Report No. 67-17B
April 1967

APPROXIMATE CONTINUOUS NONLINEAR MINIMAL-VARIANCE FILTERING

by

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FOREWORD

The research described in this report, "Approximate Continuous Nonlinear Minimal-Variance Filtering," Number 67-17B, by Lawrence Schwartz, was carried out under the direction of C. T. Leondes, E. B. Stear, and A. R. Stubberud, in the Department of Engineering, University of California, Los Angeles.

This project is sponsored in part by the National Aeronautics and Space Administration under Grant NsG-237-62 to the Institute of Geophysics and Planetary Physics of the University.

This report was the basis for a dissertation submitted by the Author.

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1. INTRODUCTION

1.1 HISTORICAL BACKGROUND

Continuous minimal-variance filtering is a form of sequential stochastic estimation, and, as such, has its roots in the early least-squares differential correction schemes for orbit determination. A fairly extensive account of the development of orbit determination methods can be found in Deutsch [11] and in Mowery [30]. Continuous optimal filtering, per se, dates back to Wiener [38], wherein the minimal-variance stochastic estimation problem is solved for the linear filtering of stationary random signals. For many years thereafter, the terms "optimal filtering" and "Wiener filtering" were used interchangeably. The first investigations into the problem of the optimal filtering of nonstationary signals and nonlinear filtering were reported about a decade later; e.g., Laning [27], Zadeh [42], Zadeh and Ragazzini [43] and Booton [5]. The feature common to the early filtering studies is the derivation of an integral equation for the optimal filter. In the special case of linear filtering for stationary statistics, the integral equation can be solved in a useful form for many applications; unfortunately, the same cannot be said for the more general cases, although Booton [5] does refer to some applications.

In 1958* Wiener [39] suggested an approach to solving the filtering problem for the general stationary-statistics case which is mechanizable and has been developed further at M. I. T. (e. g., Chesler [8], Hause [17], and Schetzen [31]) under Y. W. Lee. There are related investigations by Balakrishnan [1] and others; see Fisher [13] for further discussion. However, the nonstationary-statistics case remained essentially unsolved in a practical sense, until the revolutionary works of Kalman [22] and Kalman and Bucy [23], and the pioneering papers by Stratonovich [35, 36], all between 1959 and 1961.

Kalman and Bucy published a practical solution to the optimal linear filtering of random processes with nonstationary statistics. In order to obtain their solution, they abandoned the nearly fruitless integral equation approach and reformulated the problem so that the filter is specified by a differential equation which can be mechanized on a computer. Because of the utility of the approach the term "Kalman filter" replaced "Wiener filter" as a synonym to "optimal filter." In fact, the finite dimensional Wiener filter is just a special case of the Kalman filter. However, the Kalman filter is rigorously valid only for linear filtering, even though successful nonlinear extensions were developed heuristically for orbit determination and space navigation programs.

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*Apparently, the original announcement was made in 1949 in an internal M. I. T. memorandum.

The true nonlinear minimal-variance filter follows from the suggestion of Stratonovich that the fundamental entity in sequential estimation is the conditional probability density function of the message process given the measurement process. It is elementary to show that the minimal-variance estimate of a random variable given a set of measurements is simply the conditional expectation, which, in turn, is the first moment of the conditional density. Thus, if the density is known as a function of time, so is the minimal-variance estimate. The first efforts to exploit the idea by deriving a partial differential equation for the density, Stratonovich [36], Kashyap [24], and Wonham [41], all contained errors, and the first correct formulation is due to Kushner [26] in 1964, though his derivation is nonrigorous and incomplete and contains a mistake in the filter derivation. About a year later Bucy [7] published a note presenting an alternate and apparently more rigorous derivation including a derivation containing an algebraic mistake for an approximate filter for a scalar example. Later the same year Bass, Norum and Schwartz [2] corrected Bucy's example and extended the approximate filter equations to the vector case; an independent similar investigation was conducted by Swerling [37]. In early 1966 Jazwinsky [21] presented a derivation for a similar approximation assuming the measurements are taken at discrete points in time, and Schwartz and Bass [32] derived a filter for a different approximation from that used by Bucy [7]. The last paper in the sequence stemming from

Kushner [26] is also by Kushner* in which he points out that the derivation by Bucy [7] of the exact filter equation is not completely rigorous; he also proves a theorem giving sufficiency conditions for the validity of the exact filter equation, which is a total differential equation for the conditional mean.

There are two closely related works not in the mainstream outlined in the preceding paragraph: Mortensen [29] and Fisher [13]. In the first part of his derivation, Mortensen derives an integral equation for a quantity related to the conditional density function under fairly restrictive conditions. In his analysis he, too, questions the rigor in Bucy [7]. The approach taken by Fisher is more like that of Kushner [26]; he employs nonrigorous limiting arguments to construct a partial differential equation for the conditional density, but for a more general class of stochastic processes than are considered by the previous authors. He also derives a set of filter equations based on a somewhat different representation of the conditional density function; his filter equations contain those in Schwartz and Bass [32] as a special case.

In addition, there are alternative approaches to nonlinear filtering based on other criteria, for example: Bryson and Frazier [6], Cox [9], Bellman, Kagiwada, Kalaba, and Sridhar [3], Detchmendy and Sridhar [10], as well as others concerned with more specialized problems.

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*Dynamical Equations for Optimum Nonlinear Filtering: Unpublished memorandum.

1.2 PROBLEM CONSIDERED

The problem considered in this dissertation is the rigorous validation of the approximate filter equations. An outline of the steps in the derivation is essential to set the stage for the following discussion. The first step in the analysis of any physical situation is the specification of a mathematical model; the choice should be made carefully, since the whole analysis depends upon the characteristics of the model. For minimal variance, which is a probabilistic criterion of optimality, the manipulations leading to the filter equations are made particularly simple by assuming that the random processes are white noises.* On the other hand, the white-noise assumption adds a certain amount of complication in the interpretation of the mathematical results in light of reality. In the present investigation the white-noise assumption is made.

If the criterion of optimality is statistical, rather than probabilistic, the white-noise assumption is not germane; in fact, no particular form is postulated for the random processes. The original least-squares estimation is such a statistically optimal approach. The recent studies of Cox [9] and Detchmendy and Sridhar [10] are modern versions of the statistically optimal filter. The analytical portion of this dissertation does not consider the non-probabilistic filters, but both types of filters are simulated.

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*The definition of white noise is presented in Chapter 2.

Given the probabilistic criterion and the white-noise assumption, a natural mathematical model is the stochastic differential equation. Of course, the problem must be such that the stochastic differential equation satisfies existence and uniqueness conditions, which are different from those pertaining to non-stochastic differential equations. The essential difference stems from the fact that for white-noise models there is no bound on the forcing function and global conditions must be satisfied. From the stochastic differential equations for the system and the measurement, it is possible to derive a stochastic partial differential equation for the conditional density function; but there are no conditions yet established for the validity of the equation for the density. From the stochastic partial differential equation in turn, it is possible to derive a stochastic differential equation for the expected value of any scalar function of the state of the system; this last equation is the one which Kushner considers in his unpublished memorandum. Since the approximate filter equations are derived from the stochastic differential equations for the expectation, the lack of rigor in the derivation of the equation for the density causes no real difficulty.

The exact equation for the filter requires the instantaneous evaluation of the conditional expectation of several functions of the state of the system. To simplify the problem, the original model can be replaced by an approximating stochastic differential equation, from which an approximate filter can be derived more simply. It seems reasonable to require that the approximate model equations also satisfy existence and uniqueness conditions, particularly uniqueness; if not, the solution to the approximate system may in no way approximate the solution to the original problem. Since in no previous derivation does the approximation satisfy such conditions, the first part of the validation of the filter equations is to derive an approximation that does satisfy the desired conditions, and to find the resulting filter.

Guaranteeing existence and uniqueness for the approximate system does not quite do the same for the filter, but slightly stronger conditions on the equations suffice. At this point, one step remains in the validation: that of relating the stochastic differential equation for the filter to an ordinary differential equation for the actual mechanization. In doing so, it is shown that nonvalid filter equations similar to those previously derived by Schwartz and Bass [32] and by Fisher [13] for white-noise processes can be made computationally identical to valid filter equations if mechanized on a digital computer.

1.3 OUTLINE OF THE DISSERTATION

Chapters 2 and 3 are included for completeness, and may be skipped without loss of continuity by readers familiar with the material. Chapter 2 is an outline of the definition of stochastic differential equations, the conditions for existence and uniqueness, and related topics. Chapter 3 is a review of the derivation of the filter equations as presented in Bass, Norum and Schwartz [2] as modified by Schwartz and Bass [32].

The new theoretical results, as outlined in the preceding section, are given in Chapter 4, while the analysis pertaining to the computer simulations and the results thereof comprise Chapter 5. The discussion of the filter simulations is not intended to be exhaustive, in any sense of the word; rather it is an exploratory introduction to the ill-defined area of the dynamic characteristics of and computational considerations for nonlinear filters.

The final conclusions and recommendations for further work are presented in Chapter 6.

2. MATHEMATICAL MODEL

2.1 INTRODUCTION

The usual mathematical formulation for a dynamical problem is a differential equation, nowadays most generally written in state-vector form:

$$\frac{dx}{dt} = f\big(t, x(t), u(t)\big) \quad , \tag{1}$$

where x and f are n-vectors, u is an m-vector, and t is a scalar. The meaning of (1) is well known for most input functions u, but the ensuing analysis deals with white-noise input functions, and (1) must be reinterpreted. The present chapter explains the problems associated with white-noise inputs, and outlines the development of the necessary calculus of stochastic processes. The reader should be familiar with probability theory and random processes.

The stochastic calculus began essentially with Wiener in the early 1920's, although stochastic differential equations were first studied by Bernstein [4] a decade later. The foremost contributor to the theory of stochastic differential equations is Itô [18, 12, 20], though the present exposition follows the approach of Skorokhod [34], which is easier to follow.

*See Koval'chik [25] for further details.

The presentation opens with a discussion of white noise from an engineering point of view to help motivate the subsequent mathematical analysis, which might otherwise seem somewhat contrived. The following mathematical discussion avoids appeals to measure theory as much as possible despite the fact that probability theory is often treated as a branch of measure theory. It is hoped that the approach used allows the results to reach a larger audience. Also, the various theorems and the properties of stochastic integrals and differential equations are stated without proof; the proofs are available elsewhere, and the appropriate references are cited.

2.2 WHITE NOISE

A typical engineering treatment of white noise can be found in Laning and Battin [28] , p. 136ff, wherein it is stated that a white noise is a random process with a power spectral density which is a constant, or, equivalently, an autocorrelation function which is a Dirac δ-function. It is further noted that such a process has no physical meaning since it would require infinite signal power. The foregoing definition is valid for stationary white noise, though the autocorrelation-function definition can be extended to the nonstationary case by allowing a time-varying coefficient for the δ-function.

The nonrealizability of white-noise processes is no reason to discard them; they occupy a place with respect to the family of stochastic processes analogous to the place of the Dirac δ-function with respect to functions. Just as the δ-function can pragmatically be regarded as a limit, in some sense, of a sequence of unit-area pulses of decreasing width, a white-noise process can be considered as the limit, in a similar sense, of a sequence of processes which are step functions. Moreover, both are useful only when their integrals are considered; indeed, both can be made mathematically rigorous only in terms of their integrals. To be somewhat imprecise, the δ-function may be considered as the derivative of a unit step; with similar imprecision Laning and Battin [28] show that white noise is the derivative of Brownian motion, which they define as a one-dimensional random walk.

The practical reason for being concerned with white-noise processes is that, when differential equations are forced by white noise, the solutions are Markov processes, i.e., the future is independent of the past. In other words, the solutions to differential equations forced by white noise exhibit the stochastic analogue of the property of solutions to differential equations forced by ordinary functions: given the state of the solution at some time and the forcing function from that time on, the subsequent evolution of the solution is stochastically independent of the previous history.

The mathematical problem associated with white noise is somewhat similar to that associated with the δ-function: the meaning of the integral. The δ-function is not really a function in the ordinary sense of the word, and no theory of integration can result in a value other than zero for $\int_{-\infty}^{\infty} \delta(t)\,dt$ if the δ-function is assumed to be an ordinary function of t. However, by not ascribing values to $\delta(t)$ and considering only its integral, it is possible to construct a meaningful theory. For example, see Friedman [14] p. 136ff. Similary, no ordinary theory of integration can make sense of $\int w(t)dt$, where w is a white-noise process. Here again, if no instantaneous value is given to w(t), a useful theory of stochastic integration is possible; that theory is outlined in the following section.

2.3 STOCHASTIC INTEGRALS

The exposition in this section is merely an outline of the mathematical derivation of the stochastic integral; and it does not include any discussion of stochastic integrals of discontinuous random processes. A complete discussion of stochastic integration can be found in Skorokhod [34]. The underlying idea is the description of a white-noise process as the derivative of a Brownian motion; the major difficulty lies in the fact that a Brownian motion is almost nowhere differentiable, almost surely.* Thus, if b is a Brownian motion db/dt has no meaning and $\int g(t)(db/dt)\,dt$ is generally not defined, even for continuous functions g. But, if db is an increment of b, it has a well-defined stochastic description, and the Stieltjes integral $\int g(t)db(t)$ is a possibly meaningful alternate form. However, b is not a function of bounded variation, and even the Stieltjes integral is not defined. The stochastic integral is a stochastically meaningful generalization of the Stieltjes integral,

*In essence, the Brownian motion fails to be differentiable somewhere in every interval of nonzero length, with probability one.

in which the approximating sums are required to converge in probability to the integral, rather than to converge in the ordinary sense.

For definiteness, consider $\int_{t_o}^{t_f} g(t)\,dw(t)$ for a vector process g(t) and a Wiener process w(t), where a Wiener process is a unit Brownian motion, i.e., for $t > s$ $\mathcal{E}[w(t) - w(s)] = 0$* and $\mathcal{E}[w(t) - w(s)]^2 = t - s$; furthermore, for $q < r \leq s < t$, $w(t) - w(s)$ is independent of g(r) and of $w(r) - w(q)$. Let M denote the class of functions g such that if g is in M, then g can be assigned a probability; let M_o denote the set of step functions in M ; let M_1 denote the set of functions in M which are mean-square integrable; let M_2 denote the set of functions in M which are square integrable with probability 1. If g is in M_o, there are points $t_o < t_1 < \cdots < t_n = t_f$ such that $g(t) = g(t_i)$ for $t_i \leq t < t_{i+1}$. For such a function it is natural to define

$$\int_{t_o}^{t_f} g(t)\,dw(t) \triangleq \sum_{i=0}^{n-1} g(t_i)[w(t_{i+1}) - w(t_i)] \,. \tag{2}$$

The integral defined by (2) is a linear operation on g. From the aforementioned properties of the Wiener process, it follows that for g in both M_o and M_1

$$\mathcal{E}\int_{t_o}^{t_f} g(t)\,dw(t) = 0 \tag{3}$$

*The symbol $\mathcal{E}$ denotes expectation.

and*

$$\mathcal{E} \left| \int_{t_o}^{t_f} g(t)\, dw(t) \right|^2 = \int_{t_o}^{t_f} \mathcal{E} |g(t)|^2 \, dt. \tag{4}$$

The integral in (2) is meaningful only because of the special form of the functions in M_o; the next step is to extend the definition to all of M_1. It can be shown** that for every g in M_1 and every $\epsilon > 0$, there exists a function $\overline{g}$ in the intersection of M_o and M_1 such that

$$\int_{t_o}^{t_f} \mathcal{E} |g(t) - \overline{g}(t)|^2 \, dt < \epsilon \tag{5}$$

From (5), it follows that for every g in M_1, there exists a sequence $\{g_n\}$ in the intersection of M_o and M_1 such that

$$\lim_{n \to \infty} \int_{t_o}^{t_f} \mathcal{E} |g(t) - g_n(t)|^2 \, dt = 0. \tag{6}$$

But, by linearity and (4), (6) implies

$$\lim_{\substack{n \to \infty \\ m \to \infty}} \mathcal{E} \left| \int_{t_o}^{t_f} g_n(t)\, dw(t) - \int_{t_o}^{t_f} g_m(t)\, dw(t) \right|^2 = 0, \tag{7}$$

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*The vertical bars denote the Euclidean norm.

**Skorokhod [34] , p. 16.

and (7), in turn, implies that the sequence of random variables $\int_{t_o}^{t_f} g_n(t)\,dw(t)$ converges in probability* to some random variable, which is taken as $\int_{t_o}^{t_f} g(t)\,dw(t)$. The integral thus defined for g in M_1 is unique with probability one, is linear, and satisifies (3) and (4).

