

Article

Approximate Conservation Laws of Nonvariational Differential Equations

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Abstract: The concept of an approximate multiplier (integrating factor) is introduced. Such multipliers are shown to give rise to approximate local conservation laws for differential equations that admit a small perturbation. We develop an explicit, algorithmic and efficient method to construct both the approximate multipliers and their corresponding approximate fluxes. Our method is applicable to equations with any number of independent and dependent variables, linear or nonlinear, is adaptable to deal with any order of perturbation and does not require the existence of a variational principle. Several important perturbed equations are presented to exemplify the method, such as the approximate KdV equation. Finally, a second treatment of approximate multipliers is discussed.

Keywords: differential equations; approximate symmetries; multipliers; conservation laws

1. Motivation

The last three decades have seen the development of approximate Lie symmetries [1–3] and their subsequent applications to differential equations admitting a perturbation term. This of course led to a significant extension to approximate Noether symmetries [4] and their associated approximate conservation laws. Naturally, these methods then instigated numerous investigations into the approximate symmetry properties of perturbative equations across all scientific spheres [5–11]. Recently, we have explored the connection between geometry and approximate metrics [12] in the context of heat conduction, and in [13] we proved that approximate symmetries exist if and only if the metric that defines Kinetic energy admits a nontrivial Homothetic algebra. Following this, we established explicit and generalized geometric conditions for a family of Lagrangians that occupy a special place in the literature surrounding perturbation theory [14].

In contemporary analysis, conservation laws are fundamental mathematical expressions that have far and wide applicability. In the context of partial differential equations, they feature in the construction of robust numerical schemes, contribute to questions about existence and uniqueness of solutions and are effective predictors of linearization [15–17]. The investigation of conservation laws of the Korteweg-de Vries (KdV) equation initiated a discovery of several methods, namely the Miura transformation, inverse scattering and bi-Hamiltonian structures [18]. In consequence, a significant problem is how to determine the conservation laws for given differential equations in general, and in particular, for equations that do not possess variational principles and/or include perturbative terms.

There are now several techniques widely applied in the literature that systematically derive conservation laws, inter alia, the direct construction method, the first homotopy (integral) formula,

the second homotopy formula and the scaling formula—all of which have been successfully automated by a symbolic program in Maple [19].

Generally speaking, there is no specific criterion, which deals with approximate conservation laws of perturbative nonvariational equations, i.e., approximate equations that do not possess a Lagrangian. The premise of the approximate conservation laws reported above, depends on the application of Noether’s theorem [20], where the existence of a Lagrangian is essential. Noether proved that for self-adjoint or variational partial differential equation systems, conservation laws arise from Noether symmetries. However, multitudes of partial differential equations that model natural phenomena are not variational, including those that are perturbative.

For unperturbed differential equations without a variational principle, the available methods for obtaining conservation laws are, but not limited to: solving the divergence expression by integration, the standard multiplier approach [21,22], the multiplier approach on the solutions of the equation [23], the partial Noether approach [24], a direct construction method [25] and very significantly, the newly proposed mixed method [26]. The latter can characterize which symmetry, if any, is accountable for a given conservation law. The multiplier approach in particular, has played a crucial role in several contexts [27–31]. Moreover, some recent studies of exact multipliers and conservation laws can be found in [32–36] and references therein.

We seek to formulate a direct and schematic method—one without the involvement of an action principle to find the conservation laws of a given system of perturbed differential equations. To achieve this we prove that nontrivial approximate conserved quantities on the solution space of a given differential equation can be constructed from approximate multipliers (analogous to integrating factors), where the approximate multiplier depends on a predetermined set of variables and derivatives. At the forefront, the distinguishing property of the multipliers is that they must be a divergence expression for all functions of the dependent variable(s) and not just on the solutions of the differential system.

Consider the generalized KdV equation (the original KdV equation was presented in [37])

$$u_t + uu_x + u_{xxx} = 0. \tag{1}$$

The KdV equation has been discussed extensively in the literature, especially because of its conservation laws that arise by virtue of a recursion operator [38]. It is therefore only appropriate that we demonstrate the importance of our work with the KdV equation, coupled with a small perturbation, to address the problem of approximate conservation laws. This equation, without any perturbation, admits the multipliers [25]

$$\Lambda_{e1} = 1, \quad \Lambda_{e2} = u, \quad \Lambda_{e3} = \frac{1}{2}u^2 + u_{xx}, \tag{2}$$

$$\Lambda_{e4} = ut - x. \tag{3}$$

Each of these multipliers give rise to a nontrivial conservation law. For example, if u is regarded as wave amplitude in the KdV equation, then the multipliers $\Lambda_{e1} - \Lambda_{e3}$ give rise to conserved quantities representing the physical principles of mass, momentum and energy [39]. Of course, some conservation laws do not have a physical interpretation, but may have mathematical relevance, such as the conserved vector admitted by the multiplier (3).

