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Abstract

We study control properties of a linearized fluid-structure interaction system, where the structure is a rigid body and where the fluid is a viscoelastic material. We establish the approximate controllability and the exponential stabilizability for the velocities of the fluid and of the rigid body and for the position of the rigid body. In order to prove this, we prove a general result for this kind of systems that generalizes in particular the case without structure. The exponential stabilization of the system is obtained with a finite-dimensional feedback control acting only on the momentum equation on a subset of the fluid domain and up to some rate that depends on the coefficients of the system. We also show that, as in the case without structure, the system is not exactly null-controllable in finite time.

Keywords. Fluid-structure interaction systems, viscoelastic fluids, controllability, stabilizability, finite dimensional controls.
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Contents

1 Introduction

We study some controllability properties of a mathematical model describing the motion of a rigid body immersed in a viscoelastic fluid. First we describe the corresponding model: we denote by $\Omega \subset \mathbb{R}^3$ a bounded domain containing the fluid and the rigid body. The structure domain $\mathcal{S}(t)$ can be described through its orientation $Q(t) \in SO(3)$ and its center of mass $h(t) \in \mathbb{R}^3$, whereas the fluid domain $\mathcal{F}(t)$ is the complement of the structure domain in Ω:

$$\mathcal{S}(t) = h(t) + Q(t)\mathcal{S}_0, \quad \mathcal{F}(t) = \Omega \setminus \overline{\mathcal{S}(t)},$$

where $\mathcal{S}_0 \subset \mathbb{R}^3$ is the reference domain of the rigid body, chosen as a nonempty regular domain with center of mass 0.

Now, to describe the dynamics of the corresponding fluid-structure interaction system, we use the Newton laws to obtain the equations below on the linear and angular velocities of the rigid body $\ell \in \mathbb{R}^3$ and $\omega \in \mathbb{R}^3$:

$$\begin{cases} h' = \ell & \forall t > 0, \\ Q' = \mathbb{S}(\omega)Q & \forall t > 0, \end{cases} \tag{1.1}$$

$$\begin{cases} m\ell' = -\displaystyle\int_{\partial\mathcal{S}(t)} \tau_{\text{fluid}} n \, d\Gamma & t > 0, \\ (J\omega)' = -\displaystyle\int_{\partial\mathcal{S}(t)} (x - h) \times \tau_{\text{fluid}} n \, d\Gamma & t > 0, \end{cases} \tag{1.2}$$

where τ_{fluid} is the stress tensor of the fluid, where $m > 0$ and J are the mass and the moment of inertia of the rigid body and where

$$\mathbb{S}(\omega) = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \quad (\omega \in \mathbb{R}^3).$$

For the fluid, its velocity u and pressure p satisfy

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u - \operatorname{div} \tau_{\text{fluid}} = f\chi_{\mathcal{O}} & t > 0, \, x \in \mathcal{F}(t), \\ \operatorname{div} u = 0 & t > 0, \, x \in \mathcal{F}(t), \end{cases} \tag{1.3}$$

$$\begin{cases} u = 0 & t > 0, \; x \in \partial\Omega, \\ u(t,x) = \ell(t) + \omega(t) \times (x - h(t)) & t > 0, \; x \in \partial\mathcal{S}(t), \end{cases} \tag{1.4}$$

where f is the control of the system, acting on a part of the fluid domain $\mathcal{O} \subset \Omega$ ($\chi_{\mathcal{O}}$ is the characteristic function of $\mathcal{O}$). We assume that $\mathcal{O}$ is a nonempty domain with $\mathcal{O} \subset \mathcal{F}(t)$ for all t.

Finally, it remains to describe the stress tensor of the fluid. For that, we consider the Johnson-Segalman model for viscoelastic flows, that is

$$\tau_{\text{fluid}} = \Sigma(u,p) + \tau,$$

where

$$\Sigma(u,p) = 2\eta\mathbb{D}(u) - p\mathbb{I}_3, \quad \mathbb{D}(u) = \frac{1}{2}\left(\nabla u + \nabla u^\top\right)$$

and where

$$\frac{\partial \tau}{\partial t} + (v \cdot \nabla)\tau + g_a(\tau, \nabla u) + \lambda\tau = 2\kappa\mathbb{D}u \quad t > 0, \, x \in \mathcal{F}(t), \tag{1.5}$$

Here $a \in [-1, 1]$ is a constant and

$$g_a(\tau, \nabla u) = \tau\mathbb{W}(u) - \mathbb{W}(u)\tau - a(\mathbb{D}(u)\tau + \tau\mathbb{D}(u)), \quad \mathbb{W}(u) = \frac{1}{2}\left(\nabla u - \nabla u^\top\right).$$

In the above relations, η, λ and κ are positive constants and $\mathbb{W}(u)$ denotes the vorticity tensor. Local existence of solutions of viscoelastic fluids for arbitrary data and global existence of solutions for sufficiently small data in appropriate spaces have been established in [14, 12]. Regarding a detailed discussion on Johnson-Segalman fluid flow and viscoelastic fluids in general, we refer to [26].

The corresponding system (1.1), (1.2), (1.3), (1.4), (1.5) with initial conditions for u, τ, h, Q, ℓ and ω is already studied in [13] and [32]. They show the well-posedness of the system respectively in a $L^p - L^q$ setting and in Hilbert setting. Their methods are based on a change of variables to handle the moving and unknown fluid domain $\mathcal{F}(t)$, a linearization and a fixed point argument. Due to the viscoelastic part corresponding to τ, the system is more complex to study than the system of interaction between a rigid body and Newtonian fluid governed by the classical Navier-Stokes system (that is $\tau = 0$ in the above system). In particular in the linearization, the equation of τ is treated separately.

Here our aim is to present a linearization of the above system and to study some control properties of this linearization, seen as a first step towards some control results for the nonlinear system. A similar linearization is used for the well-posedness and the controllability results of the system composed by a rigid body and Newtonian fluid (see for instance, [15], [4], [3]). The idea is to consider a change of variables to write the above system with fixed domains $\mathcal{F}$ and $\mathcal{S}$ and then remove all the nonlinear terms coming from the model and from the change of variables. This leads to the following system

$$\left\{\begin{array}{l} \dfrac{\partial u}{\partial t} - \operatorname{div}(\Sigma(u,p) + \tau) = f\chi_{\mathcal{O}} \qquad t > 0,\ y \in \mathcal{F}, \\ \operatorname{div} u = 0 \qquad t > 0,\ y \in \mathcal{F}, \\ \dfrac{\partial \tau}{\partial t} + \lambda\tau = 2\kappa\mathbb{D}u \quad t > 0,\ y \in \mathcal{F}, \end{array}\right. \tag{1.6}$$

$$\left\{\begin{array}{l} u = 0 \qquad t > 0,\ y \in \partial\Omega, \\ u(t,y) = \ell(t) + \omega(t) \times y \qquad t > 0,\ y \in \partial\mathcal{S}, \end{array}\right. \tag{1.7}$$

$$\left\{\begin{array}{l} m\ell' = -\displaystyle\int_{\partial\mathcal{S}} (\Sigma(u,p) + \tau)\, n\, d\Gamma \qquad t > 0, \\ J_0\omega' = -\displaystyle\int_{\partial\mathcal{S}} y \times (\Sigma(u,p) + \tau)\, n\, d\Gamma \qquad t > 0, \end{array}\right. \tag{1.8}$$

$$\left\{\begin{array}{l} h' = \ell \qquad \forall t > 0 \\ \theta' = \omega \qquad \forall t > 0. \end{array}\right. \tag{1.9}$$

Note that in the above system, $\mathcal{F}$ and $\mathcal{S}$ are now time-independent regular domains with $\mathcal{F} = \Omega\backslash\overline{\mathcal{S}}$. We keep the same notation as in (1.1), (1.2), (1.3), (1.4), (1.5) for u, τ, ℓ, ω but these variables have been modified through the change of variables. J_0 is time-independent positive symmetric matrix and the rotation matrix Q has been replaced by a local chart of $SO(3)$ (for instance the Euler angles), that is $\theta(t) \in \mathbb{R}^3$. If we introduce the density $\rho_{\mathcal{S}}$ of the rigid body, and we assume it is a positive constant, we have the following relation:

$$m = \rho_{\mathcal{S}}|\mathcal{S}|, \quad J_0 = \rho_{\mathcal{S}} \int_{\mathcal{S}} \left(|y|^2\mathbb{I}_3 - y \otimes y\right) dy, \tag{1.10}$$

where $|\mathcal{S}|$ is the Lebesgue measure of $\mathcal{S}$. We can assume that 0 is the center of gravity of $\mathcal{S}$ so that

$$\int_{\mathcal{S}} y\ dy = 0. \tag{1.11}$$

In what follows, we only consider a particular case of (1.6)-(1.9), where we assume that τ is of the form $\tau = 2\mathbb{D}(v)$. Due to the incompressibility condition of u and its boundary values, we finally consider the following linear system:

$$\left\{\begin{array}{l} \dfrac{\partial u}{\partial t} - \operatorname{div}(\Sigma(u,p) + 2\mathbb{D}(v)) = f\chi_{\mathcal{O}} \qquad t > 0,\ y \in \mathcal{F}, \\ \operatorname{div} u = 0 \qquad t > 0,\ y \in \mathcal{F}, \end{array}\right. \tag{1.12}$$

$$\left\{\begin{array}{l} u = 0 \qquad t > 0,\ y \in \partial\Omega, \\ u(t,y) = \ell(t) + \omega(t) \times y \qquad t > 0,\ y \in \partial\mathcal{S}, \end{array}\right. \tag{1.13}$$

$$\left\{\begin{array}{l} m\ell' = -\displaystyle\int_{\partial\mathcal{S}} (\Sigma(u,p) + 2\mathbb{D}(v))\, n\, d\Gamma \qquad t > 0, \\ J_0\omega' = -\displaystyle\int_{\partial\mathcal{S}} y \times (\Sigma(u,p) + 2\mathbb{D}(v))\, n\, d\Gamma \qquad t > 0, \end{array}\right. \tag{1.14}$$

$$\left\{\begin{array}{l} \dfrac{\partial v}{\partial t} + \lambda v = \kappa u \qquad t > 0,\ y \in \mathcal{F}, \\ \operatorname{div} v = 0 \qquad t > 0,\ y \in \mathcal{F}, \end{array}\right. \tag{1.15}$$

$$\left\{\begin{array}{l} v = 0 \qquad t > 0,\ y \in \partial\Omega, \\ v(t,y) = k(t) + r(t) \times y \qquad t > 0,\ y \in \partial\mathcal{S}, \end{array}\right. \tag{1.16}$$

$$\begin{cases} k' + \lambda k = \kappa \ell & \forall t > 0 \\ r' + \lambda r = \kappa \omega & \forall t > 0 \end{cases} \tag{1.17}$$

$$\begin{cases} h' = \ell & \forall t > 0 \\ \theta' = \omega & \forall t > 0 \end{cases} \tag{1.18}$$

$$\begin{cases} u(0, \cdot) = u^0, \quad v(0, \cdot) = v^0 \quad \text{in } \mathcal{F}, \\ h(0) = h^0, \quad \theta(0) = \theta^0, \quad \ell(0) = \ell^0, \quad \omega(0) = \omega^0, \quad k(0) = k^0, \quad r(0) = r^0. \end{cases} \tag{1.19}$$

