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Approximate controllability of a semi-discrete 1-D wave equation

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Abstract. This article deals with the approximate boundary controllability problem for a semi-discrete 1-D wave equation. By using a Fourier method, we show that there exists a sequence of approximate controls for the semi-discrete wave equation which remains uniformly bounded when the mesh size tends to zero. These approximate controls will be explicitly constructed and estimates for their norms will be given in function of the discretization step.

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1. Introduction

The approximate boundary controllability problem for the 1-D wave equation reads as follows: given $T \geq 2$, $\varepsilon > 0$ and $(u^0, u^1) \in L^2(0,1) \times H^{-1}(0,1)$ there exists a control function $v \in L^2(0,T)$ such that the solution of the equation

$$\begin{cases} u'' - u_{xx} = 0 & \text{for } x \in (0,1),\ \ t > 0 \\ u(t,0) = 0 & \text{for } t > 0 \\ u(t,1) = v_\varepsilon(t) & \text{for } t > 0 \\ u(0,x) = u^0(x) & \text{for } x \in (0,1) \\ u'(0,x) = u^1(x) & \text{for } x \in (0,1) \end{cases} \tag{1}$$

satisfies

$$||(u,u')(T,\cdot)||_{L^2(0,1)\times H^{-1}(0,1)} \leq \varepsilon. \tag{2}$$

By $'$ we denote the time derivative.

This problem has been studied and solved some decades ago and several approaches are now available. In section 2 we briefly describe the variational method which consists on obtaining a control v by minimizing a cost functional. However, the method we shall be mainly concerned with is based on Fourier analysis. The explicit construction of an appropriate biorthogonal sequence will give a control v.

The argument used to prove the approximate controllability of the continuous wave equation will be applied to the case we are interested in, that of a semi-discrete wave equation. Let us briefly describe the problem and our main results.

We consider $N \in \mathbb{N}^*$, a step $h = \frac{1}{N+1}$ and an equidistant mesh of the interval $(0,1)$, $0 = x_0 < x_1 < ... < x_N < x_{N+1} = 1$, with $x_j = jh$, $0 \leq j \leq N+1$ and we

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introduce the following finite-difference semi-discretization of (1)

$$\begin{cases} u_j''(t) - \frac{u_{j+1}(t)+u_{j-1}(t)-2u_j(t)}{h^2} = 0 & \text{for } 1 \le j \le N, \ \ t > 0 \\ u_0(t) = 0 & \text{for } t > 0 \\ u_{N+1}(t) = v_\varepsilon(h,t) & \text{for } t > 0 \\ u_j(0) = u_j^0, \ u_j' = u_j^1 & \text{for } 1 \le j \le N \end{cases} \tag{3}$$

We study the following approximate controllability problem: given $T > 0$, $\varepsilon > 0$ and $(u_j^0, u_j^1)_{1\le j\le N} \in \mathbb{C}^{2N}$, there exists a control function $v_\varepsilon(h) \in L^2(0,T)$ such that the solution u of (3) satisfies

$$||(u_j, u_j')_{1\le j\le N}(T)||_{-1} \le \varepsilon. \tag{4}$$

The norm $|| \ ||_{-1}$ is the discrete version of the norm from $L^2(0,1) \times H^{-1}(0,1)$ and it is defined in the following way

$$\|U\|_{-1} = \sup_{\substack{W\in\mathbb{C}^{2N}\\ \|W\|=1}} |< U, W >| \tag{5}$$

where $< \ , \ >$ is a duality product defined by

$$< U, W >= h\sum_{j=1}^{N}(u_j\overline{w}_{N+j} - u_{N+j}\overline{w}_j), \quad \forall U = (u_j)_{1\le j\le 2N},\, W = (w_j)_{1\le j\le 2N} \in \mathbb{C}^{2N}.$$

The norm $|| \ ||$ will denote the discrete version of the norm in $H_0^1(0,1) \times L^2(0,1)$ and it is given by the inner product

$$(f,g) = h\left[\sum_{k=1}^{N-1} \frac{f_{k+1}-f_k}{h}\frac{\overline{g}_{k+1}-\overline{g}_k}{h} + \frac{1}{h^2}(f_1\overline{g}_1 + f_N\overline{g}_N)\right] + h\sum_{k=1}^{N} f_{N+k}\overline{g}_{N+k} \tag{6}$$

where $f = (f_k)_{1\le k\le 2N}$ y $g = (g_k)_{1\le k\le 2N}$ are two vectors from $\mathbb{C}^{2N}$.

