

APPROXIMATE APPROXIMATIONS ON NONUNIFORM GRIDS

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We present an extension of approximate quasi-interpolation on uniformly distributed nodes, to functions given on a set of nodes close to an uniform, not necessarily cubic, grid.

1. Introduction

The method of approximate quasi-interpolation and its first related results were proposed in [5] and [14]. The method is characterized by a very accurate approximation in a certain range relevant for numerical computations, but in general the approximations do not converge in rigorous sense. For that reason such processes were called *approximate approximations*.

Suppose we want to approximate a smooth function $u(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$, when we prescribe the values of u at the points of an uniform grid of mesh size h . We fix a positive parameter \mathcal{D} and we choose a sufficiently smooth and rapidly decaying at infinity function η - the generating function - such that the linear combination of dilated shifts of η forms an approximate partition of the unity *i.e.*

$$\mathcal{D}^{-n/2} \sum_{\mathbf{m} \in \mathbb{Z}^n} \eta \left(\frac{\xi - \mathbf{m}}{\sqrt{\mathcal{D}}} \right) \approx 1.$$

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The method consists in approximating the function u at the point \mathbf{x} by a linear combination of the form

$$M_{h,\mathcal{D}}u(\mathbf{x}) = \mathcal{D}^{-n/2} \sum_{\mathbf{m} \in \mathbb{Z}^n} u(h\mathbf{m}) \eta \left(\frac{\mathbf{x} - h\mathbf{m}}{h\sqrt{\mathcal{D}}} \right), \quad \mathbf{x} \in \mathbb{R}^n. \quad (1)$$

This type of formulas is known as quasi-interpolants and they have the property that $M_{h,\mathcal{D}}u(\mathbf{x})$ approximates $u(\mathbf{x})$, but $M_{h,\mathcal{D}}u(\mathbf{x})$ does not converge to $u(\mathbf{x})$ as the grid size h tends to zero. However one can fix \mathcal{D} such that the approximation error is as small as we wish so that the non-convergence is not perceptible in numerical computations (see [7], [9]). On the other hand, the simplicity of the generalizations to the multi-dimensional case together with a great flexibility in choosing the generating function η compensate the lack of convergence.

The above mentioned flexibility is important in the applications because the generating function η can be selected so that integral and pseudo-differential operators of mathematical physics applied to η have analitically known expressions, obtaining semianalytic cubature formulas for these operators (see [6], [8], [11] and the review paper [13]). In some cases, *e.g.* for potentials, the cubature formulas converge even in a rigorous sense.

Another important application of the method is the possibility to develop explicit semi-analytic time marching algorithms for initial boundary value problems for linear and non linear evolution equations (see [12], [2]).

Quasi-interpolation formulas similar to (1) preserve the fundamental properties of approximate quasi-interpolation if the grid is a smooth image of the uniform one (see [10]) or if the grid is piecewise uniform (see [1]). The method of approximate quasi-interpolation has been generalized to functions given on a set of nodes close to a uniform, not necessarily cubic, grid in [4]. More general scattered grids have been considered in [3].

To illustrate the unusual behavior of approximate approximations we assume $\eta(x) = e^{-x^2}/\sqrt{\pi}$ as generating function and the following quasi-interpolant for a function u on \mathbb{R} :

$$M_{h,\mathcal{D}}u(x) = \frac{1}{\sqrt{\pi\mathcal{D}}} \sum_{m=-\infty}^{\infty} u(hm) e^{-(x-hm)^2/(\mathcal{D}h^2)}, \quad x \in \mathbb{R}. \quad (2)$$

The application of Poisson's summation formula to the function

$$\Theta(\xi, \mathcal{D}) = \frac{1}{\sqrt{\pi\mathcal{D}}} \sum_{m=-\infty}^{\infty} e^{-(\xi-m)^2/\mathcal{D}}$$

yields to these equivalent representations for

$$\Theta(\xi, \mathcal{D}) = 1 + 2 \sum_{v=1}^{\infty} e^{-\pi^2 \mathcal{D} v^2} \cos 2\pi v \xi$$

and

$$\Theta'(\xi, \mathcal{D}) = -4\pi \sum_{v=1}^{\infty} v e^{-\pi^2 \mathcal{D} v^2} \sin 2\pi v \xi.$$

We deduce that

$$|\Theta(\xi, \mathcal{D}) - 1| \leq 2 \sum_{v=1}^{\infty} e^{-\pi^2 \mathcal{D} v^2} < 2\varepsilon(\mathcal{D});$$

$$|\Theta'(\xi, \mathcal{D})| \leq 4\pi \sum_{v=1}^{\infty} v e^{-\pi^2 \mathcal{D} v^2} < 4\pi \varepsilon(\mathcal{D})$$

with

$$\varepsilon(\mathcal{D}) = e^{-\pi^2 \mathcal{D}} + \mathcal{O}(e^{-4\pi^2 \mathcal{D}}).$$

The rapid exponential decay ensures that we can choose \mathcal{D} large enough such that $\varepsilon(\mathcal{D})$ can be made arbitrarily small, for example less than the needed accuracy or the machine precision. Therefore the integer shifts of the Gaussian $\{ \frac{e^{-(\xi-m)^2/\mathcal{D}}}{\sqrt{\pi \mathcal{D}}}, m \in \mathbb{Z} \}$ form an approximate partition of unity for large \mathcal{D} .

If the approximated function u is smooth enough, the quasi-interpolant (2) can be represented in the form (see [14])

$$M_{h,\mathcal{D}}u(x) = u(x) + u(x) \left(\Theta\left(\frac{x}{h}, \mathcal{D}\right) - 1 \right) + u'(x) \frac{h\mathcal{D}}{2} \Theta'\left(\frac{x}{h}, \mathcal{D}\right) + \mathcal{R}_{h,\mathcal{D}}(x)$$

where the remainder term admits the estimate

$$|\mathcal{R}_{h,\mathcal{D}}(x)| \leq c \mathcal{D} h^2 \max_{x \in \mathbb{R}} |u''(x)|$$

with a constant c not depending on h, \mathcal{D}, u .

