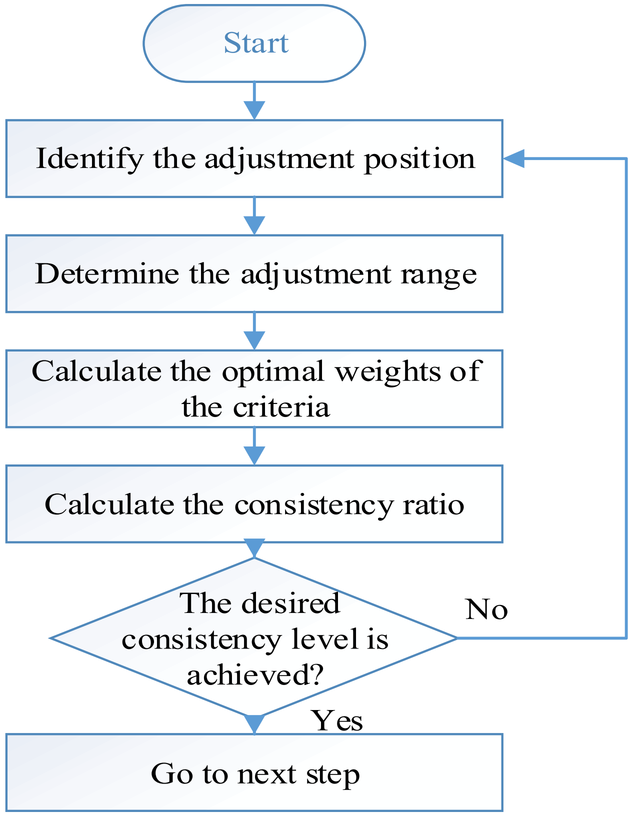

Fig. 1 The algorithm of improving the consistency of FPR

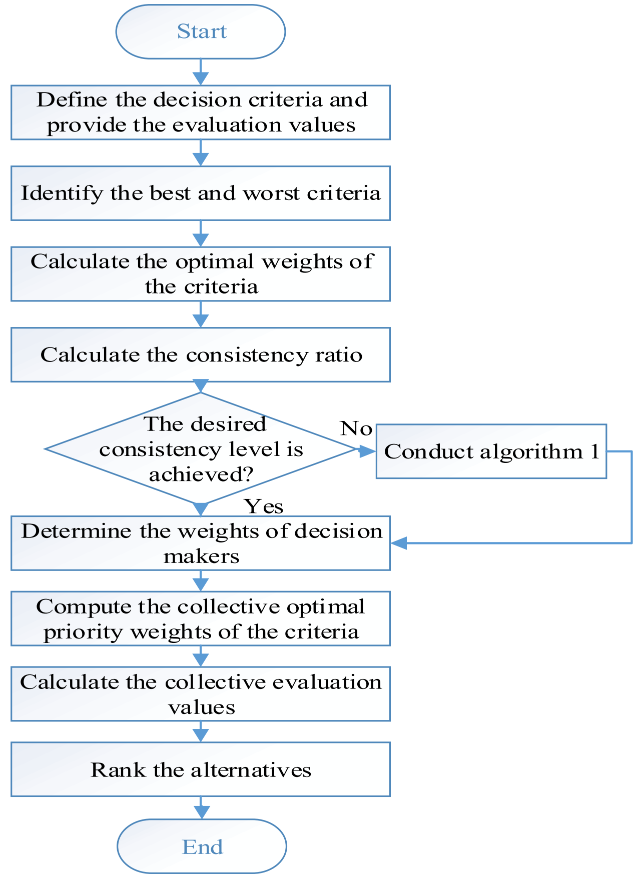

Fig. 2 The framework of the decision-making process based on hesitant fuzzy BWM

$i = 1, 2, \ldots, n$. The maximal scores of HFWA(a_i) are the best alternative.

The proposed decision-making procedure is depicted in Fig. 2.

Remark 7 In Step 2, more than two identical scores may be derived from different criteria. In this case, the best and worst criteria can be determined with respect to the deviation function that is developed in Xia and Xu [53].

Remark 8 In Step 4, since there may be more than one priority weight vector, including Eq. (7) or Eq. (11). In this case, the consistency ratio of the optimal weights can be determined according to the following formula:

$$CR_A^{\circ} = \min\left\{ \frac{\varsigma_A^k}{\delta_A^k} \middle| k = 1, 2, \ldots, l_{\circ} \right\}, \tag{19}$$

where ς_A^k, $k = 1, 2, \ldots, l_{\circ}$ is the optimal solution of model (5) or model (9) and indicates that the number of priority weight vectors included in Eq. (7) or Eq. (11) is l_0.

Remark 9 The algorithm for improving the consistency of FPRs developed in Step 5 is motivated by the idea presented in Mou et al. [57]. The proof of the algorithm's convergence is developed in a similar way.

Illustrative example

There are three subsections. The first subsection introduces the selection of the most important project problems, the second subsection illustrates the use of the proposed methods, and the comparative analysis and discussion is conducted in the last subsection.

Case description

With the development of the rural economy in China, there are an increasing number of opportunities for enterprises to develop in the countryside. Currently, a series of preferential policies have been adopted in the rural areas of China, and many large domestic companies have taken part in the investment of rural infrastructure, making rural areas that had no prospects for development previously more attractive. Currently, many enterprises in our country choose to develop in rural areas. People in every industry in China are optimistic about the development of rural areas, and everyone believes that they will earn a large profit if invested in rural industries.