System (3) consists of N linear differential equations with unknowns $u_1, u_2, ..., u_N$. $u_j(t)$ is an approximation of the solution u of (1) in (t, x_j), provided that $(u_j^0, u_j^1)_{1\le j\le N}$ approximates the continuous initial datum (u^0, u^1).

In (3), $v_\varepsilon(h)$ is a control which forces the solution to fulfill (4). $v_\varepsilon(h)$ will be called discrete control. Our interest is to see if $v_\varepsilon(h)$ may converge to a control v_ε for the continuous equation (1) when the discretization step h tends to zero.

In the last years many works have dealt with the numerical approximations for the control problem. For instance, in [2], [4] and [3], by using Hilbert Uniqueness Method, some numerical algorithms have been proposed. In these articles a bad numerical behavior of the discrete controls has been observed. Frequently, the sequence of discrete controls is not even bounded in L^2, let alone convergent. This phenomenon is due to the fact that the numerical schema introduces spurious high frequency vibrations that are not observed in the continuous problem. More precisely, as it was pointed out in [5] (see also [3]), the differences between the discrete and the continuous systems become significant for the modes of order of N.

Whereas all the works mentioned above deal with the exact controllability problem, we shall be mainly interested in the approximate controllability one.

In section 3 we explicitly construct a biorthogonal sequence and obtain a discrete control $v_\varepsilon(h)$ for (3). Like in the case of the continuous wave equation, this control will let untouched the high frequencies (with wave number greater that $n(\varepsilon)$) and will lead to zero the low ones (with wave number smaller that $n(\varepsilon)$). $n(\varepsilon)$ is a positive integer determined by the initial data (u^0, u^1) and the parameter ε. In order to obtain an uniformly bounded sequence of discrete controls $(v_\varepsilon(h))_{h>0}$, the step size h should be sufficiently small, depending of $n(\varepsilon)$. Hence, the techniques we use allow to determine an explicit relation between ε and h in order to have a uniformly bounded sequence of discrete controls.

This approach is similar and related to the one we have used in [8]. In that case, the exact controllability problem is considered and an uniformly bounded sequence of controls is obtained by eliminating from the very beginning the short wave length components of the initial datum. Like in the case of the the approximate controllability problem, the number of frequencies of the initial data uniformly controllable to zero depends on the step size h.

A variational treatment of the problem is used in [11] and [12] where the existence of a bounded sequence of controls is proved. However, based on Fourier analysis, our approach is different and shows the relation between the step size, the controllability of a given number of nodes and the norm of the approximate controls.

The rest of the paper is organized in the following way. Section 2 deals with the continuous wave equation. Whereas the variational approach is only mentioned, a detailed description of the Fourier method is given. The same ideas are used in section 3 for the semi-discrete problem. The main result of the paper, Theorem 3.2, gives a sequence of uniformly bounded discrete controls.

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2. Continuous wave equation

In this section we consider the approximate controllability problem for the linear 1-D continuous wave equation (1)-(2).

2.1. Variational approach. It is known that the approximate controllability holds if the following unique continuation principle is true

$$\varphi_x(t,1)=0 \text{ for } t\in(0,T) \quad \Rightarrow \quad \varphi(t,x)=0 \text{ for } (t,x)\in(0,T)\times(0,1) \tag{7}$$

where φ is the solution of the adjoint homogeneous equation

$$\begin{cases} \varphi''-\varphi_{xx}=0 & \text{for } x\in(0,1),\ \ t>0 \\ \varphi(t,0)=\varphi(t,1)=0 & \text{for } t>0 \\ \varphi(0,x)=\varphi^0(x) & \text{for } x\in(0,1) \\ \varphi'(0,x)=\varphi^1(x) & \text{for } x\in(0,1) \end{cases} \tag{8}$$

with $(\varphi^0,\varphi^1)\in H_0^1(0,1)\times L^2(0,1)$.

This may be simply seen by considering the minimization of the functional

$$J: H_0^1(0,1)\times L^2(0,1)\longrightarrow \mathbb{R},$$

$$J(\varphi^0,\varphi^1)=\frac{1}{2}\int_0^T|\varphi_x(t,1)|^2dt- <(u^0,u^1),(\varphi^0,\varphi^1)>_{-1,1} + \\ +\varepsilon||(\varphi^0,\varphi^1)||_{H_0^1(0,1)\times L^2(0,1)} \tag{9}$$

where $<\cdot,\cdot>_{-1,1}$ is the duality product between $L^2(0,1)\times H^{-1}(0,1)$ and $H_0^1(0,1)\times L^2(0,1)$,

$$<(u^0,u^1),(\varphi^0,\varphi^1)>_{-1,1}=\int_0^1 u^0\varphi^1- <u^1,\varphi^0>_{H^{-1}(0,1),H_0^1(0,1)}$$

and φ is the solution of (8) with initial data (φ^0, φ^1).

