

## APPROXIMATE BOUNDARY CONDITIONS

by

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Abstract: Approximate boundary conditions are a means for simulating material and surface effects in scattering and propagation. A number of conditions are discussed and criteria given for their validity.

### 1. Introduction

Approximate boundary conditions (hereafter abbreviated as abc's) can be very helpful in simplifying the analytical or numerical solution of wave problems involving complex structures. They are important in all disciplines, e.g. acoustics, hydrodynamics and electromagnetics, where boundary conditions are involved, and are becoming more so as we seek to model more complicated situations. Some versions have also been around for a long time, and while the classical condition  $E_{\text{tan}} = 0$  at the surface of a metal is often regarded as exact, it is in fact an approximation for all metals even at microwave frequencies.

In electromagnetics (to which we shall confine our attention) abc's are now widely used in scattering, propagation and waveguide analyses to simulate the material and geometric properties of the surfaces involved. Take, for example, a finite body immersed in a homogeneous medium and illuminated by an electromagnetic field. Knowing the material properties of the body it is, in principle, possible to find the scattered field outside by taking into account the propagation of the fields throughout the body. Nevertheless, the task would be greatly simplified if these properties could be simulated via a boundary condition involving only the

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external fields imposed at the outer surface, thereby converting a two (or more) media problem into a one medium one. This is the objective of an abc. Though it is generally convenient to take the surface where the abc is applied to coincide with the actual surface of the body, this is not necessary. The only requirement is that in the region of interest the field obtained using the postulated condition approximates the exact field to an adequate degree of accuracy.

If the exact solution of the original problem were known, we could always construct *a posteriori* an equivalent boundary condition which, if imposed at the surface, would reproduce precisely the fields throughout the region exterior to the body. Unfortunately the condition would almost certainly be unique to that particular situation, and to be useful an abc has to be applicable to a class of incident field and body configurations over and beyond the specific one for which it was derived.

Since an abc is needed to solve a problem whose solution cannot (in practice) be obtained without it, its use inevitably entails a certain amount of risk. In any given case there may be several different conditions that could be used, each derived by considering a simplified (canonical) problem whose solution can be found. When applied to the original problem, each will generate a solution having some degree of error. The more complicated the condition is, the greater the (presumed) accuracy of the solution; but the difficulty of computing the solution, and the restrictions on the applicability of the abc, will also be greater. Which condition to use could then be a matter of judgement, and the applicability of any abc can seldom be rigorously justified in advance. Some criteria can be specified that are necessary for an accurate approximation to the field, and others are suggested by the manner in which the abc was derived, but often those conditions which have a physical basis yield accurate results when their applicability cannot be justified. It may then be that

the only justification we have is the degree to which results obtained in analogous situations are supported by experimental and other data. If this seems rather poor grounds for proceeding, it is no worse than for the physical optics method (which is itself based on a degenerate type of abc) whose utility far exceeds its formal validity.

In the following we discuss two different types of conditions: those imposed at a plane interface or the surface of a finite body, and those which are either boundary or jump conditions and can be used to simulate the effect of a layer of electrically small thickness. For simplicity, time harmonic fields are assumed with a time dependence  $e^{i\omega t}$ , and the surrounding medium is treated as free space.

## 2. Planar Surfaces

The most common abc is an impedance boundary condition derived by considering the simple problem of a plane wave incident on a half space. We choose the interface to be the plane  $z = 0$  of a Cartesian coordinate system with  $z$  axis directed out of the material half space. If  $\underline{E}, \underline{H}$  is the field in  $z \geq 0$ , the assumed conditions at  $z = 0$  are then

$$E_x = -\eta H_y, \quad E_y = \eta H_x \quad (1)$$

i.e.,

$$\hat{n} \wedge (\hat{n} \wedge \underline{E}) = -\eta \hat{n} \wedge \underline{H} \quad (2)$$

where  $\hat{n} = \hat{z}$  is the outward unit vector normal. The condition is physically meaningful and can be justified under a variety of circumstances.  $\eta$  is the impedance looking into the half space and is a function of the material properties. In general it is also a function of the incident field direction, but if (2) is to have any reasonable applicability, it is necessary that  $\eta$  be independent of this direction.