The enterprise's board of director has to plan the development of strategy initiatives for the following several years. Suppose that there are five possible projects, denoted as (1) a_1 agricultural processing plant industry; (2) a_2 rural express industry; (3) a_3 rural logistics industry; (4) a_4 rural e-commerce industry; and (5) a_5 early childhood education in rural areas, to be evaluated. It is necessary to compare these projects to select which is the most important from the point view of their importance, taking into account four criteria suggested by the enterprise's board of director from the perspectives of (1) c_1 the future development; (2) c_2 the risk of the investment; (3) c_3 the size of revenue; and (4) c_4 impact on the environment.

The evaluation of the alternative a_i, $i = 1, 2, \ldots, 5$ with respect to four feature criteria is provided by the enterprise's board of director, which is an HFE and demonstrated in Table 4. The weights of the criteria in this decision problem are completely unknown. To derive the weights of criteria, three decision-makers from related fields are invited to take part in the decision process. First, three decision-makers are asked to provide their opinion relative to each criterion. Because of the uncertainty of the criteria, it is difficult for decision-makers to use just one value to provide their pairwise evaluation values. To facilitate the elicitation of their evaluation values, HFE is just an effective tool to address such situations. Three decision-makers provide their evaluation with HFPRs, as demonstrated in matrices 1–3.

$$R_1 = \begin{bmatrix} \{0.5\} & \{0.2\} & \{0.5, 0.6, 0.7\} & \{0.5\} \\ \{0.8\} & \{0.5\} & \{0.7\} & \{0.8, 0.9\} \\ \{0.3, 0.4, 0.5\} & \{0.3\} & \{0.5\} & \{0.6\} \\ \{0.5\} & \{0.1, 0.2\} & \{0.4\} & \{0.5\} \end{bmatrix}$$

$$R_2 = \begin{bmatrix} \{0.5\} & \{0.8, 0.9\} & \{0.8\} & \{0.7, 0.8\} \\ \{0.1, 0.2\} & \{0.5\} & \{0.5\} & \{0.7\} \\ \{0.2\} & \{0.5\} & \{0.5\} & \{0.4\} \\ \{0.2, 0.3\} & \{0.3\} & \{0.6\} & \{0.5\} \end{bmatrix} \text{ and}$$

$$R_3 = \begin{bmatrix} \{0.5\} & \{0.6\} & \{0.5, 0.6, 0.7\} & \{0.7\} \\ \{0.4\} & \{0.5\} & \{0.6\} & \{0.6\} \\ \{0.3, 0.4, 0.5\} & \{0.4\} & \{0.5\} & \{0.6\} \\ \{0.3\} & \{0.4\} & \{0.4\} & \{0.5\} \end{bmatrix}.$$

Table 4 Evaluation of the alternatives with respect to four feature criteria

Criteria Alternative	c_1	c_2	c_3	c_4
a_1	$\{0.3, 0.4, 0.5\}$	$\{0.6, 0.7\}$	$\{0.8\}$	$\{0.7, 0.8\}$
a_2	$\{0.6, 0.7, 0.8\}$	$\{0.9\}$	$\{0.2, 0.3\}$	$\{0.5\}$
a_3	$\{0.8, 0.9\}$	$\{0.2, 0.3, 0.4\}$	$\{0.7\}$	$\{0.3\}$
a_4	$\{0.5\}$	$\{0.7, 0.8\}$	$\{0.8, 0.9\}$	$\{0.3, 0.4\}$
a_5	$\{0.8, 0.9\}$	$\{0.3, 0.5\}$	$\{0.6, 0.7\}$	$\{0.4\}$

Illustration of the proposed methods

In this subsection, we only present the case in which HFPRs are provided by three decision-makers with additively consistent HFBWPRs, and a similar method can be developed when HFPRs have multiplicatively consistent HFBWPRs. The procedures for determining the most important project using the proposed methods are discussed below, and there are two cases.

Case 1: The maximum absolute differences are minimized problems with the same compromise limit constraint.

Step 1: Define the decision criteria and provide the evaluation values.

The decision criteria have been defined, and three decision-makers' evaluation values have been provided, as demonstrated in matrices 1–3.

Step 2: Identify the best and worst criteria.

According to Eq. (16), for decision-maker e_1, we have:

$s(c_1) = 1.8$; $s(c_2) = 2.85$; $s(c_3) = 1.8$ and $s(c_4) = 1.35$. Since $s(c_2) > s(c_3) = s(c_1) > s(c_4)$, we have the best criterion c_2 and the worst criterion c_4.

For decision-maker e_2, we have:

$s(c_1) = 2.9$; $s(c_2) = 1.85$; $s(c_3) = 1.6$ and $s(c_4) = 1.65$. Since $s(c_1) > s(c_2) > s(c_4) > s(c_3)$, we have the best criterion c_1 and the worst criterion c_3.

For decision-maker e_3, we have:

$s(c_1) = 2.4$; $s(c_2) = 2.1$; $s(c_3) = 1.9$ and $s(c_4) = 1.6$. Since $s(c_1) > s(c_2) > s(c_3) > s(c_4)$, we have the best criterion c_1 and the worst criterion c_4.

Step 3: Calculate the optimal weights of the criteria.