If the KdV equation is perturbed with the addition of a small dissipative term,

$$u_t + uu_x + u_{xxx} - \varepsilon u_{xx} = 0, \tag{4}$$

at first sight it is natural to proceed with the standard multiplier approach, which assumes that ϵ is simply a constant. More precisely, this assumption leads to only one multiplier, namely $\Lambda_{\epsilon 1}$. The remaining multipliers (and their associated conservation laws) are lost!

Indeed, as illustrated later in the examples of this paper, if Equation (4) is treated as a perturbed equation with perturbation parameter ϵ and subjected to our proposed method, we not only recover the lost multipliers and conservation laws, but find additional approximate conservation laws.

At this stage, we emphasize that the most prominent characteristic of multipliers is that they may be derived for equations that may or may not possess a variational principle and, each multiplier will generate a nontrivial conservation law. In this paper, we stipulate that multipliers involve a perturbation, expanded into a series, to a specified order of a perturbative parameter.

Our proposed method may be categorized into three steps: firstly, compute multipliers of the given equation by setting the perturbed term as zero, i.e., we refer to this step as the establishment of multipliers in the classical sense, and secondly, solve a divergence expression involving the multiplier of step one and a different multiplier that arises from a perturbation series to some specified order in the perturbation parameter. This procedure results in a number of approximate multipliers. Thirdly, to demonstrate the advantages of approximate multipliers, we connect these multipliers to approximate nonvariational conservation laws. Essentially, the calculation of approximate conservation laws is reduced to solving a divergence expression in terms of approximate multipliers.

We note that preliminary studies have shown that there exists a formal relationship between multipliers and (adjoint) symmetries, explained as follows: multipliers can be characterized as adjoint symmetries that satisfy an adjoint invariance condition [25], and in turn, if the adjoint symmetry determining equation is identical to the symmetry determining equation, then adjoint symmetries are symmetries. If this is the case, the given partial differential equation must possess a variational principle.

Hence this contribution provides a method to construct multipliers and conservation laws of approximate equations. To this date, the only other method in this regard is the approximate Noether’s theorem—which only applies to approximate equations that possess a Lagrangian. Our method has no such limitations. The paper is structured as follows: in the next section we propose the relevant theory on approximate multipliers and some important properties. Section 3 is devoted to the construction of approximate conservation laws, whereby the approximate multipliers of Section 2 are integral to this task. Concluding remarks are given in Section 4. In the Appendix A, we discuss an alternate formulation of multipliers for perturbative equations.

2. Theoretical Formulation of Approximate Multipliers

Firstly, for the reader’s convenience, we present the existing theory concerning the multiplier approach and thereafter introduce our method, for approximate multipliers. The indices α and i sum over $1, \dots, m$ and $i = 1, \dots, n$ respectively, unless otherwise stated. Consider an r th-order (in derivatives) system of partial differential equations of n independent variables $x = (x^1, x^2, \dots, x^n)$ and m dependent variables $u = (u^1, u^2, \dots, u^m)$,

$$G^\alpha(x, u, u_{(1)}, \dots, u_{(r)}) = 0, \tag{5}$$

where $u_{(1)}, u_{(2)}, \dots, u_{(r)}$ denote the collections of all first, second, \dots , r th-order partial derivatives.

Definition 1. The total differentiation operator D_i with respect to x^i is given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \dots, \tag{6}$$

Definition 2. The Euler operator or variational derivative is given by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u_{i_1 \cdots i_s}^\alpha}. \tag{7}$$

Definition 3. A current $\mathbf{T} = (T^1, \dots, T^n)$ is conserved if it satisfies

$$D_i T^i = 0, \tag{8}$$

along the solutions of (5). Equation (8) is called a local conservation law.

2.1. Classical Multipliers

The theory of multipliers, and their application in determining evolutionary symmetries and conservation laws for differential equations, has been concretized in the literature [21,22]—we outline the method here.

Definition 4. The multipliers Λ^α of (5) satisfies the relation

$$D_i T^i = \Lambda^\alpha G^\alpha, \tag{9}$$

for all functions u^α .

In Equation (9), the conservation law is written in characteristic form and the Λ^α are also referred to as characteristics.

Definition 5. The overdetermined equations for Λ^α are

$$\frac{\delta}{\delta u^\alpha} [\Lambda^\alpha G^\alpha] = 0. \tag{10}$$

Thus, this method relies on the fact that the variational derivative annihilates a total divergence.