If the medium occupying the half space is homogeneous and isotropic with permittivity  $\epsilon$  and permeability  $\mu$ , and if

$$|\epsilon\mu| \gg \epsilon_0\mu_0 \quad (3)$$

i.e., the (complex) refractive index is large in magnitude, it can be shown that [1,2]

$$\eta = Z \quad (4)$$

where  $Z = \sqrt{\mu/\epsilon}$  is the intrinsic impedance of the medium. In effect (3) ensures that the field in the medium is a plane wave propagating in the  $-z$  direction, and (1) then follows from the continuity of  $\hat{n} \wedge \underline{E}$  and  $\hat{n} \wedge \underline{H}$  at the boundary. As required,  $\eta = 0$  corresponds to perfect conductivity, but it is not necessary that  $|\eta|$  be small for the boundary condition to be valid. The abc (2) with (4) is usually attributed to Leontovich (see[3]), and was widely used [4,5] in Russian work on ground wave propagation during WW II.

An obvious generalization of the above result is to an inhomogeneous medium. If the medium is stratified in planes perpendicular to the  $z$  direction,  $\eta$  must be replaced by the impedance of the multilayer structure for incidence in the  $-z$  direction, and the resulting abc (2) is valid when this is also the impedance for all directions of incidence. Thus, for a metal-backed layer of thickness  $d$ ,

$$\eta = iZ \tan(k_0 d \sqrt{\epsilon\mu}) \quad (5)$$

provided (3) is satisfied, where  $k_0$  is the propagation constant of free space. On the other hand, if  $\epsilon$  and/or  $\mu$  vary continuously as functions of  $z$ , Rytov [6] has shown that to a first order the effect is to replace  $\eta$  by  $Z\{1 + O(k_0^{-1} \partial Z/\partial z)\}$ .

Lateral variations as a function of  $x$  and/or  $y$  have more effect, and a key finding [2] is that the first derivatives of  $\epsilon$  and  $\mu$  do not appear explicitly in

the boundary condition. It follows that (2) with (4) is valid as it stands to at least a first order, and can be treated as a local boundary condition with  $\eta = \eta(x,y)$ . In this form it has been used [7], for example, to analyze reflection from a periodically varying substrate beneath a coal layer, and is a vital tool in the design of radar absorbing materials for treating edges. There is no theoretical information about the maximum rate of change of  $\eta$  for which the abc remains valid.

If the material in  $z \leq 0$  is anisotropic so that  $\epsilon$  and/or  $\mu$  are tensors, (2) is still applicable under the conditions stated provided  $\eta$  is treated as a tensor. In particular, if the x and y axes coincide with the principal axes of the tensor, (1) becomes

$$E_x = -\eta_1 H_y, \quad E_y = \eta_2 H_x \quad (6)$$

with  $\eta_1 \neq \eta_2$ .

An alternative use of the impedance boundary condition is to simulate the effect of minor departures of the surface from a plane. For a perfectly conducting surface having small irregularities distributed in a statistically uniform and isotropic manner with rms height  $h$  and correlation length  $\ell$ , the field satisfies (2) at the mean surface  $z = 0$  provided  $k_0 h \ll \sqrt{k_0 \ell}$  and the surface slopes are small [5,8]. Presumably, analogous results could be obtained for a dielectric material. The same is also true for systematic departures from a plane, e.g., corrugations, and the surface can be treated as a planar impedance one provided [9] the spacing of the grooves exceeds their individual width and there are many corrugations per wavelength. The resulting abc (2) is an important tool in the analysis and design of waveguides [10] for low loss, high power applications, and for improving the radiation properties of conical and sectoral horns [11]. With the (tensor) impedance computed from a knowledge of the geometry and the field structure, the theoretical predictions are in good agreement with experimental data.