In order to study the above system, it is standard to extend u and v as functions of Ω by setting

$$u(t, y) = \ell(t) + \omega(t) \times y, \quad v(t, y) = k(t) + r(t) \times y \quad (t \geqslant 0, \ y \in \mathcal{S})$$

and similar formula for u^0 and v^0. This leads us to introduce the following space

$$\mathbb{H} = \left\{ u \in L^2(\Omega) \ ; \ \operatorname{div} u = 0 \text{ in } \Omega, \, \mathbb{D}(u) = 0 \text{ in } \mathcal{S}, \, u \cdot n = 0 \text{ on } \partial\Omega \right\}. \tag{1.20}$$

We recall (see [36, Lemma 1.1, p.18]) that $\mathbb{D}(u) = 0$ in $\mathcal{S}$, if and only if there exist $\ell_u, \omega_u \in \mathbb{R}^3$ such that

$$u(y) = \ell_u + \omega_u \times y \quad (y \in \mathcal{S}).$$

We consider the inner product on $L^2(\Omega)$ defined by

$$(u, v) = \int_{\mathcal{F}} u \cdot v \, dy + \int_{\mathcal{S}} \rho_{\mathcal{S}} u \cdot v \ dy,$$

where $\rho_{\mathcal{S}}$ is the density of the rigid body. The corresponding norm is equivalent to the usual norm in $L^2(\Omega)$ and if $u, v \in \mathbb{H}$, then we have:

$$(u, v) = \int_{\mathcal{F}} u \cdot v \, dy + m \ell_u \cdot \ell_v + J_0 \omega_u \cdot \omega_v.$$

We also define the spaces

$$\mathbb{H}_{1/2} = \left\{ u \in H^1_0(\Omega) \ ; \ \operatorname{div} u = 0 \text{ in } \Omega, \, \mathbb{D}(u) = 0 \text{ in } \mathcal{S} \right\}, \quad \text{and} \quad \mathbb{H}_1 = \left\{ u \in \mathbb{H}_{1/2} \ ; \ u_{|\mathcal{F}} \in H^2(\mathcal{F}) \right\}. \tag{1.21}$$

Then, we will show that if $u^0 \in \mathbb{H}_{1/2}$, $v^0 \in \mathbb{H}_1$, $(h^0, \theta^0) \in \mathbb{R}^6$ and $f \in L^2(0, T; L^2(\mathcal{O}))$, there exists a unique solution to (1.12)–(1.19)

$$u \in H^1(0, T; \mathbb{H}) \cap L^2(0, T; \mathbb{H}_1) \cap C^0([0, T]; \mathbb{H}_{1/2}), \quad v \in H^1(0, T; \mathbb{H}_1), \quad (h, \theta) \in H^1(0, T; \mathbb{R}^6).$$

Note that it implies in particular that

$$(\ell, \omega, k, r) \in H^1(0, T; \mathbb{R}^{12}).$$

We say that (1.12)–(1.19) is approximately controllable in time $T > 0$ if for any $u^0, u^1 \in \mathbb{H}_{1/2}$, $v^0, v^1 \in \mathbb{H}_1$, $(h^0, \theta^0), (h^1, \theta^1) \in \mathbb{R}^6$ and for any $\varepsilon > 0$, there exists $f \in L^2(0, T; L^2(\mathcal{O}))$ such that the solution to (1.12)–(1.19) satisfies

$$\left\| [u(T, \cdot), v(T, \cdot), h(T), \theta(T)] - [u^1, v^1, h^1, \theta^1 \right\|_{\mathbb{H}_{1/2} \times \mathbb{H}_1 \times \mathbb{R}^6} < \varepsilon.$$

In the earlier works on controllability of linear viscoelastic models [19, 18, 17], Leugering considered an integro-differential equation and focus on the control of the velocity but not on the residual stresses. Then, in [9], the authors addressed the problem of control for both the velocity and the stress of a viscoelastic material. Precisely, they stated an approximate controllability result for fluids of the Jeffreys kind (system (1.6)) and null approximate controllability results for fluids of the Maxwell kind (that is system (1.6) with $\eta = 0$ in $\Sigma(u, p)$). In [27], Renardy considered one-dimensional shear flows of multimode linear Maxwell and Jeffreys fluids (see (6.6)) with a distributed control. Exact controllability for single-mode Maxwell fluids, approximate controllability of multimode Maxwell and Jeffreys fluids hold when the control is restricted to a subinterval and exact controllability in the case of several relaxation modes holds if the control is on the entire interval. In [8], the authors considered higher-dimensional single mode Jeffreys fluid and established the approximate controllability result in an arbitrarily small time only for the velocity with distributed or boundary controls supported by arbitrarily small sets. The controllability properties for both the velocity and the stress of a single mode Maxwell model in higher dimension with distributed or boundary controls is analyzed in [2]: they proved the large time approximate controllability and under a geometric condition imposed on the controlled region along with an additional restriction on the constants λ and κ, a large time exact controllability was also established. Several improvements of these results on controllability of [8, 2] have been done in [5] where the authors proved the approximate controllability of both the velocity and the stress for single and multi-mode Jeffreys fluids. Under the usual geometric condition on the controlled domain, they established the exact controllability with interior controls for the single mode and the approximate controllability for multimode Maxwell models. The lack of null controllability of Jeffreys and Maxwell fluids with distributed control is considered in [21]. They established that the solution of single mode and multimode Jeffreys systems cannot reach zero. Due to the finite speed of propagation property, the single mode Maxwell system cannot be null controllable for small time whereas the multimode Maxwell system is not null controllable for any time.

In all these results for linear viscoelastic flows in higher dimensions, control of the stress tensor is possible only under the constraint that the stress is the symmetric part of a gradient. This is the main hindrance to work with nonlinear viscoelastic model and only few results are available [28, 29, 33] regarding the characterization of the set of reachable states.

Let us mention some works concerning the controllability of fluid-structure interaction system. Regarding one-dimensional viscous Burgers-particle system, Doubova and Fernández-Cara proved in [7] that the local null controllability holds by boundary controls acting on both ends of the finite interval. Later, Liu et. al. [20] improved this result by using only one control (located at one end of the interval). A simplified 2D model where the fluid equations are replaced by the Helmholtz equations and the structure is modeled by a harmonic oscillator is considered in [25] and the exact controllability is established with an internal control acting only in the fluid part. In the case of a 2D fluid-structure system where a rigid ball is moving inside a viscous, incompressible Navier-Stokes fluid, the exact controllability with an internal control in the fluid equation is proved in [16]. In [4], Boulakia and Osses obtained the same result but for a body of more general shape. These results have been extended to 3D and for a general shaped rigid body in [3]. Finally, the authors in [30] prove the local null controllability for a 2D Boussinesq flow in interaction with a rigid body by using controls acting only on the temperature equation.

The above mentioned works related to fluid-structure interaction systems correspond to the case where the control acts on the fluid. Some articles are available concerning the case where the control is supported on the structure. In [6], the authors deal with the one-dimensional case and the null controllability for the velocities of the fluid and of the particle and the approximate controllability for the position of the particle are established with a control acting only on the particle. Note that in this result, no smallness assumption is considered, but the time of controllability can be large and may depends on the initial data and the final data. In [23], the authors show that this time of controllability can be uniform with respect to the initial data. In [24], the structure is a deformable beam located at the boundary of the fluid domain and the author obtains the local stabilization of the corresponding system. In [35] (respectively [31]), an open stabilization result is proved in the case of rigid ball moving into viscous incompressible (respectively compressible) fluid with a spring-damper type control.

