Approximate controllability for a system of Schrödinger equations modelling a single trapped ion

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Outline of the talk

Introduction

Let A be the harmonic operator on $\mathbb{R}$

$$A = \frac{1}{2}\Big(- \partial_{xx}^2 + x^2 \Big).$$

Properties of A

This operator is self-adjoint, with compact resolvent.

The spectrum of A is explicit

$$A\Phi_j = \lambda_j \Phi_j, \qquad (\Phi_j) = \text{Hermite functions}, \quad \lambda_j = j + 1/2.$$

The model

Model of a single trapped ion:

$$\begin{cases} i\partial_t \psi_e = \omega A \psi_e + \dfrac{\Omega}{2}\psi_e + (\mathbf{u} + \mathbf{u}^*)\psi_g, & (t,x) \in (0,T) \times \mathbb{R}, \\ i\partial_t \psi_g = \omega A \psi_g - \dfrac{\Omega}{2}\psi_g + (\mathbf{u} + \mathbf{u}^*)\psi_e, & (t,x) \in (0,T) \times \mathbb{R}, \\ \psi_e(0,x) = \psi_e^0(x), \quad \psi_g(0,x) = \psi_g^0(x), & x \in \mathbb{R}. \end{cases} \tag{1}$$

- ω, Ω are large real numbers ! $\Omega \gg \omega \gg 1$.
- $\mathbf{u}$ is the control function, superposition of 3 lasers:

$$\mathbf{u}(t,x) = u_0 e^{i(\Omega t - \sqrt{2}\eta_0 x)} + u_r e^{i((\Omega-\omega)t - \sqrt{2}\eta_r x)} + u_b e^{i((\Omega+\omega)t - \sqrt{2}\eta_b x)}.$$

The model

Physical constraints on the control function:

$$\mathbf{u}(t,x) = u_0 e^{i(\Omega t - \sqrt{2}\eta_0 x)} + u_r e^{i((\Omega-\omega)t - \sqrt{2}\eta_r x)} + u_b e^{i((\Omega+\omega)t - \sqrt{2}\eta_b x)},$$

- $(u_0, u_b, u_r) \in \mathbb{C}^3$.
- $t \mapsto (u_0(t), u_b(t), u_r(t))$ is piecewise constant.
- at each time t, there is at most one control “on".
- η are the Lamb-Dicke parameters, assumed small

$$\eta \ll 1.$$

The model, main assumptions

$$\begin{cases} i\partial_t \psi_e = \omega A \psi_e + \dfrac{\Omega}{2}\psi_e + (\mathbf{u} + \mathbf{u}^*)\psi_g, & (t,x) \in (0,T) \times \mathbb{R}, \\ i\partial_t \psi_g = \omega A \psi_g - \dfrac{\Omega}{2}\psi_g + (\mathbf{u} + \mathbf{u}^*)\psi_e, & (t,x) \in (0,T) \times \mathbb{R}, \\ \psi_e(0,x) = \psi_e^0(x), \quad \psi_g(0,x) = \psi_g^0(x), & x \in \mathbb{R}. \end{cases}$$

$\Omega \gg \omega \gg 1, \quad \eta \ll 1.$

$$\mathbf{u}(t,x) = u_0 e^{i(\Omega t - \sqrt{2}\eta_0 x)} + u_r e^{i((\Omega-\omega)t - \sqrt{2}\eta_r x)} + u_b e^{i((\Omega+\omega)t - \sqrt{2}\eta_b x)}.$$

Problem

Can we control this systems with such controls ?