First, calculate the weights of the criteria, according to Eq. (5), for decision-maker e_1, we have:

$$\begin{aligned} &\min \quad z^k = \varsigma \\ &\text{s.t.} \begin{cases} \left|\frac{1}{2}\left(w_2^k - w_1^k\right) + 0.5 - 0.8\right| \le \varsigma \\ \left|\frac{1}{2}\left(w_2^k - w_3^k\right) + 0.5 - 0.7\right| \le \varsigma \\ \left|\frac{1}{2}\left(w_2^k - w_4^k\right) + 0.5 - \left(0.8\alpha_{24}^1 + 0.9\alpha_{24}^2\right)\right| \le \varsigma \\ \left|\frac{1}{2}\left(w_1^k - w_4^k\right) + 0.5 - 0.5\right| \le \varsigma \\ \left|\frac{1}{2}\left(w_3^k - w_4^k\right) + 0.5 - 0.6\right| \le \varsigma \\ \alpha_{24}^s\left(w_1^k + w_2^k + w_3^k + w_4^k - 1\right) = 0 \\ \alpha_{24}^1 + \alpha_{24}^2 = 1 \\ \alpha_{24}^s = 0 \vee 1, & s = 1, 2 \\ \alpha_{24}^s = 1, & s = 1, 2 \\ w_j^k \ge 0 \\ \varsigma \ge 0 \\ k = 1, 2 \end{cases} \end{aligned}$$

By solving this optimization model, when α_{24}^1 and α_{24}^2, respectively, set 1, we obtain:

z^1= 0.0; w_1^1= 0.05, w_2^1= 0.65, w_3^1= 0.25 and w_4^1= 0.05.
z^2= 0.04; w_1^2= 0.04, w_2^2= 0.72, w_3^2= 0.24 and w_4^2= 0.0.

Similarly, for decision-maker e_2, when α_{12}^1, α_{12}^2, α_{14}^1 and α_{14}^2 are set to 1, we can obtain,

z^1= 0.0; w_1^1= 0.65, w_2^1= 0.05, w_3^1= 0.05 and w_4^1= 0.25.
z^2= 0.03; w_1^2= 0.73, w_2^2= 0.0, w_3^2= 0.07 and w_4^2= 0.2.
z^3= 0.0; w_1^3= 0.65; w_2^3= 0.05; w_3^3= 0.05 and w_4^3= 0.25.
z^4= 0.03; w_1^4= 0.73; w_2^4= 0.0; w_3^4= 0.07 and w_4^4= 0.2.

Moreover, for decision-maker e_3, when α_{13}^1, α_{13}^2 and α_{13}^3, respectively, set 1, we can obtain,

z^1= 0.03; w_1^1= 0.38, w_2^1= 0.25, w_3^1= 0.32 and w_4^1= 0.05.
z^2= 0.0; w_1^2= 0.45, w_2^2= 0.25, w_3^2= 0.25 and w_4^2= 0.05.
z^3= 0.03; w_1^3= 0.52, w_2^3= 0.25, w_3^3= 0.18 and w_4^3= 0.05.

Second, derive the optimal priority weight vector.

First, utilize Eq. (6) to obtain the distance between w^k and R_k, for decision-maker e_1, we have:$d_1\left(w^1, R_1\right) = 0.07$ and $d_1\left(w^2, R_1\right) = 0.09$. Since there is only one priority weight vector included in Eq. (7), then the optimal priority weight vector can be determined as follows: $w^{1*} = (0.05, 0.65, 0.25, 0.05)$.

Similarly, for decision-maker e_2, we can obtain.

$d_1\left(w^1, R_2\right) = d_1\left(w^3, R_2\right) = 0$ and $d_1\left(w^2, R_2\right) = d_1\left(w^4, R_2\right) = 0.03$. Since there is only one priority weight vector included in Eq. (7), then the optimal priority weight vector can be determined as follows: $w^{2*} = (0.65, 0.05, 0.05, 0.25)$.

Moreover, for decision-maker e_3, we can obtain.

$d_1\left(w^1, R_3\right) = d_1\left(w^3, R_3\right) = 0.028$ and $d_1\left(w^2, R_3\right) = 0$. Since there is only one priority weight vector included in Eq. (7), then the optimal priority weight vector can be determined as follows: $w^{3*} = (0.45, 0.25, 0.25, 0.05)$.

Step 4: Calculate the consistency ratio.

According to $\mathrm{CR}_A = \frac{\varsigma_A}{\delta_A}$, for decision-maker e_1, we have $\mathrm{CR}_1 = \frac{0.0}{0.1} = 0$; for decision-maker e_2, we have $\mathrm{CR}_2 = 0$; and for decision-maker e_3, we have $\mathrm{CR}_3 = 0$.

Step 5: Improve the consistency of FPRs.

Since the consistency ratio values CR_1, CR_2 and CR_3 are equal to 0, we have that R_1, R_2 and R_3 are completely consistent, and their consistency does not need to be further improved.

Step 6: Determine the weights of decision-makers.

According to Eq. (14), the objective weights of decision-makers e_1, e_2 and e_3 are determined as follows:$\tau_1^o = 0$, $\tau_2^o = 0$ and $\tau_3^o = 1$. Suppose the subjective weights of decision-makers are provided by the enterprise's board of director, which are equal values $\tau_1^s = \tau_2^s = \tau_3^s = \frac{1}{3}$; then, according to Eq. (15), the comprehensive weights of decision-makers e_1, e_2 and e_3 are calculated as follows: $\tau_1 = \tau_2 = \frac{1}{3}$ and $\tau_3 = \frac{2}{3}$.

Step 7: Compute the collective optimal priority weights of the criteria.

With respect to Eq. (17), the collective optimal priority weights of the criteria are calculated as follows:$\lambda_1 = 0.4167$, $\lambda_2 = 0.2833$, $\lambda_3 = 0.2167$ and $\lambda_4 = 0.0833$.

Step 8: Calculate the collective evaluation values.

According to Eq. (18), the collective evaluation values are calculated by the HFWA operator as follows:

$$\mathrm{HFWA}(a_1) = \{0.5757, 0.9557, 0.9506, 0.9609, 0.9637, 0.9604, 0.9670, 0.9632, 0.9708\},$$

$$\mathrm{HFWA}(a_2) = \{0.9350, 0.9455, 0.9597, 0.9683, 0.9731\}, \quad \mathrm{HFWA}(a_3) = \{0.8516, 0.8833, 0.9414, 0.9449, 0.95\}$$

$$\mathrm{HFWA}(a_4) = \{0.8912, 0.9079, 0.9586, 0.9452, 0.9538, 0.9750, 0.9735\} \text{ and,}$$

$$\mathrm{HFWA}(a_5) = \{0.8736, 0.9073, 0.9113, 0.9389, 0.9475, 0.9519, 0.9595\}.$$

Step 9: Rank the alternatives.