Example 1. Amongst ordinary differential equations of second-order, consider a type of oscillation equation with a small parameter,

$$u'' + \omega^2 u + \epsilon M(x, u, u_x) = 0, \quad u = u(x). \tag{11}$$

Such an equation encompasses many important perturbative equations, such as the approximate Ermakov-Pinney equation, van der Pol equation, Duffing equation, etc. Suppose we first neglect the perturbative term and apply the classical multiplier approach. We choose the dependence of Λ to be $\Lambda = \Lambda(x, u)$. Thus, by the application of Definition 5, and separation by equating the coefficients of u , we find the determining system:

$$\begin{aligned} \Lambda_{uu} &= 0, & 2\Lambda_u &= 0, & 2\Lambda_{ut} &= 0, \\ \Lambda_u \omega^2 u + \Lambda \omega^2 + \Lambda_{tt} &= 0. \end{aligned} \tag{12}$$

The above system is overdetermined in the unknown function Λ , and elementary solving of (12) provides that the multipliers of (11), to the specified form of Λ , are

$$\Lambda(x, u) = C_1 \sin(\omega x) + C_2 \cos(\omega x), \tag{13}$$

where C_{1-2} are arbitrary constants.

We may split (13) as follows: $\Lambda_a = \sin(\omega x)$, $\Lambda_b = \cos(\omega x)$.

Example 2. Let us consider the nonlinear wave equation with a small perturbation

$$\square u + \lambda u^3 + \varepsilon F(u) = 0, \tag{14}$$

where \square is the d'Alembertian, λ is an arbitrary constant, $u = u(x, t)$ and $F(u)$ is an arbitrary smooth function. As before, we choose to ignore the perturbation term.

Suppose we select the dependence of Λ to be $\Lambda = \Lambda(x, t, u, u_t, u_x)$. Thus, by the application of Definition 5, we find the determining system:

$$\begin{aligned} &\Lambda_{u_x u_x} - \Lambda_{u_t u_t} = 0, \\ &-\lambda u^3 \Lambda_{u_t u_t} + \Lambda_{uu_x} u_x - \Lambda_{uu_t} u_t - \Lambda_{tu_t} + \Lambda_{xu_x} - 2\Lambda_u = 0, \\ &-2\lambda u^3 \Lambda_{u_x u_t} - 2\Lambda_{uu_x} u_t + 2\Lambda_{uu_t} u_x - 2\Lambda_{tu_x} + 2\Lambda_{xu_t} = 0, \\ &-\lambda u^3 \Lambda_{u_x u_x} + \Lambda_{uu_x} u_x - \Lambda_{uu_t} u_t - \Lambda_{tu_t} + \Lambda_{xu_x} + 2\Lambda_u = 0, \\ &-\lambda u^3 u_x \Lambda_{uu_x} - \lambda u^3 u_t \Lambda_{uu_t} - \lambda u^3 \Lambda_{tu_t} - \lambda u^3 \Lambda_{xu_x} + (u_x^2 - u_t^2) \Lambda_{uuu} \\ &\quad - 2\Lambda_{tu} u_t + 2\Lambda_{xu} u_x - \Lambda_{tt} + \Lambda_{xx} + \lambda u^3 \Lambda_u - 3\Lambda_{u_x} \lambda u^2 u_x \\ &\quad - 3\Lambda_{u_t} \lambda u^2 u_t + 3\Lambda \lambda u^2 = 0. \end{aligned} \tag{15}$$

Proceeding in the same way as described in Example 1, we find the solution of (15) provides that the multipliers of (14), to the specified form of Λ , are

$$\Lambda(x, t, u, u_t, u_x) = (C_1 t + C_3) u_x + (C_1 x + C_2) u_t, \tag{16}$$

where C_{1-3} are arbitrary constants.

We may split (16) as follows $\Lambda_c = tu_x + xu_t$, $\Lambda_d = u_t$, $\Lambda_e = u_x$.

Results, by way of the above method, will be referred to as exact multipliers since they are calculated from an unperturbed system (as defined below).

2.2. Approximate Multipliers

We begin this subsection by proposing the necessary theory in order to define approximate multipliers. In addition to the method developed below, a second treatment of approximate multipliers is exhibited in Appendix A.

To start with, the multiplier Λ is expanded in a perturbation series, that is, we shall discuss deformations of the multiplier up to first-order in the perturbation parameter ε , but the theory is easily generalized to higher-order perturbations.