The functional J is continuous and convex. Moreover, property (7) implies that J is coercive. Hence, J has a minimum $(\widehat{\varphi}^0, \widehat{\varphi}^1) \in H_0^1(0,1) \times L^2(0,1)$ which will give an approximate control $v(t) = \widehat{\varphi}_x(t,1)$ for equation (1) ($\widehat{\varphi}$ is the solution of (8) with initial data $(\widehat{\varphi}^0, \widehat{\varphi}^1)$). More details may be found in [7].

2.2. Fourier approach. If the Fourier decomposition of the solution φ of (8) is considered, we can see that (7) is related to the completeness property of the exponential functions whose exponents are the eigenvalues of the wave operator (see, for instance, [1]). Our aim is to use an explicit biorthogonal sequence to construct an approximate control v.

2.2.1. *Spectral analysis.* The eigenvalues of (8) are $i\lambda_n$ with $\lambda_n = n\pi$, $n \in \mathbb{Z}^*$, and the corresponding eigenfunctions are

$$\Phi_n(x) = \left(\frac{1}{n\pi}, i\right) \sin(n\pi x), \quad n \in \mathbb{Z}^*.$$

Proposition 2.1. *The set $(\Phi_n)_{n\in\mathbb{Z}^*}$ forms an orthonormal basis in $H_0^1(0,1)\times L^2(0,1)$ and an orthogonal basis in $L^2(0,1) \times H^{-1}(0,1)$ and $||\Phi_n||_{L^2\times H^{-1}} = 1/\lambda_n$.*

Each element $(\varphi^0, \varphi^1) \in H_0^1(0,1) \times L^2(0,1)$ has an expansion

$$(\varphi^0, \varphi^1)(x) = \sum_{n\in\mathbb{Z}^*} a_n \, \Phi_n(x)$$

with $(a_n)_{n\in\mathbb{Z}^*} \in \ell^2$ and the corresponding solution of (8) is given by

$$(\varphi, \varphi')(t,x) = \sum_{n\in\mathbb{Z}^*} a_n \, \Phi_n(x) e^{i\lambda_n t}.$$

Remark 2.1. *We obtain that (7) is equivalent to*

$$\sum_{n\in\mathbb{Z}^*} a_n \, (-1)^n e^{i\lambda_n t} = 0 \text{ for } t \in (0,T) \quad \Rightarrow \quad a_n = 0, \ \forall n \in \mathbb{Z}^*. \tag{10}$$

If the exponential family $\left(e^{i\lambda_n t}\right)_{n\in\mathbb{Z}^*}$ has a biorthogonal sequence in $L^2(0,T)$ then it is minimal in $L^2(0,T)$ (no element is obtained as linear combination of the others) and (10) holds (see [1] *and* [10]*). Remark that, although the existence of a biorthogonal is a sufficient condition for (10), it is not a necessary one. Nevertheless, the construction of a biorthogonal sequence will allow no only to prove directly the approximate controllability result for (1) but also to give an explicit approximate control.*

In what follows we construct a biorthogonal sequence to the exponential family $\left(e^{i\lambda_n t}\right)_{n\in\mathbb{Z}^*}$ in $L^2(0,T)$ and prove the approximate controllability result with an explicit approximate control.

2.2.2. *An explicit biorthogonal.* The existence of a biorthogonal is not difficult to prove if $T = 2$. Note that it is sufficient to limit ourselves to the case $T = 2$. Indeed, if $T > 2$ we simply extend the functions by zero in $(2,T)$ and obtain a biorthogonal in $L^2(0,T)$ for any $T \geq 2$.

Given any $n \in \mathbb{Z}^*$, take

$$\Theta_n(z) = \frac{\sin(z)}{z - n\pi}, \ z \in \mathbb{C}.$$

Θ_n is an entire function of exponential type 1 and $\Theta_n(x) \in L^2(\mathbb{R})$ for each $n \in \mathbb{Z}^*$. From the Paley-Wiener Theorem it follows that the Fourier transform of Θ_n,

$$\widehat{\Theta}_n(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \Theta(x) e^{-ixt},$$

belongs to $L^2(-1,1)$. Nextly, define

$$\Psi_n(t) = \widehat{\Theta}_n(t-1) e^{-i\lambda_n}. \tag{11}$$

$(\Psi_n)_{n\in\mathbb{Z}^*}$ is a biorthogonal sequence in $L^2(0,2)$ for the family of exponential functions $\left(e^{i\lambda_n t}\right)_{n\in\mathbb{Z}^*}$. Indeed, we have

- $\Psi \in L^2(0,2)$ since $\widehat{\Theta}_n \in L^2(-1,1)$.
- $\int_0^2 \Psi_n(t) e^{i\lambda_m t} dt = e^{i(\lambda_m - \lambda_n)} \int_{-1}^1 \widehat{\Theta}_n(t) e^{i\lambda_m t} dt = e^{i(\lambda_m-\lambda_n)} \Theta_n(\lambda_m) = \delta_{nm}$.