According to Eq. (16), the score values of collective evaluation values are derived as follows: $s\left(\mathrm{HFWA}\left(a_1\right)\right) = 0.7686$, $s\left(\mathrm{HFWA}\left(a_2\right)\right) = 0.7970$, $s\left(\mathrm{HFWA}\left(a_3\right)\right) = 0.7619$, $s\left(\mathrm{HFWA}\left(a_4\right)\right) = 0.8257$ and $s\left(\mathrm{HFWA}\left(a_5\right)\right) = 0.8112$. Since $s(\mathrm{HFWA}(a_4)) > s(\mathrm{HFWA}(a_5)) > s(\mathrm{HFWA}(a_2)) > s(\mathrm{HFWA}(a_1)) > s(\mathrm{HFWA}(a_3))$, the ranking order of all alternatives is obtained as $a_4 \succ a_5 \succ a_2 \succ a_1 \succ a_3$. Thus, the rural e-commerce industry is the most important project to invest.

Case 2: The maximum absolute differences are minimized problems with different compromise limit constraints.

Step 1′ and **Step** 2′ are the same as those given in **Step** 1 and **Step** 2.

Step 3′: Calculate the optimal weights of the criteria.

First, calculate the weights of the criteria, according to Eq. (12), for decision-maker e_1, we have:

$$\min \quad z^k = \sum_{i=1}^{5} \varsigma_i$$

$$\text{s.t.}\begin{cases}
\left|\frac{1}{2}\left(w_2^k - w_1^k\right) + 0.5 - 0.8\right| \le \varsigma_1 \\
\left|\frac{1}{2}\left(w_2^k - w_3^k\right) + 0.5 - 0.7\right| \le \varsigma_2 \\
\left|\frac{1}{2}\left(w_2^k - w_4^k\right) + 0.5 - \left(0.8\alpha_{24}^1 + 0.9\alpha_{24}^2\right)\right| \le \varsigma_3 \\
\left|\frac{1}{2}\left(w_1^k - w_4^k\right) + 0.5 - 0.5\right| \le \varsigma_4 \\
\left|\frac{1}{2}\left(w_3^k - w_4^k\right) + 0.5 - 0.6\right| \le \varsigma_5 \\
\alpha_{24}^s\left(w_1^k + w_2^k + w_3^k + w_4^k - 1\right) = 0 \\
\alpha_{24}^1 + \alpha_{24}^2 = 1 \\
\alpha_{24}^s = 0 \vee 1, & s = 1, 2 \\
\alpha_{24}^s = 1, & s = 1, 2 \\
w_j^k \ge 0 \\
\varsigma_j \ge 0, & j = 1, 2, \ldots, 5 \\
k = 1, 2
\end{cases}$$

By solving this optimization model, when α_{24}^1 and α_{24}^2 respectively set 1, we obtain:

$$z^1 = 0.0;\ \varsigma_1 = \varsigma_2 = \varsigma_3 = \varsigma_4 = \varsigma_5 = 0.0,$$
$$w_1^1 = 0.05,\ w_2^1 = 0.65,\ w_3^1 = 0.25 \text{ and } w_4^1 = 0.05.$$
$$z^2 = 0.1;\ \varsigma_1 = \varsigma_2 = \varsigma_4 = \varsigma_5 = 0.0,\ \varsigma_3 = 0.1,$$
$$w_1^1 = 0.05,\ w_2^1 = 0.65,\ w_3^1 = 0.25 \text{ and } w_4^1 = 0.05.$$

Similarly, for decision-maker e_2, when α_{12}^1, α_{12}^2, α_{14}^1 and α_{14}^2 are set to 1, we can obtain.

$$z^1 = 0.0;\ \varsigma_1 = \varsigma_2 = \varsigma_3 = \varsigma_4 = \varsigma_5 = 0.0,\ w_1^1 = 0.65,\ w_2^1 = 0.05,\ w_3^1 = 0.05 \text{ and } w_4^1 = 0.25.$$
$$z^2 = 0.1;\ \varsigma_2 = \varsigma_3 = \varsigma_5 = 0.0,\ \varsigma_1 = 0.07,\ \varsigma_4 = 0.03,\ w_1^2 = 0.67,\ w_2^2 = 0.0,\ w_3^2 = 0.07 \text{ and } w_4^2 = 0.27.$$
$$z^3 = 0.0;\ \varsigma_1 = \varsigma_2 = \varsigma_3 = \varsigma_4 = \varsigma_5 = 0.0,\ w_1^3 = 0.65,\ w_2^3 = 0.05,\ w_3^3 = 0.05 \text{ and } w_4^3 = 0.25.$$
$$z^4 = 0.1;\ \varsigma_1 = \varsigma_2 = \varsigma_3 = \varsigma_4 = 0.0,\ \varsigma_5 = 0.1\ w_1^4 = 0.7,\ w_2^4 = 0.1,\ w_3^4 = 0.1, \text{ and } w_4^4 = 0.1.$$

Moreover, for decision-maker e_3, when α_{13}^1, α_{13}^2 and α_{13}^3 respectively set 1, we can obtain.

$$z^1 = 0.1,\ \varsigma_1 = \varsigma_3 = \varsigma_4 = \varsigma_5 = 0.0,\ \varsigma_2 = 0.1,\ w_1^1 = 0.45,\ w_2^1 = 0.25,\ w_3^1 = 0.25 \text{ and } w_4^1 = 0.05.$$
$$z^2 = 0.0,\ \varsigma_1 = \varsigma_2 = \varsigma_3 = \varsigma_4 = \varsigma_5 = 0.0,\ w_1^2 = 0.45,\ w_2^2 = 0.25,\ w_3^2 = 0.25 \text{ and } w_4^2 = 0.05.$$
$$z^3 = 0.1,\ \varsigma_1 = \varsigma_3 = \varsigma_4 = \varsigma_5 = 0.0,\ \varsigma_2 = 0.1,\ w_1^1 = 0.45,\ w_2^1 = 0.25,\ w_3^1 = 0.25 \text{ and } w_4^1 = 0.05.$$

Second, derive the optimal priority weight vector.