The final step in the definition is its extension to all of M_2. Let f_N be defined as follows: $f_N(x) = 1$ for $|x| \le N$ and $f_N(x) = 0$ for $|x| > N$. Then for g in M_2, the function

$$g_N(t) = g(t)\, f_N\left(\int_{t_o}^{t_f} |g(s)|^2\, ds\right)$$

is in M_1. It follows** that† for $N' > N$

$$P\left\{\left|\int_{t_o}^{t_f} g_N(t)\,dw(t) - \int_{t_o}^{t_f} g_{N'}(t)\,dw(t)\right| > 0\right\} \le P\left\{\int_{t_o}^{t_f} |g(t)|^2\, dt \ge N\right\}$$

and that

$$\lim_{N\to\infty} P\left\{\int_{t_o}^{t_f} |g(t)|^2\, dt \ge N\right\} = 0.$$

Thus, $\int_{t_o}^{t_f} g_N(t)\,dw(t)$ converges in probability to some random variable which is taken as $\int_{t_o}^{t_f} g(t)\,dw(t)$. This integral exhibits the properties ascribed to the previous integrals.

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*A sequence g_n converges in probability to g if the limit as $n\to\infty$ of the probability that $|g_n - g| > \epsilon$ is zero for any $\epsilon > 0$.

**Skorokhod [34], p. 18-19.

†The symbol P{ } denotes the probability of the event in the braces.

2.4 STOCHASTIC DIFFERENTIAL EQUATIONS

The discussion contained in this section is limited to the following special case of (1):

$$\frac{dx}{dt} = f\big(t,x(t)\big) + g\big(t,x(t)\big)\,u(t), \tag{8}$$

where x and f are n-vectors, t is a scalar, u is an m-vector unit white-noise process with independent elements,* and g is an $n \times m$ matrix. While (8) is less general than (1), it is sufficient for most practical applications. Let g^i denote the i^{th} column of g and w_i the i^{th} row of the vector Wiener process from which u is derived. Then by formal multiplication of (8) by dt and integration of the resulting expression, (8) can be rewritten as

$$x(t) - x(t_o) = \int_{t_o}^{t} f\big(s,x(s)\big)\,ds + \sum_{i=1}^{m} \int_{t_o}^{t} g^i\big(s,x(s)\big)\,dw_i(s). \tag{9}$$

In (9), the first integral is an ordinary integral, and the remaining m integrals are stochastic integrals. For simplicity, if the stochastic integral equation is satisfied by a process x with probability one, then (9) is written in the form

$$dx = f(t,x)\,dt + g(t,x)\,dw(t). \tag{10}$$

The simplified form (10) is referred to as a stochastic differential equation, and is understood to be a shorthand notation for (9). It is assumed that f and all the g^i evaluated along x(t) are in M_2.

*That is, elements of u are derived from independent Wiener processes.

The fundamental existence and uniqueness theorem for the analysis in Chapter 4 is a modification of Theorem 4 on page 56 of Skorokhod [34] :

Theorem. Suppose that $x(t_o)$ is independent of the processes $w_i(t)$ and that $f(t,x)$ and $g^i(t,x)$ are defined for $t_o \leq t \leq t_f$ and for all n-vectors x, are measurable* with respect to all variables, and that they satisfy the following conditions:

1. For every $C > 0$, there exists an L_C such that

$$(t_f - t_o)|f(t,x) - f(t,y)|^2 + \sum_{i=1}^{m} |g^i(t,x) - g^i(t,y)|^2 \leq L_C^2 |x - y|^2$$

   if $|x| \leq C$ and $|y| \leq C$.

2. There exists a K at which

$$(t_f - t_o)|f(t,x)|^2 + \sum_{i=1}^{m} |g^i(t,x)|^2 \leq K(|x|^2 + 1).$$

   In such a case, (9) has a bounded continuous solution with probability one; also, if there are two solutions, with probability one both coincide at all points t.

The proof is essentially that in the reference; boundedness still follows from Theorem 3, p.51 of the reference, but continuity follows from Theorem 3, p.21 of the reference. For simplicity in the sequel, condiion 1 is referred to as the local Lipschitz condition, and condition 2 as sublinearity.

*Measurability is a regularity condition that functions of engineering interest will satisfy.

The next result is necessary for the derivation of stochastic differential equations for functions of solutions of other stochastic differential equations. The scalar version of the formula is proved in Skorokhod [34] on p. 24ff; the vector version follows quite simply. Let x satisfy (10) for $t_o \le t \le t_f$, with each f_i, g_{ij}, and $g_{ij}g_{k\ell}$ belonging to M_2. If a scalar function $\phi(t,x)$ is defined and continuous and has a continuous derivative with respect to t and continuous second cross partial derivatives with respect to the x_i for $t_o \le t \le t_f$ and for all x, then the process $y(t) = \phi\big(t,x(t)\big)$ satisfies the relation

$$dy = \left(\frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial x} f + \frac{1}{2} g^* \frac{\partial^2\phi}{\partial x^2} g\right) dt + \frac{\partial\phi}{\partial x} g\, dw , \qquad (11)$$

where $\partial(\ )/\partial x$ denotes the gradient (row) vector, $\partial^2(\ )/\partial x^2$ denotes the Hessian (matrix of cross partials), and the asterisk denotes matrix transpose.

2.5 RELATION TO THE PHYSICAL PROBLEM

There are two interfaces between the physical situation and the mathematical model in the filtering problem: The reduction of the dynamics to a stochastic differential equation and the interpretation and mechanization of the stochastic differential equation for the filter as a computational algorithm. The second interface is considered first. The interpretation of the stochastic integral in the context of the actual estimation environment is not at all a trivial matter. The filtering algorithm will be a finite-difference approximation to (8), with u represented by a sampled measurement, not a white noise. The problem of the interpretation of (8) and the approach to use for the integration is discussed at length in Gray and Caughey [16] ; they specify two approaches and propose a list of four pragmatic rules for choosing between the two approaches based on the interpretation of (8). Another treatment of the difference between the two approaches is given in Wong and Zakai [40] .

The real difference between the approaches is in the choice of whether to use the ordinary calculus or the stochastic calculus. In the nomenclature of Gray and Caughey [16] the former choice is the physical approach and the latter choice is the mathematical approach. In contrasting the two approaches, the authors are quick to state that neither approach is inherently correct; the choice should be made according to their pragmatic rules:

1. If $g(t,x)$ is not actually a function of x, both approaches provide identical results.
2. If the problem is a strictly mathematical one, the mathematical approach must be used.
3. If (8) is either an approximation to or a limit of the discrete problem $[x(t_{k+1}) - x(t_k)] / (t_{k+1} - t_k) = f\big(t_k, x(t_k)\big) + g\big(t_k, x(t_k)\big)\, u(t_k)$, then the mathematical approach must be used.

4. If (8) is either an approximation to a white-noise problem or the limit of a problem with short correlation time, then the physical approach must be used.

The computational effect of the difference between the two approaches is stated by Wong and Zakai [40] as follows: Let $\{w^n\}$ be a sequence of piecewise linear approximations to the Wiener process in (10) such that $w^n \to w$; then if $\{x^n\}$ denotes the sequence of corresponding solutions, $x^n \to z$, where z is the solution to

$$
\begin{aligned}
dz_i = {} & f_i\big(t,z(t)\big)\,dt + \frac{1}{2}\sum_{j,k} g_{kj}\big(t,z(t)\big)\,\frac{\partial g_{ij}}{\partial z_k}\big(t,x(t)\big)\,dt \\
& + \sum_j g_{ij}\big(t,z(t)\big)\,dw_j
\end{aligned}
\tag{12}
$$

They state some reservations about the correctness of (12) in the vector case, but the same form is implied by the results of Gray and Caughey [16].

There is also a problem in relating the statistics of the real data to the statistics of the white noise used in the model; this problem exists at both interfaces, and is really the only one at the first. For simplicity, consider the following special case: Let u(t) denote a sequence of pulses of width Δt and of random height given by a Gaussian distribution of zero mean and variance σ^2. The autocorrelation function for u is a triangular spike of width $2\Delta t$ and height σ^2; the area under the spike is then $\sigma^2 \Delta t$. It then seems reasonable that the equivalent white noise be specified by an impulse of weight $\sigma^2 \Delta t$. The foregoing conclusion can also be implied for more general situations by considering the limit process of a sequence of Markov chains;* for example, see Skorokhod [34] , Chapter 6. The general result is as follows: If an n-dimensional white noise is given by a covariance of the form $S(t)\delta(t-\tau)$, an n-dimensional pulse-sequence approximating the process should be chosen from a population given by a covariance of $S(t_i)/\Delta t$ for $t_i \le t < t_i + \Delta t$. The case of continuous u is not quite so direct, though an equivalent formulation can be obtained by using the concept of a correlation time τ, which is a time interval such that $u(t)$ can be considered uncorrelated with $u(t+\tau)$.

Since the mathematical model is constructed under the assumption that the Wiener process has independent elements, one final step is required to model a noise with correlated elements. Let $S(t)\delta(t-\tau)$ be the desired covariance, which implies $S(t)$ is positive semi-definite for all t. Then there exists a matrix (which may be taken as symmetric) $S^{1/2}$ such that $S^{1/2}(S^{1/2})^* = S$. If $dv \triangleq S^{1/2} dw$, the white noise derived from v has the proper covariance. For notational simplicity it may be assumed that $S^{1/2}(t)$ is incorporated into $g(t,x)$, and the formalism of (8), (9), and (10) is still valid.

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*Random sequences exhibiting the Markov property.

It should be mentioned, in conclusion, that the strong conditions for existence and uniqueness are required by the mathematical model for white noise, and not by the equations for the actual physical situation. For an ordinary differential equation, local conditions suffice. However, a stochastic differential is forced by functions that cannot be bounded, and global conditions are required.

3. REVIEW OF PRESENT APPROXIMATE FILTER EQUATIONS

3.1 INTRODUCTION

In this chapter, the derivation of the filter equations in Schwartz and Bass [32] is outlined, using the approach of Bucy [7] to obtain the formula for the exact estimator. Bucy's approach is used because it provides a good example of the use of the stochastic calculus. This chapter is included for completeness, since some of the work is not easily accessible. In particular, Bucy's note is quite sketchy and the report by Schwartz and Bass was not widely distributed.

The derivation consists of two separate parts: the first part treats the exact estimation problem, and the second considers the development of practical approximate estimators. The questions of mathematic rigor of the second part are deferred to Chapter 4, where the new theoretical results are presented.

3.2 PROBLEM STATEMENT

Let the dynamic equation of the system be given by (8), and let the measurement be given by

$$y(t) = h\big(t, x(t)\big) + r(t)\,v(t) \,, \tag{13}$$

where h is an ℓ-vector, $\ell \le n$, v is an ℓ-dimensional unit white noise and r is a nonsingular symmetric $\ell \times \ell$ matrix relating the unit white noise to the modeled white noise (equivalent to the matrix $S^{1/2}$ in Section 5 of Chapter 2). Since the mathematical model cannot handle white noise directly, it is assumed that the measurement is derived from a process z given by

$$dz = h\big(t, x(t)\big)\, dt + r(t)\, db(t) \qquad (14)$$

where b is an ℓ-dimensional Wiener process. The mathematical model of the system consists of the two vector equations (10) and (14). The problem is to find the minimal-variance estimate of x(t) given the process z(s) for $t_o \le s \le t$, that is to find the estimate $\hat{x}(t)$ such that the matrix given by

$$\mathcal{E}(x - \bar{x})(x - \bar{x})^* - \mathcal{E}(x - \hat{x})(x - \hat{x})^*$$

is positive semi-definite, where $\bar{x}$ is any estimate of x, and the processes are evaluated at t.

It is a simple exercise to show that the minimal-variance estimate of a random variable given a related quantity is simply the conditional expectation, so that

$$\hat{x}(t) = \mathcal{E}\big(x(t) \,|\, z(s),\ \ t_o \le s \le t\big), \qquad (15)$$

and the problem is to find an equation for the conditional expectation.

3.3 STOCHASTIC DIFFERENTIAL EQUATION FOR THE CONDITIONAL EXPECTATION*

The approach adopted for the derivation of the exact filter equation requires the existence of the appropriate conditional probability density function $p\big(x(t)|z(s),\ t_o \le s \le t\big)$, which, for simplicity, is denoted $p(x|z)$. Let $p(x|z_n)$ denote $p\big(x(t)|z(s_1), z(s_2), \ldots, z(s_n)\big)$, where no particular ordering is assumed among the s_i, but each s_i is in the interval $[t_o, t]$; the expression for $p(x|z)$ is found as $\lim_{n \to \infty} p(x|z_n)$. The first step is to show that the limit makes sense.

Let $\xi_n \triangleq \mathcal{E}\big(a|z(s_1), z(s_2), \ldots, z(s_n)\big)$, for some random variable a assuming $\mathcal{E}(a)$ exists. It is shown in Doob [12], p. 293, that the random variables $\xi_1, \xi_2, \ldots, a$ constitute a martingale**; furthermore, by Theorem 1.1(ii) of Chapter VII in Doob, the random variables $|\xi_1|, |\xi_2|, \ldots, |a|$ form a semi-martingale. Assume that $x(t)$ is bounded with probability one, and let $a = x(t)$, then $\mathcal{E}|\xi_n|$ is uniformly bounded. By Doob, Chapter VII, Theorem 3.1(i), $\mathcal{E}|\xi_1| \le \mathcal{E}|\xi_2| \le \cdots$, so that the sequence $\{\mathcal{E}|\xi_n|\}$ is bounded and monotone, which implies that there exists a K such that $\lim_{n \to \infty} \mathcal{E}|\xi_n| = K < \infty$. By Doob, Chapter VII, Theorem 4.1, $\lim_{n \to \infty} \xi_n = \xi_\infty$ exists with probability one and $\mathcal{E}|\xi_\infty| \le K$.

Now, let A be a Borel set in a Euclidean n-space and let Ω_A denote the set of elementary events ω such that $x(t, \omega)$ is in A.

---

*This section may be skipped without loss of continuity by readers not interested in the mathematical details of the derivation.

**For the definition and properties of martingales and semi-martingales, see Doob [12], Chapter VII.

Let I_A denote the indicator of Ω_A; then*

$$P\{x(t) \in A | z(s_1), \dots, z(s_n)\} = \mathcal{E}\{I_A | z(s_1), \dots, z(s_n)\}. \qquad (16)$$

Let ξ_n denote the right hand side of (16). Since I_A is either zero or one, $0 \le \xi_n \le 1$ and $\lim_{n \to \infty} \xi_n = \xi_\infty$ exists as a number between zero and one. Thus, (16) implies that the sequence of probabilities on the left-hand side converges as $n \to \infty$. Then, if the conditional probability measures have densities $p(x | z_n)$, $p(x | z)$ exists with probability one if the limit of (16) has a density.

The existence of the densities is guaranteed by the fact that the noise on the measurement is a non-degenerate ℓ-dimensional white-noise process, as is demonstrated by the following construction. The Bayes' rule for conditional densities is

$$p(x | z_n) = \frac{p(z_n | x)\, p(x)}{p(z_n)} \qquad (17)$$

Note that $p(z_n | x)\, p(x) = p(z_n, x)$ and that both $p(z_n)$ and $p(z_n, x)$ can be computed from $p(z_n, x_n) \triangleq p\big(z(s_1), \dots, z(s_n),\ x(s_1), \dots, x(t)\big)$, where now it is assumed that $t_o = s_1 < \cdots < s_n = t$. Indeed**

$$p(z_n, x) = \int_{R_n} \cdots \int_{R_n} p(z_n, x_n)\, dx(s_1) \cdots dx(s_{n-1})$$

$$p(z_n) = \int_{R_n} p(z_n, x)\, dx(t) \qquad (18)$$

*See Doob [12] Chapter I, Section 7.
**R_n is Euclidean n-space.

Also,

$$p(z_n, x_n) = p(z_n | x_n) p(x_n) . \tag{19}$$

From the definition of stochastic integrals and (14)

$$z(s_i) - z(s_{i-1}) = \int_{s_{i-1}}^{s_i} h\big(t, x(t)\big) dt + \int_{s_{i-1}}^{s_i} r(t) \, db(t) . \tag{20}$$

Let $\Delta z_i \triangleq z(s_{i+1}) - z(s_i)$, $\Delta b_i \triangleq b(s_{i+1}) - b(s_i)$, and $\Delta s_i = s_{i+1} - s_i$. Then

$$\Delta z_i = h\big(s_i, x(s_i)\big)\Delta s_i + r(s_i)\Delta b_i + \delta_i , \tag{21}$$

where δ_i is implicitly defined by requiring the right-hand sides of (20) and (21) to be equal. An expression for $p(x|z)$ can be derived under the assumption that δ_i is $o(\Delta s_i)$; the validity of the resulting expression can then be verified.