It might seem that the boundary condition (2) is fundamentally a vector one in that  $\underline{E}$  and  $\underline{H}$  are both involved, but if  $\eta$  is a constant scalar, simple manipulation of (2) or (1) yields [2]

$$\frac{\partial E_z}{\partial z} - ik_0 \frac{\eta}{Z_0} E_z = 0, \quad \frac{\partial H_z}{\partial z} + ik_0 \frac{Z_0}{\eta} H_z = 0 \quad (7)$$

where  $Z_0$  is the intrinsic impedance of free space. These are 'scalar' conditions, each involving only one field component, and have no counterpart where  $\eta$  is a tensor. If, for a homogeneous medium, (3) is relaxed,  $\eta$  will depend on the incident field direction in the manner displayed by the Fresnel reflection coefficients. To provide a better approximation to the field and still have the boundary conditions independent of the incident field direction, Karp and Karal [12] proposed a generalization of (7) in the form

$$\prod_{m=1}^M \left( \frac{\partial}{\partial z} - ik_0 \Gamma_m \right) u = 0 \quad (8)$$

with  $u$  either  $E_z$  or  $H_z$ . By using two or more factors in the product, constant  $\Gamma_m$ 's can be found to improve the simulation of both homogeneous and multi-layer structures.

Recently, the idea has been revived [13] in connection with the use of fictitious boundaries to limit the area of computation in field problems. In effect, the need is for a perfectly absorbing surface. Using methods quite different from those discussed here, a hierarchy of highly absorbing boundary conditions were developed which, for the scalar problem treated, are identical to (8) with  $\Gamma_m = 1$ .

### 3. Curved Surfaces

In writing the abc (1) in the vector form (2) and requiring that the surface impedance  $\eta$  be independent of the incident field direction, an objective was

to obtain a boundary condition that could be applicable to the curved surface of a finite body. The restrictions that must be placed on the type of surface to justify the application are discussed in [1-3]. In addition to (3) it is evident that the penetration depth must be small compared with the minimum thickness and minimum radius of curvature  $\rho$  of the body, and the local wavelength in the material must also be small compared with  $\rho$ . For a lossy homogeneous body, these can be summarized as

$$|\text{Im } N| k_0 \rho \gg 1 \quad (9)$$

where  $N = \sqrt{\epsilon\mu/\epsilon_0\mu_0}$  is the complex refractive index.

Evidence in support of (9) has been obtained from the exact solutions for coated and homogeneous cylinders and spheres [14,15]. If (3) is satisfied, each mode satisfies an impedance boundary condition, but only if (9) is also fulfilled are the modal impedances identical and equal to the planar value. In spite of this, (2) has been applied to edged bodies such as a lossy half plane [16] or coated ogival cylinder with the same impedance used right up to the edge. The results obtained are physically reasonable and in those cases where experimental data are available, the theoretical results give excellent agreement. For the general problem of a real (and irregular) earth, Godziński [17] has developed a number of approximate expressions for the surface impedance to simulate the effects encountered in propagation.

In many instances the numerical solution of a boundary value problem subject to the abc (2) is not significantly harder than in the corresponding perfectly conducting case for which  $\eta = 0$ . It may even be simpler. A non-zero impedance can improve the stability of a numerical solution of an integral equation, and having developed a formulation for one particular (E field) incident polarization, results for the corresponding H field excitation can be obtained using duality ( $\underline{E} \rightarrow \underline{H}$ ,  $\underline{H} \rightarrow -\underline{E}$ ,  $\eta \rightarrow 1/\eta$ ). Although the boundary supports both electric and magnetic surface currents  $\underline{K}$  and  $\underline{K}^*$  respectively, where

$$\underline{K} = \hat{n} \wedge \underline{H} , \quad \underline{K}^* = -\hat{n} \wedge \underline{E} , \quad (10)$$

the two are related via (2):

$$\underline{K} = -\frac{1}{\eta} \hat{n} \wedge \underline{K}^* . \quad (11)$$

It is therefore sufficient to solve for one of them.

Nevertheless, the vector character of the condition (2) can be a major hindrance to an analytical solution. A basic problem such as a wedge or an elliptic cylinder illuminated by a plane wave no longer yields to the method of separation of variables unless  $\eta$  varies in a manner prescribed by the metric coefficients for the appropriate coordinate system. Unfortunately, the scalar equivalents (7) have no counterparts for other than a single plane interface [18].