The difference between $M_{h,\mathcal{D}}u(x)$ and $u(x)$ can be estimated by

$$|M_{h,\mathcal{D}}u(x) - u(x)| \leq c \mathcal{D} h^2 \max_{x \in \mathbb{R}} |u''(x)| + \varepsilon(\mathcal{D}) (2|u(x)| + \frac{h\mathcal{D}}{2} |u'(x)|). \tag{3}$$

This means that, above the tolerance (3), the quasi-interpolant (2) approximates u like usual second order approximations and, if \mathcal{D} is chosen appropriately, any prescribed accuracy can be reached. Then the non-convergent part - called *saturation error* because it does not converge to 0 - can be neglected and the approximation process behaves like a second order approximation process.

2. Quasi-interpolation on uniform grids

One of the advantages of the method is that quasi-interpolants in arbitrary space dimension n with approximation order larger than two, up to some prescribed accuracy, have the same simple form as second order quasi-interpolants. The quasi-interpolant in \mathbb{R}^n has the form

$$M_{h,\mathcal{D}}u(\mathbf{x}) = \mathcal{D}^{-n/2} \sum_{\mathbf{j} \in \mathbb{Z}^n} u(h\mathbf{j}) \eta \left(\frac{\mathbf{x} - h\mathbf{j}}{h\sqrt{\mathcal{D}}} \right) \tag{4}$$

with the generating function η in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of smooth and rapidly decaying functions. Maz'ya and Schmidt have proved that formula (4) provides the following approximation result.

Theorem 2.1. ([10]) *Suppose that*

$$\int_{\mathbb{R}^n} \eta(\mathbf{y})d\mathbf{y} = 1, \int_{\mathbb{R}^n} \mathbf{y}^\alpha \eta(\mathbf{y})d\mathbf{y} = 0, \forall \alpha : 1 \leq |\alpha| < N \tag{5}$$

and $u \in W_\infty^N(\mathbb{R}^n)$. Then

$$|M_{h,\mathcal{D}}u(\mathbf{x}) - u(\mathbf{x})| \leq c_{\eta,N}(\sqrt{\mathcal{D}}h)^N \|\nabla_N u\|_{L_\infty} + \sum_{k=0}^{N-1} \left(\frac{h\sqrt{\mathcal{D}}}{2\pi} \right)^k \sum_{|\alpha|=k} \frac{|\nabla_k u(\mathbf{x})|}{\alpha!} \sum_{\mathbf{v} \in \mathbb{Z}^n \setminus \{0\}} |\partial^\alpha \mathcal{F} \eta(\sqrt{\mathcal{D}}\mathbf{v})|$$

with the constant $c_{\eta,N}$ not depending on u, h and \mathcal{D} .

Moreover for any $\varepsilon > 0$, there exists $\mathcal{D} > 0$ such that for all $\alpha, 0 \leq |\alpha| < N$,

$$\sum_{\mathbf{v} \in \mathbb{Z}^n \setminus \{0\}} |\partial^\alpha \mathcal{F} \eta(\sqrt{\mathcal{D}}\mathbf{v})| < \varepsilon.$$

$\nabla_k u(x)$ denotes the vector of all partial derivatives $\{\partial^\alpha u(x)\}_{|\alpha|=k}$ and $\mathcal{F} \eta$ denotes the Fourier transform of η . We deduce that for any $\varepsilon > 0$ there exists $\mathcal{D} > 0$ such that $M_{h,\mathcal{D}}u(\mathbf{x})$ approximates $u(\mathbf{x})$ pointwise with the estimate (see [7],[9])

$$|M_{h,\mathcal{D}}u(\mathbf{x}) - u(\mathbf{x})| \leq c_{\eta,N}(\sqrt{\mathcal{D}}h)^N \|\nabla_N u\|_{L_\infty} + \varepsilon \sum_{k=0}^{N-1} (h\sqrt{\mathcal{D}})^k |\nabla_k u(\mathbf{x})|.$$

Therefore $M_{h,\mathcal{D}}u$ behaves like an approximation formula of order N up to the saturation term that can be ignored in numerical computations if \mathcal{D} is large enough. Similar estimates are also valid for integral norms (see [6]).

Several methods to construct generating functions satisfying the moment conditions (5) for arbitrarily large N have been developed (see [9], [10]). In fact any sufficiently smooth and rapidly decaying function η with $\mathcal{F} \eta(0) \neq 0$ can be used to construct new generating functions η_N satisfying the moment conditions for arbitrary large N as shown in the next theorem.

Theorem 2.2. ([9]) *Let $\eta \in \mathcal{S}(\mathbb{R}^n)$ with $\mathcal{F}\eta(0) \neq 0$. Then*

$$\eta_N(\mathbf{x}) = \sum_{|\alpha|=0}^{N-1} \frac{\partial^\alpha (\mathcal{F}\eta(\lambda)^{-1})|_{\lambda=0}}{\alpha! (2\pi i)^{|\alpha|}} \partial^\alpha \eta(\mathbf{x})$$

satisfies the moment conditions (5).

An interesting example is given by the Gaussian function $\eta(\mathbf{x}) = e^{-|\mathbf{x}|^2}$ where the application of Theorem 2.2 leads to the generating function

$$\eta_{2M}(\mathbf{x}) = \pi^{-n/2} \sum_{j=0}^{M-1} \frac{(-1)^j}{j! 4^j} \Delta^j e^{-|\mathbf{x}|^2} = \pi^{-n/2} L_{M-1}^{(n/2)}(|\mathbf{x}|^2) e^{-|\mathbf{x}|^2}$$

with $N = 2M$ and the generalized Laguerre polynomial

$$L_k^{(\gamma)}(y) = \frac{e^y y^{-\gamma}}{k!} \left(\frac{d}{dy} \right)^k (e^{-y} y^{k+\gamma}), \gamma > -1.$$

Hence the quasi-interpolant

$$M_{h,\mathcal{D}} u(\mathbf{x}) = (\pi \mathcal{D})^{-n/2} \sum_{\mathbf{j} \in \mathbb{Z}^n} u(h\mathbf{j}) L_{M-1}^{(n/2)} \left(\left| \frac{\mathbf{x} - h\mathbf{j}}{h\sqrt{\mathcal{D}}} \right|^2 \right) e^{-\left| \frac{\mathbf{x} - h\mathbf{j}}{h\sqrt{\mathcal{D}}} \right|^2}$$

is an approximation formula of order $N = 2M$ plus the saturation term.