For simplicity, let $h_i \triangleq h\big(s_i, x(s_i)\big)$, $r_i \triangleq r(s_i)$, and $z_i \triangleq z(s_i)$. Then ignoring terms of $o(\Delta s_i)$,

$$p\big(z_1 = Z_1, z_2 = Z_2, \ldots, z_n = Z_n | x_1 = X_1, x_2 = X_2, \ldots, x_n = X_n\big) =$$

$$= p\big(z_1 = Z_1, \ldots, z_n = Z_{n-1} + h_{n-1}\Delta s_{n-1} + r_{n-1}\Delta b_{n-1} | x_1 = X_1, \ldots, x_n = X_n\big)$$

$$= p\big(z_1 = Z_1, \ldots, \Delta b_{n-1} = r_{n-1}^{-1}(Z_n - Z_{n-1} - h_{n-1}\Delta s_{n-1}) | x_1 = X_1, \ldots, x_n = X_n\big) . \tag{22}$$

Since z_1 is independent of the Δb_i and since the Δb_i are increments of a Wiener process, (22) can be rewritten as

$$p(z_n | x_n) = p(z_1) \prod_{i=1}^{n-1} \frac{1}{\sqrt{2\pi\Delta s_i}} \exp\left\{-\frac{1}{2\Delta s_i} \left| r_i^{-1}(\Delta z_i - h_i \Delta s_i)\right|^2\right\} . \quad (23)$$

where

$$\left| r_i^{-1}(\Delta z_i - h_i \Delta s_i)\right|^2 = \Delta z_i^* r_i^{-2} \Delta z_i + h_i^* r_i^{-2} h_i \Delta s_i^2 - 2h_i^* r_i^{-2} \Delta z_i \Delta s_i \; . \quad (24)$$

Substituting (24) into (23), the resulting expression into (19), and using (18) provides

$$p(z_n, x) = p(z_i) \prod_{i=1}^{n-1} \left(\frac{1}{\sqrt{2\pi\Delta s_i}} \exp\left[-\frac{1}{2} \frac{\Delta z_i^* r_i^{-2} \Delta z_i}{\Delta s_i}\right]\right) \times$$

$$\times \int_{R_n} \cdots \int_{R_n} \left\{ \exp\left[-\frac{1}{2}\sum_{i=1}^{n-1} h_i^* r_i^{-2} h_i \Delta s_i + \sum_{i=1}^{n-1} h_i^* r_i^{-2} \Delta z_i\right] \times \right. \quad (25)$$

$$\left. \times \; p(x_n) \right\} dx(s_i) \cdots dx(s_{n-1}) \; .$$

Let

$$\Phi_n \triangleq -\frac{1}{2} \sum_{i=1}^{n-1} h_i^* r_i^{-2} h_i \Delta s_i + \sum_{i=1}^{n-1} h_i^* r_i^{-2} \Delta z_i \; .$$

By definition the integral in (25) is simply $\mathcal{E}\left(e^{\Phi_n}|x\right)$; whence, using (25), (18), and (17),

$$p(x|z_n) = \frac{\mathcal{E}\left(e^{\Phi_n}|x\right) p(x)}{\mathcal{E}\left(e^{\Phi_n}\right)} \quad . \tag{26}$$

Since $\Phi_n \rightarrow \Phi$ in probability, where

$$\Phi \triangleq -\frac{1}{2}\int_{t_o}^{t} h^* r^{-2} h \, ds + \int_{t_o}^{t} h^* r^{-2} \, dz \ ,$$

it is tempting to postulate that

$$p(x|z) = \frac{\mathcal{E}\left(e^{\Phi}|x\right) p(x)}{\mathcal{E}\left(e^{\Phi}\right)} \quad , \tag{27}$$

where the expectations are now functionspace integrals over the set of functions $\{x(t)\}$. The theory of functionspace integration is discussed in Koval'chik [25] and in Shilov [33], although for present purposes a more specialized paper by Getoor [15] is more to the point. Specifically, he constructs the conditional expectations in (27) for a class of functionals that includes e^{Φ} if Φ is bounded. Since Φ is a solution to a stochastic differential equation, boundedness conditions are given by the theorem in Chapter 2.

For the derivation of the partial differential equation for the density, the equality in (27) may be relaxed; all that is needed is that (27) holds in the following sense:

For arbitrary $\epsilon > 0$

$$P\left\{\left|p(x|z) - \frac{\mathcal{E}(e^{\Phi}|x)\,p(x)}{\mathcal{E}(e^{\Phi})}\right| > \epsilon\right\} = 0 \; . \tag{28}$$

Consider the following plausibility argument for (28): let $q(x|z_n)$ denote the right-hand side of (26) and $q(x|z)$ denote the right-hand side of (27). Then

$$\begin{aligned} |p(x|z) - q(x|z)| \le\ & |p(x|z) - p(x|z_n)| + |p(x|z_n) - q(x|z_n)| \\ & + |q(x|z_n) - q(x|z)| \; . \end{aligned} \tag{29}$$

It has already been established that $p(x|z_n) \rightarrow p(x|z)$ with probability one by a martingale argument. Moreover, the results in Getoor [15] imply $q(x|z_n) \rightarrow q(x|z)$ in probability. It remains to show that $q(x|z_n) \rightarrow p(x|z_n)$ in probability.

If the term δ_i in (21) is not neglected, (24) becomes

$$\begin{aligned} |r_i^{-1}(\Delta z_i - h\Delta s_i - \delta_i)|^2 =\ & \Delta z_i^* r_i^{-2} \Delta z_i + h_i^* r_i^{-2} h_i \Delta s_i^2 + \delta_i^* r_i^{-2} \delta_i \\ & - 2h_i^* r_i^{-2} \Delta z_i \Delta s_i - 2\delta_i^* r_i^{-2} \Delta z_i \\ & + 2\delta_i^* r_i^{-2} h_i \Delta s_i \; . \end{aligned} \tag{30}$$

which results in a new functional for (26)

$$\Psi_n = \Phi_n - \frac{1}{2}\sum_{i=1}^{n-1} \frac{\delta_i^* r_i^{-2} \delta_i}{\Delta s_i} + \sum_{i=1}^{n-1} \frac{\delta_i^* r_i^{-2} \Delta z_i}{\Delta s_i} - \sum_{i=1}^{n-1} \delta_i^* r_i^{-2} h_i \; . \tag{31}$$

The desired result obtains if $\Psi_n - \Phi_n \to 0$ in probability as $n \to \infty$. Substituting (21) into (31) provides

$$\Psi_n - \Phi_n = \frac{1}{2} \sum_{i=1}^{n-1} \frac{\delta_i^* r_i^{-2} \delta_i}{\Delta s_i} + \sum_{i=1}^{n-1} \frac{\delta_i^* r_i^{-1} \Delta b_i}{\Delta s_i} . \tag{32}$$

Now, from the definition of the stochastic integral

$$P \left\{ \left| \sum_{i=1}^{n-1} \delta_i \right| > \epsilon \right\} \to 0 \tag{33}$$

From (33), the independence of the δ_i, and the uniformity of the convergence of (21) to (20), it seems reasonable that (32) vanishes in probability, although no direct proof is apparent in the literature. As noted earlier*, the manipulations resulting from the assumption that (28) holds have been rigorized by Kushner, so that it is not necessary to verify directly that (32) does indeed vanish.

The next step is to formally derive an expression for the stochastic differential of $p(x|z)$. Following Bucy [7], let $p(x|z) \triangleq Q/P$. Also, let $\theta \triangleq \mathcal{E}(e^{\Phi}|x)$. Then Q is explicitly a function of θ and $p(x)$, and θ is explicitly a function only of a random process Φ given by

$$d\Phi = \frac{1}{2} h^* r^{-2} h\, dt + h^* r^{-1}\, db . \tag{34}$$

*On p. 4.

In addition $p(x(t))$ is a function of t, while x(t) and z are assumed fixed. Then, using (11) and (34)

$$
\begin{aligned}
dQ &= \frac{\partial Q}{\partial t}\, dt + \frac{\partial Q}{\partial \Phi}\left(\frac{1}{2} h^* r^{-2} h\, dt + h^* r^{-1}\, db\right) \\
&\quad + \frac{1}{2}\frac{\partial^2 Q}{\partial \Phi^2}\, h^* r^{-2} h\, dt\ .
\end{aligned} \tag{35}
$$

Now,

$$
\begin{aligned}
\frac{\partial Q}{\partial t} &= \frac{\partial \theta}{\partial t}\, p(x(t)) + \theta\, \frac{\partial p(x(t))}{\partial t} \\
&= \theta\, \tilde{\mathfrak{L}}\, p(x(t)) = \tilde{\mathfrak{L}}\, Q\ ,
\end{aligned} \tag{36}
$$

where $\tilde{\mathfrak{L}}$ is the forward diffusion operator

$$
-\sum_{i=1}^{n} \frac{\partial (f_i\, \cdot\,)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 ([gg^*]_{ij}\, \cdot\,)}{\partial x_i\, \partial x_j}\ . \tag{37}
$$

Also

$$
\frac{\partial Q}{\partial \Phi} = \frac{\partial^2 Q}{\partial \Phi^2} = Q\ . \tag{38}
$$

Substituting (36) and (38) into (35) provides

$$
\begin{aligned}
dQ &= \tilde{\mathfrak{L}}\, Q\, dt + Q(h^* r^{-2} h\, dt + h^* r^{-1}\, db) \\
&= \tilde{\mathfrak{L}}\, Q\, dt + Q\, h^* r^{-2}\, dz\ .
\end{aligned} \tag{39}
$$

Using (18) and (15), and the definition of Q/P

$$dP = d\left(\int_{R_n} Q\,dx\right) = \int_{R_n} (dQ)\,dx$$

$$= P\left\{\tilde{\mathfrak{L}}\left[\int_{R_n}\left(\frac{Q}{P}\right)dx\right]dt + \left[\int_{R_n}\frac{Q}{P}\,h^* r^{-2}\,dx\right]dz\right\} \tag{40}$$

$$= P\widehat{h^*}r^{-2}\,dz = P\widehat{h^*}r^{-2}h\,dt + P\widehat{h^*}r^{-1}\,db\,.$$

Using (11) with $x \triangleq P$, $\phi \triangleq P^{-1}$, $f \triangleq P\hat{h}^* r^{-2} h$, $g \triangleq P\hat{h}^* r^{-1}$, and $w \triangleq b$;

$$d(P^{-1}) = -P^{-1}\widehat{h^*}\,r^{-2}(h - \hat{h})\,dt - P^{-1}\widehat{h^*}r^{-1}\,db\,.$$

Finally, using (11) with $x \triangleq (P^{-1}, Q)^*$, $\phi \triangleq QP^{-1}$, $f \triangleq (-P^{-1}\hat{h}^* r^{-2}(h-\hat{h}),$ $\tilde{\mathfrak{L}}Q + Qh^* r^{-2})^*$, $g \triangleq (-P^{-1}\hat{h}^* r^{-1}, Qh^* r^{-1})^*$, and $w \triangleq b$,

$$d\left(\frac{Q}{P}\right) = \left(\tilde{\mathfrak{L}}\left(\frac{Q}{P}\right) + h^* r^{-2} h\,\frac{Q}{P}\right)dt + h^* r^{-1}\left(\frac{Q}{P}\right)db$$

$$- \widehat{h^*}r^{-2}(h - \hat{h})\left(\frac{Q}{P}\right)dt - \widehat{h^*}r^{-1}\left(\frac{Q}{P}\right)db - \widehat{h^*}r^{-2}h\left(\frac{Q}{P}\right)dt \tag{41}$$

$$= \tilde{\mathfrak{L}}\left(\frac{Q}{P}\right)dt + (h - \hat{h})^* r^{-2}\left(\frac{Q}{P}\right)(dz - \hat{h}\,dt)\,.$$

At this point all the operations on Q/P have been formal, assuming that the conditional density has the necessary differentiability. Actually, (41) will be used only in terms of its integral, and the rigorous conditions of validity need be applied only to the integral, which is the stochastic differential equation for conditional expectations.

Let ϕ be any scalar function, twice continuously differentiable in x:

$$\hat{\phi} = \int_{R_n} \phi(x)\, p(x|z)\, dx$$

and

$$d\hat{\phi} = \int_{R_n} \phi(x)\big(dp(x|z)\big)\, dx\,. \tag{42}$$

Substituting (41) into (42) provides

$$d\hat{\phi} = \int_{R_n} \phi(x)\big(\tilde{\mathfrak{L}}p(x|z) + (h - \hat{h})^* r^{-2} p(x|z)(dz - \hat{h}\, dt)\big)\, dx\,. \tag{43}$$

Let $\mathfrak{L}$ denote the formal adjoint of $\tilde{\mathfrak{L}}$

$$\sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} [gg^*]_{ij} \frac{\partial^2}{\partial x_i \partial x_j}\;. \tag{44}$$

Then (43) becomes

$$d\hat{\phi} = \widehat{\mathfrak{L}\phi}\, dt + (\widehat{\phi h} - \hat{\phi}\hat{h})^* r^{-2}(dz - \hat{h}\, dt)\;. \tag{45}$$

In Kushner's unpublished note, he presents a proof showing that under certain assumptions* (45) is meaningful. For the purpose of this dissertation, (45) can be considered only formally correct. In any case, (45) is the key equation from which the approximate filter is derived.

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* 12 of them!

3.4 APPROXIMATE FILTER EQUATIONS

The use of (45) as a differential equation for $\hat{x}$ results in

$$d\hat{x}_i = \hat{f}_i\, dt + \left(\widehat{x_i h} - \hat{x}_i \hat{h}\right) r^{-2} (dz - \hat{h}\, dt) \,, \qquad (46)$$

which is not very practical because $\hat{f}_i$, $\hat{h}$, and $\widehat{x_i h}$ are needed continuously. As the first step in the approximation, let f and h be approximated by a second-degree expansion about $x = \hat{x}$; also, for notational simplicity, suppress the explicit appearance of t as an argument of f, g, and h, since their dependence on time is incidental to the following manipulations. Then, adopting the summation convention

$$f_i(x) \approx f_i(\hat{x}) + f_{ij}^{(1)}(\hat{x})(x_j - \hat{x}_j) + \frac{1}{2} f_{ijk}^{(2)}(\hat{x})(x_j - \hat{x}_j)(x_k - \hat{x}_k) \,, \qquad (47)$$

where $f_{ij}^{(1)} \triangleq \partial f_i / \partial x_j$ and $f_{ijk}^{(2)} \triangleq \partial^2 f_i / \partial x_j \partial x_k$. A similar expression holds for h. From (47)

$$\hat{f}_i(x) \approx f_i(\hat{x}) + \frac{1}{2} f_{ijk}^{(2)}(\hat{x}) \widehat{(x_j - \hat{x}_j)(x_k - \hat{x}_k)} \,, \qquad (48)$$

where $\widehat{(x_j - \hat{x}_j)(x_k - \hat{x}_k)}$ is the conditional covariance of x and is denoted P_{ij}. Similarly

$$\widehat{x_i f_j}(x) \approx f_{jk}^{(1)}(\hat{x}) P_{ki} + \hat{x}_i \hat{f}_j(x) \,. \qquad (49)$$

Using (48) and (49) for f and h in (46) provides

$$\begin{aligned} d\hat{x}_i \approx\; & f_i(\hat{x})\, dt + \frac{1}{2} f_{ijk}^{(2)}(\hat{x}) P_{jk}\, dt \\ & + P_{ij} h_{kj}^{(1)}(\hat{x}) r_{k\ell}^{-2} \left[dz_\ell - \left(h_\ell(\hat{x}) + \frac{1}{2} h_{\ell mn}^{(2)}(\hat{x}) P_{mn} \right) dt \right] . \end{aligned} \qquad (50)$$

The next step is to find a differential equation for P. In general, even with a second-degree expansion for the nonlinearities, an infinite sequence of differential equations is required, because all the moments are needed to describe the conditional density.* However, by assuming an appropriate form for $p(x|z)$, the sequence stops at P. The first assumption, used by Bucy [7], is that third and fourth central conditional moments be neglected. In Schwartz and Bass [32], it is shown that the assumption is reasonable for a distribution with most of the probability mass sufficiently close to the mean. If it is assumed that $p(x|z)$ is Gaussian, the sequence also stops at P, and there is no restriction on the size of the moments.

Since P_{ij} can be written $(\widehat{x_i x_j} - \hat{x}_i \hat{x}_j)$, dP_{ij} is derived in two parts. Let $\phi = x_i x_j$ and use (45) to find $d\widehat{x_i x_j}$. Since

$$\mathfrak{L}\, x_i x_j = f_i x_j + f_j x_i + g_{ik} g_{jk} \, .$$

it follows that

$$\begin{aligned} d\widehat{x_i x_j} &= \widehat{f_i x_j}\, dt + \widehat{f_j x_i}\, dt + \widehat{g_{ik} g_{jk}}\, dt \\ &\quad + (\widehat{x_i x_j h_k} - \widehat{x_i x_j}\, \hat{h}_k)\, r_{k\ell}^{-2}\, (dz_\ell - \hat{h}_\ell\, dt) \, . \end{aligned} \tag{51}$$

* Equivalently, quasi-moments may be used, see Fisher [13].