#### 4. Sheets and Layers

An abc can also be used to simulate a layer of electrically small thickness using a sheet of infinitesimal thickness. An example is a conducting layer whose thickness  $\tau$  is much greater than the penetration depth but much smaller than the free space wavelength, which can be approximated mathematically as a sheet where the condition (2) is imposed. If  $\hat{n}$  is now the outward normal to (say) the upper (positive) side, the sheet supports (total) electric and magnetic currents

$$\underline{J} = \hat{n} \wedge \underline{H} \Big|_{-}^{+} , \quad \underline{J}^* = -\hat{n} \wedge \underline{E} \Big|_{-}^{+} \quad (12)$$

respectively, with

$$\underline{J} = -\frac{1}{\eta} \hat{n} \wedge \left[ \hat{n} \wedge \{ \underline{E}(+) + \underline{E}(-) \} \right] , \quad \underline{J}^* = -\eta \hat{n} \wedge \left[ \hat{n} \wedge \{ \underline{H}(+) + \underline{H}(-) \} \right] , \quad (13)$$

where the symbol  $\Big|_{-}^{+}$  denotes the discontinuity across the sheet. The sheet is, of course, impenetrable, and there is in general no connection between  $\underline{J}$  and  $\underline{J}^*$  analogous to (11).



An impedance sheet is closely related to two other types of sheet both of which are partially transparent. Consider a thin layer of highly conducting material whose permeability is that of free space. If  $\sigma$  is the conductivity, a surface resistance  $R = (\sigma\tau)^{-1}$  can be defined, and as  $\tau \rightarrow 0$  we can imagine  $\sigma$  to increase in such a manner that  $R$  is finite in the limit. The result is a sheet whose electromagnetic properties are specified by the single measurable quantity  $R$ . Though obviously an idealization, sheets are readily available with thickness no more than about 0.1 mm presenting almost constant resistance as high as 1800  $\Omega$ /square over a wide range of frequencies. The precise value depends on the amount of carbon loading employed, and sheets of this type are of great interest for cross section reduction purposes [19].

We can arrive at the same concept using a layer of material whose permittivity is  $\epsilon$  with  $\mu = \mu_0$  as before. If the permittivity is large enough for the normal component of the polarizability current to be neglected, the layer can be approximated [20] by a sheet whose surface "resistance" is

$$R = - \frac{iZ_0}{k_0\tau(\epsilon/\epsilon_0 - 1)} , \quad (14)$$

and this has been used to analyze thin dielectric sheets, lossy and otherwise, in various configurations [20,21].

The above sheets are examples of an electrically resistive sheet whose total current strength is proportional to the tangential electric field at the surface. As originally proposed by Levi-Civita [22], the sheet is characterized by a jump discontinuity in  $\hat{n} \wedge \underline{H}$  across the surface, but no discontinuity in  $\hat{n} \wedge \underline{E}$ . It therefore supports only an electric current, and the conditions that define it are

$$\hat{n} \wedge E|_{-}^{+} = 0, \quad \hat{n} \wedge (\hat{n} \wedge E) = -R \hat{n} \wedge H|_{-}^{+}. \quad (15)$$

When  $R = 0$  the sheet is perfectly conducting and when  $R = \infty$  it is no longer there. In general  $R$  is complex, and if the material comprising the sheet is also anisotropic, e.g. if parallel wires are embedded in a dielectric,  $R$  will also be a tensor.

The electromagnetic dual is a 'magnetically conductive' sheet having conductivity  $R^*$ , a realization of which could be a layer of material having permeability  $\mu$  with  $\epsilon = \epsilon_0$ . The conditions at its surface are the dual of (15), namely

$$\hat{n} \wedge H|_{-}^{+} = 0, \quad \hat{n} \wedge (\hat{n} \wedge H) = R^* \hat{n} \wedge E|_{-}^{+}. \quad (16)$$

When  $R^* = 0$  the sheet looks like a 'perfect ferrite' whose permeability is infinite, and when  $R^* = \infty$  it no longer exists.

Planar sheets of these types are complementary in the sense that a generalization of Babinet's principle can be developed for them [23]. Though each is partially transparent, the superposition of a resistive sheet having  $R = \eta/2$  and a conductive one with  $R^* = (2\eta)^{-1}$  is mathematically equivalent [23] to an impedance sheet with surface impedance  $\eta$ . A set of sheets spaced slightly apart has been suggested as a model for a graded absorber.