Our result: Approximate controllability

Theorem

Let (ψ_e^0, ψ_g^0) and (ψ_e^1, ψ_g^1) of unit $(L^2)^2$ norm.
Then $\forall \delta > 0, \exists (\aleph,\ \eta_0,\ \rho_0)$, such that for all (ω, Ω) with $2\Omega \geq 3\omega$ and

$$\eta \leq \eta_0, \quad KT = \aleph/\eta, \quad \frac{\omega\eta}{K} \geq \rho_0,$$

there exists a control $\mathbf{u}(t, x)$ as above, furthermore satisfying

$$\sup\{|u_0(t)|, |u_r(t)|, |u_b(t)|\} \leq K,$$

such that the solution (ψ_e, ψ_g) of (1) with initial data (ψ_e^0, ψ_g^0) satisfies, for some β of modulus 1,

$$\left\| (\psi_e(T), \psi_g(T)) - \beta(\psi_e^1, \psi_g^1) \right\|_{0\times 0} \leq \delta.$$

Comments

- Many different interpretations of the conditions
$\eta \leq \eta_0, \quad KT = \aleph/\eta, \quad \frac{\omega\eta}{K} \geq \rho_0$:
  - K fixed, then $\eta \ll 1,\ T = T^*/\eta$, and $\omega \geq \omega^*/\eta$.
  - T fixed: $\eta \ll 1,\ K = K^*/\eta,\ \omega \geq \omega^*/\eta^2$.
  - ω, Ω fixed: $\eta \ll 1,\ \eta/K \gg 1,\ T = T^*/(K\eta)$.
  - $K = \eta$: $\eta \ll 1,\ \omega \gg 1,\ T = T^*/\eta^2$.
- We always have $\omega \gg K$.
- If, at time T, the control is turned off, the solution stays in a δ neighborhood of the target trajectory.
- Can be generalized for all norms $\|(\cdot,\cdot)\|_{k\times k}$, see later.

Bibliography

- Approximate controllability:
  - Through a stabilization approach: Beauchard, Coron, Mirrahimi, Rouchon, Turinici, Nersesyan...
  - Through optimal control: Baudouin, Puel, Salomon, Ito, Kunisch, ...
  - Via a geometric approach: Bloch, Brockett, Rangan, Agrachev...
  - Via a finite dimensional point of view: Chambrion, Mason, Sigalotti, Boscain. $\rightsquigarrow$ idea close to the one we shall use below.
- Exact controllability:
  - Negative results: Ball, Marsden, Slemrod, Turinici, Mirrahimi, Rouchon.
  - Positive results: Beauchard, Coron, Laurent.

The Cauchy problem

Notations

$$\|\psi\|_k = \left\| A^{k/2}\psi \right\|_{L^2(\mathbb{R})}, \qquad \forall \psi \in \mathcal{D}(A^{k/2})$$

$$\|(\psi_1, \psi_2)\|_{k\times k} = \left(\|\psi_1\|_k^2 + \|\psi_2\|_k^2 \right)^{1/2}, \qquad \forall (\psi_1, \psi_2) \in \mathcal{D}(A^{k/2})^2$$

$$\begin{cases} i\partial_t \psi_e = \omega A \psi_e + \dfrac{\Omega}{2}\psi_e + f\psi_g, & (t,x) \in (0,T)\times\mathbb{R}, \\ i\partial_t \psi_g = \omega A \psi_g - \dfrac{\Omega}{2}\psi_g + f\psi_e, & (t,x) \in (0,T)\times\mathbb{R}, \\ \psi_e(0,x) = \psi_e^0(x), \quad \psi_g(0,x) = \psi_g^0(x), & x \in \mathbb{R}. \end{cases} \tag{2}$$

Here $f = f(t,x)$ is real valued.

The Cauchy problem

Theorem

Let $T > 0$. Let $f : (0, T) \times \mathbb{R} \to \mathbb{R}$, $f \in L^\infty((0, T); C_b^0(\mathbb{R}))$. Then, for all initial data $(\psi_e^0, \psi_g^0) \in L^2(\mathbb{R})^2$, there exists a unique solution (ψ_e, ψ_g) of (2) in $C([0, T]; L^2(\mathbb{R})^2)$, and $\forall t > 0$,

$$\left\|(\psi_e(t), \psi_g(t))\right\|_{0\times 0} = \left\|(\psi_e^0, \psi_g^0)\right\|_{0\times 0}.$$

Moreover, if $(\psi_e^0, \psi_g^0) \in \mathcal{D}(A^{k/2})^2$ and $f \in L^\infty((0, T); C_b^k(\mathbb{R}))$, then $(\psi_e, \psi_g) \in C([0, T]; \mathcal{D}(A^{k/2})^2)$.