We now return to the approximate controllability property of (1).

2.2.3. *An approximate controllability result.* The following result is a consequence of the existence of the biorthogonal from the previous section.

Theorem 2.1. *Let $T \geq 2$, $\varepsilon > 0$ and $(u^0, u^1) \in L^2(0,1) \times H^{-1}(0,1)$ with*

$$(u^0, u^1)(x) = \sum_{n\in\mathbb{Z}^*} a_n^0 \, \Phi_n(x).$$

There exists a control $v_\varepsilon \in L^2(0,T)$ such that the solution (u, u') of (1) satisfies (2).

Proof. Since $(u^0, u^1) \in L^2(0,1) \times H^{-1}(0,1)$, there exists $n(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{|n|>n(\varepsilon)} |a_n^0/\lambda_n|^2 < \varepsilon^2.$$

We consider

$$(w^0, w^1)(x) = \sum_{1\leq|n|\leq n(\varepsilon)} a_n^0 \Phi_n(x), \quad (z^0, z^1)(x) = \sum_{|n|>n(\varepsilon)} a_n^0 \Phi_n(x)$$

and we solve the following equations

$$\begin{cases} w'' - w_{xx} = 0 & \text{for } x \in (0,1), \ \ t > 0 \\ w(t,0) = 0 & \text{for } t > 0 \\ w(t,1) = v_\varepsilon(t) & \text{for } t > 0 \\ w(0,x) = w^0(x) & \text{for } x \in (0,1) \\ w'(0,x) = w^1(x) & \text{for } x \in (0,1) \end{cases} \tag{12}$$

$$\begin{cases} z'' - z_{xx} = 0 & \text{for } x \in (0,1), \ \ t > 0 \\ z(t,0) = z(t,1) = 0 & \text{for } t > 0 \\ z(0,x) = z^0(x) & \text{for } x \in (0,1) \\ z'(0,x) = z^1(x) & \text{for } x \in (0,1). \end{cases} \tag{13}$$

Evidently, $u = w + z$ where w and z are the solution of (12) and (13) respectively. Moreover,

$$(z, z')(t,x) = \sum_{|n|>n(\varepsilon)} a_n^0 \Phi_n(x) e^{i\lambda_n t}$$

and

$$||(z, z')(T, \cdot)||^2_{L^2(0,1)\times H^{-1}(0,1)} = \sum_{|n|>n(\varepsilon)} |a_n^0/\lambda_n|^2 < \varepsilon^2. \tag{14}$$

On the other hand, let us consider that $(w, w')(t,x) = \sum_{n\in\mathbb{Z}^*} a_n(t) \Phi_n(x)$.

Given any $(\varphi^0, \varphi^1) \in H_0^1(0,1) \times L^2(0,1)$, if we multiply (12) by φ, the solution of (8) with initial data (φ^0, φ^1), and we integrate by parts we obtain

$$\int_0^T v(t)\varphi_x(t,1)dt - < (w^0, w^1), (\varphi^0, \varphi^1) >_{-1,1} + < (w, w')(T)), (\varphi, \varphi')(T)) >_{-1,1} = 0.$$

If (φ^0, φ^1) has the expansion $(\varphi^0, \varphi^1)(x) = \sum_{n\in\mathbb{Z}^*} \alpha_n \Phi_n(x)$, it follows that

$$\int_0^T v(t) \left(\sum_{n\in\mathbb{Z}^*} (-1)^n \alpha_n e^{i\lambda_n t} \right) dt - \sum_{1\leq|n|\leq n(\varepsilon)} \frac{1}{\lambda_n} a_n^0 \alpha_n + \sum_{n\in\mathbb{Z}^*} \frac{1}{\lambda_n} a_n(T)\alpha_n = 0. \quad (15)$$

Now, if $(\Psi_n)_{n\in\mathbb{Z}^*}$ is the biorthogonal sequence given by (11), we define the control

$$v_\varepsilon(t) = \sum_{1\leq|m|\leq n(\varepsilon)} \frac{(-1)^m}{\lambda_m} a_m^0 \Psi_m(t). \quad (16)$$