The quasi-interpolation formula and the corresponding approximation results have been generalized in [1] and [4] to the case when the values of u are given on uniform grids, not necessarily cubic, of this type

$$\Lambda_h := \{hA\mathbf{j}, \mathbf{j} \in \mathbb{Z}^n\}$$

with a real nonsingular $n \times n$ -matrix A .

Under the same assumptions on the generating function η , it is always possible to choose $\mathcal{D} > 0$ such that the quasi-interpolant

$$\mathcal{M}_{\Lambda_h} u(\mathbf{x}) := \frac{\det A}{\mathcal{D}^{n/2}} \sum_{\mathbf{j} \in \mathbb{Z}^n} u(hA\mathbf{j}) \eta \left(\frac{\mathbf{x} - hA\mathbf{j}}{\sqrt{\mathcal{D}h}} \right) \tag{6}$$

satisfies an estimate similar to that obtained in Theorem 2.1 for uniform cubic grid *i.e.*

$$|\mathcal{M}_{\Lambda_h} u(\mathbf{x}) - u(\mathbf{x})| \leq c_{\eta,N} (\sqrt{\mathcal{D}h})^N \|\nabla_N u\|_{L_\infty} + \varepsilon \sum_{k=0}^{N-1} (h\sqrt{\mathcal{D}})^k |\nabla_k u(\mathbf{x})| \tag{7}$$

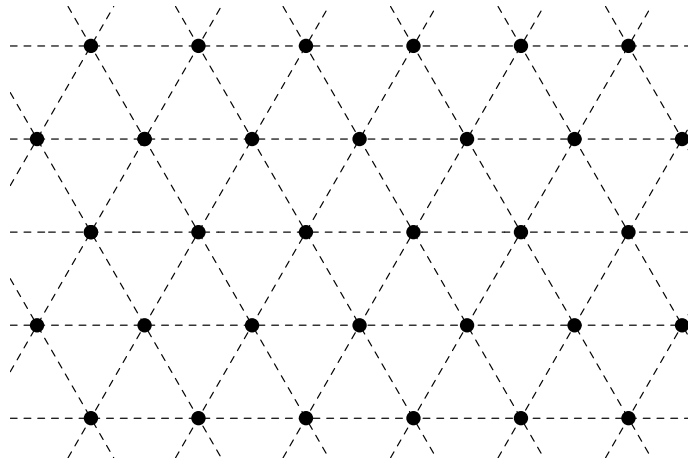


Figure 1: Tridiagonal grid

for any $\varepsilon > 0$.

The first application of formula (6) is the construction of quasi-interpolants on a regular triangular grid in the plane, as indicated in Figure 1.

The vertices \mathbf{y}_j^Δ of a partition of the plane into equilateral triangles of side length 1 are given by

$$\mathbf{y}_j^\Delta = A\mathbf{j}; \quad A = \begin{pmatrix} 1 & 1/2 \\ 0 & \sqrt{3}/2 \end{pmatrix}.$$

The application of formula (6) to the nodes of the regular triangular grid of size h

$$\Lambda_h = \{h\mathbf{y}_j^\Delta\} = \{hA\mathbf{j}\}_{\mathbf{j} \in \mathbb{Z}^2}$$

gives the following quasi-interpolant

$$\mathcal{M}_h^\Delta u(\mathbf{x}) := \frac{\sqrt{3}}{2\mathcal{D}} \sum_{\mathbf{j} \in \mathbb{Z}^2} u(h\mathbf{y}_j^\Delta) \eta\left(\frac{\mathbf{x} - h\mathbf{y}_j^\Delta}{\sqrt{\mathcal{D}}h}\right).$$

The system of functions $\left\{ \frac{\sqrt{3}}{2\mathcal{D}} \eta\left(\frac{\mathbf{x} - \mathbf{y}_j^\Delta}{\sqrt{\mathcal{D}}}\right) \right\}$, centered at the points of the uniform triangular grid, forms an approximate partition of unity. Using Poisson's summation formula one can bound the main term of the saturation error by

$$\left| 1 - \frac{\sqrt{3}}{2\mathcal{D}} \sum_{\mathbf{j} \in \mathbb{Z}^2} \eta\left(\frac{\mathbf{x} - \mathbf{y}_j^\Delta}{\sqrt{\mathcal{D}}}\right) \right| \leq \sum_{\mathbf{v} \in \mathbb{Z}^2 \setminus \{0\}} \left| \int_{\mathbb{R}^2} \eta(\mathbf{y}) e^{-2\pi i \sqrt{\mathcal{D}}(A^{-1}\mathbf{y}, \mathbf{v})} d\mathbf{y} \right|.$$

By assuming as generating function the Gaussian $\eta(\mathbf{x}) = \pi^{-1}e^{-|\mathbf{x}|^2}$ we obtain

$$\begin{aligned} & \left| 1 - \frac{\sqrt{3}}{2\pi\mathcal{D}} \sum_{\mathbf{j} \in \mathbb{Z}^2} e^{-|\mathbf{x}-\mathbf{y}_j^\Delta|^2/\mathcal{D}} \right| \\ & \leq \sum_{(v_1, v_2) \neq (0,0)} e^{-4\pi^2\mathcal{D}(v_1^2 - v_1 v_2 + v_2^2)/3} = 6e^{-4\pi^2\mathcal{D}/3} + \mathcal{O}(e^{-4\pi^2\mathcal{D}}). \end{aligned}$$

In Figure 2 the graph of the difference $\frac{\sqrt{3}}{2\pi\mathcal{D}} \sum_{\mathbf{j} \in \mathbb{Z}^2} e^{-|\mathbf{x}-\mathbf{y}_j^\Delta|^2/\mathcal{D}} - 1$ is plotted with two different values of D .