Sketch of the proof

- Step 1: Prove that the map

$$\Psi_e(\psi_e,\psi_g)(t) = S(t)e^{-i\Omega t/2}\psi_e^0 + i\int_0^t S(t-s)e^{-i\Omega(t-s)/2} f(s)\psi_g(s)\,ds,$$

$$\Psi_g(\psi_e,\psi_g)(t) = S(t)e^{i\Omega t/2}\psi_g^0 + i\int_0^t S(t-s)e^{i\Omega(t-s)/2} f(s)\psi_e(s)\,ds,$$

on $Y = C([0,T];\mathcal{D}(A^{k/2})^2)$ endowed with the norm

$$\left\|(\psi_e,\psi_g)\right\|_Y = \sup_{t\in[0,T]}\left\{e^{-\lambda t}\left\|(\psi_e(t),\psi_g(t))\right\|_{k\times k}\right\},$$

is a contraction, for a good choice of λ. ($S(t) = exp(-it\omega A)$ is the free Schrödinger semigroup).

- Step 2. A priori estimates for smooth solutions.
- Step 3. Limit for low-regularity data.

An approximate system: Law-Eberly

Let us consider the approximate system

$$\begin{cases} i\partial_t \phi_e = \left(u_0^* + v_r^* \mathbf{a} + v_b^* \mathbf{a}^\dagger\right)\phi_g, & (t,x) \in (0,T)\times\mathbb{R}, \\ i\partial_t \phi_g = \left(u_0 + v_r \mathbf{a}^\dagger + v_b \mathbf{a}\right)\phi_e, & (t,x) \in (0,T)\times\mathbb{R}, \end{cases} \tag{3}$$

with

$$\mathbf{a} = \frac{1}{\sqrt{2}}\left(x + \partial_x\right), \qquad \mathbf{a}^\dagger = \frac{1}{\sqrt{2}}\left(x - \partial_x\right).$$

Here v_r and v_b respectively correspond to $-i\eta u_r$ et $-i\eta u_b$.

Advantage: This system is exactly controllable !

An approximate system: Law-Eberly

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$$\begin{cases} i\partial_t \phi_e = \left(u_0^* + v_r^* \mathbf{a} + v_b^* \mathbf{a}^\dagger\right)\phi_g, & (t,x) \in (0,T)\times\mathbb{R}, \\ i\partial_t \phi_g = \left(u_0 + v_r \mathbf{a}^\dagger + v_b \mathbf{a}\right)\phi_e, & (t,x) \in (0,T)\times\mathbb{R}, \end{cases} \tag{3}$$

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Some spectral theory

The operators $\mathbf{a}$ and $\mathbf{a}^\dagger$ respectively are the annihilation and creation operators.

$$A = \mathbf{a}^\dagger \mathbf{a} + \frac{1}{2} = \mathbf{a}\mathbf{a}^\dagger - \frac{1}{2},$$

$$\mathbf{a}\Phi_0 = 0, \quad \begin{cases} \mathbf{a}\Phi_{n+1} & = \sqrt{n+1}\,\Phi_n, \\ \mathbf{a}^\dagger\Phi_n & = \sqrt{n+1}\,\Phi_{n+1}, \end{cases} \quad \forall n \in \mathbb{N}.$$

Notations: For $M \in \mathbb{N}$, we define

$$V_M = \operatorname{span}\left\{\Phi_j\, ; \;\; 0 \;\le j \le M\right\}.$$

Law-Eberly, a refined version

Theorem (Law Eberly revisited)