In the literature, there are no available controllability or stabilizability results concerning the motion of a rigid body inside a viscoelastic fluid. In this article, we want to explore the control properties for fluid-structure system (1.12)-(1.19). This is the first result in the context of controllability and stabilizability of viscoelastic fluid-rigid body interaction problem. More precisely, our main result is the following one

Theorem 1.1. *Assume $\mathcal{O}$ is a nonempty open subset of $\mathcal{F}$. Then the linear system* (1.12)-(1.19) *is approximately controllable. Assume moreover that*

$$0 < \beta < \lambda + \frac{\kappa}{\eta}.$$

Then the system (1.12)-(1.19) *is exponentially stabilizable with rate $-\beta$ and with a feedback of finite dimension: there exists*

$$[\phi^j, \psi^j, \alpha^j] \in \mathbb{H} \times \mathbb{H}_1 \times \mathbb{R}^6, \quad w^j \in L^2(\mathcal{O}) \quad (j = 1, \ldots, K), \quad C > 0$$

such that the system (1.12)-(1.19) *with*

$$f = \sum_{j=1}^{K} \left(\langle \phi^j, u \rangle_{\mathbb{H}} + \langle \psi^j, v \rangle_{\mathbb{H}_1} + \langle \alpha^j, (h, \theta) \rangle_{\mathbb{R}^6} \right) w^j$$

admits a unique solution $[u, v, h, \theta] \in C^0([0, \infty); \mathbb{H} \times \mathbb{H}_1 \times \mathbb{R}^6)$, *with*

$$\|u(t,\cdot)\|_{\mathbb{H}} + \|v(t,\cdot)\|_{\mathbb{H}_1} + \|(h(t),\theta(t))\|_{\mathbb{R}^6} \leqslant Ce^{-\beta t}\left(\left\|u^0\right\|_{\mathbb{H}} + \left\|v^0\right\|_{\mathbb{H}_1} + \left\|(h^0,\theta^0)\right\|_{\mathbb{R}^6}\right) \quad (t \geqslant 0).$$

This result will be a consequence of a general result on approximate controllability that we state and prove in Section 2. We start with an abstract operator form and we analyze spectral properties of the corresponding operators. The approximate controllability and the exponential stabilizability of the abstract system follow from the well-known Fattorini criterion (see, for instance [1]). We extend this abstract result from single mode to the multiple modes framework in Section 3. The analysis concerning behavior of eigenvalues and generalized eigenfunctions is more technical in this multimode part. Section 4 is devoted to the proof of the main result of the paper. The idea is to define appropriate operators and spaces corresponding to the fluid-structure interaction system so that we can apply our abstract result obtained in Section 2. Precisely, the Fattorini criterion reduces to a unique continuation problem for fluid-structure interaction system that we deal in this section. In Section 5, we show that if the support of the control is not $\mathcal{F}$, then there exists an initial data such that for any control, the solution of the fluid-structure interaction system cannot be brought to a smooth trajectory in finite time and hence that the system is not null controllable. Finally, we take advantage of our abstract result in Section 2 to mention some possible extensions of the main result in Section 6: for instance the case of controls with a null component and the case of a fluid-rigid body system when the fluid follows a linear Jeffreys model with several relaxation mode.

The main novelties that we bring in this article are :

- In the literature, only existence results in appropriate spaces for the fluid-structure interaction system where a rigid body is moving inside a viscoelastic fluid are available (see [13, 32]). At the best of our knowledge, the controllability and stabilizability properties of such system have not yet been studied in the literature.
- We prove some general results in abstract framework on approximate controllability and stabilizability of coupled systems. As an application of that abstract result, the approximate controllability and exponential stabilizability results for the interaction between a single mode Jeffreys fluid (linearized version of nonlinear viscoelastic model) and a rigid body are established. Note that our abstract results permit to recover previous results in the case of a fluid without structure.
- Further, we show that the system is not exactly null controllable and hence the approximate controllability and the exponential stabilizability are the best possible results for the system in this direction.
- We have also extended our results to the interaction between Jeffreys fluid with multiple relaxation modes and a rigid body and the case of controls with a null-component.

2 An abstract result

In this section, we state and prove a general result that will imply Theorem 1.1.

Let us consider $\mathbb{H}, \mathbb{U}$ Hilbert spaces, $A_0 : \mathcal{D}(A_0) \to \mathbb{H}$ a self-adjoint positive operator with compact resolvent. We use the following notation:

$$\mathbb{H}_\gamma := \begin{cases} \mathcal{D}(A_0^\gamma) & \text{if } \gamma \geqslant 0, \\ \mathcal{D}(A_0^{-\gamma})' & \text{if } \gamma < 0, \end{cases}$$

where V' stands for the dual space of V with respect to the pivot space $\mathbb{H}$.

We also assume $B_0 \in \mathcal{L}(\mathbb{U}, \mathbb{H})$ and $C_0 \in \mathcal{L}(\mathbb{H}, \mathbb{R}^N)$, where $N \in \mathbb{N}^*$. From standard results on parabolic equations, we deduce from the above hypotheses that if $f \in L^2(0,T;\mathbb{U})$ and $u^0 \in \mathbb{H}_{1/2}$ then there exists a unique solution

$$u \in H^1(0,T;\mathbb{H}) \cap L^2(0,T;\mathbb{H}_1) \cap C^0([0,T];\mathbb{H}_{1/2})$$

of the Cauchy problem

$$\dot{u} + A_0 u = B_0 f, \quad t \in (0,T), \quad u(0) = u^0. \tag{2.1}$$

Let us recall the definition of (A_0, B_0) approximately controllable in time $T > 0$: for any $u^0, u^1 \in \mathbb{H}_{1/2}$ and $\varepsilon > 0$, there exists $f \in L^2(0,T;\mathbb{U})$ such that the solution u of (2.1) satisfies $\|u(T) - u^1\|_{\mathbb{H}_{1/2}} < \varepsilon$. Using the standard criterion of Fattorini-Hautus that we recall below, the time $T > 0$ in the above definition can be chosen arbitrary.

We consider $\eta, \lambda, \kappa \in \mathbb{R}_+^*$ and the following system

$$\left\{\begin{array}{rcl} \dot{u} + \eta A_0 u + A_0 v &=& B_0 f, \\ \dot{v} + \lambda v &=& \kappa u, \\ \dot{a} &=& C_0 u, \end{array}\right. \quad (t>0), \quad \left\{\begin{array}{rcl} u(0) &=& u^0, \\ v(0) &=& v^0, \\ a(0) &=& a^0. \end{array}\right. \tag{2.2}$$

In the above system, $Z = [u, v, a]$ is the state and f is the control. We will see in Section 4 that the system (1.12)-(1.19) can be written under the above form.

We can write (2.2) as follows:

$$\dot{Z} = AZ + Bf, \quad t > 0, \quad Z(0) = Z^0, \tag{2.3}$$

by defining

$$\mathbb{X} := \mathbb{H} \times \mathbb{H}_1 \times \mathbb{R}^N, \quad \mathcal{D}(A) := \mathbb{H}_1 \times \mathbb{H}_1 \times \mathbb{R}^N, \quad A := \begin{bmatrix} -\eta A_0 & -A_0 & 0 \\ \kappa\,\mathrm{I} & -\lambda\,\mathrm{I} & 0 \\ C_0 & 0 & 0 \end{bmatrix}, \quad B := \begin{bmatrix} B_0 \\ 0 \\ 0 \end{bmatrix}, \tag{2.4}$$

and

$$Z = [u, v, a], \quad Z^0 = [u^0, v^0, a^0].$$

First, we have the following result on A:

Proposition 2.1. *The operator $(A, \mathcal{D}(A))$, defined by (2.4) generates an analytic semigroup in $\mathbb{X}$. Its adjoint in $\mathbb{Y} := \mathbb{H} \times \mathbb{H}_{-1} \times \mathbb{R}^N$ is given by*

$$\mathcal{D}(A^*) := \{[\phi, \psi, \alpha] \in \mathbb{Y} \ ; \ -\eta A_0 \phi + \kappa \psi \in \mathbb{H}\}, \tag{2.5}$$

$$A^* = \begin{bmatrix} -\eta A_0 & \kappa\,\mathrm{I} & C_0^* \\ -A_0 & -\lambda\,\mathrm{I} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{2.6}$$

Proof. First we split A into two parts:

$$A = A_1 + A_2, \quad A_1 := \begin{bmatrix} -\eta A_0 & 0 & 0 \\ \kappa\,\mathrm{I} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 := \begin{bmatrix} 0 & -A_0 & 0 \\ 0 & -\lambda\,\mathrm{I} & 0 \\ C_0 & 0 & 0 \end{bmatrix}$$

and we note that $A_2 \in \mathcal{L}(\mathbb{X})$. Then, using that A_0 is self-adjoint and positive, we deduce that there exists $\vartheta \in (\pi/2, \pi)$ such that for all $\mu \in \mathbb{C}^*$, $|\arg \mu| < \vartheta$,

$$\left\| (\mu\,\mathrm{I} - A_1)^{-1} \right\|_{\mathcal{L}(\mathbb{X})} \leqslant \frac{C}{|\mu|}$$

which yields that A_1 generates an analytic semigroup in $\mathbb{X}$. We then apply Theorem 2.1 in [22, p.80], to deduce that $A = A_1 + A_2$ also generates an analytic semigroup in $\mathbb{X}$.

Formulas (2.5) and (2.6) are obtained by standard computation. □

From the above result and standard results on parabolic equations, we deduce that if

$$Z^0 = [u^0, v^0, a^0] \in \mathcal{D}(A^{1/2}) = \mathbb{H}_{1/2} \times \mathbb{H}_1 \times \mathbb{R}^N, \quad f \in L^2(0, T; \mathbb{U}),$$

then (2.2) admits a unique solution

$$u \in H^1(0, T; \mathbb{H}) \cap L^2(0, T; \mathbb{H}_1) \cap C^0([0, T]; \mathbb{H}_{1/2}), \quad v \in H^1(0, T; \mathbb{H}_1), \quad a \in H^1(0, T; \mathbb{R}^N).$$

We say that (2.2) is approximately controllable in time $T > 0$ if for any

$$u^0, u^1 \in \mathbb{H}_{1/2}, \quad v^0, v^1 \in \mathbb{H}_1, \quad a^0, a^1 \in \mathbb{R}^N,$$

and $\varepsilon > 0$, there exists $f \in L^2(0, T; \mathbb{U})$ such that the solution $[u, v, a]$ of (2.2) satisfies

$$\|u(T) - u^1\|_{\mathbb{H}_{1/2}} + \|v(T) - v^1\|_{\mathbb{H}_1} + \|a(T) - a^1\|_{\mathbb{R}^N} < \varepsilon.$$

We are also interested by stabilization results: we say that the system (2.2) is stabilizable in $\mathbb{X}$ with a rate lower than $-\beta$ if there exists $F_\beta \in \mathcal{L}(\mathbb{X}, \mathbb{U})$ such that $A_\beta := A + BF_\beta$ with domain $\mathcal{D}(A_\beta) = \mathcal{D}(A)$ is the infinitesimal generator of an analytic and exponentially stable semigroup on $\mathbb{X}$ of type lower than $-\beta$. In our main result, we obtain a feedback operator F_β of finite rank K. In that case, there exists

$$[\phi^j, \psi^j, \alpha^j] \in \mathbb{X}, \quad w^j \in \mathbb{U} \quad (j = 1, \ldots, K), \quad C > 0$$

such that for any

$$Z^0 = [u^0, v^0, a^0] \in \mathbb{X},$$

the system (2.2) with

$$f = \sum_{j=1}^{K} \langle [\phi^j, \psi^j, \alpha^j], [u, v, a] \rangle_{\mathbb{X}} w^j$$

admits a unique solution $[u, v, a] \in C^0([0, \infty); \mathbb{X})$, with

$$\| [u(t), v(t), a(t)] \|_{\mathbb{X}} \leqslant Ce^{-\beta t} \left\| [u^0, v^0, a^0] \right\|_{\mathbb{X}} \quad (t \geqslant 0).$$