Next, let $\phi = \hat{x}_i\hat{x}_j$ and use (11) and (46) to obtain

$$d\hat{x}_i\hat{x}_j = \frac{\partial(\hat{x}_i\hat{x}_j)}{\partial\hat{x}_k}\left[\hat{f}_k\,dt + (\widehat{x_k h_\ell} - \hat{x}_k\hat{h}_\ell)r^{-2}_{\ell m}(dz_m - \hat{h}_m dt)\right]$$

$$+\frac{1}{2}\frac{\partial^2(\hat{x}_i\hat{x}_j)}{\partial\hat{x}_k\partial\hat{x}_\ell}(\widehat{h_m x_k} - \hat{h}_m\hat{x}_k)r^{-2}_{mn}(\widehat{h_n x_\ell} - \hat{h}_n\hat{x}_\ell)\,dt . \qquad (52)$$

Combining (51) and (52) provides

$$dP_{ij} = \left(\widehat{f_i x_j} - \hat{f}_i\hat{x}_j + \widehat{f_j x_i} - \hat{f}_j\hat{x}_i + \widehat{g_{ik}g_{jk}}\right)dt$$

$$-\left(\widehat{x_i h_k} - \hat{x}_i\hat{h}_k\right)r^{-2}_{k\ell}\left(\widehat{x_j h_\ell} - \hat{x}_j\hat{h}_\ell\right)dt \qquad (53)$$

$$+\left[\widehat{x_i x_j h_k} - \widehat{x_i x_j}\hat{h}_k - \hat{x}_j\widehat{h_k x_i} - \hat{x}_i\widehat{h_k x_j} + 2\hat{x}_i\hat{x}_j\hat{h}_k\right]r^{-2}_{k\ell}\left(dz_\ell - \hat{h}_\ell\,dt\right),$$

which is an exact equation. The derivation of the approximate form of (53) is a tedious algebraic exercise, and is covered in Bass, Norum and Schwartz [2]. For completeness, several intermediate results follow: For either assumption about the conditional density,

$$\widehat{x_i x_j(x_k - \hat{x}_k)} - \widehat{x_i(x_k - \hat{x}_k)}\hat{x}_j - \widehat{x_j(x_k - \hat{x}_k)}\hat{x}_i = 0 .$$

For the first assumption,

$$\widehat{x_i x_j h_k} - \widehat{x_i x_j}\hat{h}_k - \hat{x}_j\widehat{h_k x_i} - \hat{x}_i\widehat{h_k x_j} + 2\hat{x}_i\hat{x}_j\hat{h}_k = -\frac{1}{2}P_{ij}\left(h^{(2)}_{k\ell m}(\hat{x})P_{\ell m}\right). \qquad (54)$$

For the other assumption, the right-hand side of (54) is

$$\frac{1}{2}\left(P_{i\ell} P_{mj} + P_{im} P_{\ell i}\right) h^{(2)}_{k\ell m}(\hat{x}) \quad . \tag{55}$$

For simplicity, let the term in parentheses in (55) be denoted by $T_{ijk\ell}$. Then, if gg^* is also expanded in a second-degree approximation, the following equations are obtained:

$$\begin{aligned} dP_{ij} \approx\; & P_{ik} f^{(1)}_{jk}(\hat{x})\, dt + f^{(1)}_{ik}(\hat{x}) P_{kj}\, dt - P_{ik} h^{(1)}_{\ell k}(\hat{x})\, r^{-2}_{\ell m} h^{(1)}_{mn} P_{nj}\, dt \\ & + g_{ik} g_{jk}(\hat{x})\, dt + \frac{1}{2}[g_{ik} g_{jk}]^{(2)}_{mn}(\hat{x}) P_{mn}\, dt \\ & - \frac{1}{2} P_{k\ell} h^{(2)}_{mk\ell}(\hat{x}) r^{-2}_{mn}\left(dz_n - h_n(\hat{x}) - \frac{1}{2} h^{(2)}_{npq}(\hat{x}) P_{pq}\right) P_{ij} \end{aligned} \tag{56}$$

and

$$\begin{aligned} dP_{ij} \approx\; & P_{ik} f^{(1)}_{jk}(\hat{x})\, dt + f^{(1)}_{ik}(\hat{x}) P_{kj}\, dt - P_{ik} h^{(1)}_{\ell k}(\hat{x})\, r^{-2}_{\ell m} h^{(1)}_{mn}(\hat{x}) P_{nj}\, dt \\ & + g_{ik}(\hat{x}) g_{jk}(\hat{x})\, dt + \frac{1}{2}[g_{ik} g_{jk}]^{(2)}_{mn}(\hat{x}) P_{mn}\, dt \\ & + \frac{1}{2} T_{ijk\ell} h^{(2)}_{mk\ell}(\hat{x})\, r^{-2}_{mn}\left(dz_n - h_n(\hat{x}) - \frac{1}{2} h^{(2)}_{npq}(\hat{x}) P_{pq}\right), \end{aligned} \tag{57}$$

where (56) uses (54) and (57) uses (55).

The problems associated with the above formulation for the filter equations are discussed in the next chapter.

4. DERIVATION OF VALID FILTER EQUATIONS

4.1 INTRODUCTION

This chapter begins with a critical re-evaluation of the approximation procedure used in Chapter 3 to generate the filter equations. No matter how rigorous the derivation is through (45), the resulting filter equations are unsatisfactory from the standpoint of mathematical validity. A modification to the approximation is made which results in a set of filter equations that are valid in the sense that they satisfy existence and uniqueness conditions (if the system nonlinearities satisfy certain assumptions) and that they are derived from an approximate system that also satisfies these conditions. The new equations, which contain a parameter k such that they are valid for any $k > 0$, and are almost identical to the previous equations when $k = 0$; this fact leads to a definition of computational validity which is satisfied by a slightly modified version of the previous equations.

4.2 CRITICISMS OF PREVIOUS DERIVATIONS

The first step in the approximation procedure is the replacement of the exact system model by a power series expansion about the instantaneous value of the conditional mean. The resulting differential equation cannot satisfy the conditions of the existence and uniqueness theorem given in Chapter 2, because a polynomial of degree 2 or more is inherently not sublinear. To put the problem in its proper perspective, it should be noted that the conditions of the theorem are only sufficient conditions, and it is not claimed that they are necessary. Moreover the problem stems from the white-noise model; it is not inherent in the physical problem. However, given that the original mathematical system equations have a unique, bounded, continuous solution with probability one, it is highly desirable that the approximate system also exhibit the same characteristics.

For the filtering problem, uniqueness in any of the equations is more than a desideratum, it is a necessity. If the approximate system of equations with a given set of initial conditions can be satisfied by an infinitude of solutions, in what sense can a solution be considered an approximation to the solution of the original system with the same initial conditions? Obviously, then, it is necessary to guarantee the uniqueness of the solution of the approximating equations. Moreover, for the same reason, it is equally necessary to guarantee that the filter equations themselves possess a unique solution, while the present filters do not satisfy the presently known existence and uniqueness conditions.

4.3 MODIFIED APPROXIMATION

The first requirement noted in the previous section is that the approximate system equations satisfy existence and uniqueness conditions. One difficulty arises in relating the filter equations to a set of system equations: the expansion of gg^* in a second-degree power series to obtain the equation for dP may not correspond to any real coefficient matrix for dw in an approximate equation for dx. To avoid the aforementioned difficulty, a linear expansion for g in the system equation is assumed. The change in the approximation to g is the slight modification alluded to in Section 4.1. The effect of the change can be inferred from the following: the quadratic expansion of gg^* about $\hat{x}$ is given by

$$g_{ik}(\hat{x})g_{jk}(\hat{x}) + \left(\frac{\partial g_{ik}(\hat{x})}{\partial x_\ell} g_{jk}(\hat{x}) + g_{ik}(\hat{x}) \frac{\partial g_{jk}(\hat{x})}{\partial x_\ell}\right)(x_\ell - \hat{x}_\ell)$$

$$+ \frac{1}{2}\left(\frac{\partial^2 g_{ik}(\hat{x})}{\partial x_\ell \partial x_m} g_{jk}(\hat{x}) + 2\frac{\partial g_{ik}(\hat{x})}{\partial x_\ell} \frac{\partial g_{jk}(\hat{x})}{\partial x_m}\right.$$

$$\left. + g_{ik}(\hat{x}) \frac{\partial^2 g_{jk}(\hat{x})}{\partial x_\ell \partial x_m}\right)(x_\ell - \hat{x}_\ell)(x_m - \hat{x}_m) \quad ,$$

where as the product of the linear approximation to g is

$$g_{ik}(\hat{x})g_{jk}(\hat{x}) + \left(\frac{\partial g_{ik}(\hat{x})}{\partial x_\ell} g_{jk}(\hat{x}) + g_{ik}(\hat{x}) \frac{\partial g_{jk}(\hat{x})}{\partial x_\ell}\right)(x_\ell - \hat{x}_\ell)$$

$$+ \frac{\partial g_{ik}(\hat{x})}{\partial x_\ell} \frac{\partial g_{jk}(\hat{x})}{\partial x_m}(x_\ell - \hat{x}_\ell)(x_m - \hat{x}_m) \quad .$$

For simplicity, let $\partial g_{ij}/\partial x_k$ be denoted by $g_{ijk}^{(1)}$.

To meet the existence and uniquess conditions, the following expansion* is suggested to replace (47), for k > 0:

$$f_i^k(x) = f_i(\hat{x}) + e^{-k|x-\hat{x}|^2}\left[f_{ij}^{(1)}(\hat{x})(x_j - \hat{x}_j) + \frac{1}{2} f_{ij\ell}^{(2)}(\hat{x})(x_j - \hat{x}_j)(x_\ell - \hat{x}_\ell)\right], \quad (58)$$

plus the following expansion for g:

$$g_{ij}^k(x) = g_{ij}(\hat{x}) + e^{-k|x-\hat{x}|^2} g_{ij\ell}^{(1)}(\hat{x})(x_\ell - \hat{x}_\ell) \quad (59)$$

Clearly, f^k and g^k are sublinear and satisfy a local Lipschitz condition, therefore the approximate joint system

$$\begin{bmatrix} dx \\ dz \end{bmatrix} = \begin{bmatrix} f^k(x) \\ h^k(x) \end{bmatrix} dt + \begin{bmatrix} g^k(x) & 0 \\ 0 & r(t) \end{bmatrix} \begin{bmatrix} dw \\ db \end{bmatrix} \quad (60)$$

satisfies existence and uniqueness conditions. Note that (58) approaches (47) as $k \to 0$, so that the approximations can be made arbitrarily close to each other, in a sense that is made precise later in this chapter.

The filter equations, derived in the coming section, are based on the assumption that the conditional density may be adequately approximated by a Gaussian density.

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* Note: (47) and (58) are close in a neighborhood of $\hat{x}$, but have different asymptotic behaviors.

4.4 MODIFIED FILTER

The following definition will be useful in the derivation of the filter equations:

$$\left[\mu_k^m(P)\right]_{i_1 \cdots i_m} \triangleq$$

$$\triangleq \int_{R_n} e^{-k|x-\hat{x}|^2} \prod_{j=1}^{m} (x_{i_j} - \hat{x}_{i_j}) \frac{\exp\left[-\frac{1}{2}(x-\hat{x})^* P^{-1}(x-\hat{x})\right]}{(2\pi)^{n/2}(\det P)^{1/2}} dx \; . \qquad (61)$$

The right-hand side of (61) can be written as

$$\frac{\left[\det(2kI_n + P^{-1})^{-1}\right]^{1/2}}{(\det P)^{1/2}} \int_{R_n} \prod_{j=1}^{m} (x_{i_j} - \hat{x}_{i_j}) \frac{\exp\left[-\frac{1}{2}(x-\hat{x})^*(2kI_n + P^{-1})(x-\hat{x})\right]}{(2\pi)^{n/2}\left[\det(2kI_n + P^{-1})^{-1}\right]^{1/2}} dx$$

so that the tensor whose components are given by (61)

$$\mu_k^m(P) = \left[\det(2kP + I_n)\right]^{-1/2} \mu_o^m\left[(2kI_n + P^{-1})^{-1}\right] . \qquad (62)$$

For simplicity, let $P^k \triangleq \mu_k^2(P)$ and $c_k \triangleq \left[\det(2kP + I_n)\right]^{-1/2}$.

The basic steps in deriving the filter equations are those outlined in Chapter 3. From (58) and (61)

$$\hat{f}_i(x) \approx f_i(\hat{x}) + \frac{1}{2} f^{(2)}_{ij\ell}(\hat{x}) P^k_{j\ell} \; , \qquad (63)$$

and

$$\widehat{x_i f_j}(x) \approx f^{(1)}_{j\ell}(\hat{x}) P^k_{\ell j} + \hat{x}_i \hat{f}_j(x) \; . \qquad (64)$$

Then (50) becomes

$$d\hat{x}_i \approx f_i(\hat{x})\,dt + \frac{1}{2} f^{(2)}_{ij\ell}(\hat{x}) P^k_{j\ell}\,dt$$
$$+ P^k_{ij} h^{(1)}_{\ell j}(\hat{x}) r^{-2}_{\ell m}\left[dz_m - \left(h_m(\hat{x}) + \frac{1}{2} h^{(2)}_{mnp}(\hat{x}) P^k_{np}\right) dt\right]. \quad (65)$$

To compute P^k, which is needed for (65), note that (62) implies

$$P^k = \left[\det(2kP + I_n)\right]^{-\frac{1}{2}} (2kI_n + P)^{-1}. \quad (66)$$

Thus, it suffices to compute P, as before.

The derivation of the expression for dP also follows the pattern established in Chapter 3, and (53) still holds. The new expression for (55) is

$$\frac{1}{2} c_k^{-1}\left[(P^k_{ij} - c_k P_{ij}) P^k_{pq} + P^k_{iq} P^k_{pj} + P^k_{ip} P^k_{qj}\right] h^{(2)}_{mpq}(\hat{x}). \quad (67)$$

To prove (67), the following manipulations are useful

$$\mu^4_k(P) = c_k \mu^4_o\left[(2kI_n + P^{-1})^{-1}\right]. \quad (68)$$

But it is shown in Laning and Battin [28] p. 83 that

$$[\mu^4_o]_{ijpq} = [\mu^2_o]_{ij}[\mu^2_o]_{pq} + [\mu^2_o]_{iq}[\mu^2_o]_{pj} + [\mu^2_o]_{ip}[\mu^2_o]_{pj}, \quad (69)$$

where the argument, $(2kI_n + P^{-1})^{-1}$, has been suppressed.

Also

$$\mu_o^2\left[(2kI_n + P^{-1})^{-1}\right] = c_k^{-1}\mu_k^2(P) = c_k^{-1}P^k . \tag{70}$$

Repeated substitution of (70) into (69), and substitution of (69) into (68) provides

$$\mathcal{E}\left(e^{-k|x-\hat{x}|^2}(x_i-\hat{x}_i)(x_j-\hat{x}_j)(x_\ell-\hat{x}_\ell)(x_m-\hat{x}_m)\,|\,z\right) \approx \approx c_k^{-1}\left(P_{ij}^kP_{\ell m}^k + P_{i\ell}^kP_{jm}^k + P_{im}^kP_{j\ell}^k\right) . \tag{71}$$

The remaining manipulations are straightforward, but tedious.

For simplicity, let the bracketed factor in (67) be denoted T_{ijpq}^k. The modified version of (57) is then

$$\begin{aligned} dP_{ij} \approx\ & P_{i\ell}^k f_{j\ell}^{(1)}(\hat{x})\,dt + f_{i\ell}^{(1)}(\hat{x})P_{\ell j}^k\,dt \\ & - P_{i\ell}^k h_{m\ell}^{(1)}(\hat{x})\, r_{mn}^{-2}\, h_{np}^{(1)}(\hat{x})P_{pj}^k\,dt + g_{i\ell}(\hat{x})\,g_{j\ell}(\hat{x})\,dt \\ & + g_{i\ell m}^{(1)}(\hat{x})\,g_{i\ell n}^{(1)}(\hat{x})P_{mn}^k\,dt \\ & + \tfrac{1}{2}c_k^{-1}\,T_{ij\ell m}^k\,h_{n\ell m}^{(2)}(\hat{x})\,r_{np}^{-2}\left[dz_p - \left(h_p(\hat{x}) + \tfrac{1}{2}h_{pqr}^{(2)}(\hat{x})P_{qr}^k\right)dt\right]. \end{aligned} \tag{72}$$

To determine if the filter system (65) and (72) satisfies existence and uniqueness conditions, it is convenient to recast the equations in the form of a single vector equation, as follows:

Let s be the vector of dimension $n(n+3)/2$ with components defined by $s_i = \hat{x}_i$ for $i = 1, \ldots, n$; and $s_i = P_{pq}$ where $i = q + np - p(p-1)/2$ for $q = 1, \ldots, p$; $p = 1, \ldots, n$. It is easy to verify that s contains each element of $\hat{x}$ and each nonredundant element of P. It is more convenient to use the notation s_{pq} in place of s_i for $i > n$. The filter system can then be rewritten in the form

$$ds = a^k(t, s)\,dt + b^k(t, s)\,dz , \tag{73}$$

where, restoring t as an explicit argument, from (65), for i=1,..., n

$$\begin{aligned} a_i^k(t,s) &= f_i(t,s) + \frac{1}{2} f_{ij\ell}^{(2)}(t,s) P_{j\ell}^k(s) - P_{ij}^k(s) h_{\ell j}^{(1)}(t,s) r_{\ell m}^{-2}(t) h_m(t,s) \\ &\quad - \frac{1}{2} P_{ij}^k(s) h_{\ell j}^{(1)}(t,s) r_{\ell m}^{-2}(t) h_{mnp}^{(2)}(t,s) P_{np}^k(s) \\ b_i^k(t,s) &= P_{ij}^k(s) h_{\ell j}^{(1)}(t,s) r_{\ell m}^{-2}(t) \end{aligned} \tag{74}$$

and, from (72) and (59), for i > n

$$\begin{aligned} a_{ij}^k(t,s) &= P_{i\ell}^k(s) f_{j\ell}^{(1)}(t,s) + P_{\ell j}^k(s) f_{i\ell}^{(1)}(t,s) \\ &\quad - P_{i\ell}^k(s) P_{pj}^k h_{\ell m}^{(1)}(t,s) h_{np}^{(1)}(t,s) r_{mn}^{-2}(t) + g_{i\ell}(t,s) g_{j\ell}(t,s) \\ &\quad + P_{\ell m}^{2k}(s) g_{in\ell}^{(1)}(t,s) g_{jnm}^{(1)}(t,s) \\ &\quad - \frac{1}{2} c_k^{-1}(s) T_{ij\ell m}^k(s) h_{n\ell m}^{(2)}(t,s) r_{np}^{-2}(t) h_p(t,s) \\ &\quad - \frac{1}{4} c_k^{-1}(s) T_{ij\ell m}^k(s) h_{n\ell m}^{(2)}(t,s) r_{np}^{-2}(t) \end{aligned} \tag{75a}$$

and

$$b_{ij}^{k}(t,s) = \frac{1}{2} c_k^{-1}(s) T_{ij\ell m}^{k}(s) h_{n\ell m}^{(2)}(t,s) r_{np}^{-2}(t) \ . \qquad (75b)$$

Note: If any of the terms in (74) and (75) include a factor of s_{ij} with $j > 1$, by symmetry, s_{ji} may be substituted so that only the upper triangular elements of P are used.