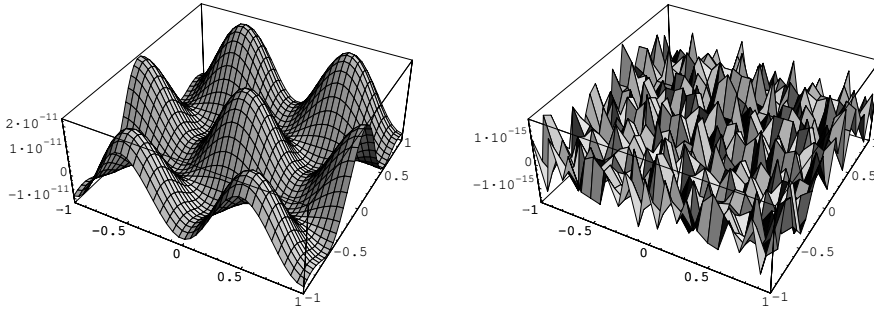


Figure 2: The graph of $\frac{\sqrt{3}}{2\pi\mathcal{D}} \sum_{\mathbf{j} \in \mathbb{Z}^2} e^{-|\mathbf{x}-\mathbf{y}_j^\Delta|^2/\mathcal{D}} - 1$ when $D = 2$ (on the left) and $D = 3$ (on the right).

As second example we construct quasi-interpolants with functions centered at the nodes of a regular hexagonal grid in the plane, as depicted in Figure 3. We obtain a hexagonal grid if, from the nodes of a regular triangular grid of side length 1, the nodes of another triangular grid of side length $\sqrt{3}$ are removed (see Figure 4). Therefore the set of nodes \mathbf{X}^\diamond of the regular hexagonal grid are given by

$$\mathbf{X}^\diamond = \{A\mathbf{j}\}_{\mathbf{j} \in \mathbb{Z}^2} \setminus \{B\mathbf{j}\}_{\mathbf{j} \in \mathbb{Z}^2}$$

where

$$B = \begin{pmatrix} 3/2 & 0 \\ \sqrt{3}/2 & \sqrt{3} \end{pmatrix}$$

and $B\mathbf{j}$, $\mathbf{j} \in \mathbb{Z}^2$, denote the removed nodes.

The quasi-interpolant on the h -scaled hexagonal grid

$$h\mathbf{X}^\diamond = \{hA\mathbf{j}\}_{\mathbf{j} \in \mathbb{Z}^2} \setminus \{hB\mathbf{j}\}_{\mathbf{j} \in \mathbb{Z}^2} \tag{8}$$

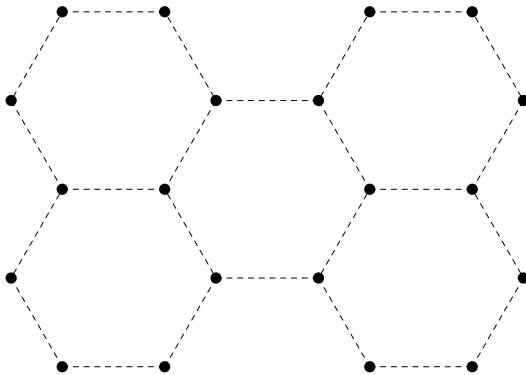


Figure 3: Hexagonal grid

is defined as

$$\mathcal{M}_h^\diamond u(\mathbf{x}) := \frac{3\sqrt{3}}{4\mathcal{D}} \sum_{\mathbf{y}^\diamond \in \mathbf{X}^\diamond} u(h\mathbf{y}^\diamond) \eta\left(\frac{\mathbf{x} - h\mathbf{y}^\diamond}{\sqrt{\mathcal{D}h}}\right).$$

For (8) the quasi-interpolant $\mathcal{M}_h^\diamond u$ can be written in an equivalent way

$$\mathcal{M}_h^\diamond u(\mathbf{x}) = \frac{3\sqrt{3}}{4\mathcal{D}} \left(\sum_{\mathbf{j} \in \mathbb{Z}^2} u(hA\mathbf{j}) \eta\left(\frac{\mathbf{x} - hA\mathbf{j}}{\sqrt{\mathcal{D}h}}\right) - \sum_{\mathbf{j} \in \mathbb{Z}^2} u(hB\mathbf{j}) \eta\left(\frac{\mathbf{x} - hB\mathbf{j}}{\sqrt{\mathcal{D}h}}\right) \right),$$

Therefore we derive that under the decay conditions and the moment conditions on η the quasi-interpolant $\mathcal{M}_h^\diamond u$ provides the estimate (7) for sufficiently large \mathcal{D} .

From Poisson's summation formula

$$\sum_{\mathbf{j} \in \mathbb{Z}^2} \eta\left(\frac{\mathbf{x} - A\mathbf{j}}{\sqrt{\mathcal{D}}}\right) = \frac{\mathcal{D}}{\det A} \left(1 + \sum_{\mathbf{v} \in \mathbb{Z}^2 \setminus \{0\}} \mathcal{F} \eta(\sqrt{\mathcal{D}}(A^t)^{-1}\mathbf{v}) e^{2\pi i(\mathbf{x}, (A^t)^{-1}\mathbf{v})} \right),$$

we obtain an approximate partition of unity centered at the hexagonal grid:

$$\frac{3\sqrt{3}}{4\mathcal{D}} \sum_{\mathbf{y}^\diamond \in \mathbf{X}^\diamond} \eta\left(\frac{\mathbf{x} - \mathbf{y}^\diamond}{\sqrt{\mathcal{D}}}\right) - 1 = \sum_{\mathbf{j} \in \mathbb{Z}^2} \eta\left(\frac{\mathbf{x} - A\mathbf{j}}{\sqrt{\mathcal{D}}}\right) - \sum_{\mathbf{j} \in \mathbb{Z}^2} \eta\left(\frac{\mathbf{x} - B\mathbf{j}}{\sqrt{\mathcal{D}}}\right) - 1 =$$

$$\frac{3}{2} \sum_{\mathbf{v} \in \mathbb{Z}^2 \setminus \{0\}} \mathcal{F} \eta(\sqrt{\mathcal{D}}(A^t)^{-1}\mathbf{v}) e^{2\pi i(\mathbf{x}, (A^t)^{-1}\mathbf{v})} -$$

$$\frac{1}{2} \sum_{\mathbf{v} \in \mathbb{Z}^2 \setminus \{0\}} \mathcal{F} \eta(\sqrt{\mathcal{D}}(B^t)^{-1}\mathbf{v}) e^{2\pi i(\mathbf{x}, (B^t)^{-1}\mathbf{v})}.$$

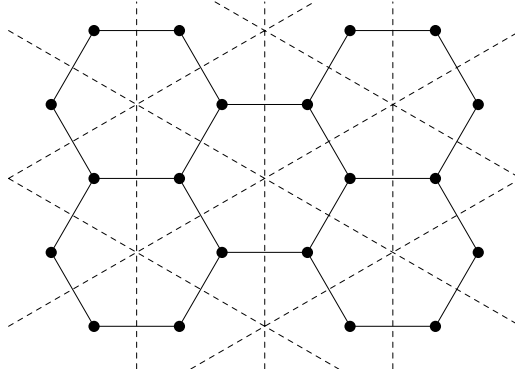


Figure 4: Nodes of a hexagonal grid. The eliminated triangular grid Bj is depicted with dashed lines.

In the case of the exponential $\eta(\mathbf{x}) = \pi^{-1} e^{-|\mathbf{x}|^2}$ we have estimated the main term of the saturation error by

$$\begin{aligned}
 & \left| 1 - \frac{3\sqrt{3}}{4\pi\mathcal{D}} \sum_{\mathbf{y}^\circ \in \mathbf{X}^\circ} e^{-|\mathbf{x}-\mathbf{y}^\circ|^2/\mathcal{D}} \right| \tag{9} \\
 & \leq \frac{1}{2} \sum_{(v_1, v_2) \neq (0,0)} (3e^{-4\pi^2\mathcal{D}(v_1^2 - v_1v_2 + v_2^2)/3} + e^{-4\pi^2\mathcal{D}(v_1^2 - v_1v_2 + v_2^2)/9}) \\
 & = 3e^{-4\pi^2\mathcal{D}/9} + \mathcal{O}(e^{-4\pi^2\mathcal{D}/3}).
 \end{aligned}$$

In Figure 5 the difference (9) is depicted for two different values of \mathcal{D} .

3. Results for nonuniform grids

Next we consider an extension of the approximate quasi-interpolation formulas on uniform grid to the case that the data are given on a set of scattered nodes $\mathbf{X} = \{\mathbf{x}_j\} \subset \mathbb{R}^n$ close to a uniform grid in the sense that we specify in Condition 3.1.

Proposition 3.1. *There exists a uniform grid Λ such that the quasi-interpolants*

$$\mathcal{M}_{h,\mathcal{D}} u(\mathbf{x}) = \mathcal{D}^{-n/2} \sum_{\mathbf{y}_j \in \Lambda} u(h\mathbf{y}_j) \eta\left(\frac{\mathbf{x} - h\mathbf{y}_j}{h\sqrt{\mathcal{D}}}\right)$$

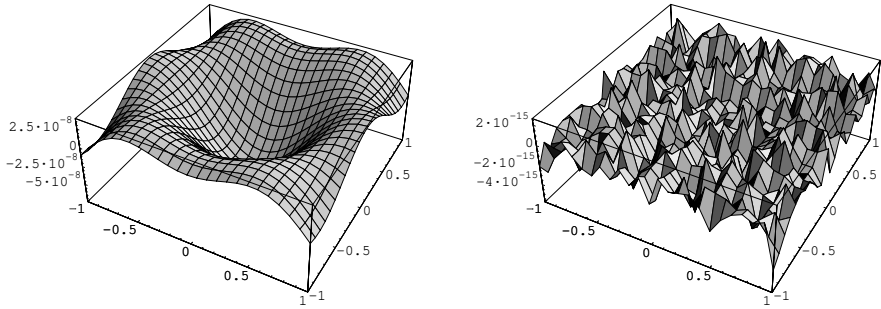


Figure 5: The graph of $\frac{3\sqrt{3}}{4\pi\mathcal{D}} \sum_{\mathbf{y}^\diamond \in \mathbf{X}^\diamond} e^{-|\mathbf{x}-\mathbf{y}^\diamond|^2/\mathcal{D}} - 1$ when $D = 4$ (on the left) and $D = 8$ (on the right).

approximate sufficiently smooth functions u with the error

$$|\mathcal{M}_{h,\mathcal{D}}u(\mathbf{x}) - u(\mathbf{x})| \leq c_{N,\eta} (h\sqrt{\mathcal{D}})^N \|\nabla_N u\|_{L^\infty(\mathbb{R}^n)} + \varepsilon \sum_{k=0}^{N-1} (h\sqrt{\mathcal{D}})^k |\nabla_k u(\mathbf{x})| \quad (10)$$

for any $\varepsilon > 0$.

Let \mathbf{X}_h be a sequence of grids with the property that for $\kappa_1 > 0$ not depending on h and any $\mathbf{y}_j \in \Lambda$ the ball $B(h\mathbf{y}_j, h\kappa_1)$ contains nodes of \mathbf{X}_h .

For example, if η satisfies the conditions of Theorem 2.1, we may assume as Λ the cubic grid $\{\mathbf{j}\}$ or, in the plane, the triangular grid $\{\mathbf{y}^\Delta\}$ or the hexagonal grid $\{\mathbf{y}^\diamond\}$.