Our main result in this section is the following theorem:

Theorem 2.2. *Assume (A_0, B_0) is approximately controllable and that $B_0^* A_0^{-1} C_0^* : \mathbb{R}^N \to \mathbb{U}$ is injective. Then the system (2.2) is approximately controllable for any $T > 0$.*

Assume

$$0 < \beta < \lambda + \frac{\kappa}{\eta}. \tag{2.7}$$

Then, the system (2.2) is stabilizable in $\mathbb{X}$ with a rate lower than $-\beta$. More precisely, by denoting

$$K := \max\{\dim \ker(\mu \operatorname{I} - A) \; ; \; \mu \in \sigma(A), \; \operatorname{Re}(\mu) \geqslant -\beta\} < \infty,$$

there exists $F_\beta \in \mathcal{L}(\mathbb{X}, \mathbb{U})$ with $\operatorname{rank}(F_\beta) = K$ such that $A_\beta := A + BF_\beta$ with domain $\mathcal{D}(A_\beta) = \mathcal{D}(A)$ is the infinitesimal generator of an analytic and exponentially stable semigroup on $\mathbb{X}$ of type lower than $-\beta$.

The proof of Theorem 2.2 mainly relies on the application of the Fattorini criterion, see [1]. Let us recall the hypotheses of [1].

($\mathcal{H}_1$) The spectrum of A consists of isolated eigenvalues with finite algebraic multiplicity.

($\mathcal{H}_2$) The family of root vectors of A is complete in $\mathbb{X}$.

($\mathcal{H}_3$) The semigroup (e^{tA}) is analytic.

Then [1, Theorem 1.3] states as follows:

Theorem 2.3. *Assume ($\mathcal{H}_1$)–($\mathcal{H}_3$). Then (A, B) is approximately controllable if and only if*

$$A^* \xi = \mu \xi \quad \text{and} \quad B^* \xi = 0 \implies \xi = 0. \tag{2.8}$$

Adding the following hypothesis on the spectrum of A, we can obtain a stabilization result.

($\mathcal{H}_4$) There exists $\beta_0 > 0$ such that the spectrum of A has no cluster point in

$$\{z \in \mathbb{C} \; ; \; \operatorname{Re} z > -\beta_0\}.$$

More precisely, [1, Theorem 1.6] implies the following result:

Theorem 2.4. *Assume ($\mathcal{H}_1$), ($\mathcal{H}_3$) and ($\mathcal{H}_4$). Let us consider $\beta \in (0, \beta_0)$ and*

$$K := \max\{\dim \ker(\mu \operatorname{I} - A) \; ; \; \mu \in \sigma(A), \; \operatorname{Re}(\mu) \geqslant -\beta\} < \infty.$$

Assume

$$\forall \mu \in \mathbb{C}, \; \operatorname{Re} \mu \geqslant -\beta, \quad A^* \xi = \mu \xi \quad \text{and} \quad B^* \xi = 0 \implies \xi = 0. \tag{2.9}$$

Then, there exists $F_\beta \in \mathcal{L}(\mathbb{X}, \mathbb{U})$ with $\operatorname{rank}(F_\beta) = K$ such that $A_\beta := A + BF_\beta$ with domain $\mathcal{D}(A_\beta) = \mathcal{D}(A)$ is the infinitesimal generator of an analytic and exponentially stable semigroup on $\mathbb{X}$ of type lower than $-\beta$.

Remark 2.5. *In fact [1, Theorem 1.6] is stated in the case where there is no cluster point in the spectrum of A, but the proof of [1, Theorem 1.6] implies the above result.*

In order to prove Theorem 2.2, let us start by studying the spectral properties of A^*. With the properties of A_0, the spectrum of A_0 is reduced to a nondecreasing sequence of eigenvalues $\{\Lambda_k\}_{k\geqslant 1} \subset \mathbb{R}_+^*$ such that $\Lambda_k \to \infty$ as $k \to \infty$. Moreover, there exists a corresponding orthonormal basis of $\mathbb{H}$ composed of eigenvectors w_k with

$$A_0 w_k = \Lambda_k w_k \quad (k \geqslant 1). \tag{2.10}$$

Using the above notation, we can obtain the eigenvalues of A^*. More precisely, let us define

$$\delta_k := (\lambda + \eta\Lambda_k)^2 - 4\Lambda_k(\eta\lambda + \kappa), \tag{2.11}$$

$$K_- := \{k \geqslant 1 \ ; \ \delta_k < 0\}, \quad K_0 := \{k \geqslant 1 \ ; \ \delta_k = 0\}, \quad K_+ := \{k \geqslant 1 \ ; \ \delta_k > 0\}. \tag{2.12}$$

Note that, since $\Lambda_k \to \infty$, K_- and K_0 are finite sets whereas K_+ is an infinite set.

Then we write

$$\mu_0 := 0 \tag{2.13}$$

$$\mu_k^1 := \frac{-(\lambda + \eta\Lambda_k) + i\sqrt{-\delta_k}}{2}, \quad \mu_k^2 := \frac{-(\lambda + \eta\Lambda_k) - i\sqrt{-\delta_k}}{2}, \quad (k \in K_-) \tag{2.14}$$

$$\mu_k := \frac{-(\lambda + \eta\Lambda_k)}{2}, \quad (k \in K_0), \tag{2.15}$$

and

$$\mu_k^1 := \frac{-(\lambda + \eta\Lambda_k) + \sqrt{\delta_k}}{2}, \quad \mu_k^2 := \frac{-(\lambda + \eta\Lambda_k) - \sqrt{\delta_k}}{2}, \quad (k \in K_+). \tag{2.16}$$

One can check that

$$\mu_k < 0 \quad (k \in K_0), \quad \operatorname{Re} \mu_k^j < 0 \quad (k \in K_- \cup K_+, \ j = 1, 2), \tag{2.17}$$

$$\mu_k + \lambda \neq 0 \quad (k \in K_0), \quad \mu_k^j + \lambda \neq 0 \quad (k \in K_- \cup K_+, \ j = 1, 2), \tag{2.18}$$

$$\lim_{k\to\infty} \mu_k^1 = -\left(\lambda + \frac{\kappa}{\eta}\right), \quad \lim_{k\to\infty} \mu_k^2 = -\infty. \tag{2.19}$$

We also define

$$\xi_0^{*j} = \begin{bmatrix} \lambda A_0^{-1} C_0^* e_j \\ -C_0^* e_j \\ (\lambda\eta + \kappa) e_j \end{bmatrix} \quad (1 \leqslant j \leqslant N), \quad \xi_k^{*j} = \begin{bmatrix} \lambda + \mu_k^j \\ -\Lambda_k \\ 0 \end{bmatrix} w_k \quad (k \in K_- \cup K_+, \ j = 1, 2), \tag{2.20}$$

$$\xi_k^{*1} = \begin{bmatrix} \lambda + \mu_k \\ -\Lambda_k \\ 0 \end{bmatrix} w_k, \quad \xi_k^{*2} = \begin{bmatrix} 0 \\ (\lambda + \mu_k)/\kappa \\ 0 \end{bmatrix} w_k \quad (k \in K_0). \tag{2.21}$$

Then we have the following property

Proposition 2.6. *The eigenvalues of A^* are given by*

$$\mu_0, \quad \mu_k^j \quad (k \in K_- \cup K_+, j = 1, 2), \quad \mu_k \quad (k \in K_0).$$

A corresponding family of generalized eigenvectors of A^* associated with the above eigenvalues is given by (2.20), (2.21). *More precisely, we have the following formula*

$$A^* \xi_0^{*j} = 0 \quad (1 \leqslant j \leqslant N), \quad A^* \xi_k^{*j} = \mu_k^j \xi_k^{*j} \quad (k \in K_- \cup K_+, \ j = 1, 2), \tag{2.22}$$

$$A^* \xi_k^{*1} = \mu_k \xi_k^{*1}, \quad A^* \xi_k^{*2} = \mu_k \xi_k^{*2} + \xi_k^{*1}, \quad (k \in K_0). \tag{2.23}$$

If $A^* \xi^* = \mu \xi^*$, then ξ^* is a linear combination of

$$\xi_0^{*j} \ (1 \leqslant j \leqslant N), \quad \xi_k^{*j} \ (k \in K_- \cup K_+, j = 1, 2), \quad \textit{and} \quad \xi_k^{*1} \ (k \in K_0). \tag{2.24}$$

Proof. Assume that

$$A^* \begin{bmatrix} \phi \\ \psi \\ \alpha \end{bmatrix} = \mu \begin{bmatrix} \phi \\ \psi \\ \alpha \end{bmatrix}. \tag{2.25}$$

First assume that $\mu \neq 0$. Then we have $\alpha = 0$, $\kappa\psi = \mu\phi + \eta A_0 \phi$ and

$$-\left(\eta(\lambda + \mu) + \kappa\right) A_0 \phi = \mu(\lambda + \mu)\phi.$$

- Case 1: $\phi = 0$. Then $[\phi, \psi, \alpha] = 0$.
- Case 2: $\phi \neq 0$ and $\eta(\lambda + \mu) + \kappa = 0$. Then $\mu(\lambda + \mu) = 0$ and since $\mu \neq 0$, $\kappa > 0$, this leads to a contradiction.
- Case 3: $\phi \neq 0$ and $\eta(\lambda + \mu) + \kappa \neq 0$. Then, from (2.10), we deduce that μ satisfy for some $k \geqslant 1$ the equation

$$(\mu + \lambda)(\mu + \eta\Lambda_k) + \kappa\Lambda_k = 0. \tag{2.26}$$

Then solving this equation in μ leads to formula (2.11), (2.14)-(2.16). Standard computation yields that $[\phi, \psi, \alpha]$ is a linear combination of

$$\xi_k^{*j} \ (k \in K_- \cup K_+, j = 1, 2), \quad \text{and} \quad \xi_k^{*1} \ (k \in K_0).$$