The validity of the filter, in the sense adopted herein, depends upon the characteristics of a^k and b^k, which, in turn, are given by (74) and (75). A certain amount of preliminary analysis is required to determine if a^k and b^k satisfy existence and uniqueness conditions. The quantity P^k occurs quite often in the expanded equations. From (66), P^k can be seen to be bounded for nonnegative definite P; since P is a covariance matrix, it is required to be nonnegative definite. The quantity c_k^{-1} is essentially of degree n/2 in s, at least when $|P|$ becomes large*, and violates sublinearity, but the product $c_k^{-1} T^k$ is sublinear, as is implied by (70). Then, by inspection, the following conditions are sufficient for validity:

1. f is twice differentiable everywhere, such that each f_i, $f_{ij}^{(1)}$, and $f_{ijk}^{(2)}$ is sublinear.
2. h is twice differentiable everywhere, such that $h^{(1)}$, $h^{(2)}$, and $|h| \cdot |h^{(2)}|$ are bounded.

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* $|P|$ is the norm of P.

3. g is differentiable everywhere such that each $g_{ij}g_{k\ell}$ and $g^{(1)}_{ijk}\,g^{(1)}_{\ell mn}$ is sublinear.

It is assumed from this point on that conditions 1-3 are met, so that (73) is a valid stochastic differential equation for the filter.

One of the conditions required in the derivation is that P be nonnegative definite. If the filter equation were exact and the solution to the P equation were truly the conditional covariance, the condition would be met automatically, as noted above, because of the properties of a covariance matrix. However, in the present case, the solution to the P equation is merely an approximation to the conditional covariance, and is not guaranteed to be semi-definite. At this point in the state of knowledge of stochastic differential equations, the problem of determining conditions under which the computed value of P is nonnegative definite is unsolved. The following pragmatic rule is suggested: If, in the course of an actual computation, P should fail to be nonnegative definite, the approximation must be considered improper.

4.5 COMPUTATIONAL VALIDITY

It has just been shown that the modified filter is valid for any $k > 0$; for $k = 0$, the equations are essentially the previous invalid filter equations. The results to this point are true for the mathematical problem derived from the physical situation. It is shown in this section that when the mathematical solution is transformed into a computational algorithm for use on a digital computer, the equation for $k = 0$ is also valid in the sense that there is an equation for some $k > 0$ that computes the same values of ds/dt as the equation for $k = 0$.

It is interesting to note that while the vector s was constructed to allow the use of the vector form of the theorem in Chapter 2, an actual computer program would probably be coded in terms of the vector s, since most differential equation integration routines are designed to integrate a first-order vector equation, and since it is inefficient to compute the redundant elements of P. The formal conversion of (73) to a computer algorithm would entail writing the related ordinary differential equation

$$\frac{ds}{dt} = a^k(t, s) + b^k(t, s) y \ , \tag{76}$$

where, as in Section 3.2 dz/dt is taken as y, since y is now a physically measured signal.

It can be seen with little difficulty that for $k = 0$ the filter equation (73) reduces to the vector form of (50) and (57) except that $g^{(1)}g^{(1)^*}$ replaces $[gg^*]^{(2)}/2$, as discussed earlier in this chapter. Also, for fixed t and s, a^k and b^k are continuous functions of k. Furthermore, for a computer mechanization y is bounded and there is a difference threshold within which two numbers are considered identical. Let Y be the bound on y and let ϵ be the difference threshold. By the continuity of a^k and b^k, for a given t and s there exists a $\kappa > 0$ such that

$$|a^{\kappa}(t, s) - a^{o}(t, s)| + |b^{\kappa}(t, s) - b^{o}(t, s)| Y < \epsilon \quad . \qquad (77)$$

Moreover, since t and s are also bounded for computer applications, there exists a $\kappa > 0$ such that (77) holds for all attainable t and s. Thus, the value of ds/dt computed for $0 < k \leq \kappa$ is identical to that computed for $k = 0$.

5. SIMULATIONS

5.1 INTRODUCTION

There are a number of reasons for the simulation study. First of all, despite the strong flavor of basic mathematics in the research into nonlinear filtering, the technique is essentially computational and is useless if it is not mechanizable. Secondly, the excuse for studying nonlinear minimal-variance filtering is the hope that it is an improvement over other filtering approaches in the computational environment. Finally, there is still some question about how to mechanize the integration. In particular, if the integral is mechanized as an ordinary algorithm like Runge-Kutta or Adams, should the mathematical approach still be used, or is the mathematical approach good only for rectangular rule integration?*

The numerical investigation was conducted for eight different approaches to filtering, for two sets of dynamics, for two measurement schemes each. The two systems were chosen such that one satisfied existence and uniqueness conditions for stochastic differential equations while the other did not. No difficulties were

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* Recall the discussion beginning on page 18.

anticipated in connection with filtering for the second system in the actual computer environment,* and none were encountered.

5.2 FILTER EQUATIONS

Because of the large number of cases involved in the study, only first-order systems are considered. The first system is

$$\text{(D1)} \qquad \frac{dx}{dt} = -\frac{x}{1+x^2} + v$$

$$\text{(M11)} \qquad y = \tan^{-1} x + w \qquad (78)$$

$$\text{(M12)} \qquad y = x + w$$

where v and w are white noises. With either measurement scheme (M11) or (M12), the overall system satisfies existence and uniqueness. The second system, which does not, is given by

$$\text{(D2)} \qquad \frac{dx}{dt} = -x^3 + v$$

$$\text{(M21)} \qquad y = x + x^3 + w \qquad (79)$$

$$\text{(M22)} \qquad y = x + w$$

The filtering schemes are outlined below. For the outline, f and h are the system nonlinearities as in (8) and (13). The

*Because of the boundedness noted in Section 4.5, the nonlinear functions are forced to be sublinear as mechanized; therefore the apparent violation of existence and uniqueness conditions does not really occur.

white-noise processes, v and w, are assumed stationary with covariances σ_v^2 and σ_w^2, respectively so that $g = \sigma_v$ and $r = \sigma_w$. The subscript 'n' denotes nominal, $\hat{x}$ is the estimate of x, p is the approximate covariance, and a prime denotes differentiation. Note that for linear measurements, M12 and M22, certain terms vanish and there are only three different filters.

1. Linear

The linear filtering algorithm can be applied to a set of equations linearized about an a priori nominal motion. The equations, derived by Kalman and Bucy [23] are

$$\frac{d\hat{x}}{dt} = f_n' \hat{x} + \sigma_w^{-2} p h_n' (y - h_n' \hat{x}) \, ,$$

$$\frac{dp}{dt} = 2pf_n' - \sigma_w^{-2} p^2 h_n'^2 + \sigma_v^2 \, . \qquad (80)$$

2. Quasi-Moment Minimal-Variance

This is the filter derived by Schwartz and Bass [32], and independently by Fisher [13].

$$\frac{d\hat{x}}{dt} = f(\hat{x}) + \frac{1}{2} f''(\hat{x}) p + \sigma_w^{-2} p h'(\hat{x}) \left[y - h(\hat{x}) - \frac{1}{2} p h''(\hat{x}) \right],$$

$$\frac{dp}{dt} = 2pf'(\hat{x}) - \sigma_w^{-2} p^2 h'^2(\hat{x}) + \sigma_w^{-2} p^2 h''(\hat{x}) \left[y - h(\hat{x}) - \frac{1}{2} p h''(\hat{x}) \right] + \sigma_v^2 \, . \qquad (81)$$

3. Truncated Minimal-Variance

This is the filter derived by Bass, Norum, and Schwartz [2].

$$\frac{d\hat{x}}{dt} = f(\hat{x}) + \frac{1}{2}f''(\hat{x})p + \sigma_w^{-2}ph''(\hat{x})\left[y - h(\hat{x}) - \frac{1}{2}ph''(\hat{x})\right] ,$$

$$\frac{dp}{dt} = 2pf'(\hat{x}) - \sigma_w^{-2}p^2h'^2(\hat{x}) - \frac{1}{2}\sigma_w^{-2}p^2h''(\hat{x})\left[y - h(\hat{x}) - \frac{1}{2}ph''(\hat{x})\right] + \sigma_v^2 . \tag{82}$$

4. Modified Minimal-Variance

This filter is a compromise between (81) and (82) which is based on the difference in the driving terms in the p equation. By dropping the driving term, the filter is simpler, yet the response falls between the responses for the two preceding filters.

$$\frac{d\hat{x}}{dt} = f(\hat{x}) + \frac{1}{2}f''(\hat{x})p + \sigma_w^{-2}ph''(\hat{x})\left[y - h(\hat{x}) - \frac{1}{2}ph''(\hat{x})\right] ,$$

$$\frac{dp}{dt} = 2pf'(\hat{x}) - \sigma_w^{-2}p^2h'^2(\hat{x}) + \sigma_v^2 . \tag{83}$$

5. Maximum-Principle Least Squares

This filter is derived by Detchmendy and Sridhar [10] for minimizing an integral-square-estimation-error criterion, using deterministic techniques. By the use of Pontriagin's maximum principle the minimization is "reduced" to a two-point boundary-value problem which is solved by an invariant imbedding technique using an approximation to one boundary condition.

$$\frac{d\hat{x}}{dt} = f(\hat{x}) + \sigma_w^{-2} p h'(\hat{x}) [y - h(\hat{x})] \, ,$$

$$\frac{dp}{dt} = 2pf'(\hat{x}) - \sigma_w^{2} p^2 h'^2(\hat{x}) + \sigma_w^{-2} p^2 h''(\hat{x}) [y - h(\hat{x})] + \sigma_v^2 \, . \tag{84}$$

Actually, (84) is a special case of the derivation in Detchmendy and Sridhar [10]; they used arbitrary weighting functions in the criterion integrand, while (84) corresponds to the particular set of weighting functions that result in the Kalman filter for linear dynamics.

6. Dynamic Programming Least Squares

This filter is derived by Cox [9] for a criterion which is similar to that used for the previous filter. The minimization is effected by dynamic programming using a quadratic approximation to the cost function.

$$\frac{d\hat{x}}{dt} = f(\hat{x}) + \sigma_w^{-2} p h'(\hat{x}) [y - h(\hat{x})] \, ,$$

$$\frac{dp}{dt} = 2pf'(\hat{x}) - \sigma_w^{-2} p^2 h'^2(\hat{x}) + \sigma_v^2 \, . \tag{85}$$

A look at (85) shows that this filter is essentially equivalent to using linear filtering about the computed mean, a technique that had been used heuristically previously.

7. Discrete-Measurement Minimal-Variance

This filter is derived by Jazwinsky [21] for a minimal-variance criterion under the assumption that the measurements arrive at isolated instants. The form presented here is the limiting form for continuous measurements.

$$\frac{d\hat{x}}{dt} = f(\hat{x}) + \frac{1}{2}f''(\hat{x})p + \sigma_w^{-2}ph'(\hat{x})[y - h(\hat{x})] \; ,$$

$$\frac{dp}{dt} = 2pf'(\hat{x}) - \sigma_w^{-2}p^2h'^2(\hat{x}) - \frac{1}{2}\sigma_w^{-2}h''(\hat{x})[y - h(\hat{x})] + \sigma_v^2 \; . \tag{86}$$

Since the evolution of the system itself is considered continuous, the portion of the equations related to updating the estimate in the absence of measurements agrees with (81), (82), and (83). The loss of the term in $h''(\hat{x})$ is due to the difference between the physical and mathematical approaches discussed in Chapter 2. In the derivation of (86), Jazwinsky uses the approximation to the conditional density that is used to derive (82).

8. Modified Discrete-Measurement Minimal-Variance

This filter is related to (86) in the same way that (81) is related to (82), i.e., the conditional density is assumed to be Gaussian.

$$\frac{d\hat{x}}{dt} = f(\hat{x}) + \frac{1}{2}f''(\hat{x})p + \sigma_w^{-2}ph'(\hat{x})[y - h(\hat{x})] \; ,$$

$$\frac{dp}{dt} = 2pf'(\hat{x}) - \sigma_w^{-2}p^2h'^2(\hat{x}) + \sigma_w^{-2}p^2h''(\hat{x})[y - h(\hat{x})] + \sigma_v^2 \; . \tag{87}$$

The application of (80)-(87) to (78) and (79) is straightforward and is not carried out herein.

5.3 SIMULATION DETAILS

The first simulation program was used for familiarization and preliminary investigation of certain basic questions. For example, the analysis discussed on p. 20 relating the white-noise model to the simulated white-noise sequence was substantiated by showing that the filter performance deteriorates drastically if the factor of Δt is ignored. Also, it did not seem to make an essential difference if the integration was performed by a Runge-Kutta scheme or rectangular rule.

The computer program that was finally used for the simulation study included a simple rectangular-rule integration with constant step-size. The use of constant step-size allows each filter to be compared on the basis of the same pseudo-random sequence. The random number generator is a combination of a standard uniform random sequence routine plus an approximate transformation to a Gaussian random sequence. The program has two modes of operation: in one mode, only a single estimation is made with a prespecified initial condition for the state; in the other mode, several runs are made for random initial conditions, and the statistics of the estimation errors are computed. The output from the program is a computer-prepared plot showing the time-history of the filter response or the error statistics. The plots show every 20th point with linear interpolation between.

5.4 RESULTS

As noted earlier, the simulation study was not intended to be a complete investigation of the computational characteristics of nonlinear filters. Two standard cases were used, which were computationally docile when used with the step-size chosen. For both dynamical equations (D1) of (78) and (D2) of (79), the standard run consists of 5000 points 0.001 second apart for a total problem time of five seconds, with $\sigma_v^2 = 1$ and $\sigma_w^2 = 10$. Only the initial condition differs. Figures 1 and 2 show the state x and the measurement y for the two standard cases. The figures show the nonlinear measurement, but the noise is so large that there is not too much apparent difference between the linear and nonlinear measurements.