First, utilize Eq. (6) to obtain the distance between w^k and R_k, for decision-maker e_1, we have: $d_1\left(w^1, R_1\right) = d_1\left(w^2, R_1\right) = 0.07$. Since there is only one priority weight vector including in Eq. (7), then the optimal priority weight vector can be determined as follows:

$$w^{1*} = (0.05, 0.65, 0.25, 0.05).$$

Similarly, for decision-maker e_2, we can obtain. $d_1\left(w^1, R_2\right) = d_1\left(w^3, R_2\right) = 0$ and $d_1\left(w^2, R_2\right) = d_1\left(w^4, R_2\right) = 0.03$. Since there is only one priority weight vector including in Eq. (7), then the optimal priority weight vector can be determined as follows:

$$w^{2*} = (0.65, 0.05, 0.05, 0.25).$$

Moreover, for decision-maker e_3, we can obtain, $d_1\left(w^1, R_3\right) = d_1\left(w^3, R_3\right) = 0.028$ and $d_1\left(w^2, R_3\right) = 0$. Since there is only one priority weight vector including in Eq. (7), then the optimal priority weight vector can be determined as follows: $w^{3*} = (0.45, 0.25, 0.25, 0.05)$.

Step $4'$: Calculate the consistency ratio.

According to $\mathrm{CR}_A = \frac{\varsigma_A}{\delta_A}$, for decision-maker e_1, we have: $CR_1 = \frac{0.0}{0.1} = 0$; for decision-maker e_2, we have: $CR_2 = 0$; for decision-maker e_3, we have: $CR_3 = 0$.

Step $5'$: Improve the consistency of FPRs.

Since the consistency ratio values CR_1, CR_2 and CR_3 are equal to 0, we have R_1, R_2 and R_3 are completely consistency, the consistency of them do not need to be further improved.

Since the collective optimal priority weights of the criteria are the same as in case 1, Steps $6'$ to $9'$ are the same as those given in Steps 6–9, and the best choice is the same as in case 1.

Comparative analysis and discussion

To validate the feasibility of the proposed method, we conducted a comparative study with other methods based on the same illustrative example.

Mi and Liao [35] investigated the BWM with hesitant fuzzy information and then developed three different models to derive the priority weights of criteria with respect to diverse objectives. That is: (1) score-based method. The score values of HFEs are used to denote the most possible values of HFEs; (2) normalization-based method. The HFEs are extended to those with equal lengths according to decision-makers' attitudes. (3) Regression-based method. Each possible value of pairwise comparisons is traversed without ignoring or adding any information. The shortcomings of the score-based method are that the HFEs are translated into different score values, which may lead to the loss of information. The shortcomings of the normalization-based method are that different values are added into shorter HFEs, which may distort the original information provided by decision-makers. The regression-based method is the most similar to the methods proposed in this study. To save space, only the regression-based method is conducted with a comparative study with the proposed methods. The calculation process is shown as follows.

Step 1: Calculate the individual weights of the criteria.

Suppose the HFPRs are provided by three decision-makers with additive consistency. According to the method presented in [35], for decision-maker e_1, we have:

$$\min \quad z = \varsigma$$

$$\text{s.t.} \begin{cases} \left|\frac{1}{2}\left(w_2 - w_1\right) + 0.5 - 0.8\right| \le \varsigma \\ \left|\frac{1}{2}\left(w_2 - w_3\right) + 0.5 - 0.7\right| \le \varsigma \\ \left|\frac{1}{2}\left(w_2 - w_4\right) + 0.5 - \left(0.8\alpha_{24}^1 + 0.9\alpha_{24}^2\right)\right| \le \varsigma \\ \left|\frac{1}{2}\left(w_1 - w_4\right) + 0.5 - 0.5\right| \le \varsigma \\ \left|\frac{1}{2}\left(w_3 - w_4\right) + 0.5 - 0.6\right| \le \varsigma \\ w_1 + w_2 + w_3 + w_4 = 1 \\ \alpha_{24}^1 + \alpha_{24}^2 = 1 \\ \alpha_{24}^s = 0 \vee 1, \qquad s = 1, 2 \\ w_j \ge 0 \\ \varsigma \ge 0 \end{cases}$$

By solving this optimization model, we obtain:

z= 0.0; w_1= 0.05, w_2= 0.65, w_3= 0.25 and w_4= 0.05.

Similarly, for decision-maker e_2, we can obtain,

z= 0.0; w_1= 0.65, w_2= 0.05, w_3= 0.05 and w_4= 0.25.

Moreover, for decision-maker e_3, we can obtain,

z= 0.0; w_1= 0.45; w_2= 0.25; w_3= 0.25 and w_4= 0.05.

Step 2: Compute the collective optimal priority weights of the criteria.

Suppose the decision-makers' weights are the same as those given in Step 6, that is, $\tau_1 = \tau_2 = \frac{1}{3}$ and $\tau_3 = \frac{2}{3}$. With respective to Eq. (17), the collective optimal priority weights of the criteria are calculated as follows: $\lambda_1 = 0.4167$, $\lambda_2 = 0.2833$, $\lambda_3 = 0.2167$ and $\lambda_4 = 0.0833$.