Second, assume that $\mu = 0$ in (2.25). Then, if $\alpha = 0$, we deduce that $[\phi, \psi, \alpha] = 0$. Else, we can check that $[\phi, \psi, \alpha]$ is a linear combination of ξ_0^{*j} $(1 \leqslant j \leqslant N)$ given by (2.20). ☐

Similarly, we can define

$$\xi_0^j = \begin{bmatrix} 0 \\ 0 \\ e_j \end{bmatrix} \quad (1 \leqslant j \leqslant N), \quad \xi_k^j = \begin{bmatrix} \overline{\mu_k^j}\left(\lambda + \overline{\mu_k^j}\right) w_k \\ \kappa \overline{\mu_k^j} w_k \\ \left(\lambda + \overline{\mu_k^j}\right) C_0 w_k \end{bmatrix} \quad (k \in K_- \cup K_+, j = 1, 2), \tag{2.27}$$

$$\xi_k^1 = \begin{bmatrix} \mu_k \left(\lambda + \mu_k\right) w_k \\ \kappa \mu_k w_k \\ \left(\lambda + \mu_k\right) C_0 w_k \end{bmatrix}, \quad \xi_k^2 = \begin{bmatrix} \left(\lambda + \mu_k\right) w_k \\ (\kappa\lambda)\left(\lambda + \mu_k\right)^{-1} w_k \\ 0 \end{bmatrix} \quad (k \in K_0). \tag{2.28}$$

Then standard computation shows that

$$A\xi_0^j = 0 \quad j = 1, \ldots, N, \quad A\xi_k^j = \overline{\mu_k^j}\xi_k^j \quad (k \in K_- \cup K_+, j = 1, 2), \tag{2.29}$$

$$A\xi_k^1 = \mu_k \xi_k^1, \quad A\xi_k^2 = \mu_k \xi_k^2 + \xi_k^1, \quad (k \in K_0), \tag{2.30}$$

Moreover we have the following result:

Proposition 2.7. *The family of root vectors of A is complete:*

$$\operatorname{span}\left(\left\{\xi_k^j\right\}_{k \geqslant 1, j = 1, 2} \cup \left\{\xi_0^j\right\}_{j = 1, \ldots, N}\right)$$

is dense in $\mathbb{X}$.

Proof. Assume $[f, g, \alpha] \in \mathbb{Y}$ is such that

$$\langle [f, g, \alpha], \xi_k^j \rangle = 0 \quad (k \geqslant 1, j = 1, 2), \quad \langle [f, g, \alpha], \xi_0^j \rangle = 0 \quad (j = 1, \ldots, N). \tag{2.31}$$

From (2.27), we deduce that $\alpha = 0$.

Now, we decompose f, g in the orthogonal basis (w_k) :

$$f = \sum_{k \geqslant 1} f_k w_k, \quad g = \sum_{k \geqslant 1} g_k w_k,$$

and we deduce from (2.27) and (2.28) the following relations:

$$\begin{bmatrix} \overline{\mu_k^j}\left(\lambda + \overline{\mu_k^j}\right) \\ \kappa \overline{\mu_k^j} \end{bmatrix} \cdot \begin{bmatrix} f_k \\ g_k \end{bmatrix} = 0 \quad (k \in K_- \cup K_+, j = 1, 2), \tag{2.32}$$

$$\begin{bmatrix} \mu_k \left(\lambda + \mu_k\right) \\ \kappa \mu_k \end{bmatrix} \cdot \begin{bmatrix} f_k \\ g_k \end{bmatrix} = 0, \quad \begin{bmatrix} \left(\lambda + \mu_k\right) \\ (\kappa\lambda)\left(\lambda + \mu_k\right)^{-1} \end{bmatrix} \cdot \begin{bmatrix} f_k \\ g_k \end{bmatrix} = 0 \quad (k \in K_0). \tag{2.33}$$

Using (2.14)–(2.16), we deduce $f_k = g_k = 0$ for all $k \geqslant 1$. ☐

We are now in a position to prove Theorem 2.2:

Proof of Theorem 2.2. We apply Theorem 2.3 (that is [1, Theorem 1.3]). First, since (A_0, B_0) is approximately controllable and satisfies $(\mathcal{H}_1)$–$(\mathcal{H}_3)$, we deduce that $B_0^* w_k \neq 0$ for all $k \geqslant 1$.

Now, from Proposition 2.1, Proposition 2.6 and Proposition 2.7, (A, B) satisfies $(\mathcal{H}_1)$–$(\mathcal{H}_3)$. We can thus apply again Theorem 2.3 and use the Fattorini criterion to prove the approximate controllability of (2.2).

From (2.4), (2.20)-(2.21) and (2.18), we have

$$B^* \xi_0^{*j} = \lambda B_0^* A_0^{-1} C_0^* e_j \ \neq 0 \quad j = 1, \dots, N,$$

$$B^* \xi_k^{*j} = (\lambda + \mu_k^j) B_0^* w_k \neq 0 \quad (k \in K_+ \cup K_-, j = 1, 2),$$
$$B^* \xi_k^{*1} = (\lambda + \mu_k) B_0^* w_k \neq 0 \quad (k \in K_0).$$

Using Proposition 2.6, we deduce that (2.8) holds true and that (A, B) is approximately controllable.

To obtain the stabilizability result, we apply Theorem 2.4 (that is, [1, Theorem 1.6]) combined with (2.19). □

3 Case of multiple modes

We can generalize the result obtained in Section 2. Let us consider the same hypotheses on $\mathbb{H}, \mathbb{U}$, A_0, B_0 and C_0. We consider $\eta > 0$, $M \in \mathbb{N}^*$, $\lambda_i, \kappa_i \in \mathbb{R}_+^*$ $(i \in \{1, \dots, M\}$. We assume that λ_i are distinct and for instance we assume

$$0 < \lambda_1 < \ldots < \lambda_M. \tag{3.1}$$

We consider the following generalization of (2.2):

$$\left\{ \begin{array}{rcl} \dot{u} + \eta A_0 u + \displaystyle\sum_{i=1}^{M} A_0 v_i & = & B_0 f, \\ \dot{v}_i + \lambda_i v_i & = & \kappa_i u \quad (i \in \{1, \dots, M\}), \\ \dot{a} & = & C_0 u. \end{array} \right. \quad (t > 0), \quad \left\{ \begin{array}{rcl} u(0) & = & u^0, \\ v_i(0) & = & v_i^0 \quad (i \in \{1, \dots, M\}), \\ a(0) & = & a^0. \end{array} \right. \tag{3.2}$$

Setting $Z = [u, v_1, \dots, v_M, a]$, we can write (3.2) under the form

$$\dot{Z} = AZ + Bf, \quad t > 0, \quad Z(0) = Z^0$$

by defining

$$\mathbb{X} := \mathbb{H} \times \mathbb{H}_1^M \times \mathbb{R}^N, \quad \mathcal{D}(A) := \mathbb{H}_1 \times \mathbb{H}_1^M \times \mathbb{R}^N, \tag{3.3}$$

$$A := \begin{bmatrix} -\eta A_0 & -A_0 & -A_0 & \cdots & -A_0 & 0 \\ \kappa_1 \,\mathrm{I} & -\lambda_1 \,\mathrm{I} & 0 & \cdots & 0 & 0 \\ \kappa_2 \,\mathrm{I} & 0 & -\lambda_2 \,\mathrm{I} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \kappa_M \,\mathrm{I} & 0 & 0 & \cdots & -\lambda_M \,\mathrm{I} & 0 \\ C_0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad B := \begin{bmatrix} B_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{3.4}$$

and

$$Z = [u, v_1, \dots, v_M, a], \quad Z^0 = [u^0, v_1^0, \dots, v_M^0, a^0]$$

In order to state our main result, let us consider the function

$$\mu \mapsto \eta + \sum_{j=1}^{M} \frac{\kappa_j}{\lambda_j + \mu}. \tag{3.5}$$

One can check that it has exactly M roots $(\mu^j)_{j=1}^M$ that are real and satisfy

$$\mu^M < -\lambda_M < \mu^{M-1} < -\lambda_{M-1} < \cdots < \mu^1 < -\lambda_1.$$

The definitions of approximate controllability and stabilization are the same as in the previous section. Our main result in this section is the following theorem:

Theorem 3.1. *Assume (A_0, B_0) is approximately controllable and that $B_0^* A_0^{-1} C_0^* : \mathbb{R}^N \to \mathbb{U}$ is injective. Then the system (3.2) is approximately controllable for any $T > 0$.*

Moreover, for any

$$0 < \beta < -\mu^1 \tag{3.6}$$

the system (3.2) is stabilizable in $\mathbb{X}$ with a rate lower than $-\beta$. More precisely, by denoting

$$K := \max\{\dim\ker(\mu \mathrm{I} - A) \ ; \ \mu \in \sigma(A), \ \mathrm{Re}(\mu) \geqslant -\beta\} < \infty,$$

there exists $F_\beta \in \mathcal{L}(\mathbb{X}, \mathbb{U})$ with $\mathrm{rank}(F_\beta) = K$ such that $A_\beta := A + BF_\beta$ with domain $\mathcal{D}(A_\beta) = \mathcal{D}(A)$ is the infinitesimal generator of an analytic and exponentially stable semigroup on $\mathbb{X}$ of type lower than $-\beta$.