In the response to the standard inputs, there is a large difference between the response of the linear systems and the nonlinear systems, while the various nonlinear systems are remarkably similar. Figures 3 and 4 show the estimation error for the two standard cases. As might be expected, linear measurements help the linear system. What might not be expected is that the relative error performance of the nonlinear filters within the shaded region is different for the two cases. To show that the results in Figures 3 and 4 are not peculiar to the particular initial condition and pseudo-random sequences, three representative filters were chosen for a set of ten runs. The mean and mean-square errors are shown in Figures 5 and 6 for initial conditions with variance one for

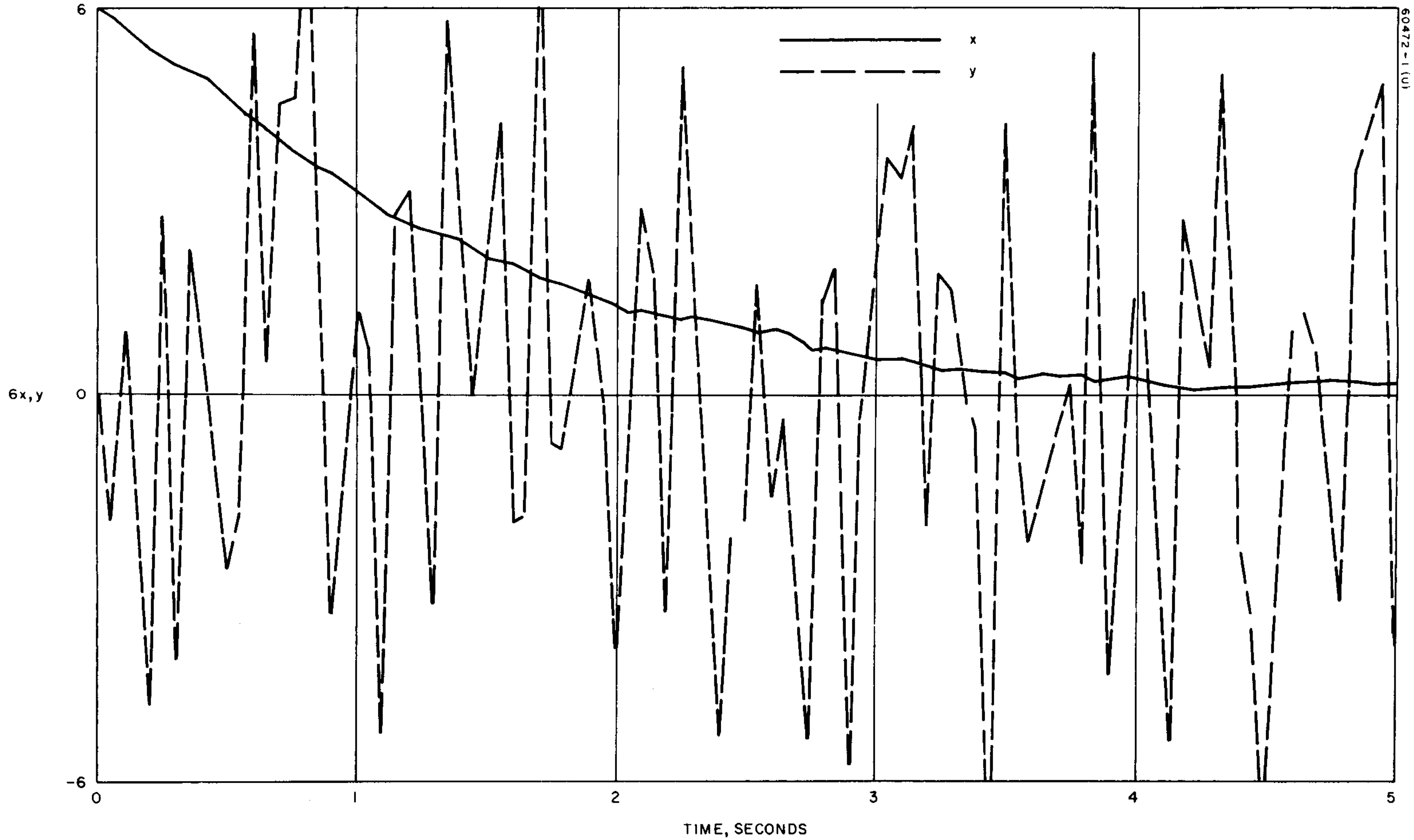

Figure 1. Standard Response for Dynamics D1 With Measurement M11

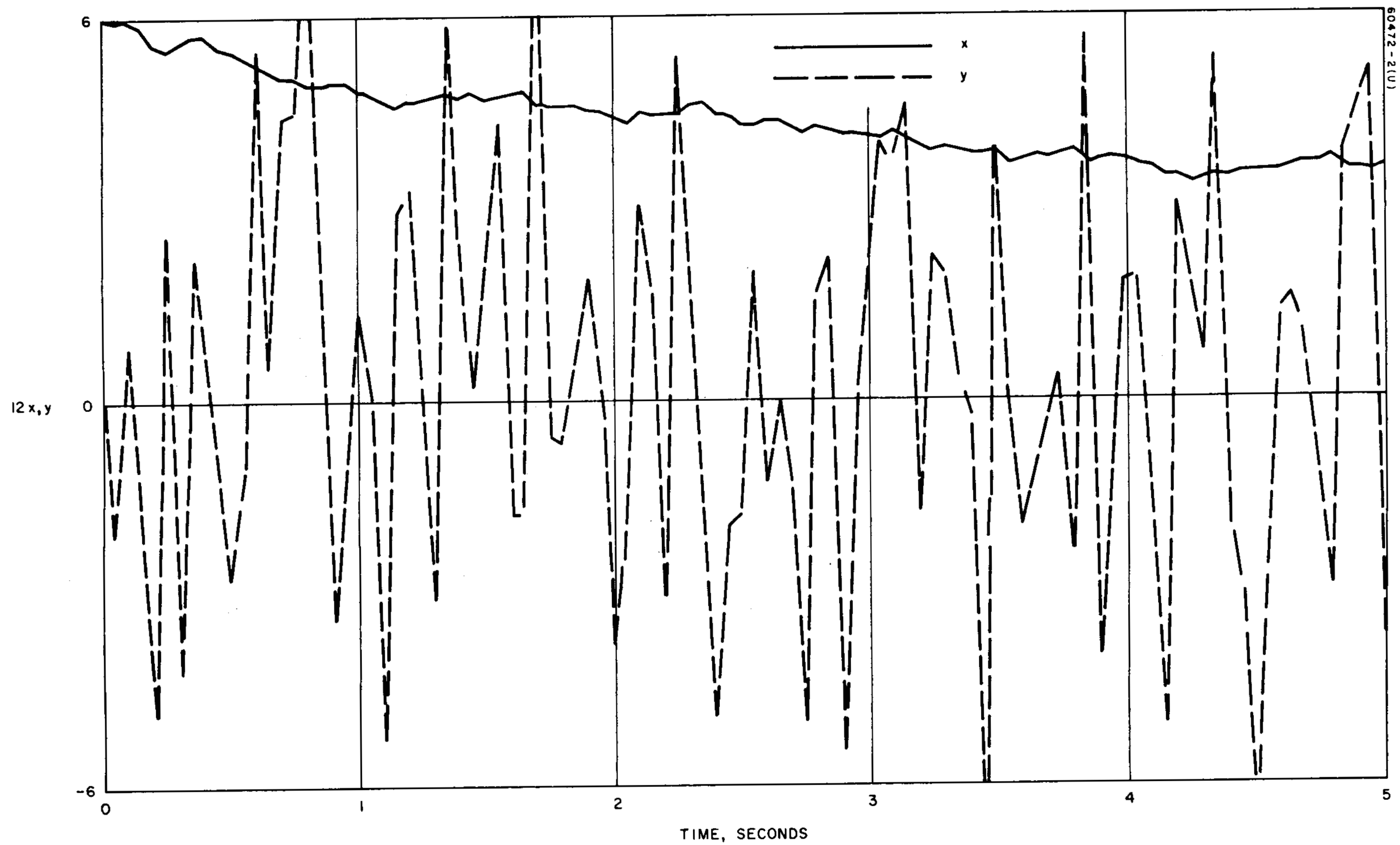

Figure 2. Standard Response for Dynamics D2 With Measurement M21

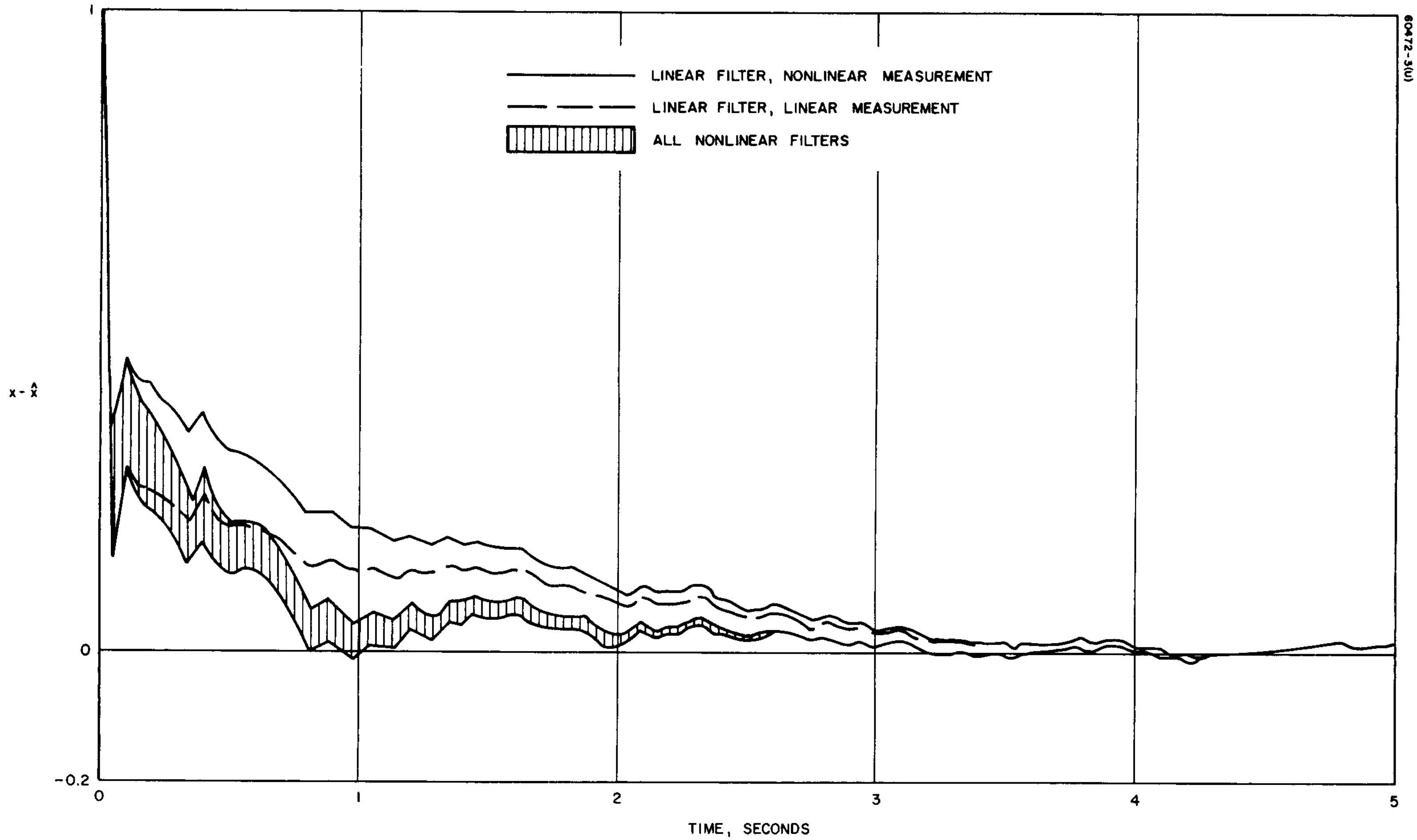

Figure 3. Estimation Error for Standard Case for Dynamics D1

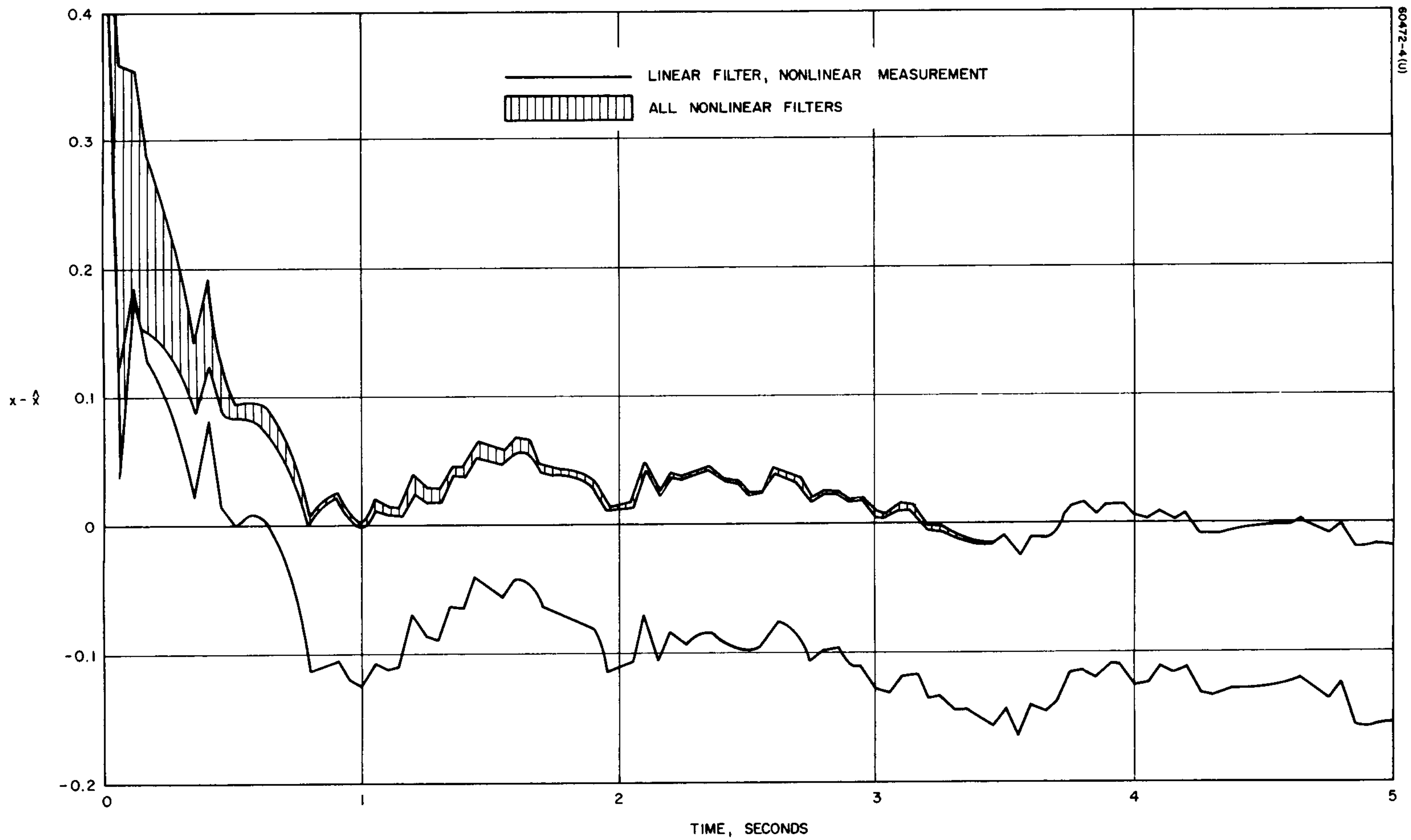

Figure 4. Estimation Error for Standard Case for Dynamics D2

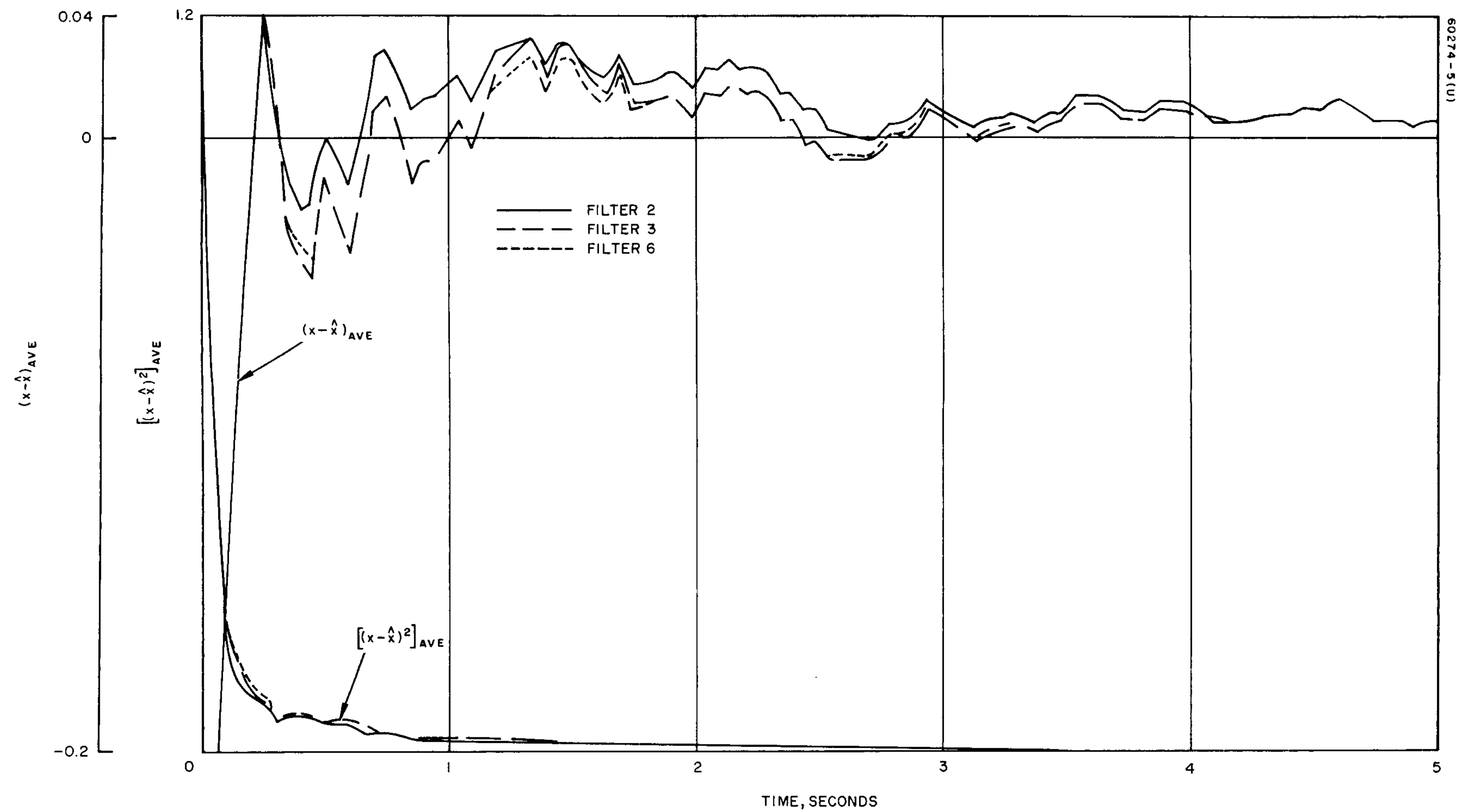

Figure 5. Error Statistics for Ten Runs of Dynamics D1 and Measurement M11

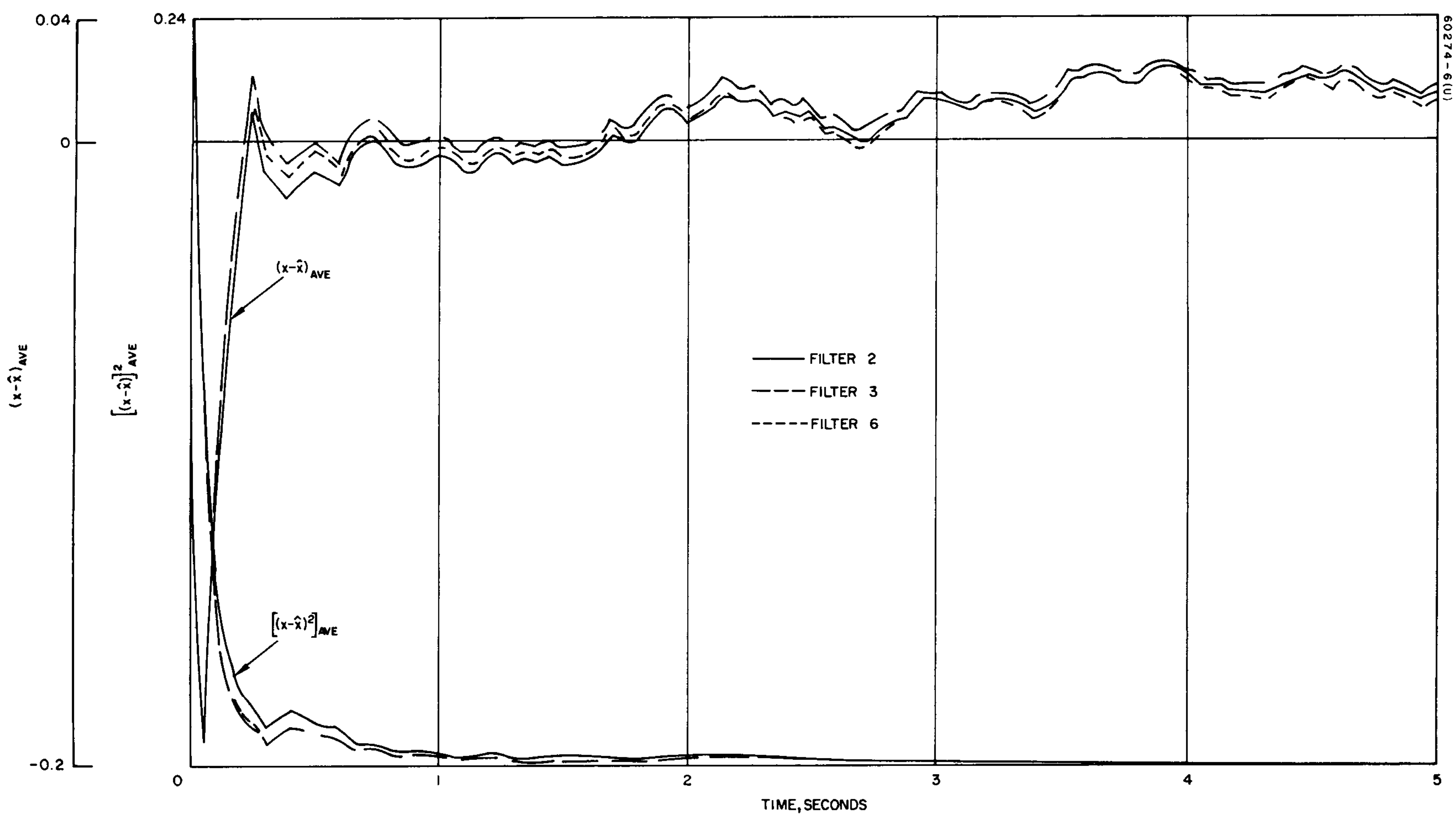

Figure 6. Error Statistics for Ten Runs of Dynamics D2 and Measurement M21

the first case and variance 1/4 for the second case. It can be seen that the error statistics are still quite close, practically indistinguishable from a mean-square error point of view. Moreover, the relative error performance of the chosen filters is qualitatively unchanged.

While on the topic of statistical runs, it is interesting to note the comparison between the output of the p equation for an initial condition of one standard deviation and the mean-square error for ten runs for one case, as shown in Figure 7 for filter 2, dynamics D1, and measurement M11. Moreover, it was found that the change in the mean-square error for a larger number of cases, up to 50, while noticeable, was relatively small.

The effects of changing the statistics of the random sequences are similar for all the filters, and are intuitively reasonable. Changing σ_v has little effect on the initial response, though the higher σ_v the worse the ultimate following. Contrariwise, increasing σ_w slows the response noticeably without affecting the ultimate tracking. The error responses for filter 2 with D1 and M11 are shown in Figures 8 and 9.

The runs, mentioned earlier, demonstrating the correctness of the noise model, show the effects of improperly matching the filter parameters to the noise statistics. The effect of the mismatch on $x-\hat{x}$ and p are shown in Figures 10 and 11. For all the curves shown, the actual pseudo-random inputs were taken from identical populations, only the filter parameters varied.