Consider the system of approximate partial differential equations, perturbed up to first-order of precision in the perturbation parameter ε ,

$$\bar{G}^\alpha \equiv G_0^\alpha(x, u, u_{(1)}, \dots, u_{(r)}) + \varepsilon G_1^\alpha(x, u, u_{(1)}, \dots, u_{(r)}) \approx 0, \quad \alpha = 1, \dots, m. \tag{17}$$

Definition 6. A current $\mathbf{T} = (T_0^1 + \varepsilon T_1^1, \dots, T_0^n + \varepsilon T_1^n)$ is conserved if it satisfies

$$D_i T^i = o(\varepsilon), \tag{18}$$

along the solutions of (17).

Theorem 1. *The multipliers*

$$\bar{\Lambda}^\alpha = \Lambda_0^\alpha + \varepsilon \Lambda_1^\alpha + O(\varepsilon^2), \tag{19}$$

of (17) satisfies the relation

$$D_i T_0^i = \Lambda_0^\alpha G_0^\alpha, \tag{20}$$

$$D_i T_1^i = \Lambda_0^\alpha G_1^\alpha + \Lambda_1^\alpha G_0^\alpha, \tag{21}$$

for all functions u^α .

Proof. Equation (20) and (21) are obtained by substituting (17) and (19) into the determining Equation (9) and considering the result in the first-order of precision with respect to ε . \square

Corollary 1. *The conservation law (18) is equivalent to*

$$D_i (T_0^i + \varepsilon T_1^i) = \varepsilon^2 (-\Lambda_1^\alpha G_1^\alpha) \tag{22}$$

Proof. A substitution of Equation (20) and (21) of Theorem 1 into the LHS of Equation (22), followed by a replacement of G_0^α with $-\varepsilon G_1^\alpha$ from Equation (17), yields the desired result. \square

It should be clear that Equation (22) is zero, up to order ε .

Comparing Equation (20) with the expression of exact multipliers Equation (9), we obtain the following.

Theorem 2. *If Equation (17) admits an approximate multiplier $\bar{\Lambda}^\alpha = \Lambda_0^\alpha + \varepsilon \Lambda_1^\alpha$, where $\Lambda_0^\alpha \neq 0$, then the multiplier Λ_0^α is an exact multiplier of the equation*

$$G_0^\alpha(x, u, u_{(1)}, \dots, u_{(r)}) = 0. \tag{23}$$

Remark 1. *Based on Equation (20) and (21), if Λ_0^α is an exact multiplier of Equation (23), then the multipliers (19) satisfying Equation (21) are called first-order approximate multipliers admitted by Equation (17). If T_0^i is an exact conserved quantity of Equation (23), then the $T^i = T_0^i + \varepsilon T_1^i$ are called first-order approximate conserved quantities of Equation (17).*

Accordingly, application of the variational derivative, analogous to Equation (10), provides the determining equations for the multipliers, viz.

$$\frac{\delta}{\delta u^\alpha} [\bar{\Lambda}^\alpha \bar{G}^\alpha] = o(\varepsilon). \tag{24}$$

Hence, Theorem 1 and Equation (24) provide a scheme for calculating approximate multipliers (19) for differential equations with a small parameter. Implementation of the method requires the following two steps.

1st step. Calculation of the exact multipliers Λ_0^α of the unperturbed equation Equation (23), by solving the determining equation:

$$\frac{\delta}{\delta u^\alpha} [\Lambda_0^\alpha G_0^\alpha] = 0, \tag{25}$$

2nd step. Calculation of the Λ_1^α by solving the determining equation for deformations:

$$\frac{\delta}{\delta u^\alpha} \left[\Lambda_0^\alpha G_1^\alpha + \Lambda_1^\alpha G_0^\alpha \right] = 0. \tag{26}$$

Definition 7. Equation (23) and (17) are termed as the unperturbed equation and the perturbed equation, respectively.

Mimicking the terminology from approximate symmetry theory, we establish the following definitions.

Definition 8. $\bar{\Lambda}^\alpha = \Lambda_0^\alpha + \varepsilon \Lambda_1^\alpha$ is a trivial multiplier if $\Lambda_0^\alpha = 0$.

Definition 9. If the approximate multiplier $\bar{\Lambda}^\alpha = \Lambda_0^\alpha + \varepsilon \Lambda_1^\alpha$, is such that the exact multiplier $\Lambda_0^\alpha \neq 0$, we say that the multiplier Λ_0^α of the unperturbed equation is stable with respect to the perturbation considered. Hence, the perturbed Equation (17) inherits the multipliers of the unperturbed Equation (23).

