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Approximate Controllability of Neutral Functional Integro-Differential Equations with State-Dependent Delay and Non-Instantaneous Impulses

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Abstract: In this manuscript, we investigate the issue of approximate controllability for a certain class of abstract neutral integro-differential equations having non-instantaneous impulsions and being subject to state-dependent delay. Our methodology relies on the utilization of resolvent operators in conjunction with Darbo's fixed point theorem. To exemplify the practical implications of our findings, we provide an illustration.

Keywords: approximate controllability; fixed point theorem; infinite delay; integrodifferential equation; neutral system; measures of noncompactness; mild solution; resolvent operator

MSC: 93B05; 47H10; 45J05; 47H08; 34K45; 34K40

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1. Introduction

Controllability is an essential and fundamental issue that must be addressed in control systems operating in both finite and infinite-dimensional spaces. Infinite-dimensional systems can be classified into several controllability notions, which include exact controllability, approximate controllability and null controllability. Exact controllability allows for steering the system towards an arbitrary final state; however, this concept is usually too stringent and therefore limited in applicability in infinite-dimensional spaces (see [1–6] and references therein). On the other hand, approximate controllability is often sufficient for most infinite-dimensional systems in applications as it permits steering the system towards a small neighborhood of the final state (see [1] and references therein).

Bashirov and Mahmudov demonstrated in article [7] that the approximate controllability of semi-linear systems is indicated by the approximate controllability of its linear element under the right conditions on a resolvent operator. Since it is practical for applications, several articles have examined the approximative controllability of nonlinear differential equations using this resolvent condition (see [1,8]).

Impulsive events, which frequently occur in both nature and human activity, are the results of a sudden change in a system's state brought on by external disturbance. The phenomena of this type are split into two groups depending on the duration of the change. One is that this alteration only lasts for a brief period of time in comparison to the entire process, known as the instantaneous impulse, and when the effects are continuous, this means they can begin at any fixed point and last for a set amount of time. It is known as a non-instantaneous impulse. The theory of instantaneous impulsive differential equations has experienced significant advancements and has played a crucial role in modern applied mathematical models for real-world phenomena. Current research on impulsive evolution equations can be found in [9–11] and the related literature. However, there are scenarios where the dynamics of certain evolution processes cannot be accurately described by instantaneous impulses. For instance, the introduction of drugs into the bloodstream and their subsequent absorption by the body during hemodynamic equilibrium is a gradual and continuous process. Another example of non-instantaneous impulses is the sudden introduction of insulin into the bloodstream, followed by a gradual absorption over a finite period. Non-instantaneous impulses have been extensively studied by researchers, as demonstrated by works such as [2,12–19].

In several areas of practical mathematics, impulsive neutral integro-differential equations are encountered. For example, the system of rigid heat conduction with finite wave speeds, investigated in [20], can be described in the form of integrodifferential equations of neutral type with delay. As a result, these equations have attracted a lot of attention (see for instance, [14,21]). On the other hand, due to their frequent appearance in applications as equation models, functional-differential equations with delays have drawn a lot of interest in recent years; see, for instance, [3,10] and the references therein.

In [13], the authors studied the attractivity and exact controllability of the following impulsive integrodifferential equation with unbounded delay:

$$\begin{cases} x'(\theta) = \aleph x(\theta) + \Psi(\theta, x_\theta, (\mathcal{K}x)(\theta)) + \int_0^\theta \Lambda(\theta - s)x(s)ds + \mathcal{W}u(\theta), \text{ if } \theta \in I_\jmath, \jmath \in N_0^m, \\ x(\theta) = \Upsilon_\jmath\left(\theta, x\left(\theta_\jmath^-\right)\right), \text{if } \theta \in \mho_\jmath, \jmath \in N_1^m, \\ x(\theta) = \Phi(\theta), \text{ if } \theta \in \mathbb{R}_-. \end{cases} \tag{1}$$

Our work in this paper is a direct continuation of the research mentioned in [13], where we build on the existing framework to address the approximate controllability for a system (1) by applying the technique concerning the resolvent condition. Furthermore, we aim to extend the results to the case of neutral partial functional integro-differential equations with unbounded delay described in the form:

$$\begin{cases} \frac{d}{d\theta}(x(\theta) - G(\theta, x_\theta)) = \aleph(x(\theta) - G(\theta, x_\theta)) + \Psi(\theta, x_\theta, (\mathcal{K}x)(\theta)) \\ \qquad\qquad + \int_0^\theta \Lambda(\theta - s)(x(s) - G(s, x_s))ds + \mathcal{W}u(\theta), \text{ if } \theta \in I_\jmath, \jmath \in N_0^m, \\ x(\theta) = \Upsilon_\jmath\left(\theta, x\left(\theta_\jmath^-\right)\right), \text{if } \theta \in \mho_\jmath, \jmath \in N_1^m, \\ x(\theta) = \Phi(\theta), \text{ if } \theta \in \mathbb{R}_-, \end{cases} \tag{2}$$