For all $(\phi_e^0, \phi_g^0), (\phi_e^1, \phi_g^1) \in V_M^2$ of same $(L^2)^2$-norm, there exist $T > 0$ and a control $t \mapsto (u_0(t), v_r(t), v_b(t))$ such that

- (ϕ_e, ϕ_g) solution of (3) with initial data (ϕ_e^0, ϕ_g^0) satisfies $(\phi_e(T), \phi_g(T)) = \beta(\phi_e^1, \phi_g^1)$ for some $\beta \in \mathbb{C}$ of unit modulus.
- $\forall t \in [0, T], (\phi_e(t), \phi_g(t)) \in V_M^2$.
- There is at most one control “ON"
- There are at most $2M$ switching times.
- Imposing $|u_0| \leq K_0$ and $|v_r|, |v_b| \leq K_1$, then one can take any T s. t. $T \geq T^*$, with

$$T^* = \frac{(M+1)\pi}{K_0} + \frac{\pi}{K_1} \sum_{j=1}^{M} \frac{1}{\sqrt{j}}$$

Sketch of the proof

- if u_0 is the only active control:

$$i\partial_t \phi_e = u_0^* \phi_g, \quad i\partial_t \phi_g = u_0 \phi_e, \quad (t,x) \in (0,T) \times \mathbb{R}.$$

The ratio of populations $< \phi_e, \Phi_n >$ and $< \phi_g, \Phi_n >$ oscillate at frequency $|u_0|$.

- If v_r is the only active control:

$$i\partial_t \phi_e = v_r^* \mathbf{a} \phi_g, \quad i\partial_t \phi_g = v_r \mathbf{a}^\dagger \phi_e, \quad (t,x) \in (0,T) \times \mathbb{R}.$$

The ratio of populations $< \phi_e, \Phi_n >$ and $< \phi_g, \Phi_{n+1} >$ oscillate at frequency $|v_r|\sqrt{n}$.

$\rightsquigarrow$ Idea: Put everything on $(\phi_e, \phi_g) = (0, \Phi_0)$, and use reversibility to conclude.

Formal derivation : Lamb-Dicke approximation

$\rightsquigarrow$ Step 1: $\eta \ll 1 \Longrightarrow e^{-i\sqrt{2}\eta x} \simeq 1 - \sqrt{2}i\eta x$:

$$\mathbf{u}_{LD}(t,x) = \left(u_0 e^{i\Omega t} + u_r e^{i(\Omega-\omega)t} + u_b e^{i(\Omega+\omega)t}\right)(1 - i\sqrt{2}\eta x).$$

$\longrightarrow$ Approximate system $(\tilde{\psi}_e, \tilde{\psi}_g)$.

$\rightsquigarrow$ Step 2: Interaction frame. Let $\tilde{\phi}_e = S(-t)e^{i\Omega t/2}\tilde{\psi}_e,\ \tilde{\phi}_g = S(-t)e^{-i\Omega t/2}\tilde{\psi}_g$. The system becomes

$$\begin{cases} i\partial_t\tilde{\phi}_e = e^{i\Omega t}S(-t)(\mathbf{u}_{LD} + \mathbf{u}_{LD}^*)S(t)\tilde{\phi}_g, & (t,x) \in (0,T) \times \mathbb{R}, \\ i\partial_t\tilde{\phi}_g = e^{-i\Omega t}S(-t)(\mathbf{u}_{LD} + \mathbf{u}_{LD}^*)S(t)\tilde{\phi}_e, & (t,x) \in (0,T) \times \mathbb{R}, \\ \tilde{\phi}_e(0,x) = \psi_e^0(x), \quad \tilde{\phi}_g(0,x) = \psi_g^0(x), & x \in \mathbb{R}. \end{cases}$$