Evidently, $v_\varepsilon \in L^2(0,T)$. Moreover, (15) implies that

$$\sum_{n\in\mathbb{Z}^*} \frac{1}{\lambda_n} a_n(T)\alpha_n = 0. \quad (17)$$

Since (φ^0, φ^1) is arbitrary it follows that $a_n(T) = 0$ for all $n \in \mathbb{Z}^*$. Hence,

$$w(T, \cdot\,) = w'(T, \cdot\,) = 0. \quad (18)$$

From (14) and (18) it follows that the initial data (u^0, u^1) is approximately controlled with a control v given by (16). □

Remark 2.2. *The control give by (16) does not affect the high modes. Indeed, the initial data (w^0, w^1) has the property*

$$((w^0, w^1), \Phi^n) = 0, \quad \forall n > n(\varepsilon)$$

and the same is true for the solution $(w, w')(T)$ at time T. Hence v leads to zero the modes with $1 \leq |n| \leq n(\varepsilon)$ but does not affect the modes with $|n| > n(\varepsilon)$. This property is due to the particular form of our control obtained from the biorthogonal $(\Psi_n)_{n\in\mathbb{Z}^*}$.

Remark 2.3. *The biorthogonal $(\Psi_n)_{n\in\mathbb{Z}^*}$ is bounded in $L^2(0,T)$ and*

$$||v||_{L^2(0,T)} \leq C \sum_{1\leq|n|\leq n(\varepsilon)} \frac{|a_n^0|}{|\lambda_n|}. \quad (19)$$

Hence, the sequence $\{v(\varepsilon)\}_{\varepsilon>0}$ is uniformly bounded in ε if the last series is convergent. In this case, an exact control of the wave equation may be obtained by passing to the limit when ε goes to zero.

3. Semi-discrete wave equation

Let us now pass to the controllability problem for the discrete system (3)-(4).

Suppose that the continuous initial data (u^0, u^1) is given by

$$(u^0, u^1)(x) = \sum_{n\in\mathbb{Z}^*} a_n^0 \Phi^n(x) \quad (20)$$

where $\Phi^n(x) = (\frac{1}{i\,n\pi}, -1)\sin(n\pi x)$ and $(a_n^0)_{n\in\mathbb{Z}^*}$ is such that $\sum_{n\in\mathbb{Z}^*} \frac{|a_n^0|^2}{n^2\pi^2} < \infty$.

The corresponding discrete initial data of (3), $(u_j^0, u_j^1)_{1\le j\le N}$, is given by

$$(u_j^0, u_j^1)_{1\le j\le N} = \sum_{n\in\mathbb{Z}^*} a_n^0 \Phi^n(n\pi jh) \tag{21}$$

and it is obtained by simply discretizing the eigenfunctions $\Phi^n(x)$ in (20).

Remark 3.1. *In order to ensure the convergence of the series in (21) a more restrictive condition must be imposed on the coefficients of* (u^0, u^1). *Hence, we shall consider that there exists* $r > 1$ *such that*

$$\sum_{n\in\mathbb{Z}^*} n^r |a_n^0|^2 < \infty. \tag{22}$$

Condition (22) implies that

$$\sum_{n\in\mathbb{Z}^*} |a_n^0 \sin(n\pi jh)| \le \left(\sum_{n\in\mathbb{Z}^*} |a_n^0|^2 n^r\right)^{\frac{1}{2}} \left(\sum_{n\in\mathbb{Z}^*} \frac{|\sin(n\pi jh)|^2}{n^r}\right)^{\frac{1}{2}} < \infty$$

and the series from (21) is absolutely convergent.

On the other hand, note that (22) implies also that

$$\sum_{n\in\mathbb{Z}^*} |a_n^0| \le \left(\sum_{n\in\mathbb{Z}^*} |a_n^0|^2 n^r\right)^{\frac{1}{2}} \left(\sum_{n\in\mathbb{Z}^*} \frac{1}{n^r}\right)^{\frac{1}{2}} < \infty. \tag{23}$$