where $I_0 = [0, \theta_1], I_\jmath = (s_\jmath, \theta_{\jmath+1}]$ and $\mho_\jmath = (\theta_\jmath, s_\jmath]$, $N_1^m = \{1, \dots, m\}$, and $N_0^m = N_1^m \cup \{0\}$ with $0 = s_0 < \theta_1 \le s_1 \le \theta_2 < \dots < s_{m-1} \le \theta_m \le s_m \le \theta_{m+1} = T, \mho = [0, T], \widetilde{\mho} = (-\infty, T]$, and $\aleph : D(\aleph) \subset F \to F$ is the infinitesimal generator of a strongly continuous semigroup $\{T(\theta)\}_{t \ge 0}$, $\Lambda(\theta)$ is a closed linear operator with domain $D(\aleph) \subset D(\Lambda(\theta))$, the operator $\mathcal{K}$ is defined by

$$(\mathcal{K}x)(\theta) = \int_0^T g(\theta, s, x(s))ds,$$

the nonlinear terms $\Psi : \mho \times \Bbbk \times F \to F$, $G : \mho \times \Bbbk \to F$, $\Upsilon_\jmath : \mho_\jmath \times F \to F$, $\jmath \in N_1^m$, $\Phi : \mathbb{R}_- \to F$, are given functions, and the control function u is given in $L^2(\mho, U)$ a Banach space of admissible controls with U as a Banach space. $\mathcal{W}$ is a bounded linear operator from U into F, and $(F, \|\cdot\|)$ is a Banach space. This is a significant expansion because the system under consideration has additional complications that necessitate new methods to solve. Our work adds to current knowledge and techniques while also introducing new perspectives and methods to the study of controllability in this class of systems.

The rest of this work is organized as follows: in the next section, we mention some results and notations referents resolvent of operators, abstract phase spaces, and measures of noncompactness needed to establish our results. The approximate controllability of the system (1) is studied in Section 3. We present and prove the existence and approximate controllability of solutions for the problem (2) in Section 4. Finally, an example is given to show the applications of the obtained results.

2. Preliminaries

Let $C(\mho, \digamma)$ be the Banach space of continuous functions ϑ mapping $\mho := [0, T]$ into $\digamma$, with

$$\|\vartheta\|_\infty = \sup_{\theta \in \mho} \|\vartheta(\theta)\|.$$

Let us denote by $L^1(\mho, \digamma)$ the Banach space of measurable functions $\vartheta : \mho \to \digamma$ which are Bochner integrable [22], with the norm

$$\|\vartheta\|_{L^1} = \int_0^T \|\vartheta(\theta)\| d\theta.$$

We consider the following Cauchy problem:

$$\begin{cases} x'(\theta) = \aleph x(\theta) + \int_0^\theta \Lambda(\theta - s)x(s)ds; \text{ for } \theta \in \mho, \\ x(0) = x_0 \in \digamma. \end{cases} \tag{3}$$

The existence and properties of a resolvent operator has been discussed in [23,24]. The underlying assertions are assumed in what precedes:

(H_1) $\aleph$ is the infinitesimal generator of a uniformly continuous semigroup $\{T(\theta)\}_{\theta>0}$,

(H_2) For all $\theta \geq 0, \Lambda(\theta)$ is closed linear operator from $D(\aleph)$ to $\digamma$ and $\Lambda(\theta) \in \Lambda(D(\aleph), \digamma)$. For any $\vartheta \in D(\aleph)$, the map $\theta \to \Lambda(\theta)\vartheta$ is bounded, differentiable and the derivative $\theta \to \Lambda'(\theta)\vartheta$ is bounded uniformly continuous on $\mathbb{R}^+$.

Theorem 1 ([24])**.** *Suppose that $(H_1) - (H_2)$ hold. Then, there exists a unique resolvent operator for the Cauchy problem (3).*

Let us introduce the following space:

$$\widehat{\digamma} = PC(\widetilde{\mho}, \digamma) = \Big\{ \vartheta : \widetilde{\mho} \to \digamma \; : \; \vartheta|_{\mathbb{R}^-} \in \Bbbk, \vartheta|_{\mho_J} = \Upsilon_J; \, J \in N_1^m, \vartheta|_{I_J}; \, J \in N_0^m, \text{ is continuous}, \; \vartheta\left(s_J^+\right), \vartheta\left(\theta_J^-\right) \text{ and } \vartheta\left(\theta_J^+\right) \text{ exists with } \vartheta(s_J^+) = \Upsilon_J(s_J, \vartheta(s_J^-)) \text{ and } \vartheta(\theta_J^-) = \Upsilon_J(\theta_J, \vartheta(\theta_J^-)) \Big\},$$

with

$$\|\vartheta\|_{PC} = \sup_{\theta \in \widetilde{\mho}} \{\|\vartheta(\theta)\|\}.$$

Let the state space $(\Bbbk, \|\cdot\|_\Bbbk)$ be a seminormed linear space of functions mapping $(-\infty, 0]$ into $\mathbb{R}$, and verifying the following [25]:

(A_1) If $\vartheta \in PC$ and $\vartheta_0 \in \Bbbk$, then for every $\theta \in \mho$, the following hold:

- (i) $\vartheta_\theta \in \Bbbk$,
- (ii) There exists $\wp > 0$ such that $|\vartheta(\theta)| \leq \wp \|\vartheta_\theta\|_\Bbbk$,
- (iii) There exist two functions $\widetilde{\wp}(\cdot)$ and $\widehat{\wp}(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$independent of ϑ with $\widetilde{\wp}$ continuous and bounded and $\widehat{\wp}$ locally bounded where:

$$\|\vartheta_\theta\|_\Bbbk \leq \widetilde{\wp}(\theta) \sup\{|\vartheta(s)| : 0 \leq s \leq \theta\} + \widehat{\wp}(\theta)\|\vartheta_0\|_\Bbbk.$$

(A_2) For the function ϑ in (A_1), ϑ_θ is a $\Bbbk$-valued continuous function on $\mathbb{R}^+ \setminus \mho_J$.