Formal derivation: the averaging approximation

$\rightsquigarrow$ Computation of $S(-t)xS(t)$. Remark that $\sqrt{2}x = \mathbf{a} + \mathbf{a}^\dagger$ and that $e^{it\omega A}(\mathbf{a} + \mathbf{a}^\dagger)e^{-it\omega A} = e^{-i\omega t}\mathbf{a} + e^{i\omega t}\mathbf{a}^\dagger$. Hence

$$\begin{aligned}
&e^{i\Omega t} S(-t)(\mathbf{u}_{LD} + \mathbf{u}_{LD}^*)S(t) \\
&= u_0 e^{2i\Omega t}\Big(1 - i\eta\Big(e^{-i\omega t}\mathbf{a} + e^{i\omega t}\mathbf{a}^\dagger\Big)\Big) + u_0^*\Big(1 + i\eta\Big(e^{-i\omega t}\mathbf{a} + e^{i\omega t}\mathbf{a}^\dagger\Big)\Big) \\
&+ u_r e^{i(2\Omega-\omega)t}\Big(1 - i\eta\Big(e^{-i\omega t}\mathbf{a} + e^{i\omega t}\mathbf{a}^\dagger\Big)\Big) \\
&\qquad + u_r^* e^{i\omega t}\Big(1 + i\eta\Big(e^{-i\omega t}\mathbf{a} + e^{i\omega t}\mathbf{a}^\dagger\Big)\Big) \\
&+ u_b e^{i(2\Omega+\omega)t}\Big(1 - i\eta\Big(e^{-i\omega t}\mathbf{a} + e^{i\omega t}\mathbf{a}^\dagger\Big)\Big) \\
&\qquad + u_b^* e^{-i\omega t}\Big(1 + i\eta\Big(e^{-i\omega t}\mathbf{a} + e^{i\omega t}\mathbf{a}^\dagger\Big)\Big).
\end{aligned}$$

$\rightsquigarrow$ Averaging: Cancel all oscillatory terms !
Yields Law-Eberly equations by setting $v_b = -i\eta u_b, \; v_r = -i\eta u_r$ as announced.

Our approach

Our approach is as follows:

1. To precisely measure the error done in the previous approximations for initial and target data in V_M^2.
2. To truncate initial and target data to go back to the previous item

Approximate controllability in V_M^2

Let $M \in \mathbb{N}$.

From Law-Eberly's theorem, the time should be

$$T \geq T^* = \frac{(M+1)\pi}{K_0} + 2\frac{\pi}{K_1}\sum_{j=1}^{M}\frac{1}{\sqrt{j}}$$

under the constraints $|u_0| \leq K_0$ et $|v_b|, |v_r| \leq K_1$.

We want to consider the constraints $|u_0|, |u_r|, |u_b| \leq K$. Since $v_b = -i\eta u_b$ and $v_r = -i\eta u_r$, we take $K_0 = K$ and $K_1 = \eta K$: for $\eta \leq 1/2\sqrt{M}$,

$$TK = \frac{3\pi\sqrt{M}}{\eta}.$$

Fix T as above.

Approximate controllability in V_M^2

Let (ψ_e^0, ψ_g^0) and (ψ_e^1, ψ_g^1) in V_M^2 of unit $(L^2)^2$-norm.

Define $(\phi_e^0, \phi_g^0) = (\psi_e^0, \psi_g^0)$ and

$$(\phi_e^1, \phi_g^1) = \big(S(-T)\exp(i\Omega T/2)\psi_e^1, S(-T)\exp(-i\Omega T/2)\psi_g^1\big)$$

Then Law-Eberly's theorem provides a control **u** that steers solutions of Law-Eberly approximate equations (3) from (ϕ_e^0, ϕ_g^0) to $\beta(\phi_e^1, \phi_g^1)$, for $\beta \in \mathbb{C}$ of unit modulus.

In the sequel, we always consider this control !