In order to construct an approximate quasi-interpolant which use the data at the nodes of \mathbf{X}_h we introduce the following definition.

Definition 3.2. Let $\mathbf{x}_j \in \mathbf{X}_h$. A collection of $m_N = \frac{(N-1+n)!}{n!(N-1)!} - 1$ nodes $\mathbf{x}_k \in \mathbf{X}_h$ will be called *star* of \mathbf{x}_j and denoted by $\text{st}(\mathbf{x}_j)$ if the Vandermonde matrix

$$V_{j,h} = \left\{ \left(\frac{\mathbf{x}_k - \mathbf{x}_j}{h} \right)^\alpha \right\}, \quad |\alpha| = 1, \dots, N-1,$$

is not singular.

Proposition 3.3. Denote by $\tilde{\mathbf{x}}_j \in \mathbf{X}_h$ the node closest to $h\mathbf{y}_j \in h\Lambda$. There exists $\kappa_2 > 0$ such that for any $\mathbf{y}_j \in \Lambda$ the star $\text{st}(\tilde{\mathbf{x}}_j) \subset B(\tilde{\mathbf{x}}_j, h\kappa_2)$ with $|\det V_{j,h}| \geq c > 0$ uniformly in h .

Let us denote by $\{b_{\alpha,k}^{(j)}\}$, $|\alpha| = 1, \dots, N-1$, $\mathbf{x}_k \in \text{st}(\tilde{\mathbf{x}}_j)$, the elements of the inverse matrix of $V_{j,h}$, and consider the functional

$$F_{j,h}(u) = u(\tilde{\mathbf{x}}_j) \left(1 - \sum_{|\alpha|=1}^{N-1} \left(\mathbf{y}_j - \frac{\tilde{\mathbf{x}}_j}{h} \right)^\alpha \sum_{\mathbf{x}_k \in \text{st}(\tilde{\mathbf{x}}_j)} b_{\alpha,k}^{(j)} \right) + \sum_{\mathbf{x}_k \in \text{st}(\tilde{\mathbf{x}}_j)} u(\mathbf{x}_k) \sum_{|\alpha|=1}^{N-1} b_{\alpha,k}^{(j)} \left(\mathbf{y}_j - \frac{\tilde{\mathbf{x}}_j}{h} \right)^\alpha.$$

The functional $F_{j,h}(u)$ depends on the values of u at the nodes of $\text{st}(\tilde{\mathbf{x}}_j) \cup \tilde{\mathbf{x}}_j$ i.e. $m_N + 1$ points close to $h\mathbf{y}_j$.

Let us define the following quasi-interpolant which uses the values of u on \mathbf{X}_h

$$\mathbb{M}_{h,\mathcal{D}}u(\mathbf{x}) = \mathcal{D}^{-n/2} \sum_{\mathbf{y}_j \in \Lambda} F_{j,h}(u) \eta \left(\frac{\mathbf{x} - h\mathbf{y}_j}{h\sqrt{\mathcal{D}}} \right). \tag{11}$$

The following theorem states that, under the above mentioned conditions on the grid, $\mathbb{M}_{h,\mathcal{D}}u$ has the same behavior as in the case of uniform grids.

Theorem 3.4. ([4]) *Under the Conditions 3.1 and 3.3, for any $\varepsilon > 0$ there exists $\mathcal{D} > 0$ such that the quasi-interpolant (11) approximates any $u \in W_\infty^N(\mathbb{R}^n)$ with*

$$|\mathbb{M}_{h,\mathcal{D}}u(\mathbf{x}) - u(\mathbf{x})| \leq c_{N,\eta,\mathcal{D}} h^N \|\nabla_N u\|_{L_\infty(\mathbb{R}^n)} + \varepsilon \sum_{k=0}^{N-1} (h\sqrt{\mathcal{D}})^k |\nabla_k u(\mathbf{x})|,$$

where $c_{N,\eta,\mathcal{D}}$ does not depend on u and h .

One of the motivations of approximate approximations is the construction of cubature formulas for integral operators of convolution type

$$\mathcal{H}u(\mathbf{x}) = \int_{\mathbb{R}^n} k(\mathbf{x} - \mathbf{y})u(\mathbf{y}) d\mathbf{y}. \tag{12}$$

A cubature formula of the multi-dimensional integral (12) can be obtained if the density u is replaced by the quasi-interpolant $\mathbb{M}_{h,\mathcal{D}}u$. Then

$$\begin{aligned} \mathcal{H} \mathbb{M}_{h,\mathcal{D}}u(\mathbf{x}) &= \mathcal{D}^{-n/2} \sum_{\mathbf{y}_j \in \Lambda} F_{j,h}(u) \int_{\mathbb{R}^n} k(\mathbf{x} - \mathbf{y}) \eta \left(\frac{\mathbf{y} - h\mathbf{y}_j}{h\sqrt{\mathcal{D}}} \right) d\mathbf{y} \\ &= h^n \sum_{\mathbf{y}_j \in \Lambda} F_{j,h}(u) \int_{\mathbb{R}^n} k \left(h\sqrt{\mathcal{D}} \left(\frac{\mathbf{x} - h\mathbf{y}_j}{h\sqrt{\mathcal{D}}} - \mathbf{y} \right) \right) \eta(\mathbf{y}) d\mathbf{y} \end{aligned}$$

is a cubature formula for (12) with a generating function η chosen such that $\mathcal{H} \eta$ can be computed analytically or at least by some efficient quadrature method.

In (11) the generating function is centered at the nodes of the uniform grid $h\Lambda$. This can be helpful to design fast methods for the approximation of (12). If we define

$$a_{k-j}^{(h)} = \int_{\mathbb{R}^n} k(h(\mathbf{y}_k - \mathbf{y}_j - \sqrt{\mathcal{D}}\mathbf{y})) \eta(\mathbf{y}) d\mathbf{y}.$$

we reduce to the computation of the following sums

$$\mathcal{K} \mathbb{M}_{h,\mathcal{D}} u(h\mathbf{y}_k) = h^n \sum_{\mathbf{y}_j \in \Lambda} F_{j,h}(u) a_{k-j}^{(h)}$$

which provide an approximation of (12) at the mesh points $h\mathbf{y}_k$.