(A_3) The space $\Bbbk$ is complete.

Denote

$$\widetilde{\wp}_* = \sup\{\widetilde{\wp}(\theta) : \theta \in \mho\},$$
$$\widehat{\wp}_* = \sup\{\widehat{\wp}(\theta) : \theta \in \mho\},$$

and

$$\nabla = \max\{\widetilde{\wp}_*, \widehat{\wp}_*\}.$$

Now, let $p \in N_0^m$ and $(\varepsilon_\jmath)_{\jmath \in N_1^m}$ be a sequence defined by

$$\varepsilon_\jmath = \begin{cases} \varepsilon_{p+1} - \theta, & \text{if } \jmath = 2p+1,\ \theta \in \mathbb{R}^-, \\ s_p - \theta, & \text{if } \jmath = 2p,\ \theta \in \mathbb{R}^-. \end{cases}$$

Then, for $I_\varepsilon = \mathbb{R}^- \setminus \{\varepsilon_\jmath \ : \ \jmath \in N_1^m\}$, we define the space

$$PC_\varepsilon(\mathbb{R}^-, F) = \{\vartheta : \mathbb{R}^- \to F :\ \vartheta|_{I_\varepsilon} \text{ is continuous and } \vartheta(\varepsilon_\jmath^-),\ \vartheta(\varepsilon_\jmath^+) \text{ exist with } \vartheta(\varepsilon_\jmath^-) = \vartheta(\varepsilon_\jmath)\},$$

and the space

$$C_\varepsilon := \{x \in PC_\varepsilon(\mathbb{R}^-, F) : \lim_{\bar{\varepsilon} \to -\infty} x(\bar{\varepsilon}) \ \text{ exist in } \ F\},$$

endowed with the norm

$$\|x\|_\varepsilon = \sup\{|x(\bar{\varepsilon})| : \bar{\varepsilon} \le 0\}.$$

Then, the axioms $(A_1) - (A_3)$ are verified in the space C_ε. Let $\Bbbk = C_\varepsilon$.

Definition 1 ([26]). *Let $\widetilde{F}$ be a Banach space and $\Omega_{\widetilde{F}}$ the bounded subsets of $\widetilde{F}$. The Kuratowski measure of noncompactness is the map $\chi : \Omega_{\widetilde{F}} \to [0, \infty]$ defined by*

$$\chi(\Theta) = \inf\{\iota > 0 : \Theta \subseteq \cup_{i=1}^n \Theta_i \text{ and } diam(\Theta_i) \le \iota\};\ \text{here } \Theta \in \Omega_{\widetilde{F}},$$

where

$$diam(\Theta_i) = sup\{\|\vartheta - \varkappa\|_F : \vartheta, \varkappa \in \Theta_i\}.$$

Lemma 1 ([27]). *If $\mathcal{E}$ is a bounded subset of a Banach space $\widetilde{F}$, then for each $\iota > 0$, there is a sequence $\{\vartheta_\jmath\}_{\jmath=1}^\infty \subset \mathcal{E}$ such that*

$$\chi(\mathcal{E}) \le 2\chi\left(\{\vartheta_\jmath\}_{\jmath=1}^\infty\right) + \iota.$$

Lemma 2 ([26,28]). *If $\{\vartheta_\jmath\}_{\jmath=0}^\infty \subset L^1$ is uniformly integrable, then the function $t \to \alpha(\{\vartheta_\jmath(\theta)\}_{\jmath=0}^\infty)$ is measurable and*

$$\chi\left(\left\{\int_0^\theta \vartheta_\jmath(s)ds\right\}_{\jmath=0}^\infty\right) \le 2\int_0^\theta \chi\left(\{\vartheta_\jmath(s)\}_{\jmath=0}^\infty\right)ds.$$

Theorem 2 (Darbo's fixed point theorem [29]). *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space $\widetilde{F}$ and let $P : \Omega \to \Omega$ be a continuous mapping. Assume that there exists a constant $\jmath \in [0, 1)$, such that*

$$\chi(P(M)) \le \jmath\chi(M),$$

for any nonempty subset M of Ω. Then, P has a fixed point in set Ω.