Approximate controllability in V^2_M

Theorem

Let (ψ_e^0, ψ_g^0) and (ψ_e^1, ψ_g^1) in V^2_M and the above control:
$\forall \delta > 0, \exists \eta_0 = \eta_0(\delta, M), \exists \rho_0 = \rho_0(\delta, M)$, s. t. $\forall K$, $\forall (\omega, \Omega)$ with $2\Omega \geq 3\omega$ and

$$\eta \leq \eta_0, \quad TK = \frac{3\pi\sqrt{M}}{\eta}, \quad \frac{\omega\eta}{K} \geq \rho_0,$$

the solution of the complete system (1) with initial data (ψ_e^0, ψ_g^0) satisfies

$$\left\| (\psi_e(T), \psi_g(T)) - \beta(\psi_e^1, \psi_g^1) \right\|_{0\times 0} \leq \delta.$$

Sketch of the proof

In the interaction frame:

$$\xi_e = S(-t)e^{i\Omega t/2}\psi_e, \qquad \xi_g = S(-t)e^{-i\Omega t/2}\psi_g.$$

Compare these with the functions (ϕ_e, ϕ_g).

In the interaction frame, the equations read as follows:

$$\begin{cases} i\partial_t \xi_e = e^{i\Omega t} S(-t)(\mathbf{u}+\mathbf{u}^*)S(t)\xi_g, \\ i\partial_t \xi_g = e^{-i\Omega t} S(-t)(\mathbf{u}+\mathbf{u}^*)S(t)\xi_e, \end{cases}$$

Sketch of the proof

We set $\epsilon_e(t,x) = \xi_e - \phi_e, \epsilon_g(t,x) = \xi_g - \phi_g$, and we have to study
(with $f = \mathbf{u} + \mathbf{u}^*$, $f_{LD} = \mathbf{u}_{LD} + \mathbf{u}^*_{LD}$ with Lamb-Dicke approximation, and f_{LE} Law-Eberly approximation)

$$\begin{cases} i\partial_t \epsilon_e = e^{i\Omega t} S(-t) f(t,x) S(t) \epsilon_g + e^{i\Omega t} S(-t)(f - f_{LD}) S(t) \phi_g \\ \quad + \left(e^{i\Omega t} S(-t) f_{LD} S(t) - f_{LE}(t,x)^* \right) \phi_g, \quad (t,x) \in (0,T) \times \mathbb{R}, \\ i\partial_t \epsilon_g = e^{-i\Omega t} S(-t) f(t,x) S(t) \epsilon_e + e^{-i\Omega t} S(-t)(f - f_{LD}) S(t) \phi_e \\ \quad + \left(e^{-i\Omega t} S(-t) f_{LD} S(t) - f_{LE}(t,x) \right) \phi_g, \quad (t,x) \in (0,T) \times \mathbb{R}, \\ \epsilon_e(0) = 0, \quad \epsilon_g(0) = 0. \end{cases}$$

- blue: Error coming from the Lamb-Dicke approximation.
- red: Error coming from the averaging approximation.

Sketch of the proof

Can be put under the form

$$\begin{cases} i\partial_t \epsilon_e = e^{i\Omega t} S(-t) f(t,x) S(t) \epsilon_g + h_{LDe}(t,x) + \partial_t h_{me}(t,x), \\ i\partial_t \epsilon_g = e^{-i\Omega t} S(-t) f(t,x) S(t) \epsilon_e + h_{LDg}(t,x) + \partial_t h_{mg}(t,x). \end{cases}$$

with

$$h_{LDe}(t,x) = e^{i\Omega t} S(-t)(f(t) - f_{LD}(t)) S(t) \phi_g(t),$$

$$h_{me}(t,x) = \int_0^t \left(e^{i\Omega s} S(-s) f_{LD}(s,x) S(s) - f_{LE}(s,x)^* \right) \phi_g(s)\, ds, \cdots$$

We prove $\|h_{LD}\|_0 \le C\eta^2 K(M+1)$ and $\|h_{me}\|_0 \le C\dfrac{K}{\omega - K}(M+1)^{3/2}$.