For a better comparison, the results obtained by Mi and Liao [35]'s method and the proposed methods are summarized in Table 5.

As shown in Table 5, the ranking values are the same when compared with Mi and Liao [35] method and the proposed methods. This confirms the feasibility of the proposed methods. The possible reasons for the consistency are explained as follows: Mi and Liao [35] method and the proposed methods both integrate the consistent HFPRs and BWM and determine the best priority weight vector. However, in the review of the calculation process, the definitions of the consistent HFPRs are different from Mi and Liao [35] method and the proposed methods, In Mi and Liao [35] method, the consistency definitions based on one FPR derived from HFPRs, that is, optimistic consistency, while the proposed methods the consistency definitions are based on the derived FPRs for each value in HFEs. The ways to determine the best priority weight vector are also different. The best priority weight vector was derived from the mathematical programming model in Mi and Liao [35] method, while the proposed methods were derived from the proposed distance formulas. Moreover, the proposed methods consider the consistency checking and improving process, while Mi and Liao [35] method fails to this.

In addition, although the ranking values are the same when compared with the proposed methods with the same and different compromise limit constraints, in the review of the calculation process, we find that the weights of the criteria are different for some FPRs. This confirms the necessity of considering different compromise limit constraints. For a better comparison, the results obtained by Mi and Liao [35] method and the proposed methods are summarized in Table 6.

According to the comparison analysis, the method proposed in this study has the following advantages over other existing methods.

1. The consistency measures from the perspectives of additive consistent and multiplicatively consistent HFBWPRs are defined based on the relationships between each fixed value and their corresponding priority weight vector. It can then avoid information loss and distortion, and the ranking result obtained from the proposed methods seems more reasonable.
2. The ways to determine the best priority weight vector developed in this study consider the case when more than one priority weight vector has the same minimum distance.

Table 5 The ranking results of the different methods

Methods		Ranking values				Ranking results
		λ_1	λ_2	λ_3	λ_4	
Mi and Liao [35] method	Regression-based method	0.4167	0.2833	0.2167	0.0833	$c_1 \succ c_2 \succ c_3 \succ c_4$
The proposed methods	Same compromise limit	0.4167	0.2833	0.2167	0.0833	$c_1 \succ c_2 \succ c_3 \succ c_4$
	Different compromise limit	0.4167	0.2833	0.2167	0.0833	$c_1 \succ c_2 \succ c_3 \succ c_4$

Table 6 The weights of criteria derived from different methods

Methods		Decision makers	Weight vectors w_1	w_2	w_3	w_4	The number of weight vectors
Mi and Liao [35]'s method	Regression-based method	e_1	0.05	0.65	0.25	0.05	1
		e_2	0.65	0.05	0.05	0.25	1
		e_3	0.45	0.25	0.25	0.05	1
The proposed methods	Same compromise limit	e_1	0.05	0.65	0.25	0.05	2
		e_2	0.65	0.05	0.05	0.25	2
			0.67	0.0	0.07	0.27	1
			0.7	0.1	0.1	0.1	1
		e_3	0.45	0.25	0.25	0.05	3
	Different compromise limit	e_1	0.65	0.05	0.05	0.25	1
			0.04	0.72	0.24	0.0	1
		e_2	0.65	0.05	0.05	0.25	2
			0.73	0.0	0.07	0.2	2
		e_3	0.38	0.25	0.32	0.05	1
			0.45	0.25	0.25	0.05	1
			0.52	0.25	0.18	0.05	1

3. The proposed methods consider the consistency checking and improving process, and the ranking result obtained from the proposed methods seems more convincing.

Conclusion

To address the situation where HFPRs are necessary, this paper develops several decision-making models integrating HFPRs with BWM. First, consistency measures from the perspectives of additive/multiplicatively consistent HFBWPRs are introduced. Second, several decision-making models are developed in view of the proposed additive/multiplicatively consistent HFBWPRs. Third, an absolute programming model is developed to obtain the decision-makers' objective weights utilizing the optimal priority weight vector information, and the calculation of comprehensive weights of decision-makers is provided. Finally, a framework of the MCDM procedure based on hesitant fuzzy BWM is introduced, and an illustrative example in conjunction with comparative analysis is used to demonstrate that the proposed models are feasible and efficient for practical MCDM problems.

The present study provides several significant contributions to MCDM problems with HFPRs. They are summarized as follows: (1) consistency measures from the perspectives of additive/multiplicatively consistent HFBWPRs are introduced, which integrate the advantages of HFPRs and BWM. The proposed hesitant fuzzy BWM based on HFPRs provides us with a very useful method for MCDM in fuzzy environments. (2) Several decision-making models are developed in view of the proposed additive/multiplicatively consistent HFBWPRs and consider the same and different compromise limit constraints. In view of this, the MCDM methods developed in this study have wide practical applications. (3) An absolute programming model is developed to obtain the decision-makers' objective weights utilizing the optimal priority weight vector information. It provides us with a new way to derive decision-makers' weights. In our future research, the proposed methods are extended to hesitant fuzzy linguistic preference relations, and the proposed methods are applied to solve other practical MCDM problems. In addition, since the results given in this study are consistent, inconsistency is not considered. How to solve the weight vector by minimizing the inconsistency of the hesitant fuzzy BWM is also an area of future research.

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Declarations

Conflict of interest All the authors declare that they have no conflict of interest.

Ethical approval This article does not contain any studies with human participants performed by any of the authors.