A generalization of the method approximate approximations to functions with values given on a rather general grid was obtained in [3].

4. Numerical Experiments

The quasi-interpolant $\mathbb{M}_{h,\mathcal{D}} u$ in (11) was tested by one- and two-dimensional experiments and the results of the numerical experiments confirm the predicted approximation orders. In all cases the grid \mathbf{X}_h is chosen such that any ball $B(h\mathbf{j}, h/2)$, $\mathbf{j} \in \mathbb{Z}^n$, $n = 1$ or $n = 2$, contains one randomly chosen node, which we denote by \mathbf{x}_j .

The one-dimensional case. Figures 6 – 9 show the graphs of $\mathbb{M}_{h,\mathcal{D}} u - u$ for different smooth functions u using the basis function $\eta(x) = \pi^{-1/2}e^{-x^2}$ (Fig. 6 and 7) for which $N = 2$, and $\eta(x) = \pi^{-1/2}(3/2 - x^2)e^{-x^2}$ (Fig. 8 and 9) for which $N = 4$, for different values of h . We have chosen the parameter $\mathcal{D} = 4$ in order to keep the saturation error less than 10^{-16} .

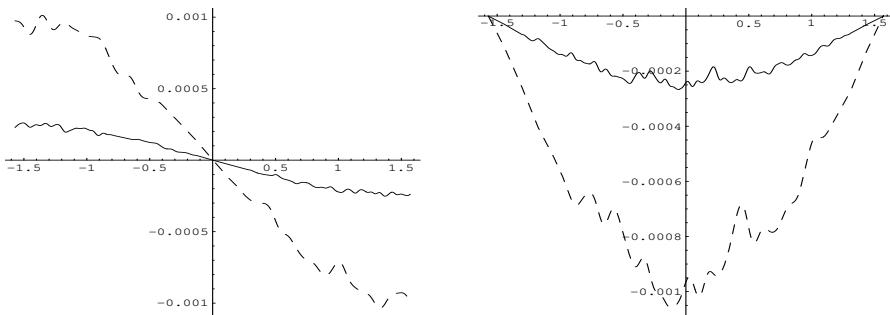


Figure 6: The graphs of $\mathbb{M}_{h,\mathcal{D}} u - u$ with $\eta(x) = \pi^{-1/2}e^{-x^2}$, $\mathcal{D} = 4$, $st(x_j) = \{x_{j+1}\}$, when $u(x) = \sin(x)$ (on the left) and $u(x) = \cos(x)$. Dashed and solid lines correspond to $h = 1/32$ and $h = 1/64$.

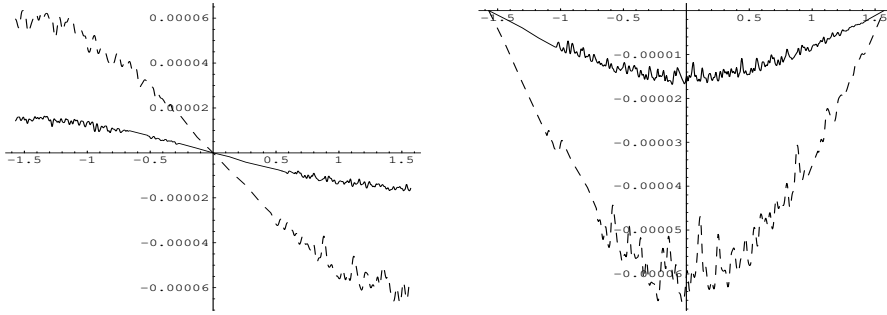


Figure 7: The graphs of $\mathbb{M}_{h, \mathcal{D}} u - u$ with $\eta(x) = \pi^{-1/2} e^{-x^2}$, $\mathcal{D} = 4$, $st(x_j) = \{x_{j+1}\}$, when $u(x) = \sin(x)$ (on the left) and $u(x) = \cos(x)$. Dashed and solid lines correspond to $h = 1/128$ and $h = 1/256$.

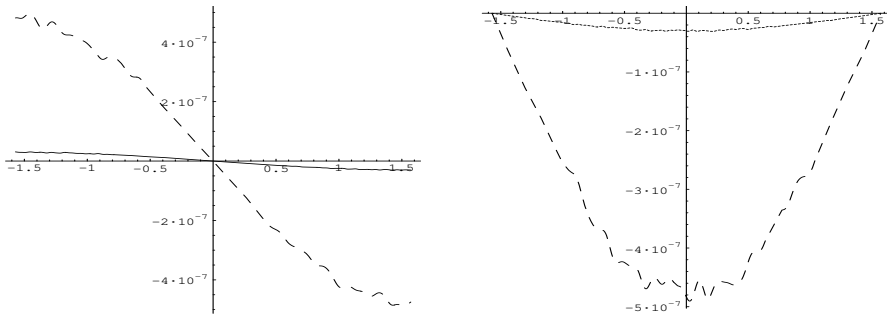


Figure 8: The graphs of $\mathbb{M}_{h, \mathcal{D}} u - u$ with $\eta(x) = \pi^{-1/2} (3/2 - x^2) e^{-x^2}$, $\mathcal{D} = 4$, $st(x_j) = \{x_{j-2}, x_{j-1}, x_{j+1}\}$, when $u(x) = \sin(x)$ (on the left) and $u(x) = \cos(x)$. Dashed and solid lines correspond to $h = 1/32$ and $h = 1/64$.