3. Integro-Differential Equations with Infinite Delay

Existence and Controllability Results

For a nonempty bounded subset S of the space $\widehat{F}$ and $\varkappa \in S$, $\iota > 0$, $\nu_1, \nu_2 \in [-\kappa, \kappa]$ such that $|\nu_1 - \nu_2| \le \iota$, we denote $\mu^\kappa(\varkappa, \iota)$ the modulus of continuity of the function $\varkappa$ on the interval $[-\kappa, \kappa]$, namely,

$$\begin{array}{lcl} \mu^{\kappa}(\varkappa, \iota) & = & \sup\{\|e^{-\nu_1}\varkappa(\nu_1) - e^{-\nu_2}\varkappa(\nu_2)\| \, ; \, \nu_1, \nu_2 \in [-\kappa, \kappa]\}, \\ \mu^{\kappa}(S, \iota) & = & \sup\{\mu^{\kappa}(\varkappa, \iota) \, ; \, \varkappa \in S\}, \\ \mu_0(S) & = & \lim_{\iota \to 0}\{\mu^{\kappa}(S, \iota)\}. \end{array}$$

See [30], for more information. Let χ_{PC} be given on the family of subset of $\widehat{F}$ by

$$\chi_{PC}(S) = \mu_0(S) + \sup_{\theta \in \widetilde{\mho}}\{\chi(S(\theta))\}.$$

Similar to the proof presented in [31], it can be demonstrated that the function χ_{PC} represents a sublinear measure of noncompactness on the space $\widehat{F} = PC(\widetilde{\mho}, F)$.

Definition 2. *$x \in \widehat{F}$ is a mild solution of (1) if it verifies*

$$x(\theta) = \begin{cases} \mathcal{Q}(\theta)\Phi(0) + \int_0^{\theta} \mathcal{Q}(\theta - s)(\Psi(s, x_s, (\mathcal{K}x)(s)) + \mathcal{W}\vartheta(s))ds; \text{ if } \theta \in I_0, \\ \mathcal{Q}(\theta - s_\jmath)\Big[\Upsilon_\jmath(s_\jmath, x(\theta_\jmath^-))\Big] \\ + \int_{s_\jmath}^{\theta} \mathcal{Q}(\theta - s)(\Psi(s, x_s, (\mathcal{K}x)(s)) + \mathcal{W}\vartheta(s))ds; \text{ if } \theta \in I_\jmath, \ \jmath \in N_1^m, \\ \Upsilon_\jmath(\theta, x(\theta_\jmath^-)); \text{ if } \theta \in \mho_\jmath, \ \jmath \in N_1^m, \\ \Phi(\theta); \text{ if } \theta \in \mathbb{R}_-. \end{cases}$$

To guarantee the existence of mild solutions, we need the following assumptions:

(C_1) $\Psi : \mho \times \Bbbk \times F \to F$ is a Carathéodory function and there exist $p_\Psi > 0$, $q_\Psi > 0$ and continuous nondecreasing functions $\psi_1, \psi_2 : \mho \to (0, +\infty)$ such that

$$||\Psi(\theta, x_1, x_2)|| \leq p_\Psi \psi_1(\|x_1\|_{\Bbbk}) + q_\Psi \psi_2(\|x_2\|), \quad \text{for } x_1 \in \Bbbk, \ x_2 \in F.$$

Additionally, there exists a positive constant l_Ψ, such that for any bounded set $\Theta \subset \widehat{F}$, and $\Theta_\theta \in \Bbbk$ we have

$$\chi(\Psi(\theta, \Theta_\theta, \mathcal{K}(\Theta(\theta)))) \leq l_\Psi \chi(\Theta(\theta)).$$

(C_2) The function $g : D_g \times F \to F$ is continuous and there exists $\alpha_g > 0$, such that

$$\|g(\theta, s, x_1) - g(\theta, s, x_2)\| \leq \alpha_g \|x_1 - x_2\|, \ \text{ for each } (\theta, s) \in D_g \text{ and } x_1, \ x_2 \in F.$$

$$\sup_{D_g}\{\|g(\theta, s, 0)\|\} = g_0^* < \infty.$$

(C_3) $\Upsilon_\jmath : \mho_\jmath \times F \to F$ are continuous and there exist functions $L_{\Upsilon_\jmath} > 0$, $\jmath \in N_1^m$, such that

$$\|\Upsilon_\jmath(\theta, x_1) - \Upsilon_\jmath(\theta, x_2)\| \leq L_{\Upsilon_\jmath}\|x_1 - x_2\|, \quad \text{for all } x_1, x_2 \in F, \ \jmath \in N_1^m,$$

and

$$\Upsilon_\jmath^0 = \|\Upsilon_\jmath(\theta, 0)\|, \ \max_{\jmath \in N_1^m}\{L_{\Upsilon_\jmath}, \ \jmath \in N_1^m\} = L^*_{\Upsilon_\jmath} < +\infty.$$

(C_4) Assume that $(H_1) - (H_2)$ hold, and there exist $K_\mathcal{Q} \geq 1$, $\chi \geq 0$, and $M_\mathcal{W} > 0$, such that

$$\|\mathcal{Q}(\theta)\|_{D(F)} \leq K_\mathcal{Q} e^{-\chi\theta}, \qquad \|\mathcal{W}\| = M_\mathcal{W}.$$

Theorem 3. *Suppose that* (C_1)–(C_4) *are verified. If* $K_\mathcal{Q} L^*_{\Upsilon_\jmath} < 1$, *then (1) admit at least one mild solution.*

Proof. The proof of this theorem is analogous to that of Theorem 2 and Theorem 4 in [13], and hence, we shall omit it here. □

Next, we investigate the controllability of system (1). First, we provide a definition of the approximation controllability idea.
Let $x(\kappa;\Phi,\vartheta)$ be the state-value of (1) at terminal time κ corresponding to $\Phi \in \Bbbk$. To define the notion of approximate controllability, we introduce the following set:

$$\mathcal{R}(\kappa,\Phi) = \left\{ x(\kappa,\Phi,\vartheta), \vartheta(\cdot) \in L^2(\mho;U) \right\},$$

which is called the reachable set of system (1) at terminal time κ. Its closure in $\digamma$ is denoted by $\overline{\mathcal{R}(\kappa,\Phi)}$.