Examples of Approximate Multipliers

Example 3. Consider the introductory example, the perturbed KdV Equation (4). The exact multipliers (2)–(3) admitted by this equation may be written in the form (B_{1-4} are arbitrary constants),

$$\Lambda_0(x, t, u, u_x, u_t, u_{xx}) = B_1 ut + \frac{1}{2} B_2 u^2 + B_2 u_{xx} - B_1 x + B_3 u + B_4. \tag{27}$$

Using (26), we have the determining equation for $\Lambda_1(x, t, u, u_x, u_t, u_{xx})$:

$$\frac{\delta}{\delta u} \left[\Lambda_0(-u_{xx}) + \Lambda_1(u_t + uu_x + u_{xxx}) \right] = 0,$$

which is equivalent to the determining system:

$$\begin{aligned} &\Lambda_{1,u_t} = 0, \quad \Lambda_{1,u_{xx}u_{xx}} = 0, \\ &-u_{xx}\Lambda_{1,u_xu_{xx}} - u_x\Lambda_{1,u,u_{xx}} - 2\Lambda_{1,u_x} - \Lambda_{1,xu_{xx}} = 0, \\ &-4u_{xx}\Lambda_{1,xu_xu_{xx}} - 4u_x\Lambda_{1,x,u,u_{xx}} - 2u_{xx}^2\Lambda_{1,u_xu_xu_{xx}} - 2\Lambda_{1,xxu_{xx}} \\ &-4u_{xx}\Lambda_{1,u_xu_x} - 4\Lambda_{1,xu_x} - 4u_x\Lambda_{1,u,u_x} - 2u_{xx}\Lambda_{1,u,u_{xx}} = 0, \\ &-\Lambda_{1,u_xu_xu_x}u_{xx}^3 - 4u_{xx}(-1/2uu_x - 1/2u_t)\Lambda_{1,xu_xu_{xx}} \\ &-2u_{xx}^2(-1/2uu_x - 1/2u_t)\Lambda_{1,u_xu_xu_{xx}} - 3\Lambda_{1,xu_xu_x}u_{xx}^2 - B_2u_x^2 \\ &-3\Lambda_{1,uu}u_xu_{xx} + 3\Lambda_{1,u_{xx}}u_xu_{xx} - 6\Lambda_{1,xuu_x}u_xu_{xx} - \Lambda_{1,xxx} - 3\Lambda_{1,xuu}u_x^2 \\ &+ (2uu_{xx} + 2u_x^2)\Lambda_{1,xu_{xx}} - 3\Lambda_{1,xxu_x}u_{xx} - 4u_x(-1/2uu_x - 1/2u_t)\Lambda_{1,xuu_{xx}} - \\ &\Lambda_{1,xu} + (-2B_1t - 2B_2u - 2B_3)u_{xx} + (uu_x + u_t) \\ &\Lambda_{1,xxu_{xx}} + (-uu_xu_{xx} - u_tu_{xx})\Lambda_{1,u_xu_x} + (-uu_x - u_t)\Lambda_{1,xu_x} \\ &+ (-uu_x^2 - u_tu_x - 3u_{xx}^2)\Lambda_{1,uu_x} + (2uu_{xx}^2 + 2u_x^2u_{xx})\Lambda_{1,u_xu_{xx}} \\ &+ (3uu_xu_{xx} + 2u_x^3 + u_tu_{xx})\Lambda_{1,u,u_{xx}} + (-2uu_{xx} - u_x^2)\Lambda_{1,u_x} - \Lambda_{1,t} \\ &-3\Lambda_{1,u_x}u_{xx} - \Lambda_{1,uuu}u_x^3 - 3\Lambda_{1,u_{xx}}u_x = 0. \end{aligned} \tag{28}$$

The solution of this system is possible if $B_2 = 0$, which leads us to

$$\begin{aligned} \Lambda_1(x, t, u, u_x, u_t, u_{xx}) = &\frac{1}{2} (2B_1t^2 + 4B_3t + C_1)u^2 \\ &+ \frac{1}{2} ((-4B_1x - 2C_2)t - 4xB_3 + 2C_3)u \\ &+ 2B_1t^2u_{xx} + 4B_3tu_{xx} + B_1x^2 + C_1u_{xx} \\ &+ C_2x + C_4, \end{aligned} \tag{29}$$

where C_{1-4} are arbitrary constants. Thus, the first-order nontrivial approximate multipliers are

$$\begin{aligned} \bar{\Lambda}_{a1} &= u + \varepsilon(2tu^2 - 2xu + 4tu_{xx}), \\ \bar{\Lambda}_{a2} &= ut - x + \varepsilon(t^2u^2 - 2uxt + 2t^2u_{xx} + x^2), \end{aligned} \tag{30}$$

while the trivial first-order approximate multipliers are

$$\begin{aligned} \bar{\Lambda}_{a3} &= \varepsilon, \\ \bar{\Lambda}_{a4} &= \varepsilon u, \\ \bar{\Lambda}_{a5} &= \varepsilon(ut - x), \\ \bar{\Lambda}_{a6} &= \varepsilon\left(\frac{1}{2}u^2 + u_{xx}\right). \end{aligned} \tag{31}$$