The two-dimensional case. We depict in Figures 10 and 11 the quasi-interpolation error $\mathbb{M}_{h, \mathcal{D}} u - u$ for the function $u(\mathbf{x}) = (1 + |\mathbf{x}|^2)^{-1}$ and different h if generating functions of second (with $\mathcal{D} = 2$) and fourth (with $\mathcal{D} = 4$) order of approximation are used. The h^2 - and respectively h^4 -convergence of the corresponding two-dimensional quasi-interpolants are confirmed by the L_∞ - errors which are given in Table 1.

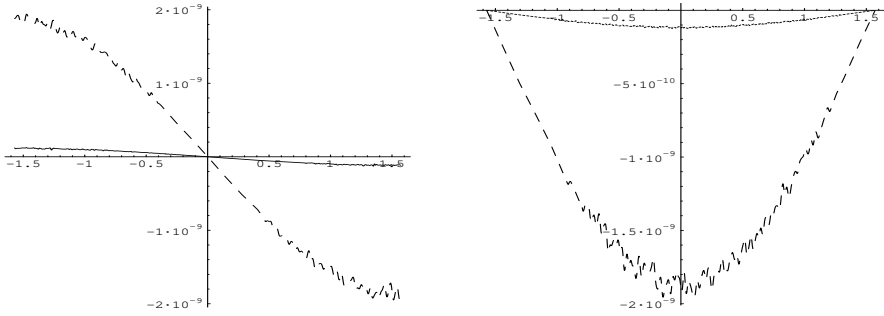


Figure 9: The graphs of $\mathbb{M}_{h, \mathcal{D}} u - u$ with $\eta(x) = \pi^{-1/2}(3/2 - x^2)e^{-x^2}$, $\mathcal{D} = 4$, $\text{st}(x_j) = \{x_{j-2}, x_{j-1}, x_{j+1}\}$, when $u(x) = \sin(x)$ (on the left) and $u(x) = \cos(x)$. Dashed and solid lines correspond to $h = 1/128$ and $h = 1/256$.

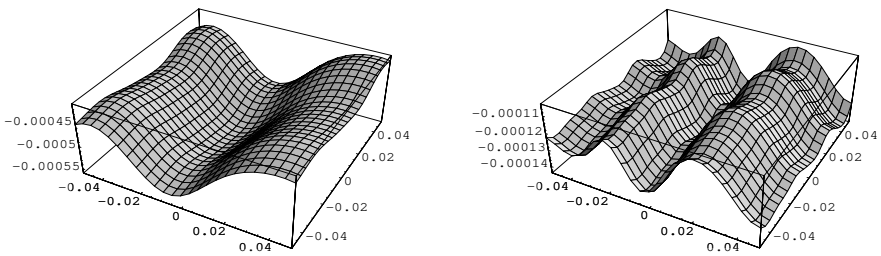


Figure 10: The graph of $\mathbb{M}_{h, \mathcal{D}} u - u$ with $\mathcal{D} = 2$, $\eta(\mathbf{x}) = \pi^{-1}e^{-|\mathbf{x}|^2}$, $N = 2$, $u(\mathbf{x}) = (1 + |\mathbf{x}|^2)^{-1}$, $h = 2^{-6}$ (on the left) and $h = 2^{-7}$ (on the right).

h	$\mathcal{D} = 2$	$\mathcal{D} = 4$
2^{-4}	$8.75 \cdot 10^{-3}$	$1.57 \cdot 10^{-2}$
2^{-5}	$2.21 \cdot 10^{-3}$	$4.00 \cdot 10^{-3}$
2^{-6}	$5.51 \cdot 10^{-4}$	$1.01 \cdot 10^{-3}$
2^{-7}	$1.42 \cdot 10^{-4}$	$2.52 \cdot 10^{-4}$
2^{-8}	$3.56 \cdot 10^{-5}$	$6.50 \cdot 10^{-5}$

h	$\mathcal{D} = 4$	$\mathcal{D} = 6$
2^{-4}	$4.42 \cdot 10^{-4}$	$9.59 \cdot 10^{-4}$
2^{-5}	$2.95 \cdot 10^{-5}$	$6.61 \cdot 10^{-5}$
2^{-6}	$1.92 \cdot 10^{-6}$	$4.24 \cdot 10^{-6}$
2^{-7}	$1.24 \cdot 10^{-7}$	$2.68 \cdot 10^{-7}$
2^{-8}	$7.80 \cdot 10^{-9}$	$1.71 \cdot 10^{-8}$

Table 1: L_∞ approximation error for the function $u(\mathbf{x}) = (1 + |\mathbf{x}|^2)^{-1}$ using $\mathbb{M}_{h, \mathcal{D}} u$ with $\eta(\mathbf{x}) = \pi^{-1}e^{-|\mathbf{x}|^2}$, $N = 2$ (on the left), and $\eta(\mathbf{x}) = \pi^{-1}(2 - |\mathbf{x}|^2)e^{-|\mathbf{x}|^2}$, $N = 4$ (on the right).

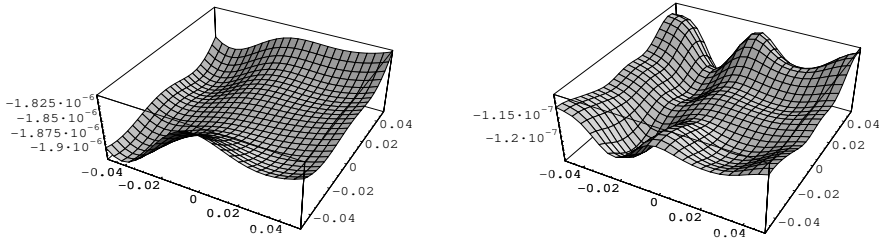


Figure 11: The graph of $\mathbb{M}_h \mathcal{D}u - u$ with $\mathcal{D} = 4$, $\eta(\mathbf{x}) = \pi^{-1}(2 - |\mathbf{x}|^2)e^{-|\mathbf{x}|^2}$, $N = 4$, $u(\mathbf{x}) = (1 + |\mathbf{x}|^2)^{-1}$, $h = 2^{-6}$ (on the left) and $h = 2^{-7}$ (on the right).

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