Definition 3. *System (1) is said to be approximately controllable on the interval $\mho = [0,\kappa]$ if $\mathcal{R}(\kappa,\Phi)$ is dense in $\digamma$, i.e., $\overline{\mathcal{R}(\kappa,\Phi)} = \digamma$.*

To study the approximate controllability of system (1) we introduce the following operators

$$\Gamma_{s_\jmath}^{\theta_{\jmath+1}} = \int_{s_\jmath}^{\theta_{\jmath+1}} \mathcal{Q}(\theta_{\jmath+1}-s)\mathcal{W}\mathcal{W}^*\mathcal{Q}^*(\theta_{\jmath+1}-s)ds, R\left(\lambda,\Gamma_{s_\jmath}^{\theta_{\jmath+1}}\right) = \left(\lambda I + \Gamma_{s_\jmath}^{\theta_{\jmath+1}}\right)^{-1},$$

where $s_0 = 0,\ \theta_{\jmath+1} = \kappa;\ \jmath = 0,\ldots,m$, $\mathcal{W}^*$ and $\mathcal{Q}^*$ denote the adjoints of the operators $\mathcal{W}$ and $\mathcal{Q}$, respectively. It is straightforward to see that the operator $\Gamma_{s_\jmath}^{\theta_{\jmath+1}}$ is a linear bounded operator. So, we assume that for all $\jmath \in N_0^m$, the operator $R\left(\lambda,\Gamma_{s_\jmath}^{\theta_{\jmath+1}}\right)$ satisfies

(C_0) $\lambda R\left(\lambda,\Gamma_{s_\jmath}^{\theta_{\jmath+1}}\right) \longrightarrow 0$ as $\lambda \longrightarrow 0^+$ in the strong operator topology.
From [32], hypothesis (C_0) is equivalent to the fact that the linear control system corresponding to system (1) is approximately controllable on $[0,\kappa]$.

Theorem 4. *The following statements are equivalent:*

(i) *The linear control system corresponding to system (1) is approximately controllable on* $[0,\kappa]$.
(ii) *If $\mathcal{W}^*\mathcal{Q}^*(\theta)z = 0$ for all $\theta \in [0,\kappa]$, then $z = 0$.*
(iii) *The condition (C_0) holds.*

Proof. The proof of this theorem is similar to that of ([7], Theorem 2) and ([32], Theorem 4.4.17), so we omit it here. □

Let us now study the approximate controllability of (1).
For any given $\delta^{\theta_{\jmath+1}} \in \digamma$ and $\lambda \in (0,1]$, we take the control function $\vartheta^\lambda(\theta)$ as follows:

$$\vartheta^\lambda(\theta) = \mathcal{W}^*\mathcal{Q}^*(\theta_{\jmath+1}-s)R\left(\lambda,\Gamma_{s_\jmath}^{\theta_{\jmath+1}}\right)\Delta(\delta^{\theta_{\jmath+1}},\theta),$$

where

$$\Delta(\delta^{\theta_{\jmath+1}},\theta) = \delta^{\theta_{\jmath+1}} - \Delta_\jmath(\theta) - \int_{s_\jmath}^{\theta} \mathcal{Q}(\theta-s)\Psi(s,x_s,(\mathcal{K}x)(s))ds,$$

and

$$\Delta_\jmath(\theta) = \begin{cases} \mathcal{Q}(\theta)\Phi(0);\ \text{if } \jmath = 0, \\ \mathcal{Q}(\theta-s_\jmath)\left[\Upsilon_\jmath(s_\jmath,x(\theta_\jmath^-))\right];\ \text{if } \jmath \in N_1^m. \end{cases}$$

Theorem 5. *Assume that the hypotheses (C_0)–(C_4) are satisfied and in addition, the function f is uniformly bounded. Then, Equation (1) is approximately controllable on $[0,\kappa]$.*