Example 4. Consider an equation that arises in various applied problems [40],

$$u_{tt} + \varepsilon u_t = (\phi(u) u_x)_x. \tag{32}$$

Following the classification given in [1] (some variations of this model exist and can be found in, for instance [41]), the approximate symmetry algebra may be enlarged according to various functions of $\phi(u)$. We are interested to see if the number of approximate multipliers increase with these choices of $\phi(u)$, and if so, what are these approximate multipliers explicitly. Application of the method described above and as illustrated in the previous examples, the results of this example are summarized in Table 1, i.e., multipliers of Equation (32) in exact and approximate constituents are displayed. For example, for $\phi(u) = ku^{-4}$, Equation (32) admits two exact multipliers $\Lambda_0 = tx$, $\Lambda_0 = x$; three approximate parts that are $\Lambda_1 = tx$, $\Lambda_1 = x$, and $\Lambda_1 = \frac{1}{2}t^2x$, so that the first-order nontrivial approximate multiplier, corresponding to B_1 is

$$\bar{\Lambda}_{b1} = tx + \varepsilon \frac{1}{2}t^2x,$$

and the trivial approximate multipliers are

$$\bar{\Lambda}_{b2} = \varepsilon tx \text{ and } \bar{\Lambda}_{b3} = \varepsilon x.$$

The remaining cases in Table 1 are analyzed similarly but the lengthy details are omitted. Additionally, we observe that Equation (32) with $\phi(u)$ arbitrary, admits the approximate multipliers if $B_5 = 0$.

Table 1. Comparative table of exact and approximate multipliers (k, σ, C_i, B_i are arbitrary parameters).

	$\phi(u)$	Λ_0	Λ_1
1	Arbitrary function	$(B_1t + B_2)x + tB_3 + B_4 + \frac{u_x u_t B_5}{(-u_x^2 \phi(u) + u_t^2)^2}$	$\left(\frac{1}{2} B_1 t^2 + C_1 t + C_2\right) x + \frac{1}{2} B_3 t^2 + C_3 t + C_4 + \frac{u_t C_5 u_x}{(-u_x^2 \phi(u) + u_t^2)^2}$
2	ku^σ	$(B_1x + B_3)t + xB_2 + B_4$	$\frac{1}{2} (B_1x + B_3) t^2 + (C_1x + C_3) t + xC_2 + C_4$
3	$ku^{-4/3}$	$(B_1x + B_2) t$	$\frac{t}{2} [(B_1x + B_2) t + 2C_1x + 2C_2]$
4	ku^{-4}	$(B_1t + B_2) x$	$\left(\frac{1}{2} B_1 t^2 + C_1 t + C_2\right) x$

Example 5. Recall the nonlinear wave Equation (14) with exact multipliers (16). Ultimately, the above procedure shows that Equation (14) only admits trivial first-order approximate multipliers.

3. Approximate Conserved Quantities

As previously stated, every multiplier gives rise to a conservation law. Likewise, every approximate multiplier $\bar{\Lambda}^\alpha = \Lambda_0^\alpha + \varepsilon \Lambda_1^\alpha$ will give rise to an approximate conservation law (18), see Definition 6. In order to calculate the approximate conservation laws, we have to compute T_1^i . To this end, it is the solution of Equation (21), from Theorem 1 that provides T_1^i , a solution that is obtainable by integration. In situations of complicated integration, it is fruitful to adopt an ansatz concerning the functional dependence of the conserved fluxes, i.e., if the maximal order of derivatives present in the multipliers is l and the maximal order of the given equation is h , then without loss of generality, one may assume that the fluxes contain derivatives up to order p , where $p = \max(l, h)$ [42].

To exemplify the construction of our approximate conservation laws, a particular case is discussed below.

Example 6. Suppose we revisit the KdV Equation (4) to illustrate the admitted approximate conservation laws. For the first-order nontrivial approximate multiplier $\bar{\Lambda}_{a1}$ in (30), Equation (21) becomes

$$D_t T_1^t + D_x T_1^x = -uu_{xx} + \left(2tu^2 - 2xu + 4tu_{xx}\right) (u_t + uu_x + u_{xxx}). \tag{33}$$

For the solution of Equation (33), we may use the ansatz described above or in this example, manipulation and standard integration of (33) easily provides the T_1^i components:

$$\begin{aligned} T_1^t &= -2tu_x^2 + \frac{2}{3}tu^3 - xu^2, \\ T_1^x &= 2tu^2u_{xx} - 2xuu_{xx} + 2tu_{xx}^2 + uu_x + 4tu_xu_t + xu_x^2 + \frac{1}{2}tu^4 - \frac{2}{3}xu^3. \end{aligned}$$