3.1. Fourier analysis. The eigenvalues of the operator from (3) are $i\,\lambda_n(h)$, where

$$\lambda_n(h) = \frac{2}{h}\sin(\frac{n\pi h}{2}), \quad -N \le n \le N,\ n \ne 0,$$

and the corresponding eigenvectors are

$$\Phi^n(h) = \begin{pmatrix} \frac{h}{2i\,\sin(\frac{n\pi h}{2})}\varphi^n(h) \\ -\varphi^n(h) \end{pmatrix} = \begin{pmatrix} \frac{1}{i\lambda_n(h)}\varphi^n(h) \\ -\varphi^n(h) \end{pmatrix}, \quad -N \le n \le N,\ n \ne 0$$

where $\varphi^n(h) = (\sin(jn\pi h))_{1\le n\le N}$. We have that

Proposition 3.1. *The set of vectors* $(\Phi^n(h))_{\substack{|n|\le N\\ n\ne 0}} \subset \mathbb{C}^{2N}$ *forms an orthonormal basis in* $\mathbb{C}^{2N}$ *with respect to the inner product (6).*

Remark 3.2. *The initial data of (3) given by (21) may be written as*

$$(u_j^0, u_j^1)_{1\le j\le N} = \sum_{\substack{|n|\le N\\ n\ne 0}} a_n^0(h)\Phi^n(h) \tag{24}$$

where $a_n^0(h) = \frac{1}{2}\left(\frac{\lambda_n(h)}{n\pi} + 1\right) a_n^0 + \frac{1}{2}\left(\frac{\lambda_n(h)}{n\pi} - 1\right) a_{-n}^0$.

Evidently,

$$|a_n^0(h)| \le |a_n^0| + |a_{-n}^0|, \quad 1 \le |n| \le N. \tag{25}$$

The adjoint of (3) is given by

$$\begin{cases} w_j''(t) - \frac{w_{j+1}(t)+w_{j-1}(t)-2w_j(t)}{h^2} = 0 & \text{for } 1\le j\le N,\ \ t>0 \\ w_0(t) = w_{N+1}(t) = 0 & \text{for } t>0 \\ w_j(0) = w_j^0,\ w_j' = w_j^1 & \text{for } 1\le j\le N. \end{cases} \tag{26}$$

We can now give a Fourier decomposition of the solutions of (26).

Let us denote $Z(t) = (W(t), W'(t)) = ((w_j(t))_{1\leq j\leq N}, (w_j(t))_{1\leq j\leq N}) \in \mathbb{C}^{2N}$. If the initial datum $Z^0 = (W^0, W^1)$ of (26) is such that

$$Z^0 = \sum_{\substack{|n|\leq N\\ n\neq 0}} \alpha_n^0 \Phi^n(h) \tag{27}$$

then the corresponding solution, $Z(t) = (W(t), W'(t))$, is

$$Z(t) = \sum_{\substack{|n|\leq N\\ n\neq 0}} \alpha_n^0 e^{i\,\lambda_n(h)t} \Phi^n(h). \tag{28}$$

3.2. A biorthogonal sequence. Let Λ be the family of the exponential functions $\left(e^{i\lambda_n(h)t}\right)_{\substack{|n|\leq N\\ j\neq 0}}$. The following result is proved in [8].

Theorem 3.1. *If $T > 0$ is sufficiently large (but independent of N), there exists a sequence $(\Theta_m(h))_{\substack{|m|\leq N\\ m\neq 0}}$, biorthogonal in $L^2(0,T)$ to Λ, such that*

$$||\Theta_m(h)||_{L^2(0,T)} \leq C|\lambda_m(h)| \exp\left(\alpha \frac{|\lambda_m(h)|^2}{N}\right), \text{ for } m = \pm 1, \pm 2, ..., \pm N \tag{29}$$

where C and α are two positive constants which do not depend on m and N.

Remark 3.3. *Let us remark that Theorem 3.1 implies that there exists a biorthogonal sequence $(\Theta_m(h))_{\substack{|m|\leq N\\ m\neq 0}}$, such that*

$$||\Theta_m(h)||_{L^2(0,T)} \leq C'|\lambda_m(h)|, \text{ for any } 1 \leq |m| \leq \sqrt{N} \tag{30}$$

where C' is a constant which does not depend on N. Indeed, since

$$|\lambda_m(h)| = \left|\frac{2}{h}\sin\left(\frac{m\pi h}{2}\right)\right| \leq \frac{2}{h}\frac{|m|\pi h}{2} = |m|\pi$$

from (29) follows that, for any $1 \leq |m| \leq \sqrt{N}$,

$$||\Theta_m(h)||_{L^2(0,T)} \leq C|\lambda_m(h)| \exp\left(\alpha\pi^2 \frac{|m|^2}{N}\right) \leq C|\lambda_m(h)|.$$

3.3. Fourier approach for the approximate controllability. By using the biorthogonal sequence given in Theorem 3.1 an approximate control for (3) will be constructed.