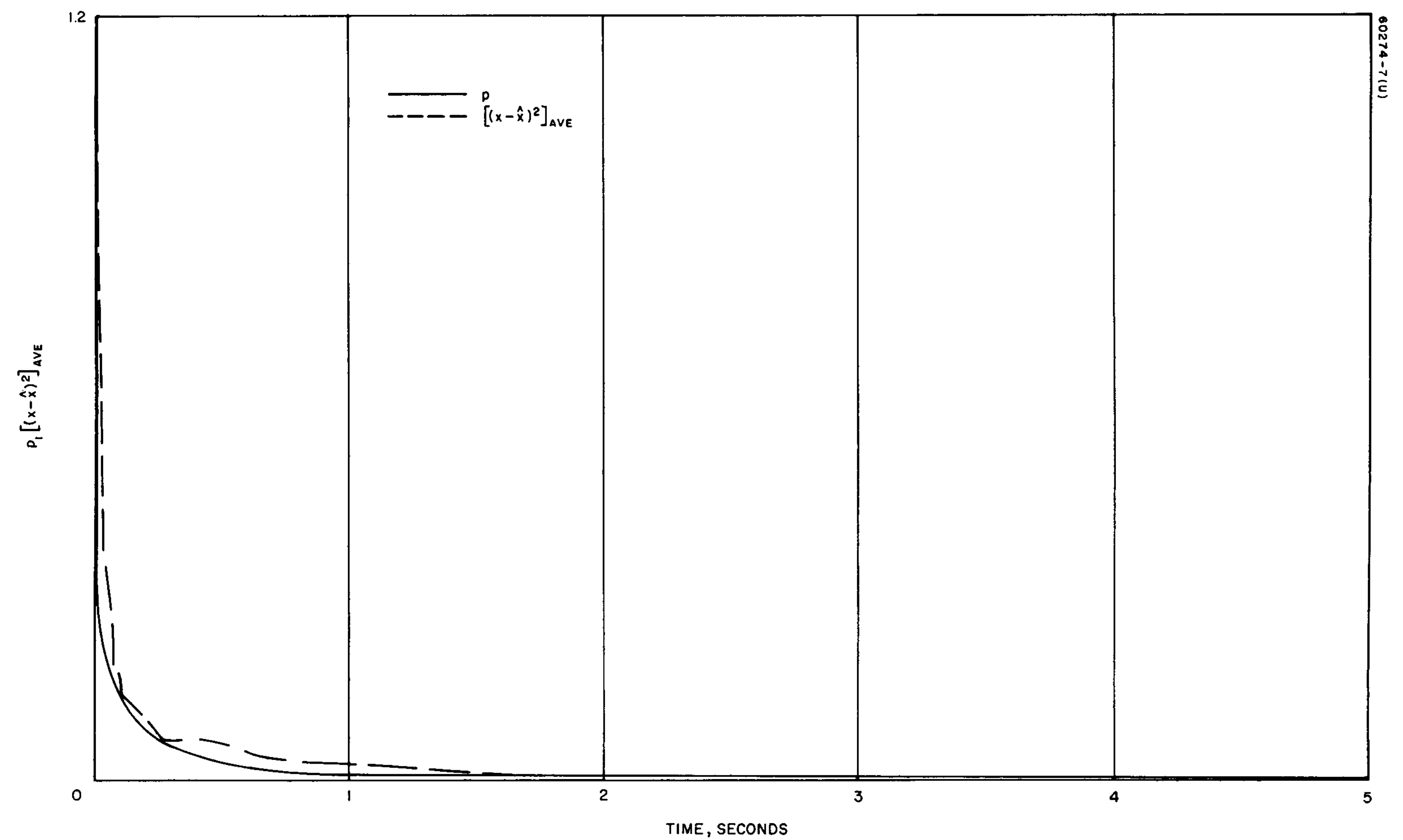

Figure 7. Comparison of p With Measured Mean-Square Error

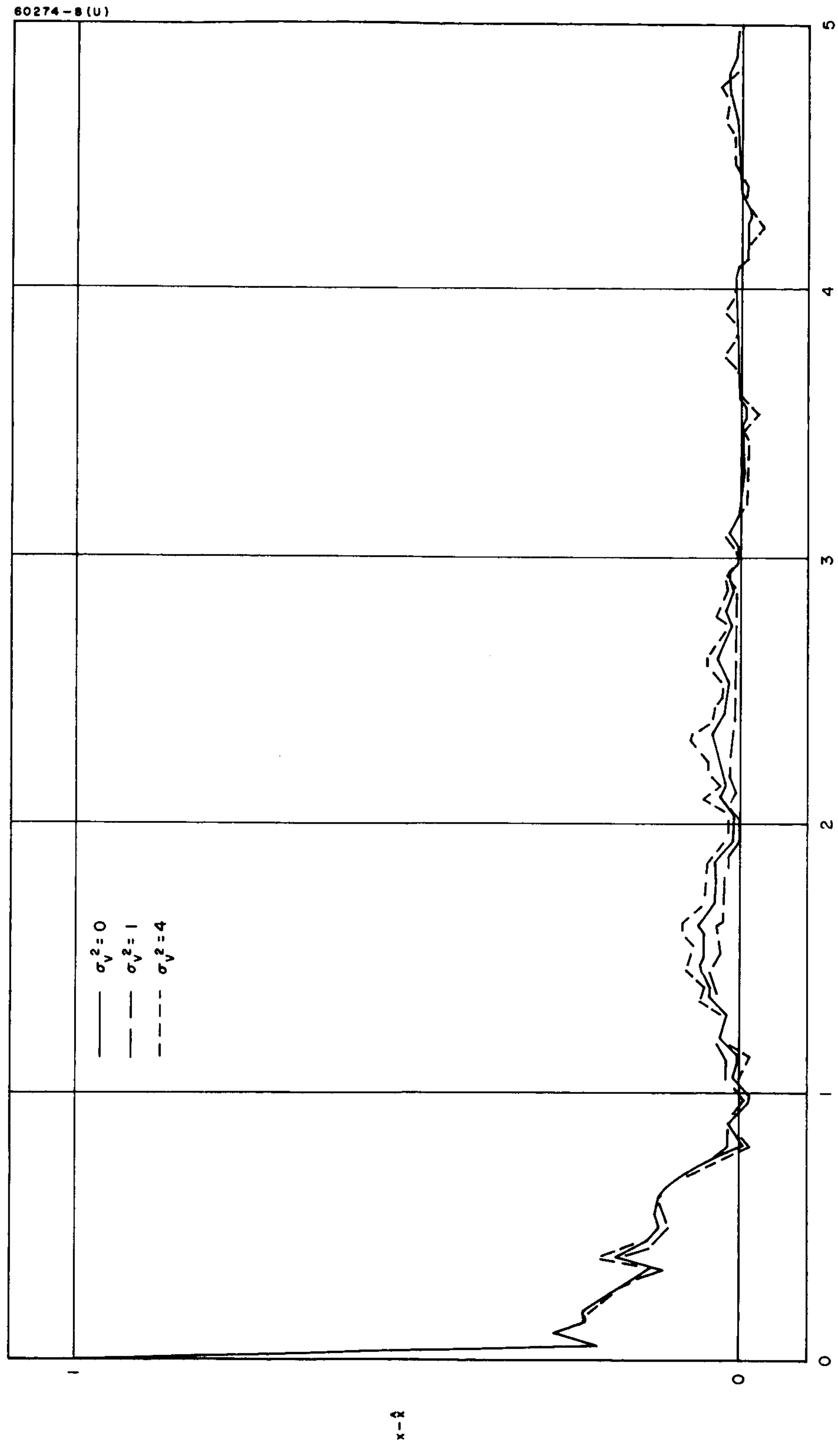

Figure 8. Effect of Varying σ_v

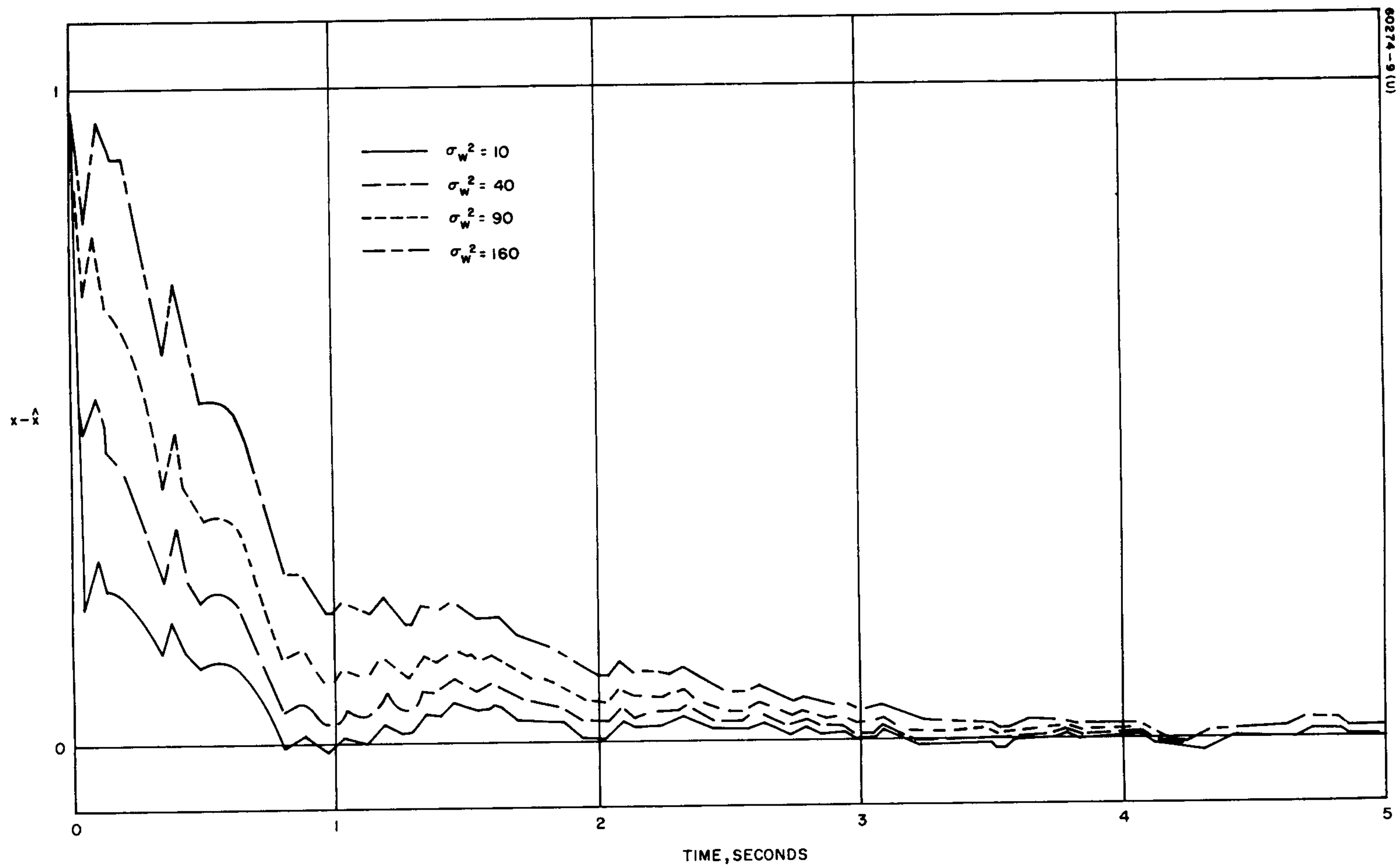

Figure 9. Effect of Varying σ_w

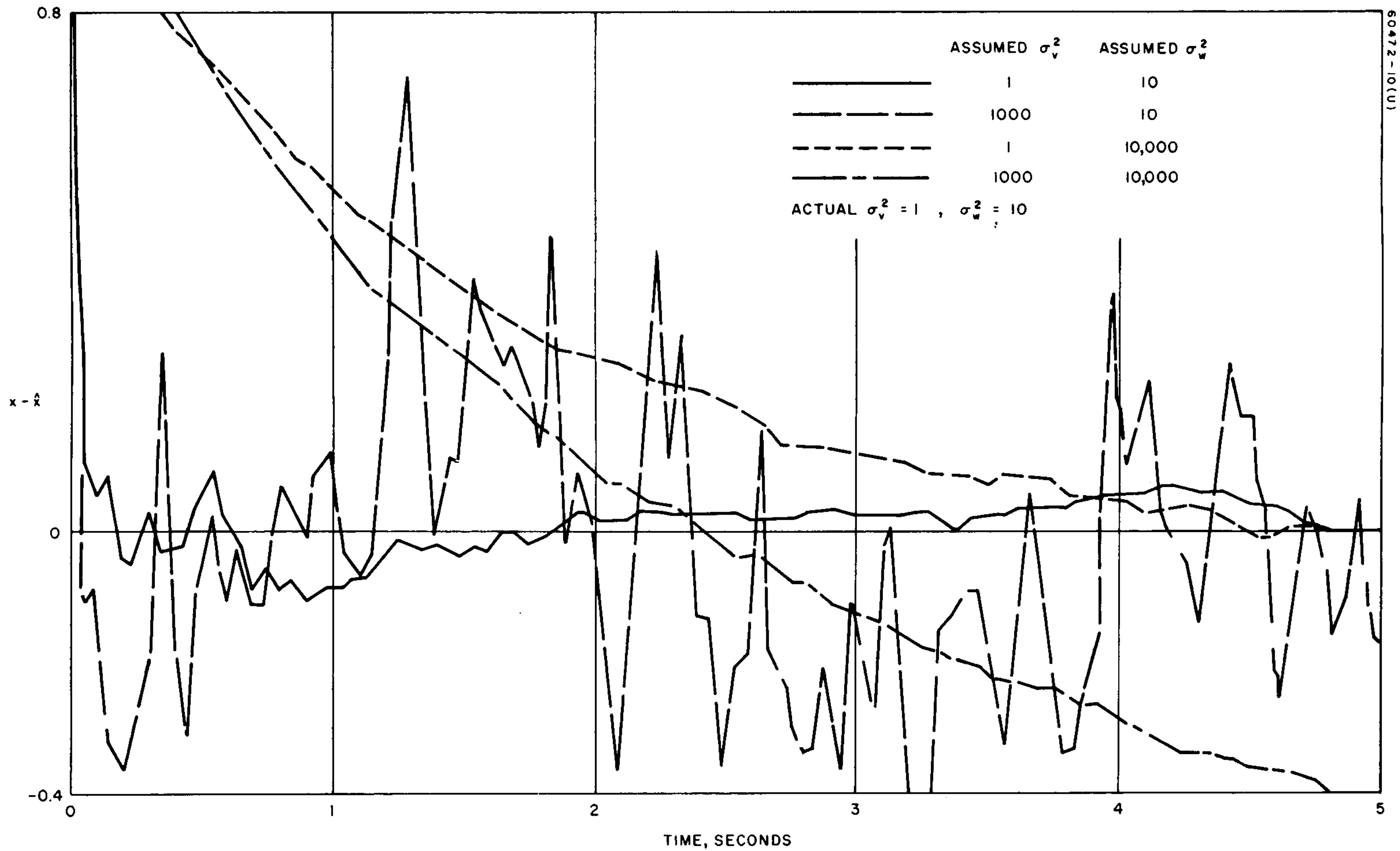

Figure 10. Effect of Statistical Mismatch on Estimation Error

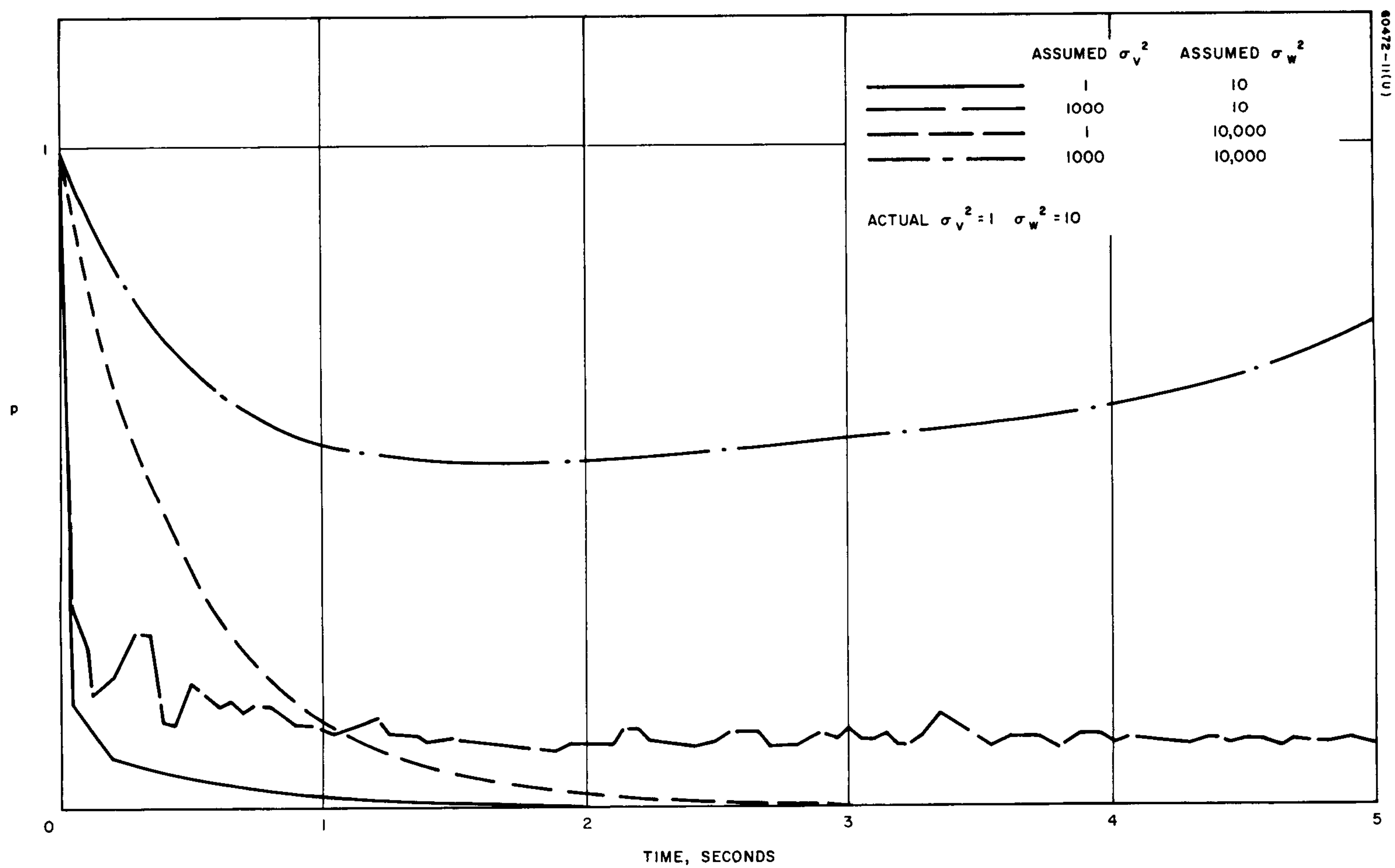

Figure 11. Effect of Statistical Mismatch on Computed Covariance

Evidently a statistical mismatch can seriously affect the performance of the filter. It also appears that the value of p generated by the filter is a good index of the performance, even when the statistics are poorly matched.

The final aspect that was studied on the computer is the effect of sampling the data. Two approaches to sampling were considered: sample-and-hold and pulse sampling. For the sample-and-hold runs, the effective noise variance was obtained using a Δt equal to the sampling period, rather than the computation interval. For the pulse-sampling runs, σ_w^{-1} was made zero for those intervals during which no measurements were made. Both types of sampling were applied to filter 2 with dynamics D1 and measurements M11. The response plots are shown in Figure 12. As should be expected, sampling is detrimental to the initial response of the filter, though ultimately the estimate settles in to the proper value. From the one case considered, sample-and-hold appears somewhat better than impulse sampling, which may be due to the smoothing effect of the zero-order hold.

There is one problem in mechanizing the filters, which has not yet been discussed, that requires much more investigation: the effects of the filter parameters and the step-size on computational stability. For some combinations of parameters σ_v and σ_w and initial covariance $p(t_o)$, the step-size must be made small to stabilize the computation during the initial portion of the run. The reason for the instability is the large value of the derivatives and

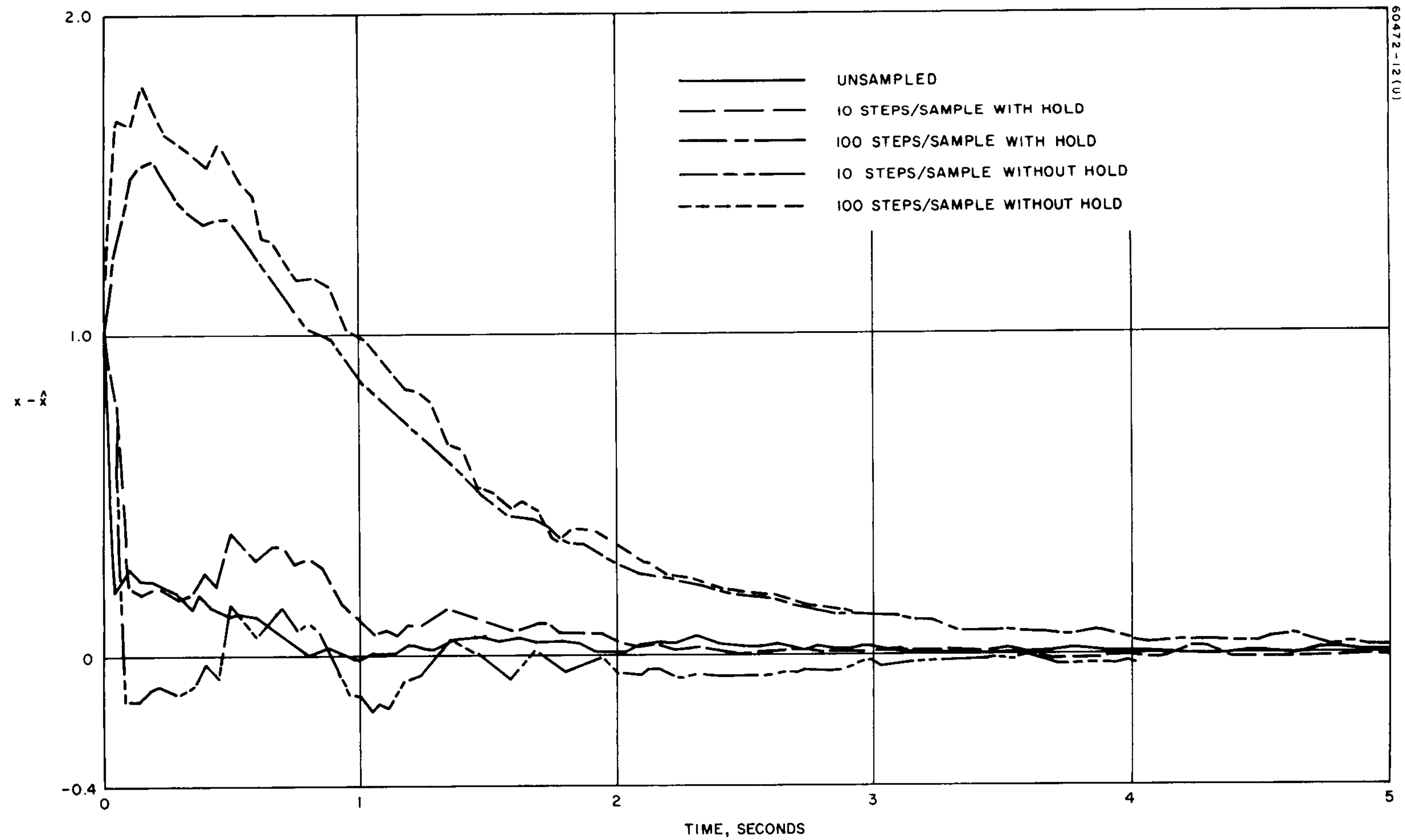

Figure 12. Error Response for Sampled Input

the consequent large truncation errors introduced in the integration. A particularly insidious form of computational instability arose in the course of the simulation study: it is possible to be in a conditionally stable region such that one pseudo-random sequence with a given statistical description results in a stable response, while another sequence drawn from the same population does not. However, because of the nature of the computational stability problem, it is most effectively studied for a particular equation; general results are difficult to find.

6. SUMMARY AND SUGGESTIONS FOR FURTHER WORK

6.1 SUMMARY AND CONCLUSIONS

The preceding treatment of the problem of the construction of approximate continuous nonlinear minimal-variance filters can be divided into three parts. The first three chapters are introductory in nature, and cover a general background of the problem, the mathematical preliminaries required for the analysis, and a detailed account of the previous work leading up to the dissertation.