Then, by energy techniques, $\sup_{t\in[0,T]} \left\|(\epsilon_e(t), \epsilon_g(t))\right\|_{0\times 0}$ is small for η small and $\omega\eta/K$ large.

In $(L^2)^2$

For $\delta > 0$, we set M large enough so that dist$((\psi_e^0, \psi_g^0), V_M^2)$ and dist$((\psi_e^1, \psi_g^1), V_M^2)$ are small. We then look at $(\tilde{\psi}_e^0, \tilde{\psi}_g^0)$, $(\tilde{\psi}_e^1, \tilde{\psi}_g^1)$ in $(V_M^2)^2$ of unit norm s.t.

$$\left\| (\psi_e^0, \psi_g^0) - (\tilde{\psi}_e^0, \tilde{\psi}_g^0) \right\|_{0\times 0} \leq \frac{\delta}{3}, \quad \left\| (\psi_e^1, \psi_g^1) - (\tilde{\psi}_e^1, \tilde{\psi}_g^1) \right\|_{0\times 0} \leq \frac{\delta}{3}.$$

We then apply the previous theorem to $(\tilde{\psi}_e^0, \tilde{\psi}_g^0)$, $(\tilde{\psi}_e^1, \tilde{\psi}_g^1)$, with the parameters as in the previous theorem

$$\left\| (\psi_e(T), \psi_g(T)) - \beta(\psi_e^1, \psi_g^1) \right\|_{0\times 0} \leq \delta.$$

Indeed, the truncature error stays constant.

In $\mathcal{D}(A^{k/2})^2$

Again, two main steps:

- In V_M^2 for $\|(\cdot,\cdot)\|_{k\times k}$
- For data in $\mathcal{D}(A^{k/2})$, a truncation argument with M large enough.

In V_M^2

Theorem

For (ψ_e^0, ψ_g^0) and (ψ_e^1, ψ_g^1) in V_M^2, and the control constructed above:
$\forall \delta > 0, \exists \eta_k = \eta_k(\delta, M), \exists \rho_k = \rho_k(\delta, M)$, s.t. $\forall K$, $\forall (\omega, \Omega)$ with $2\Omega \geq 3\omega$ and

$$\eta \leq \eta_k, \quad TK = \frac{3\pi\sqrt{M}}{\eta}, \quad \frac{\omega\eta}{K} \geq \rho_k,$$

the solution of the exact system (1) with initial data (ψ_e^0, ψ_g^0) satisfies

$$\left\| (\psi_e(T), \psi_g(T)) - \beta(\psi_e^1, \psi_g^1) \right\|_{k\times k} \leq \delta.$$

Sketch of the proof

We do as before, except that we need stronger estimates on the error terms:

$$\|h_{LD}(t,x)\|_k \leq C_1(k)\eta^2 K(M+1)^{(k+2)/2},$$

$$\|h_m(t,x)\|_k \leq C_2(k)\frac{K}{(\omega-K)}(M+1)^{(k+3)/2}.$$

$\rightsquigarrow$ Yields the proof similarly by standard energy estimates. (induction on k).

In $\mathcal{D}(A^{k/2})$

Theorem

Let (ψ_e^0, ψ_g^0) and (ψ_e^1, ψ_g^1) in $\mathcal{D}(A^{k/2})^2$ of unit $(L^2)^2$-norm. Then $\forall \delta > 0, \exists(\aleph,\ \eta_k,\ \rho_k)$, such that for (ω, Ω) with $2\Omega \geq 3\omega$ and

$$\eta \leq \eta_k, \quad KT = \aleph/\eta, \quad \frac{\omega\eta}{K} \geq \rho_k,$$

there exists a control $\mathbf{u}(t, x)$ as above, satisfying the additional constraints

$$\sup\{|u_0(t)|, |u_r(t)|, |u_b(t)|\} \leq K.$$

such that the solution (ψ_e, ψ_g) of (1) with initial data (ψ_e^0, ψ_g^0) satisfies, for some $\beta \in \mathbb{C}$ of modulus 1,

$$\left\|(\psi_e(T), \psi_g(T)) - \beta(\psi_e^1, \psi_g^1)\right\|_{k\times k} \leq \delta.$$

A word on the proof

Again, we do a truncation argument, but this is more subtle !