The exact components of the conservation laws are given by solving Equation (20),

$$D_t T_0^t + D_x T_0^x = u (u_t + uu_x + u_{xxx}), \tag{34}$$

for which we find

$$\begin{aligned} T_0^t &= \frac{1}{2}u^2 + u^2tu_x, \\ T_0^x &= uu_{xx} - \frac{1}{2}u_x^2 - u^2tu_t. \end{aligned}$$

Together, they form the approximate conserved vector $I(a1) : (T_0^t + \varepsilon T_1^t, T_0^x + \varepsilon T_1^x)$. A quick check of Corollary 1, verifies that

$$D_t (T_0^t + \varepsilon T_1^t) + D_x (T_0^x + \varepsilon T_1^x) = \varepsilon^2 \left(u_{xx} \left(2tu^2 - 2xu + 4tu_{xx} \right) \right) \approx 0,$$

up to order ε , along the solutions of Equation (4).

Similarly for $\bar{\Lambda}_{a2}$ in (30), the approximate conserved vector components are

$$\begin{aligned}
 I(a2) : \quad T^t &= -\frac{1}{2}u^2t^2u_x + u_x + uu_xxt + \varepsilon \left(-t^2u_x^2 + \frac{1}{3}t^2u^3 - xt^2u^2 + x^2u \right), \\
 T^x &= \frac{1}{2}u_x^2t + u_{xx}x - u_x - utxu_t - uu_{xx}t + \frac{1}{2}u^2t^2u_t \\
 &\quad + \varepsilon \left(t^2u^2u_{xx} - 2xtuu_{xx} + t^2u_{xx}^2 + x^2u_{xx} + tuu_x - xu_x + 2t^2u_xu_t \right. \\
 &\quad \left. + xtu_x^2 + \frac{1}{4}t^2u^4 - \frac{2}{3}xtu^3 + \frac{1}{2}x^2u^2 + u \right),
 \end{aligned}$$

and for the trivial approximate multipliers (31) we find, respectively,

$$\begin{aligned}
 I(e_1) : \quad T^t &= \varepsilon (uu_xt + u), \\
 T^x &= \varepsilon (-uu_t + u_{xx}), \\
 I(e_2) : \quad T^t &= \varepsilon T_0^t, \\
 T^x &= \varepsilon T_0^x, \\
 I(e_3) : \quad T^t &= \varepsilon \left(\frac{1}{6}u^3 + \frac{1}{2}u_xtu^3 - \frac{1}{2}u_x^2 \right), \\
 T^x &= \varepsilon \left(\frac{1}{2}u^2u_{xx} + u_xu_t + \frac{1}{2}u_{xx}^2 - \frac{1}{2}u^3u_t \right), \\
 I(e_4) : \quad T^t &= \varepsilon \left(-\frac{1}{2}u^2t^2u_x + ux + uu_xxt \right), \\
 T^x &= \varepsilon \left(\frac{1}{2}u_x^2t + u_{xx}x - u_x - utxu_t - uu_{xx}t + \frac{1}{2}u^2t^2u_t \right).
 \end{aligned}$$

as conserved quantities.

4. Summary and Concluding Remarks

The conservation laws or quantities left dynamically invariant form important foundations at a mathematical and physical level in describing model dynamics. The chief motivation of this work was to develop a systematic procedure to obtain approximate conservation laws of differential equations that admit a small perturbation, without the requirement of a variational principle. One of the basic reasons of obtaining multipliers is to derive conservation laws of classical equations—in essence, this idea led us to the notion of an approximate multiplier of their perturbed counterpart.

Once established, the knowledge of an approximate multiplier, because of its connection to the total divergence, allowed us to derive a construction formula for approximate conservation laws. This formula, in the determination of approximate conservation laws requires standard integration. The method, benchmarked with a procedure for first-order perturbations, in principle, is applicable to problems with arbitrarily many equations, functions, variables and perturbations.

We highlight that approximate multipliers may be more useful as compared to approximate Noether symmetries—in the sense that they may be applied to equations that may or may not possess Lagrangians. In addition, the presence of a perturbation in a differential equation can lead to an increase in the complexity of the multiplier computations. However, in terms of practical application of the method, due to its algorithmic nature, it may be easily implemented into symbolic software. Indeed, we have designed an interactive Maple worksheet that automates many of the computations [43].

As a final comment, we observe that there is a second approach to approximate multipliers, which we discuss in Appendix A.