Theorem 3.2. *Let $T > 0$ sufficiently large (as in Theorem 3.1), $\varepsilon > 0$ and a continuous initial data $(u^0, u^1) \in L^2(0,1) \times H^{-1}(0,1)$ as in (20) with the Fourier coefficients $(a_n^0)_{n\in\mathbb{Z}^*}$ satisfying (22) for some $r > 1$.*

There exists $n(\varepsilon) \in \mathbb{N}^*$ such that for any $N \geq (n(\varepsilon))^2$, if $(u_j^0, u_j^1)_{1\leq j\leq N}$ is the initial data of (3) given by (24), there exists a control $v_\varepsilon(h) \in L^2(0,T)$ for (3). Moreover, there exists a constant $C > 0$, independent of N, such that

$$||v_\varepsilon(h)||_{L^2(0,T)} \leq C, \text{ for all } h = \frac{1}{N+1} \text{ with } N \geq (n(\varepsilon))^2. \tag{31}$$

Proof. If the initial data is given by (24), the corresponding solution of (3) is

$$U(t) = \sum_{\substack{|n|\leq N\\ n\neq 0}} a_n(h,t) e^{i\,\lambda_n(h)t} \Phi^n(h). \tag{32}$$

There exists $n(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{|n|>n(\varepsilon)} \left| \frac{a_n^0}{n\pi} \right|^2 \leq \frac{\varepsilon^2 \pi^2}{4}. \tag{33}$$

From (25) it follows that

$$\sum_{|n|>n(\varepsilon)} \left| \frac{a_n^0(h)}{\lambda_n(h)} \right|^2 \leq \varepsilon^2. \tag{34}$$

We chose

$$v_\varepsilon(h) = \sum_{\substack{|n|\leq n(\varepsilon) \\ n\neq 0}} \frac{a_n^0(h)}{\lambda_n(h)} \frac{i\,(-1)^{n+1}}{\cos(\frac{n\pi h}{2})} \Theta_n(h) \tag{35}$$

where $(\Theta_m(h))_{\substack{|m|\leq N \\ m\neq 0}}$ is the biorthogonal sequence given by Theorem 3.1.

We write $U^0 = X^0 + Y^0$ where

$$X^0 = \sum_{\substack{|n|\leq n(\varepsilon) \\ n\neq 0}} a_n^0(h)\Phi^n(h), \quad Y^0 = \sum_{|n|>n(\varepsilon)} a_n^0(h)\Phi^n(h).$$

The corresponding solution U of (3) may be written as

$$U = X + Y$$

where $X(t) = (x_j(t), x_j'(t))_{1\leq|j|\leq N}$ is the solution of the equation

$$\begin{cases} x_j''(t) - \frac{x_{j+1}(t)+x_{j-1}(t)-2x_j(t)}{h^2} = 0 & \text{for } 1 \leq j \leq N, \ t > 0 \\ x_0(t) = 0 & \text{for } t > 0 \\ x_{N+1}(t) = v_\varepsilon(h,t) & \text{for } t > 0 \\ (x_j, x_j')_{1\leq|j|\leq N}(0) = X^0 & \end{cases} \tag{36}$$

and $Y(t) = (y_j(t), y_j'(t))_{1\leq|j|\leq N}$ is the solution of the equation

$$\begin{cases} y_j''(t) - \frac{y_{j+1}(t)+y_{j-1}(t)-2y_j(t)}{h^2} = 0 & \text{for } 1 \leq j \leq N, \ t > 0 \\ y_0(t) = 0 & \text{for } t > 0 \\ y_{N+1}(t) = 0 & \text{for } t > 0 \\ (y_j, y_j')_{1\leq|j|\leq N}(0) = Y^0. & \end{cases} \tag{37}$$

It is easy to see that the solution Y of (37) is given by

$$Y(t) = \sum_{|n|>n(\varepsilon)} a_n^0(h) e^{i\,\lambda_n t} \Phi^n(h) \tag{38}$$

and

$$||Y(T)||_{-1}^2 = \sum_{|n|>n(\varepsilon)} \left| \frac{a_n^0(h)}{\lambda_n(h)} \right|^2 \leq \varepsilon^2. \tag{39}$$