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References

- [1] M. A. Leontovich, "Investigations on radiowave propagation, part II," Academy of Sciences, Moscow, 1948.
- [2] T.B.A. Senior, "Impedance boundary conditions for imperfectly conducting surfaces," Appl. Sci. Res., vol. 8(B), 418-436, 1960.
- [3] V. A. Fock, "Solution of the problem of propagation of electromagnetic waves along the earth's surface by the method of parabolic equation," J. Phys. USSR, vol. 10, 13-24, 1946.
- [4] Ia. L. Alpert, "Application to losses in waveguides," J. Tech. Phys. USSR, vol. 10, 1358- , 1940.
- [5] E. L. Feinberg, "On the propagation of radio waves along an imperfect surface," J. Phys. USSR, vol. 8, 317-330, 1944.
- [6] S. M. Rytov, J. Exp. Theor. Phys. USSR, vol. 10, No. 2, 180, 1940 (in Russian).
- [7] J. R. Wait and D. C. Chang, "Theory of electromagnetic scattering from a layered medium with a laterally varying substrate," Radio Sci., vol. 11, No. 3, 221-229, 1976.
- [8] T.B.A. Senior, "Impedance boundary conditions for statistically rough surfaces," Appl. Sci. Res., vol. 8(B), 437-462, 1960.
- [9] M. S. Narasimhan and V. V. Rao, "Radiation characteristics of corrugated E-plane sectoral horns," IEEE Trans. Antennas Propagat., vol. AP-21, No. 3, 320-327, 1973.
- [10] R. B. Dybdal, L. Peters, Jr., and W. H. Peake, "Rectangular waveguides with impedance walls," IEEE Trans. Microwave Theory Tech., vol. MTT-19, No. 1, 2-9, 1971.
- [11] M. S. Narasimhan and M. S. Sheshadri, "GTD analysis of the radiation patterns of wide-flare corrugated conical and E-plane sectoral horns," IEEE Trans. Antennas Propagat., vol. AP-27, No. 2, 276-279, 1979.

- [12] S. N. Karp and F. C. Karal, Jr., "Generalized impedance boundary conditions with applications to surface wave structures," in "Electromagnetic Wave Theory (Pt. I)," ed. J. Brown, Pergamon Press, London, 1967.
- [13] B. Engquist and A. Majda, "Absorbing boundary conditions for the numerical simulation of waves," *Math. Comput.*, vol. 31, No. 139, 1-24, 1977.
- [14] J. R. Wait, "Exact surface impedance for a cylindrical conductor," *Electron. Lett.*, vol. 15, No. 20, 659-660, 1979.
- [15] J. R. Wait, "Exact surface impedance for a spherical conductor," *Proc. IEEE*, vol. 68, No. 2, 279-281, 1980.
- [16] T.B.A. Senior, "Diffraction by a semi-infinite metallic sheet," *Proc. Roy. Soc. (London)*, vol. 213(A), 436-458, 1952.
- [17] Z. Godziński, "The surface impedance concept and the structure of radio waves over real earth," *Proc. IEE (London)*, Part C, vol. 108, 362-373, 1961.
- [18] T.B.A. Senior, "Some problems involving imperfect half planes," in "Electromagnetic Scattering," ed. P.L.E. Uslenghi, Academic Press, NY, 1978.
- [19] T.B.A. Senior, "Backscattering from resistive strips," *IEEE Trans. Antennas Propagat.*, vol. AP-27, No. 6, 808-813, 1979.
- [20] R. F. Harrington and J. R. Mautz, "An impedance sheet approximation for thin dielectric shells," *IEEE Trans. Antennas Propagat.*, vol. AP-23, No. 4, 531-534, 1975.
- [21] T.B.A. Senior and H. Weil, "Electromagnetic scattering and absorption by thin walled dielectric cylinders with application to ice crystals," *Appl. Opt.*, vol. 16, 2979-2985, 1977.
- [22] H. Bateman, "Electrical and optical wave motion," Cambridge University Press, Cambridge, 1915 (p. 19).
- [23] T.B.A. Senior, "Some extensions of Babinet's principle in electromagnetic theory," *IEEE Trans. Antennas Propagat.*, vol. AP-25, No. 3, 417-420, 1977.