References

1. Ding Z, Chen X, Dong Y, Herrera F (2019) Consensus reaching in social network DeGroot Model: the roles of the self-confidence and node degree. Inf Sci 486:62–72
2. Mittal K, Jain A, Vaisla KS, Castillo O, Kacprzyk J (2020) A comprehensive review on type 2 fuzzy logic applications: Past, present and future. Eng Appl Artif Intell 95:103916
3. Vij S, Jain A, Tayal D, Castillo O (2020) Scientometric inspection of research progression in hesitant fuzzy sets. J Intell Fuzzy Syst 38:619–626
4. Pujahari A, Sisodia DS (2019) Modeling side information in preference relation based restricted boltzmann machine for recommender systems. Inf Sci 490:126–145
5. Tanino T (1984) Fuzzy preference orerings in group decision-making. Fuzzy Sets Syst 12:117–131
6. Saaty TL (1980) The analytic hierarchy process. McGraw-Hill, New York
7. Xie W, Xu Z, Ren Z, Herrera-Viedma E (2020) A new multi-criteria decision model based on incomplete dual probabilistic linguistic preference relations. Appl Soft Comput 91:106237
8. Chu J, Wang Y, Liu X, Liu Y (2020) Social network community analysis based large-scale group decision making approach with incomplete fuzzy preference relations. Inf Fusion 60:98–120
9. Li J, Wang JQ (2019) Multi-criteria decision-making with probabilistic hesitant fuzzy information based on expected multiplicative consistency. Neural Comput Appl 31:8897–8915
10. Li J, Wang JQ, Hu JH (2019) Multi-criteria decision-making method based on dominance degree and BWM with probabilistic hesitant fuzzy information. Int J Mach Learn Cybern 10:1671–1685
11. Gong Z, Guo W, Herrera-Viedma E, Gong Z, Wei G (2020) Consistency and consensus modeling of linear uncertain preference relations. Eur J Oper Res 283:290–307
12. Wan S-P, Yuan H-W, Dong J-Y (2021) Decision making with incomplete interval multiplicative preference relations based on stochastic program and interval category. Inf Sci 570:403–427
13. Torra V, Narukawa Y (2009) On hesitant fuzzy sets and decision. In: IEEE International Conference on Fuzzy Systems, 2009. Fuzz-IEEE, pp 1378–1382
14. Mishra AR, Rani P, Krishankumar R, Ravichandran KS, Kar S (2021) An extended fuzzy decision-making framework using hesitant fuzzy sets for the drug selection to treat the mild symptoms of Coronavirus Disease 2019 (COVID-19). Appl Soft Comput 103:107155
15. Mardani A, Saraji MK, Mishra AR, Rani P (2020) A novel extended approach under hesitant fuzzy sets to design a framework for assessing the key challenges of digital health interventions adoption during the COVID-19 outbreak. Appl Soft Comput 96:106613
16. Zindani D, Maity SR, Bhowmik S (2021) Extended TODIM method based on normal wiggly hesitant fuzzy sets for deducing optimal reinforcement condition of Agro-waste fibers for green product development. J Clean Prod 301:126947
17. Xia M, Xu Z (2013) Managing hesitant information in GDM problems under fuzzy and multiplivative referene relations. Int J Uncertain Fuzziness Knowl Based Syst 21:865–897
18. Zhang Z, Kou X, Dong Q (2018) Additive consistency analysis and improvement for hesitant fuzzy preference relations. Expert Syst Appl 98:118–128
19. Meng F, Tang J, Pedrycz W, An Q (2020) Optimal interaction priority calculation from hesitant fuzzy preference relations based on the monte carlo simulation method for the acceptable consistency and consensus. IEEE Trans Cybern. https://doi.org/10.1109/TCYB.2019.2962095
20. Li J, Wang ZX (2019) Deriving priority weights from hesitant fuzzy preference relations in view of additive consistency and consensus. Soft Comput 23:13691–13707
21. Meng F, Chen S-M, Tang J (2020) Group decision making based on acceptable multiplicative consistency of hesitant fuzzy preference relations. Inf Sci 524:77–96
22. Li J, Wang JQ, Hu JH (2019) Consensus building for hesitant fuzzy preference relations with multiplicative consistency. Comput Ind Eng 128:387–400
23. Wan S, Zhong L, Dong J (2019) A new method for group decision making with hesitant fuzzy preference relations based on multiplicative consistency. IEEE Trans Fuzzy Syst 28:1449–1463
24. Zhang Z, Pedrycz W (2020) Iterative algorithms to manage the consistency and consensus for group decision-making with hesitant multiplicative preference relations. IEEE Trans Fuzzy Syst 28:2944–2957
25. Zhang Z, Wu C (2014) A decision support model for group decision making with hesitant multiplicative preference relations. Inf Sci 282:136–166
26. Song Y, Li G (2019) Handling group decision-making model with incomplete hesitant fuzzy preference relations and its application in medical decision. Soft Comput 23:6657–6666
27. Zhu B, Xu Z, Xu J (2014) Deriving a ranking from hesitant fuzzy preference relations under group decision making. IEEE Trans Cybern 44:1328–1337
28. Zhang Z, Wang C, Tian X (2015) A decision support model for group decision making with hesitant fuzzy preference relations. Knowl Based Syst 86:77–101
29. Liu H, Xu Z, Liao H (2016) The multiplicative consistency index of hesitant fuzzy preference relation. IEEE Trans Fuzzy Syst 24:82–93
30. Mo X, Zhao H, Xu Z (2020) Feature-based hesitant fuzzy aggregation method for satisfaction with life scale. Appl Soft Comput 94:106493
31. Zhang Z, Kou X, Yu W, Guo C (2018) On priority weights and consistency for incomplete hesitant fuzzy preference relations. Knowl Based Syst 143:115–126
32. Meng F, An Q (2017) A new approach for group decision making method with hesitant fuzzy preference relations. Knowl Based Syst 127:1–15
33. Tang J, An Q, Meng F, Chen X (2017) A natural method for ranking objects from hesitant fuzzy preference relations. Int J Inf Technol Decis Mak 16:1611–1646
34. Rezaei J (2015) Best-worst multi-criteria decision-making method. Omega 53:49–57
35. Mi X, Liao H (2019) An integrated approach to multiple criteria decision making based on the average solution and normalized