Proof. According to Theorem 3, we can know that system (1) has at least one mild solution x^{λ}. Then, we have

$$\begin{aligned} x^{\lambda}(\theta_{J+1}) &= \Delta_J(\theta_{J+1}) + \int_{s_J}^{\theta_{J+1}} \mathcal{Q}(\theta_{J+1}-s)(\Psi(s,x_s,(\mathcal{K}x)(s)) + \mathcal{W}\vartheta(s))ds \\ &= \Delta_J(\theta_{J+1}) + \int_{s_J}^{\theta_{J+1}} \mathcal{Q}(\theta_{J+1}-s)(\Psi(s,x_s,(\mathcal{K}x)(s)))ds \\ &\quad + \int_{s_J}^{\theta_{J+1}} \mathcal{Q}(\theta_{J+1}-s)\Big(\mathcal{W}^*\mathcal{Q}^*(\theta_{J+1}-s)R\Big(\lambda,\Gamma_{s_J}^{\theta_{J+1}}\Big)\Delta(\delta^{\theta_{J+1}},\theta_{J+1})\Big)ds \\ &= \delta^{\theta_{J+1}} + (\Gamma_{s_J}^{\theta_{J+1}}R\Big(\lambda,\Gamma_{s_J}^{\theta_{J+1}}\Big) - I)\Delta(\delta^{\theta_{J+1}},\theta_{J+1}) \\ &= \delta^{\theta_{J+1}} + \lambda R\Big(\lambda,\Gamma_{s_J}^{\theta_{J+1}}\Big)\Delta(\delta^{\theta_{J+1}},\theta_{J+1}). \end{aligned}$$

Thus,

$$\begin{aligned} \|x^{\lambda}(\theta_{J+1}) - \delta^{\theta_{J+1}}\| \le{}& \left\|R\Big(\lambda,\Gamma_{s_J}^{\theta_{J+1}}\Big)\Big[\delta^{\theta_{J+1}} - \Delta_J(\theta_{J+1})\Big]\right\| \\ &+ \left\|R\Big(\lambda,\Gamma_{s_J}^{\theta_{J+1}}\Big)\left[\int_{s_J}^{\theta_{J+1}} \mathcal{Q}(\theta_{J+1}-\varkappa)\Psi(\varkappa,x_{\varkappa},(\mathcal{K}x)(\varkappa))d\varkappa\right]\right\|. \end{aligned}$$

We infer from the uniform boundedness of $\Psi(\cdot,\cdot,\cdot)$ that there exists $M_{\Psi} > 0$, such that

$$\int_0^{\kappa} \|\Psi(s,x_s^{\lambda},(\mathcal{K}x^{\lambda})(s))\|^2 ds \le \kappa(M_{\Psi})^2.$$

Therefore, the sequence $\{\Psi(s,x_s^{\lambda},(\mathcal{K}x^{\lambda})(s))\}_{\lambda}$ is bounded in $L^2(\mho,\digamma)$. Then, there exists subsequence still denoted by $\Big\{\Psi(s,x_s^{\widetilde{\lambda}},(\mathcal{K}x^{\lambda})(s))\Big\}_{\lambda}$ that weakly converge to the limit $\widetilde{\Psi}(s)$ in $L^2(\mho,\digamma)$. Further, we have

$$\int_0^{\kappa} \|\Psi(s,x_s^{\lambda},(\mathcal{K}x^{\lambda})(s)) - \widetilde{\Psi}(s)\| ds \xrightarrow[\lambda\to 0]{} 0.$$

So,

$$\begin{aligned} \|x^{\lambda}(\theta_{J+1}) - \delta^{\theta_{J+1}}\| \le{}& \left\|R\Big(\lambda,\Gamma_{s_J}^{\theta_{J+1}}\Big)\Big[\delta^{\theta_{J+1}} - \Delta_J(\theta_{J+1})\Big]\right\| + \Bigg\|R\Big(\lambda,\Gamma_{s_J}^{\theta_{J+1}}\Big) \\ &\times \left[\int_{s_J}^{\theta_{J+1}} \mathcal{Q}(\theta_{J+1}-s)\Big(\Psi(s,x_s,(\mathcal{K}x)(s)) - \widetilde{\Psi}(s)\Big)ds\right]\Bigg\| \\ &+ \left\|R\Big(\lambda,\Gamma_{s_J}^{\theta_{J+1}}\Big)\left[\int_{s_J}^{\theta_{J+1}} \mathcal{Q}(\theta_{J+1}-s)\widetilde{\Psi}(s)ds\right]\right\| \xrightarrow[\lambda\to 0]{} 0. \end{aligned}$$

Thus, $x^{\lambda}(\theta_{J+1}) \to \delta^{\theta_{J+1}}$ holds, and consequently, we obtain the approximate controllability of system (1). □

4. Neutral Functional Integro-Differential Equations

4.1. Existence Result

Definition 4. *A function $x \in \widehat{\digamma}$ is called a mild solution of problem (2) if it satisfies*

$$x(\theta) = \begin{cases} \mathcal{Q}(\theta)\ (\Phi(0) - G(0,\Phi)) + G(\theta,x_\theta) \\ + \int_0^{\theta} \mathcal{Q}(\theta-s)(\Psi(s,x_s,(\mathcal{K}x)(s)) + \mathcal{W}\vartheta(s))ds;\ \textit{if}\ \theta \in I_0, \\ \\ \mathcal{Q}(\theta-s_J)\Big[\mathrm{Y}_J(s_J,x(\theta_J^-)) - G(s_J,x_{s_J})\Big] + G(\theta,x_\theta) \\ + \int_{s_J}^{\theta} \mathcal{Q}(\theta-s)(\Psi(s,x_s,(\mathcal{K}x)(s)) + \mathcal{W}\vartheta(s))ds;\ \textit{if}\ \theta \in I_J,\ J \in N_1^m, \\ \\ \mathrm{Y}_J(\theta,x(\theta_J^-));\ \textit{if}\ \theta \in \mho_J,\ J \in N_1^m, \\ \\ \Phi(\theta);\ \textit{if}\ \theta \in \mathbb{R}_-. \end{cases}$$