The proof of Theorem 3.1 follows closely the proof of Theorem 2.2. First, following the proof of Proposition 2.1, we deduce

Proposition 3.2. *The operator $(A, \mathcal{D}(A))$, defined by (3.3)–(3.4) generates an analytic semigroup in $\mathbb{X}$. Its adjoint in $\mathbb{Y} := \mathbb{H} \times \mathbb{H}_{-1}^M \times \mathbb{R}^N$ is given by*

$$\mathcal{D}(A^*) := \left\{ [\phi, \psi_1, \ldots, \psi_M, \alpha] \in \mathbb{Y} \ ; \ -\eta A_0 \phi + \sum_{j=1}^M \kappa_j \psi_j \in \mathbb{H} \right\}, \tag{3.7}$$

$$A^* = \begin{bmatrix} -\eta A_0 & \kappa_1 \mathrm{I} & \kappa_2 \mathrm{I} & \cdots & \kappa_M \mathrm{I} & C_0^* \\ -A_0 & -\lambda_1 \mathrm{I} & 0 & \cdots & 0 & 0 \\ -A_0 & 0 & -\lambda_2 \mathrm{I} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -A_0 & 0 & 0 & \cdots & -\lambda_M \mathrm{I} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{3.8}$$

Then, we use again the families (Λ_k, w_k) associated with A_0 (see (2.10)), but we do not have here explicit formula such as (2.13)–(2.16). We have the following result:

Proposition 3.3. *The eigenvalues of A^* are $\mu_0 = 0$ and the roots μ_k^j ($j = 1, \ldots, M+1$, $k \geqslant 1$) of the function*

$$\mu \mapsto \frac{\mu}{\Lambda_k} + \eta + \sum_{i=1}^M \frac{\kappa_i}{\mu + \lambda_i}. \tag{3.9}$$

There exists $k_0 \geqslant 1$ such that for $k \geqslant k_0$ these roots are real negative with

$$\mu_k^{M+1} < -G < \mu_k^M < -\lambda_M < \mu_k^{M-1} < -\lambda_{M-1} < \cdots < \mu_k^1 < -\lambda_1, \tag{3.10}$$

where $G > 0$ is independent of k. We have

$$\mu_k^j \to \mu^j \quad (j = 1, \ldots, M), \quad \mu_k^{M+1} \sim -\eta \Lambda_k. \tag{3.11}$$

The eigenvectors of A^* are of the form

$$\left[\phi, -\frac{A_0 \phi}{\mu + \lambda_1}, \ldots, -\frac{A_0 \phi}{\mu + \lambda_M}, 0 \right] \tag{3.12}$$

with ϕ an eigenvector of A_0 or

$$\left[A_0^{-1} C_0^* \alpha, -\frac{C_0^* \alpha}{\lambda_1}, \ldots, -\frac{C_0^* \alpha}{\lambda_M}, \left(\eta + \sum_{i=1}^M \frac{\kappa_i}{\lambda_i} \right) \alpha \right]. \tag{3.13}$$

Finally, if $\mu_k^j = \mu_{k'}^{j'}$, then $\Lambda_k = \Lambda_{k'}$ and in particular, the algebraic multiplicity of the eigenvalues of A^* are finite.

Proof. Assume that

$$A^* \begin{bmatrix} \phi \\ \psi_1 \\ \vdots \\ \psi_M \\ \alpha \end{bmatrix} = \mu \begin{bmatrix} \phi \\ \psi_1 \\ \vdots \\ \psi_M \\ \alpha \end{bmatrix}. \tag{3.14}$$

Case 1: assume that $\mu \neq 0$. Then we have $\alpha = 0$. If there exists $i_0 \in \{1, \dots, M\}$ such that $\mu = -\lambda_{i_0}$, then $[\phi, \psi_1, \dots, \psi_M, \alpha] = 0$. We thus assume that for all $i \in \{1, \dots, M\}$, $\mu + \lambda_i \neq 0$. Again, if $\phi = 0$, then $[\phi, \psi_1, \dots, \psi_M, \alpha] = 0$. We thus obtain that

$$\psi_i = -\frac{A_0 \phi}{\mu + \lambda_i},$$

and that there exists k such that μ is a root of (3.9) and such that ϕ is an eigenvector associated with Λ_k.

In particular, μ is a root of the polynomial

$$P_k(X) := \left(\frac{X}{\Lambda_k} + \eta\right) \prod_{i=1}^{M} (X + \lambda_i) + \sum_{i=1}^{M} \kappa_i \prod_{\ell \neq i}^{M} (X + \lambda_\ell).$$

One can check that P_k admits a real root in the interval $(-\lambda_{i+1}, -\lambda_i)$, $i = 1, \dots, M-1$. Since $\Lambda_k \to \infty$, there exists $k_0 \geqslant 1$ such that

$$\Lambda_k > \max\left(\frac{2^{M+2}}{\eta^2} \sum_{i=1}^{M} \kappa_i, \frac{4\lambda_M}{\eta}\right) \quad (k \geqslant k_0). \tag{3.15}$$

Let us consider G such that

$$\max\left(\frac{2^{M+1}}{\eta} \sum_{i=1}^{M} \kappa_i, 2\lambda_M\right) < G < \frac{\eta \Lambda_{k_0}}{2}. \tag{3.16}$$

With this choice, we can check that P_k has a real root in the interval $(-G, -\lambda_M)$. In that case (that is for $k \geqslant k_0$), the last root of P_k is also real and from the change of signs of P_k in the intervals $(-\lambda_{i+1}, -\lambda_i)$ and $(-G, -\lambda_M)$, we see that this last root belongs to $(-\infty, -G)$. We thus deduce (3.10) and the limits of μ_k^j for $j \leqslant M$.

For $\left(\mu_k^{M+1}\right)_{k \geqslant k_0}$, if it admits a bounded subsequence, we see that it converges towards a root of (3.5) so that for k large enough, it belongs to one of the intervals $(-\lambda_{i+1}, -\lambda_i)$ and $(-G, -\lambda_M)$ which is false. Therefore,

$$\mu_k^{M+1} \to -\infty$$

and from (3.9), we deduce that

$$\mu_k^{M+1} \sim -\eta \Lambda_k.$$

Case 2: assume $\mu = 0$. Then standard computation yields (3.13). □

One can check that the eigenvectors of A are of the form

$$\left[0, 0, \dots, 0, a\right], \quad a \in \mathbb{R}^N, a \neq 0$$

for the eigenvalue $\mu = 0$ and

$$\left[w_k, \frac{\kappa_1}{\mu + \lambda_1} w_k, \dots, \frac{\kappa_M}{\mu + \lambda_M} w_k, \frac{1}{\mu} C_0 w_k\right] \tag{3.17}$$

for the eigenvalues solutions of (3.9). Using this, we can show the following result:

Proposition 3.4. *The family of root vectors of A is complete.*

Proof. Assume $F = [f, g_1, g_2, \dots, g_M, \alpha] \in \mathbb{Y} = \mathbb{H} \times \mathbb{H}_{-1}^M \times \mathbb{C}^N$ is such that

$$\langle F, \xi \rangle = 0 \tag{3.18}$$

for any ξ root vector of A.

We decompose f and g_i in the orthogonal basis (w_k) :

$$f = \sum_{k \geqslant 1} f_k w_k, \quad g_i = \sum_{k \geqslant 1} g_{i,k} w_k, \quad i = 1, 2, \ldots, M,$$

and we deduce from (3.17), that for $k \geqslant k_0$ (that is when we have (3.10)),

$$\left\langle \left[f_k, g_{1,k}, \ldots, g_{M,k} \right], \left[1, \frac{\kappa_1}{\mu_k^j + \lambda_1}, \ldots, \frac{\kappa_M}{\mu_k^j + \lambda_M} \right] \right\rangle = 0, \quad \forall k \geqslant k_0, \ j = 1, 2, \ldots, M+1$$

for the usual scalar product of $\mathbb{C}^{M+1}$. Noting that $\left[1, \frac{\kappa_1}{\mu_k^j + \lambda_1}, \ldots, \frac{\kappa_M}{\mu_k^j + \lambda_M} \right]$, $j = 1, 2, \ldots, M+1$ are independent vectors of $\mathbb{C}^{M+1}$ for $k \geqslant k_0$, we deduce that

$$F \in (\mathbb{W}_{k_0})^{M+1} \times \mathbb{C}^N, \quad \text{where} \quad \mathbb{W}_{k_0} = \operatorname{span} \{ w_k, \ k < k_0 \}.$$

From the expression (3.4) of A, we see that it is a linear operator of the finite-dimensional subspace $(\mathbb{W}_{k_0})^{M+1} \times \mathbb{C}^N$. From classical result of linear algebra, we deduce that the corresponding linear operator admits a basis of root vectors in $(\mathbb{W}_{k_0})^{M+1} \times \mathbb{C}^N$ and thus we deduce that $F = 0$. $\square$

Combining Proposition 3.2, Proposition 3.3 and Proposition 3.4 and following the proof of Theorem 2.2, we can prove Theorem 3.1 in a completely similar way. We thus skip the corresponding proof.

4 Proof of Theorem 1.1

In order to show Theorem 1.1, we are going to apply Theorem 2.2. First we show that (1.12)–(1.19) can be written under the form (2.2). We recall that $\mathbb{H}$ is defined by (1.20). Now we define A_0 as follows:

$$\mathcal{D}(A_0) = \left\{ u \in H_0^1(\Omega) \mid u|_{\mathcal{F}} \in H^2(\mathcal{F}), \ \operatorname{div} u = 0 \text{ in } \Omega, \ \mathbb{D}(u) = 0 \text{ in } \mathcal{S} \right\}, \tag{4.1}$$

$$\mathcal{A}_0 u = \begin{cases} -\Delta u & \text{in } \mathcal{F} \\ \left[\frac{1}{m} \int_{\partial \mathcal{S}} 2\mathbb{D}(u) n \, d\Gamma \right] + \left[J_0^{-1} \int_{\partial \mathcal{S}} \widetilde{y} \times 2\mathbb{D}(u) n \, d\Gamma_{\widetilde{y}} \right] \times y & \text{in } \mathcal{S} \end{cases} \quad (u \in \mathcal{D}(A_0)), \tag{4.2}$$

$$A_0 u = \mathbb{P} \mathcal{A}_0 u \quad (u \in \mathcal{D}(A_0)) \tag{4.3}$$

where $\mathbb{P}$ is the orthogonal projector from $L^2(\Omega)$ onto $\mathbb{H}$.