$\rightsquigarrow$ The truncature (ϵ_e, ϵ_g) is solution of
$i\partial_t \epsilon_e = e^{i\Omega t} S(-t) f(t,x) S(t) \epsilon_g,\ i\partial_t \epsilon_g = e^{-i\Omega t} S(-t) f(t,x) S(t) \epsilon_e$.
But the equation is not isometric on $\mathcal{D}(A^{k/2})^2$!

One shall do a commutator estimate

$$< f(t,x)\psi_1, A^k \psi_2 >=< f(t,x) A^{k/2} \psi_1, A^{k/2}\psi_2 > + \text{a reminder.}$$

To bound the reminder

To bound the reminder, we use

$$\sup\left\{ \|\partial_x f\|_{L^\infty((0,T)\times\mathbb{R})}, \cdots, \left\|\partial_x^k f\right\|_{L^\infty((0,T)\times\mathbb{R})} \right\} \leq C\ K\eta,$$

But the time is given $T \simeq \dfrac{\sqrt{M}}{K\eta}$.

We then obtain

$$\sup_{t\in[0,T]} \left\|(\epsilon_e(t),\epsilon_g(t))\right\|_{\ell\times\ell} \leq C\sqrt{M} \sup_{t\in[0,T]} \left\|(\epsilon_e(t),\epsilon_g(t))\right\|_{(\ell-1)\times(\ell-1)} + \left\|(\epsilon_e(0),\epsilon_g(0))\right\|_{\ell\times\ell}.$$

Miracle !

$$\sqrt{M} \sup_{t\in[0,T]} \left\|(\epsilon_e(t),\epsilon_g(t))\right\|_{(\ell-1)\times(\ell-1)} \simeq \frac{\delta}{M^{(k-\ell)/2}},$$

$$\left\|(\epsilon_e(0),\epsilon_g(0))\right\|_{\ell\times\ell} \simeq \frac{\delta}{M^{(k-l)/2}}.$$

Conclusion

- Constructive method in the limits

$$\eta \ll 1, \quad KT = \aleph/\eta, \quad \frac{\omega\eta}{K} \gg 1,$$

based on the finite dimension.

- Remark: We have estimates on $\aleph$ when the initial and target states are in $\mathcal{D}(A^{k/2})^2$ and we control approximately in $\mathcal{D}(A^{\ell/2})^2$ pour $\ell < k$.

Conclusion

- Can we do better than Law Eberly on the simplified model ? *For instance, we used only two controls.*
  $\rightsquigarrow$ Problem in combinatorics and graph theory.
  Cf Brockett's talk
- More intricate models ? Two ions coupled through oscillations...

Conclusion

- Can we do local exact controllability ?
  Preliminary question : Consider

  $$i\partial_t \psi = (-\partial_{xx} + x^2)\psi + f(t)\eta(x)\psi, \quad (t,x) \in (0,T) \times \mathbb{R}.$$

  Local exact controllability of this equation around the ground state trajectory $\exp(-i\lambda_0 t)\Psi_0$?

  Can we choose a profile function $\eta = \eta(x)$ such that any initial data ψ_0 near Ψ_0, there exists a real valued control function $f = f(t)$, such that the solution ψ satisfies $\psi(T) = \exp(-i\lambda_0 T)\Psi_0$?

Thanks

Thank you for the attention !

Based on
S. Ervedoza and J.-P. Puel. Approximate controllability for a system of Schrödinger equations modeling a single trapped ion. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26(6):2111–2136, Nov.–Dec. 2009