We introduce the following assumptions:

(C_5) (i) $G : \mho \times \Bbbk \to \digamma$ is continuous and for any bounded set $\Theta \in \Bbbk$, $\{\theta \to G(\theta, x_\theta),\ x \in \Theta\}$ is equicontinuous. Also let $L_G > 0$, where

$$\|G(\theta, x_1) - G(\theta, \widehat{x_1})\| \leq L_G \|x_1 - \widehat{x_1}\|_{\Bbbk}, \text{ for all } x_1, \widehat{x_1} \in \Bbbk,$$

with

$$G^* = \|G(0, w_0)\|.$$

(ii) There exists $\varepsilon > 3$, such that $\varepsilon K_{\mathcal{Q}}(L^*_{\Upsilon_\jmath} + L_G) < 1$.

Theorem 6. *Suppose that* $(C_1) - (C_5)$ *are met. Then, (2) admit at least one mild solution.*

Proof. First we define on $\widehat{\digamma}$ measures of non compactness by

$$\chi_{PC}(\Pi) = \mu_0(\Pi) + \sup_{\theta \in \widetilde{\mho}} \left\{ e^{-\varepsilon \Sigma(\theta)} \chi(\Pi(\theta)) \right\},$$

with $\varepsilon > 3$, $\Sigma(\theta) = 4K_{\mathcal{Q}} l_\Psi \theta$ and $\Pi(\theta) = \{x(\theta) \in \digamma \,;\, x \in \Pi\}$.
Transform the problem (2) into a fixed point problem and consider the operator $\Xi : \widehat{\digamma} \to \widehat{\digamma}$ defined by:

$$\Xi x(\theta) = \begin{cases} \mathcal{Q}(\theta)\,(\Phi(0) - G(0, \Phi)) + G(\theta, x_\theta) \\ + \int_0^\theta \mathcal{Q}(\theta - s)(\Psi(s, x_s, (\mathcal{K}x)(s)) + \mathcal{W}\vartheta(s))ds; \text{ if } \theta \in I_0, \\ \\ \mathcal{Q}(\theta - s_\jmath)\left[\Upsilon_\jmath(s_\jmath, x(\theta_\jmath^-)) - G(s_\jmath, x_{s_\jmath})\right] + G(\theta, x_\theta) \\ + \int_{s_\jmath}^\theta \mathcal{Q}(\theta - s)(\Psi(s, x_s, (\mathcal{K}x)(s)) + \mathcal{W}\vartheta(s))ds; \text{ if } \theta \in I_\jmath,\ \jmath \in N_1^m, \\ \\ \Upsilon_\jmath(\theta, x(\theta_\jmath^-)); \text{ if } \theta \in \mho_\jmath,\ \jmath \in N_1^m, \\ \\ \Phi(\theta); \text{ if } \theta \in \mathbb{R}_-. \end{cases}$$

Let $x(\cdot) : (-\infty, \kappa] \to \digamma$ be the function defined by:

$$x(\theta) = \begin{cases} \mathcal{Q}(\theta)\,(\Phi(0) - G(0, \Phi)), & \text{if } \theta \in I_0, \\ \\ 0, & \text{if } \theta \in (\theta_1, \kappa], \\ \\ \Phi(\theta), & \text{if } \theta \in \mathbb{R}_-. \end{cases}$$

Then, $x_0 = \Phi$, and for each $\varpi \in \widehat{\digamma}$, with $\varpi(0) = 0$, we denote by $\overline{\varpi}$ the function

$$\overline{\varpi}(\theta) = \begin{cases} \varpi(\theta), & \text{if } \theta \in \mho, \\ \\ 0, & \text{if } \theta \in \mathbb{R}_-. \end{cases}$$

If x verifies Definition 4, then we can decompose it as $x(\theta) = \varpi(\theta) + x(\theta)$, which implies $x_\theta = \varpi_\theta + x_\theta$, and the function $\varpi(\cdot)$ satisfies

$$\varpi(\theta) = \begin{cases} \int_0^\theta \mathcal{Q}(\theta - s)(\Psi(s, \varpi_s + x_s, \mathcal{K}(\varpi + x)(s)) + \mathcal{W}\vartheta(s))ds \\ + G(\theta, \varpi_\theta + x_\theta), \quad \text{if } \theta \in I_0, \\ \\ \mathcal{Q}(\theta - s_\jmath)\left[\Upsilon_\jmath(s_\jmath, \varpi(\theta_\jmath^-)) - G(s_\jmath, \varpi_{s_\jmath} + x_{s_\jmath})\right] + G(\theta, \varpi_\theta + x_\theta) \\ + \int_{s_\jmath}^\theta \mathcal{Q}(\theta - s)(\Psi(s, \varpi_s + x_s, (\mathcal{K}\varpi)(s)) + \mathcal{W}\vartheta(s))ds; \text{ if } \theta \in I_\jmath,\ \jmath \in N_1^m, \\ \\ \Upsilon_\jmath(\theta, \varpi(\theta_\jmath^-)); \text{ if } \theta \in \mho_\jmath,\ \jmath \in N_1^m. \end{cases}$$