Chapter 4 constitutes the second part, which is the construction of a mathematically valid filter. Here, the term "mathematically valid" implies that the stochastic differential equation for the filter satisfies existence and uniqueness conditions, and also that the filter is derived on the basis of an approximate dynamical representation that satisfies the same conditions. Since previous derivations violate the requirements for validity, certain modifications are necessary, particularly in the way the differential equation for the system is approximated. Then, given a valid formulation for the filter, it is shown that in the actual computational environment the nonvalid quasi-moment minimal-variance filter is also valid in the sense that there exist mathematically valid filters that are computationally identical to it.

The final part, presented in Chapter 5, is a computational investigation of nonlinear filtering, including filters derived by different techniques and for other criteria. Apparently, nonlinear filtering is superior to linear filtering, although no one nonlinear filter offers any clear-cut advantage over any other from a performance point of view. It also appears that the filters derived from a continuous model are quite amenable to sampled-data use.

The mathematical problem solved in Chapter 4 arises from the probabilistic criterion and the white-noise model; it does not appear in the statistical approaches which use a deterministic criterion and which need no postulate about the form of the noise. There does exist the problem of choosing a weighting matrix in the criterion integrand, but that is not at all a theoretical problem.* Moreover, in view of the success of the heuristic nonlinear filter obtained by continuously updating the nominal for the linearization in the p equation and using a nonlinear updating in the $\hat{x}$ equation, it is difficult to justify from a practical point of view all the mathematical difficulties that arise using the probabilistic nonlinear approach. Granted, the computer investigation reported in Chapter 5 is insufficient evidence to support any final conclusion, but it does offer food for thought.

*It appears that the covariances of the equivalent white-noise processes are a good choice for the weighting matrices, even though the white-noise assumption is not made.

6.2 FURTHER WORK

The derivation in Chapter 4 used a truncated quasi-moment expansion for the conditional density, in which only the mean and covariance were included, plus a modified second-degree expansion for the system nonlinearity. It would be interesting to extend the derivation to include higher quasi-moments and higher-degree expansions for the system, should the probabilistic approach still be of interest.

No derivation, probabilistic or statistical, seems to allow for a state-dependent measurement noise, yet there are practical situations in which the noise is state-dependent. The variation of the noise with state could be included heuristically, but it would be desirable to include the effect in a theoretical derivation.

In view of the comments at the end of the previous section, an attempt should be made to compare the various approaches analytically. It is not at all satisfying to base a general conclusion on a limited sample of computer runs!

A number of areas are open for numerical investigation, the most important being the numerical stability of the equations. Another problem is the simulation of higher order problems, and a third is a study of the effects of modeling errors.

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