Let us analyze the solution X of (36). If $(w_j(t), w_j'(t))_{1\leq|j|\leq N}$ is the solution of

$$\begin{cases} w_j''(t) - \frac{w_{j+1}(t)+w_{j-1}(t)-2w_j(t)}{h^2} = 0 & \text{for } 1 \leq j \leq N, \ t > 0 \\ w_0(t) = w_{N+1}(t) = 0 & \text{for } t > 0 \\ w_j(0) = w_j^0, \ w_j'(0) = w_j^1 & \text{for } 1 \leq j \leq N. \end{cases} \tag{40}$$

and we multiply equation j of (36) by $\overline{w}_j$, integrate by parts and add the relations, we obtain that

$$h \sum_{1\leq j\leq N} (x_j^0\overline{w}_j^1 - x_j^1\overline{w}_j^0) - h \sum_{1\leq j\leq N} (x_j(T)\overline{w}_j'(T) - x_j'(T)\overline{w}_j'(T)) = \frac{1}{h}\int_0^T v_\varepsilon(h,t)\overline{w}_N(t)dt. \tag{41}$$

If the Fourier expansion of the initial data of (40) is considered,

$$(w_j^0, w_j^1)_{1\leq j\leq N} = \sum_{\substack{|n|\leq n(\varepsilon)\\ n\neq 0}} \alpha_n \Phi^n(h),$$

it follows from (41) and (35) that

$$\sum_{\substack{|n|\leq n(\varepsilon)\\ n\neq 0}} \frac{1}{\lambda_n(h)} a_n^0(h)\overline{\alpha}_n - \sum_{\substack{|n|\leq N\\ n\neq 0}} \frac{1}{\lambda_n(h)} a_n^0(h,T)\overline{\alpha}_n e^{-i\lambda_n(h)T} =$$

$$= \frac{1}{h}\int_0^T v_\varepsilon(h,t)\overline{w}_N(t)dt =$$

$$= \frac{1}{h}\int_0^T \left(\sum_{\substack{|n|\leq n(\varepsilon)\\ n\neq 0}} \frac{a_n^0(h)}{\lambda_n(h)} \frac{i\,(-1)^{n+1}}{\cos(\frac{n\pi h}{2})}\Theta_n(t)\right)\left(\sum_{\substack{|n|\leq N\\ n\neq 0}} \overline{\alpha}_n \overline{\Phi}_N^n(h) e^{-i\lambda_n(h)t}\right) =$$

$$= \frac{1}{h}\sum_{\substack{|n|\leq n(\varepsilon)\\ n\neq 0}} \frac{a_n^0(h)}{\lambda_n(h)} \frac{i\,(-1)^{n+1}}{\cos(\frac{n\pi h}{2})}\overline{\alpha}_n \frac{h}{2i\,\sin(\frac{n\pi h}{2})}(-1)^{n+1} sin(n\pi h) = \sum_{\substack{|n|\leq n(\varepsilon)\\ n\neq 0}} \frac{a_n^0(h)}{\lambda_n(h)}\overline{\alpha}_n.$$

Hence,

$$-\sum_{\substack{|n|\leq N\\ n\neq 0}} \frac{1}{\lambda_n(h)} a_n^0(h,T)\overline{\alpha}_n e^{-i\lambda_n(h)T} = 0.$$

Since $(w_j^0, w_j^1)_{1\leq j\leq N} \in \mathbb{C}^{2N}$ is arbitrary it follows that solution X of (36) satisfies

$$X(T) = 0. \tag{42}$$

From (39) and (42) we obtain that (4) is satisfied if the control $v_\varepsilon(h)$ is given by (35). Moreover,

$$||v_\varepsilon(h)||_{L^2(0,T)} \leq \sum_{\substack{|n|\leq n(\varepsilon)\\ n\neq 0}} \frac{|a_n^0(h)|}{|\lambda_n(h)|}\frac{1}{|\cos(\frac{n\pi h}{2})|}||\Theta_n(h)||_{L^2(0,T)}.$$

Until now N was arbitrary in $\mathbb{N}^*$. In order to obtain an uniformly bounded sequence of discrete controls, we shall take N sufficiently large and more precisely,

$$n(\varepsilon) < \sqrt{N}. \tag{43}$$

It follows from Theorem 3.1 that $||\Theta_n(h)||_{L^2(0,T)} \leq C|\lambda_n(h)|$ for any n with $1 \leq |n| \leq n(\varepsilon)$. Consequently,

$$||v_\varepsilon(h,t)||_{L^2(0,T)} \leq C \sum_{\substack{|n|\leq N\\ n\neq 0}} |a_n^0(h)| \tag{44}$$

where C is a constant independent of N.

From (23) it follows that the sequence $(v_\varepsilon(h))_{h>0}$ given by (35) under the assumption (43) is uniformly bounded when h tends to zero. □

Remark 3.4. *Theorem 3.2 ensures the existence of an uniformly bounded sequence of discrete controls if the condition (43) is satisfied. Note that, (43) tells us how to choose the step size if we want to control to zero the first $n(\varepsilon)$ modes.*

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