The operator $C_0 \in \mathcal{L}(\mathbb{H}, \mathbb{R}^6)$ is defined by

$$C_0 u = (\ell_u, \omega_u) \quad \text{if } u(y) = \ell_u + \omega_u \times y \quad (y \in \mathcal{S}). \tag{4.4}$$

We also set $\mathbb{U} := L^2(\mathcal{O})$ and the control operator $B_0 \in \mathcal{L}(L^2(\mathcal{O}), \mathbb{H})$ is given by

$$B_0 f = \mathbb{P}(f \chi_{\mathcal{O}}) \quad f \in \mathbb{U}. \tag{4.5}$$

We set the initial conditions as:

$$u^0 = \begin{cases} u^0 & \text{in } \mathcal{F} \\ \ell^0 + \omega^0 \times y & \text{in } \mathcal{S}, \end{cases} \quad v^0 = \begin{cases} v^0 & \text{in } \mathcal{F} \\ k^0 + r^0 \times y & \text{in } \mathcal{S}, \end{cases} \quad a0 = [h^0, \theta^0], \tag{4.6}$$

where u^0, v^0, ℓ^0, ω^0, k^0, r^0, h^0 and θ^0 are as given in (1.19).

With this notation, we see that we can write (1.12)–(1.19) as (2.2), where $a = [h, \theta] \in \mathbb{R}^6$. We are now in a position to prove Theorem 1.1:

Proof of Theorem 1.1. We recall (see, for instance, [34, Proposition 5.3]) that A_0 is positive and self-adjoint. The couple (A_0, B_0) is approximately controllable. Indeed, by using the Fattorini criterion (that is Theorem 2.3), we only need to show that if

$$\begin{cases} -\Delta u + \nabla p = \mu u & \text{in } \mathcal{F}, \\ \operatorname{div} u = 0 & \text{in } \mathcal{F}, \\ u = 0 & \text{on } \partial\Omega, \\ u(y) = \ell + \omega \times y & y \in \partial\mathcal{S}, \\ m\mu\ell = \displaystyle\int_{\partial\mathcal{S}} \Sigma_1(u,p) n \, d\Gamma, \\ J_0 \mu\omega = \displaystyle\int_{\partial\mathcal{S}} y \times \Sigma_1(u,p) n \, d\Gamma, \end{cases} \tag{4.7}$$

and if $B_0^* u = u_{|\mathcal{O}} \equiv 0$, then $u \equiv 0$. In the above system,

$$\Sigma_1(u,p) = 2\mathbb{D}(u) - p\mathbb{I}_3.$$

By the standard unique continuation property on the Stokes system (see [11]), then $u \equiv 0$ in $\mathcal{F}$ and thus $\ell = \omega = 0$ by using the trace of u on $\partial\mathcal{S}$.

It remains to show that $B_0^* A_0^{-1} C_0^* : \mathbb{R}^6 \to \mathbb{U}$ is injective. We endow $\mathbb{R}^6$ with the scalar product

$$\langle (k,r), (k',r') \rangle = mk \cdot k' + J_0 r \cdot r'.$$

Then, using (1.10) and (1.11)

$$\langle (k,r), (k',r') \rangle = \int_{\mathcal{S}} \rho_{\mathcal{S}} (k + r \times y) \cdot (k' + r' \times y) \; dy.$$

It yields that if $\xi = (k,r) \in \mathbb{R}^6$, then

$$C_0^* \xi = \mathbb{P}\left[(k + r \times y) \, \chi_{\mathcal{S}} \right],$$

where $\chi_{\mathcal{S}}$ is the characteristic function of $\mathcal{S}$.

Thus, with (4.1)–(4.3), we deduce that if $B_0^* A_0^{-1} C_0^* \xi = 0$, then

$$\begin{cases} -\Delta u + \nabla p = 0 & \text{in } \mathcal{F}, \\ \operatorname{div} u = 0 & \text{in } \mathcal{F}, \\ u = 0 & \text{on } \partial\Omega, \\ u(y) = \ell + \omega \times y & y \in \partial\mathcal{S}, \\ mk = \displaystyle\int_{\partial\mathcal{S}} \Sigma_1(u,p) n \, d\Gamma, \\ J_0 r = \displaystyle\int_{\partial\mathcal{S}} y \times \Sigma_1(u,p) n \, d\Gamma, \\ u \equiv 0 & \text{in } \mathcal{O}. \end{cases} \tag{4.8}$$

By using again the unique continuation property of the Stokes system, then $u \equiv 0$ in $\mathcal{F}$ and $\nabla p \equiv 0$ in $\mathcal{F}$. Thus $k = r = 0$ and we deduce the injectivity of $B_0^* A_0^{-1} C_0^*$. Applying Theorem 2.2, we deduce Theorem 1.1. □

5 Lack of null controllability

The goal of this section is to show that if $\mathcal{O}$ is a proper open subset of $\mathcal{F}$ then there exists an initial condition $[u^0, v^0, (h^0, \theta^0)] \in \mathbb{H}_{1/2} \times \mathbb{H}_1 \times \mathbb{R}^6$ such that for any finite time $T > 0$ and for any control f, the solution $[u, v, (\ell, \omega), (k, r), (h, \theta)]$ of (1.12)-(1.19) cannot be driven to 0 for the fluid velocity.

The proof is based on a localization procedure in the fluid and thus can be deduced from results obtained in [21]. The statement is the following one:

Theorem 5.1. *Assume $\mathcal{O}$ is an open subset of $\mathcal{F}$, $\mathcal{O} \neq \mathcal{F}$. There exists $[u^0, v^0, (h^0, \theta^0)] \in \mathbb{H}_{1/2} \times \mathbb{H}_1 \times \mathbb{R}^6$ such that for any $T > 0$ and for any $f \in L^2(0,T; L^2(\mathcal{O}))$, the solution $[u, v, (\ell, \omega), (k, r), (h, \theta)]$ of (1.12)-(1.19) satisfies*

$$u(T, \cdot) \not\equiv 0.$$

In order to prove the above result, we have the following results:

Proposition 5.2. *Assume $\mathcal{O}$ is an open subset of $\mathcal{F}$, $\mathcal{O} \neq \mathcal{F}$. and let us consider $f \in L^2(0,T; L^2(\mathcal{O}))$. Then the solution of $[u, v, (\ell, \omega), (k, r), (h, \theta)]$ of (1.12)-(1.19) with $u^0 \equiv 0$, $v^0 \equiv 0$ satisfies*

$$u, v \in C^\infty\left((0,T] \times \left(\mathcal{F} \setminus \overline{\mathcal{O}}\right)\right). \tag{5.1}$$

Proof. The proof is based on Theorem 3.3 and the proof of Theorem 3.1 in [21]. First, from Proposition 2.1, we deduce that

$$u \in H^1(0,T; L^2(\mathcal{F})) \cap L^2(0,T; H^2(\mathcal{F})) \cap C^0([0,T]; H^1(\mathcal{F})), \quad v \in H^1(0,T; H^2(\mathcal{F})).$$

Second, let us consider $x_0 \in \mathcal{F} \setminus \overline{\mathcal{O}}$ and $r_0 > r_1 > 0$ with $B(x_0, r_0) \subset \mathcal{F} \setminus \overline{\mathcal{O}}$. We also take $\widetilde{\chi}_0$ a smooth function with support in $B(x_0, r_0)$ and equal to 1 in $B(x_0, r_1)$. Then we deduce from (1.12)–(1.19) that

$$\vartheta = \widetilde{\chi}_0 \operatorname{curl} u, \quad \zeta = \widetilde{\chi}_0 \operatorname{curl} v,$$

satisfy

$$\begin{cases} \dfrac{\partial \vartheta}{\partial t} - \eta \Delta \vartheta - \Delta \zeta = F & \text{in } (0,T) \times B(x_0, r_0), \\ \dfrac{\partial \zeta}{\partial t} + \lambda \zeta = \kappa \vartheta & \text{in } (0,T) \times B(x_0, r_0), \\ \vartheta = \zeta = 0 & \text{on } (0,T) \times \partial B(x_0, r_0), \\ \vartheta(0, \cdot) = \zeta(0, \cdot) = 0 & \text{in } B(x_0, r_0), \end{cases} \tag{5.2}$$

where

$$F = -2(\eta(\nabla \operatorname{curl} u) + (\nabla \operatorname{curl} v)) \nabla \chi_0 - \Delta \chi_0 (\eta \operatorname{curl} u + \operatorname{curl} \widetilde{v}) \in L^2(0,T; L^2(\mathcal{F})).$$

Using again Proposition 2.1 or directly Theorem 3.3 in [21], we deduce that

$$\vartheta \in H^1(0,T; L^2(B(x_0, r_0))) \cap L^2(0,T; H^2(B(x_0, r_0))), \quad \zeta \in H^1(0,T; H^2(B(x_0, r_0))).$$

Proceeding by induction as in the proof of of Theorem 3.1 in [21] (here we see that the equations of the rigid body are not used), we can show that that

$$\vartheta, \zeta \in C^\infty\left((0,T] \times B(x_0, r)\right)$$

for some $r \in (0, r_1)$.

Then using that

$$-\Delta u = \operatorname{curl} \vartheta \quad \text{and} \quad -\Delta v = \operatorname{curl} \zeta \quad \text{in } B(x_0, r),$$

we deduce the result from the interior regularity of the laplacian (see, for instance, [10, Theorem 3, p.334]). □

Then, we have the following result:

Proposition 5.3. *Assume $\mathcal{O}$ is an open subset of $\mathcal{F}$, $\mathcal{O} \neq \mathcal{F}$. and let us consider $x_0 \in \mathcal{F} \setminus \overline{\mathcal{O}}$. Then there exists $[u^0, v^0, (h^0, \theta^0)] \in \mathbb{H}_{1/2} \times \mathbb{H}_1 \times \mathbb{R}^6$ such that the solution $[u, v, (\ell, \omega), (k, r), (h, \theta)]$ of (1.12)-(1.19) with $f \equiv 0$ satisfies that for any $t > 0$, $u(t, \cdot)$ is not C^∞ at x_0.*

Proof. We start with the system

$$\begin{cases} \dfrac{\partial \vartheta}{\partial t} - \eta \Delta \vartheta - \Delta \zeta = 0 & \text{in } \mathbb{R}_+^* \times \mathbb{R}^3, \\ \dfrac{\partial \zeta}{\partial t} + \lambda \zeta = \kappa \vartheta & \text{in } \mathbb{R}_+^* \times \mathbb{R}^3, \\ \vartheta(0, \cdot) = \vartheta^0, \quad \zeta(0, \cdot) = \zeta^0 & \text{in } \mathbb{R}^3. \end{cases} \tag{5.3}$$

From Theorem 3.5 in [21], there exists $u_*^0 \in H^1(\mathbb{R}^3)$ and $v_*^0 \in H^2(\mathbb{R}^3)$ such that the solution of the above system with $\vartheta^0 = \operatorname{curl} u_*^0$ and $\zeta^0 = \operatorname{curl} v_*^0$ satisfies for any $t > 0$,

$$\vartheta(t,\cdot), \zeta(t,\cdot) \in C^\infty(\mathbb{R}^3 \setminus \{x_0\})$$

and $\vartheta(t,\cdot)$ is not C^∞ at x_0.