weights of criteria deduced by the hesitant fuzzy best worst method. Comput Ind Eng 133:83–94
36. Xu Y, Zhu X, Wen X, Herrera-Viedma E (2021) Fuzzy best-worst method and its application in initial water rights allocation. Appl Soft Comput 101:107007
37. Dong J, Wan S, Chen S-M (2021) Fuzzy best-worst method based on triangular fuzzy numbers for multi-criteria decision-making. Inf Sci 547:1080–1104
38. Shoji Y, Kim H, Kubo T, Tsuge T, Aikoh T, Kuriyama K (2021) Understanding preferences for pricing policies in Japan's national parks using the best–worst scaling method. J Nat Conserv 60:125954
39. Labella Á, Dutta B, Martínez L (2021) An optimal Best-Worst prioritization method under a 2-tuple linguistic environment in decision making. Comput Ind Eng 155:107141
40. Alidoosti Z, Ahmad S, Govindan K, Pishvaee MS, Mostafaeipour A, Hossain AK (2021) Social sustainability of treatment technologies for bioenergy generation from the municipal solid waste using best worst method. J Clean Prod 288:125592
41. Liu P, Zhu B, Wang P (2021) A weighting model based on best–worst method and its application for environmental performance evaluation. Appl Soft Comput 103:107168
42. Wan S, Dong J, Chen S-M (2021) Fuzzy best-worst method based on generalized interval-valued trapezoidal fuzzy numbers for multi-criteria decision-making. Inf Sci. https://doi.org/10.1016/j.ins.2021.1003.1038
43. Ali A, Rashid T (2019) Hesitant fuzzy best-worst multi-criteria decision-making method and its applications. Int J Intell Syst 34:1953–1967
44. Ming Y, Luo L, Wu X, Liao H, Lev B, Jiang L (2020) Managing patient satisfaction in a blood-collection room by the probabilistic linguistic gained and lost dominance score method integrated with the best-worst method. Comput Ind Eng 145:106547
45. Karimi H, Sadeghi-Dastaki M, Javan M (2020) A fully fuzzy best–worst multi attribute decision making method with triangular fuzzy number: A case study of maintenance assessment in the hospitals. Appl Soft Comput 86:105882
46. Chen Z, Ming X (2020) A rough–fuzzy approach integrating best–worst method and data envelopment analysis to multi-criteria selection of smart product service module. Appl Soft Comput 94:106479
47. Liang F, Brunelli M, Rezaei J (2020) Consistency issues in the best worst method: measurements and thresholds. Omega 96:102175
48. Mohammadi M, Rezaei J (2020) Bayesian best-worst method: a probabilistic group decision making model. Omega 96:102075
49. Mi X, Tang M, Liao H, Shen W, Lev B (2019) The state-of-the-art survey on integrations and applications of the best worst method in decision making: why, what, what for and what's next? Omega 87:205–225
50. Zhang Z (2016) Deriving the priority weights from incomplete hesitant fuzzy preference relations based on multiplicative consistency. Appl Soft Comput 46:37–59
51. Zhang Z, Wang C, Tian X (2015) Multi-criteria group decision making with incomplete hesitant fuzzy preference relations. Appl Soft Comput 36:1–23
52. Orlovsky SA (1978) Decision-making with a fuzzy preference relation. Fuzzy Sets Syst 1:155–167
53. Xia MM, Xu ZS (2011) Hesitant fuzzy information aggregation in decision making. Int J Approx Reason 52:395–407
54. Xu Y, Cabrerizo FJ, Herrera-Viedma E (2017) A consensus model for hesitant fuzzy preference relations and its application in water allocation management. Appl Soft Comput 58:265–284
55. Xu Y, Chen L, Rodríguez RM, Herrera F, Wang H (2016) Deriving the priority weights from incomplete hesitant fuzzy preference relations in group decision making. Knowl Based Syst 99:71–78
56. Zhu B, Xu Z (2014) Consistency measures for hesitant fuzzy linguistic preference relations. IEEE Trans Fuzzy Syst 22:35–45
57. Mou Q, Xu Z, Liao H (2016) An intuitionistic fuzzy multiplicative best-worst method for multi-criteria group decision making. Inf Sci 374:224–239
58. Fei L, Lu J, Feng Y (2020) An extended best-worst multi-criteria decision-making method by belief functions and its applications in hospital service evaluation. Comput Ind Eng 142:106355
59. Amiri M, Hashemi-Tabatabaei M, Ghahremanloo M, Keshavarz-Ghorabaee M, Zavadskas EK, Antucheviciene J (2020) A new fuzzy approach based on BWM and fuzzy preference programming for hospital performance evaluation: a case study. Appl Soft Comput 92:106279
60. Liang F, Brunelli M, Rezaei J (2019) Consistency issues in the best worst method: Measurements and thresholds. Omega 96:102175
61. Liao H, Mi X, Yu Q, Luo L (2019) Hospital performance evaluation by a hesitant fuzzy linguistic best worst method with inconsistency repairing. J Clean Prod 232:657–671
62. Wu P, Zhu J, Zhou L, Chen H (2019) Local feedback mechanism based on consistency-derived for consensus building in group decision making with hesitant fuzzy linguistic preference relations. Comput Ind Eng 137:106001