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Appendix A. Approximate Multipliers via Perturbed Dependent Variable/s

The approximate method proposed by Fushchych and Shtelen [2] uses a perturbative technique whereby the dependent variable/s are expanded in a perturbation series before being substituted into the given equation; this results in a coupled system. As expected, this makes for some tedious computations. This section is aimed at providing explicit examples of extending this method to multipliers. In this regard, the approximate multipliers of a given differential system are the exact multipliers of the new coupled system.

Example A1. Consider the van der Pol equation, which has been used historically to illustrate many methods of perturbation theory in the literature, that is, let $\omega = 1$ and $M = (1 - u^2) u'$ in (11), i.e.,

$$u'' + u - \varepsilon (1 - u^2) u' = 0. \tag{A1}$$

Expanding the dependent variables to the 1st order in ε , we have $u = u_0 + \varepsilon u_1$, and substituting into (A1) gives

$$\begin{aligned} O(1) : \quad & u_0'' + u_0 = 0, \\ O(\varepsilon) : \quad & u_1'' + u_1 - u_0' + u_0' u_0^2 = 0, \end{aligned} \tag{A2}$$

by the separation of order ε . The approximate multipliers of the differential Equation (A1) are the exact multipliers of the coupled system (A2). We select the dependency of the multipliers to be

$$\Lambda_1 = \Lambda_1(x, u_0, u_1)$$

and

$$\Lambda_2 = \Lambda_2(x, u_0, u_1).$$

The next step is to apply the classical multiplier method, Equations (9) and (10),

$$\frac{\delta}{\delta u_0} \left[\Lambda_1 (u_0'' + u_0) + \Lambda_2 (u_1'' + u_1 - u_0' + u_0' u_0^2) \right] = 0, \tag{A3}$$

and

$$\frac{\delta}{\delta u_1} \left[\Lambda_1 (u_0'' + u_0) + \Lambda_2 (u_1'' + u_1 - u_0' + u_0' u_0^2) \right] = 0, \tag{A4}$$

which produces a determining system by Definition 5 (we omit the details since this type of calculation has been exemplified above). Once solved, we find the multipliers

$$\begin{aligned} \Lambda_1(x, u_0, u_1) &= -u_1 C_1 + \sin(x) C_2 + \cos(x) C_3, \\ \Lambda_2(x, u_0, u_1) &= u_0 C_1, \end{aligned}$$

where C_{1-3} are arbitrary constants.

Example A2. Recall the nonlinear wave Equation (14) of Example 2. Expanding the dependent variables to the 1st order in ϵ , we have $u = u_0 + \epsilon u_1$, and substituting into (14), gives

$$\begin{aligned} O(1) : \quad & \square u_0 - \lambda u_0^3 = 0, \\ O(\epsilon) : \quad & \square u_1 - 3\lambda u_0^2 u_1 - F(u_0) = 0, \end{aligned} \tag{A5}$$

once we separate by order of ϵ . The approximate multipliers of the differential Equation (14) are the exact multipliers of the coupled system (A5). Suppose, we let the multipliers be of the form

$$\Lambda_1 = \Lambda_1(x, t, u_0, u_{0,x}, u_{0,t}, u_{0,tt}, u_1, u_{1,x}, u_{1,t}, u_{1,tt})$$

and

$$\Lambda_2 = \Lambda_2(x, t, u_0, u_{0,x}, u_{0,t}, u_{0,tt}, u_1, u_{1,x}, u_{1,t}, u_{1,tt}),$$

where $u_{\mu,\nu} = \frac{\partial u_\mu}{\partial x^\nu}$.

The application of the classical multiplier method gives

$$\frac{\delta}{\delta u_0} \left[\Lambda_1 (\square u_0 - \lambda u_0^3) + \Lambda_2 (\square u_1 - 3\lambda u_0^2 u_1 - F(u_0)) \right] = 0, \tag{A6}$$

and

$$\frac{\delta}{\delta u_1} \left[\Lambda_1 (\square u_0 - \lambda u_0^3) + \Lambda_2 (\square u_1 - 3\lambda u_0^2 u_1 - F(u_0)) \right] = 0, \tag{A7}$$

which produces a determining system by Definition 5 as before. Once solved, we find the multipliers

$$\begin{aligned} \Lambda_1 &= x(u_{0,t}C_2 + u_{1,t}C_1) + tC_2u_{0,x} + C_6u_{0,t} + C_3u_{0,x} + C_4u_{1,t} \\ &\quad + (C_1t + C_5)u_{1,x}, \\ \Lambda_2 &= (C_1t + C_5)u_{0,x} + (C_1x + C_4)u_{0,t}, \end{aligned}$$

where C_{1-6} are arbitrary constants.

Conservation laws corresponding to these multipliers correspond to Equation (8) in Definition 3, and such calculations are straightforward.

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