Then, we apply Lemma 3.7 in [21] to deduce the existence of $u^0 \in H_0^1(\mathcal{F})$ and $v^0 \in H^2(\mathcal{F}) \cap H_0^1(\mathcal{F})$, with $\operatorname{div} u^0 = \operatorname{div} v^0 = 0$ such that

$$\operatorname{curl} u^0 = \vartheta^0 \quad \text{and} \quad \operatorname{curl} v^0 = \zeta^0$$

in a neighborhood of x_0. Extending u^0 and v^0 by 0 in $\mathcal{S}$, we deduce that $u^0 \in \mathbb{H}_{1/2}$ and $v^0 \in \mathbb{H}_1$. Taking $(h^0, \theta^0) \in \mathbb{R}^6$ arbitrarily we can thus consider the solution $[u, v, (\ell, \omega), (k, r), (h, \theta)]$ of (1.12)-(1.19) with $f \equiv 0$. Then

$$\widetilde{\vartheta} := \operatorname{curl} u - \vartheta, \quad \widetilde{\zeta} := \operatorname{curl} v - \zeta$$

satisfy

$$\begin{cases} \dfrac{\partial \widetilde{\vartheta}}{\partial t} - \eta \Delta \widetilde{\vartheta} - \Delta \widetilde{\zeta} = 0 & \text{in } (0,T) \times B(x_0, r), \\ \dfrac{\partial \widetilde{\zeta}}{\partial t} + \lambda \widetilde{\zeta} = \kappa \widetilde{\vartheta} & \text{in } (0,T) \times B(x_0, r), \\ \widetilde{\vartheta} = g_1, \quad \widetilde{\zeta} = g_2 & \text{on } (0,T) \times \partial B(x_0, r), \\ \widetilde{\vartheta}(0,\cdot) = \widetilde{\zeta}(0,\cdot) = 0 & \text{in } B(x_0, r), \end{cases} \tag{5.4}$$

for $r > 0$ small enough, where $g_1 = (\operatorname{curl} u - \vartheta)|_{(0,T)\times\partial B(x_0,r)}$ and $g_2 = (\operatorname{curl} v - \zeta)|_{(0,T)\times\partial B(x_0,r)}$. As for (5.2), one can show that

$$\widetilde{\vartheta}, \widetilde{\zeta} \in C^\infty\left((0,T] \times B(x_0, \widetilde{r})\right)$$

for $\widetilde{r}$ small enough. This yields the result. □

From the above two propositions, we can deduce Theorem 5.1.

Proof of Theorem 5.1. We consider the initial condition $[u^0, v^0, (h^0, \theta^0)] \in \mathbb{H}_{1/2} \times \mathbb{H}_1 \times \mathbb{R}^6$ of Proposition 5.3 and we assume $f \in L^2(0,T; L^2(\mathcal{O}))$. We can then decompose the corresponding solution $[u, v, (\ell, \omega), (k, r), (h, \theta)]$ of (1.12)-(1.19) into a solution with null initial conditions and a solution with null source. Gathering Proposition 5.2 and Proposition 5.3, we deduce that $u(T,\cdot)$ is not C^∞ at x_0 and thus is not null. □

6 Some extensions of Theorem 1.1

Using the proof and the framework of Section 4 to obtain Theorem 1.1, we see that we can deduce several extensions of Theorem 1.1.

For instance, we can obtain the approximate controllability and stabilization of (1.12)–(1.19) with a control f with one component that cancels. For instance, we assume that $f_3 = 0$ and we thus control the fluid-structure system with only two scalar controls (f_1 and f_2). We thus replace the equation (1.12) by

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}\left(\Sigma(u,p) + 2\mathbb{D}(v)\right) = (f_1, f_2, 0)\chi_{\mathcal{O}} & t > 0,\ y \in \mathcal{F}, \\ \operatorname{div} u = 0 & t > 0,\ y \in \mathcal{F}, \end{cases} \tag{6.1}$$

Theorem 6.1. *Assume that $\mathcal{O}$ is a nonempty open subset of $\mathcal{F}$. Then the system* (6.1), (1.13)-(1.19) *is approximately controllable. Assume moreover that*

$$0 < \beta < \lambda + \frac{\kappa}{\eta}.$$

Then the system (6.1), (1.13)-(1.19) *is exponentially stabilizable with rate lower than $-\beta$ and with a feedback of finite dimension.*

Proof. The proof is completely similar to the proof of Theorem 1.1, we only replace (4.5) by

$$B_0(f_1, f_2) = \mathbb{P}((f_1, f_2, 0)\chi_{\mathcal{O}}), \quad (f_1, f_2) \in \mathbb{U}. \tag{6.2}$$

Then, applying the Fattorini criterion to show that (A_0, B_0) is approximately controllable, we consider a solution (u, p) of (4.7) with $B_0^* u = (u_{1|\mathcal{O}}, u_{2|\mathcal{O}}) \equiv 0$. Combining this with $\operatorname{div} u = 0$, we deduce

$$\frac{\partial u}{\partial x_3} \equiv 0 \quad \text{in } \mathcal{O}.$$

Moreover, $\left(\frac{\partial u}{\partial x_3}, \frac{\partial p}{\partial x_3}\right)$ satisfies the following system:

$$\begin{cases} -\Delta\left(\frac{\partial u}{\partial x_3}\right) + \nabla\left(\frac{\partial p}{\partial x_3}\right) = \lambda\left(\frac{\partial u}{\partial x_3}\right) & \text{in } \mathcal{F}, \\ \operatorname{div}\left(\frac{\partial u}{\partial x_3}\right) = 0 \quad \text{in } \mathcal{F}, \end{cases}$$

and from [11], we deduce

$$\frac{\partial u}{\partial x_3} \equiv 0 \quad \text{in } \mathcal{F}.$$

By applying the Poincaré inequality, the above relation yields

$$u \equiv 0 \qquad \text{in } \mathcal{F}$$

and thus $\ell = \omega = 0$ by using the trace of u on $\partial\mathcal{S}$.

The proof of the injectivity of $B_0^* A_0^{-1} C_0^* : \mathbb{R}^6 \to \mathbb{U}$ is done similarly. Applying Theorem 2.2, we thus deduce Theorem 6.1. □

We can also obtain the approximate controllability and stabilization of a linear fluid-rigid body system for a fluid modeled by a linear Jeffreys model with several relaxation mode:

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}\left(\Sigma(u, p) + 2\mathbb{D}(v)\right) = f\chi_{\mathcal{O}} & t > 0,\ y \in \mathcal{F}, \\ \operatorname{div} u = 0 \quad t > 0,\ y \in \mathcal{F}, \end{cases} \tag{6.3}$$

$$\begin{cases} u = 0 \quad t > 0,\ y \in \partial\Omega, \\ u(t, y) = \ell(t) + \omega(t) \times y \quad t > 0,\ y \in \partial\mathcal{S}, \end{cases} \tag{6.4}$$

$$\begin{cases} m\ell' = -\int\limits_{\partial\mathcal{S}} \left(\Sigma(u, p) + 2\mathbb{D}(v)\right) n \, d\Gamma & t > 0, \\ J_0\omega' = -\int\limits_{\partial\mathcal{S}} y \times \left(\Sigma(u, p) + 2\mathbb{D}(v)\right) n \, d\Gamma & t > 0, \end{cases} \tag{6.5}$$

$$\begin{cases} \frac{\partial v_i}{\partial t} + \lambda_i v_i = \kappa_i u \quad t > 0,\ y \in \mathcal{F},\ i \in \{1, \ldots, M\} \\ \operatorname{div} v_i = 0 \quad t > 0,\ y \in \mathcal{F},\ i \in \{1, \ldots, M\} \end{cases} \tag{6.6}$$

$$\begin{cases} v_i = 0 \quad t > 0,\ y \in \partial\Omega,\ i \in \{1, \ldots, M\} \\ v_i(t, y) = k_i(t) + r_i(t) \times y \quad t > 0,\ y \in \partial\mathcal{S},\ i \in \{1, \ldots, M\} \end{cases} \tag{6.7}$$

$$\begin{cases} k_i' + \lambda_i k_i = \kappa_i \ell \quad \forall t > 0,\ i \in \{1, \ldots, M\} \\ r_i' + \lambda_i r_i = \kappa_i \omega \quad \forall t > 0,\ i \in \{1, \ldots, M\} \end{cases} \tag{6.8}$$

$$\begin{cases} h' = \ell \quad \forall t > 0 \\ \theta' = \omega \quad \forall t > 0 \end{cases} \tag{6.9}$$

$$\begin{cases} u(0, \cdot) = u^0, \quad v_i(0, \cdot) = v_i^0,\ i \in \{1, \ldots, M\} \quad \text{in } \mathcal{F}, \\ h(0) = h^0, \quad \theta(0) = \theta^0, \quad \ell(0) = \ell^0, \quad \omega(0) = \omega^0, \quad k_i(0) = k_i^0, \quad r_i(0) = r_i^0,\ i \in \{1, \ldots, M\}. \end{cases} \tag{6.10}$$

Our main result is the following one

Theorem 6.2. *Assume $\mathcal{O}$ is a nonempty open subset of $\mathcal{F}$. Assume also that $0 < \lambda_1 < \ldots < \lambda_M$. Then the system* (6.3)-(6.10) *is approximately controllable. Assume moreover that*

$$0 < \beta < -\mu^1$$

where μ^1 is the largest root of (3.5)*. Then the system* (6.3)-(6.10) *is exponentially stabilizable with rate lower than $-\beta$ and with a feedback of finite dimension.*

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