Set
$$\Omega = \{\varpi \in \widehat{F} \ : \ \varpi(0) = 0\}.$$
Let the operator $\widehat{\Xi} : \Omega \to \Omega$ defined by:
$$\widehat{\Xi}\varpi(\theta) = \begin{cases} \int_0^\theta \mathcal{Q}(\theta - s)(\Psi(s, \varpi_s + x_s, \mathcal{K}(\varpi + x)(s)) + \mathcal{W}\vartheta(s))ds \\ +G(\theta, \varpi_\theta + x_\theta), \quad \text{if } \theta \in I_0, \\ \\ \mathcal{Q}(\theta - s_\jmath)\Big[\Upsilon_\jmath(s_\jmath, \varpi(\theta_\jmath^-)) - G(s_\jmath, \varpi_{s_\jmath} + x_{s_\jmath})\Big] + G(\theta, \varpi_\theta + x_\theta) \\ + \int_{s_\jmath}^\theta \mathcal{Q}(\theta - s)(\Psi(s, \varpi_s + x_s, (\mathcal{K}\varpi)(s)) + \mathcal{W}\vartheta(s))ds; \text{ if } \theta \in I_\jmath, \ \jmath \in N_1^m, \\ \\ \Upsilon_\jmath(\theta, \varpi(\theta_\jmath^-)); \text{ if } \theta \in \mho_\jmath, \ \jmath \in N_1^m. \end{cases}$$

We will use Theorem 2 to demonstrate that $\widehat{\Xi}$ has a fixed point.
Let $\Omega_\rho = \{\varpi \in \Omega : \|\varpi\|_\Omega \leq \rho\}$, with
$$0 < \max\left\{\rho_1^*, \rho_2^*, \rho_3^*\right\} \leq \rho,$$
such that
$$\begin{aligned} \rho_1^* &= L_G\delta_1^* + G^* + K_\mathcal{Q}\Big(p_\Psi\kappa\psi_1(\delta_1^*) + q_\Psi\kappa\psi_2(\delta_2^*) + M_\mathcal{W}\kappa^{\frac{1}{2}}\|\vartheta\|_{L^2}\Big), \\ \rho_2^* &= \frac{\Big(L_G\widetilde{\delta}_1^* + G^*\Big)(1 + K_\mathcal{Q}) + K_\mathcal{Q}(\Upsilon_\jmath^0 + p_\Psi\kappa\psi_1(\widetilde{\delta}_1^*) + q_\Psi\kappa\psi_2(\widetilde{\delta}_2^*) + M_\mathcal{W}\kappa^{\frac{1}{2}}\|\vartheta\|_{L^2})}{1 - K_\mathcal{Q}L_{\Upsilon_\jmath}^*}, \\ \rho_3^* &= L_{\Upsilon_\jmath}^*\rho + \Upsilon_\jmath^0, \end{aligned}$$
and δ_1^*, δ_2^*, $\widetilde{\delta}_2^*$, $\widetilde{\delta}_1^*$ are to be given later.
The set Ω_ρ is bounded, closed, and convex.

Step 1 : $\widehat{\Xi}(\Omega_\rho) \subset \Omega_\rho$.
For $\theta \in I_0$, $\varpi \in \Omega_\rho$ and from $(C_1) - (C_3)$, it follows that
$$\begin{aligned} \|\varpi_\theta + x_\theta\|_\Bbbk &\leq \|\varpi_\theta\|_\Bbbk + \|x_\theta\|_\Bbbk \\ &\leq \widetilde{\wp}(\theta)|\varpi(\theta)| + \widetilde{\wp}(\theta)(K_\mathcal{Q}(\|\Phi(0)\| + G^*)) + \widehat{\wp}(\theta)(\|\Phi\|_\Bbbk) \\ &\leq \nabla(\rho + (K_\mathcal{Q} + 1)\|\Phi\|_\Bbbk + G^*) = \delta_1^*, \end{aligned}$$
and
$$\|\mathcal{K}(\varpi + x)(s)\| \leq \kappa(\alpha_g(\rho + K_\mathcal{Q}(\|\Phi\|_\Bbbk + G^*)) + g_0^*) = \delta_2^*.$$
Then, we have
$$\begin{aligned} \|\widehat{\Xi}\varpi(\theta)\| &\leq L_G\delta_1^* + G^* + K_\mathcal{Q}\int_0^\theta (p_\Psi\psi_1(\delta_1^*) + q_\Psi\psi_2(\delta_2^*) + \|\mathcal{W}\vartheta(s)\|)ds \\ &\leq L_G\delta_1^* + G^* + K_\mathcal{Q}\Big(p_\Psi\kappa\psi_1(\delta_1^*) + q_\Psi\kappa\psi_2(\delta_2^*) + M_\mathcal{W}\kappa^{\frac{1}{2}}\|\vartheta\|_{L^2}\Big) \\ &\leq \rho. \end{aligned}$$
Now, if $\theta \in I_\jmath$ and for each $\varpi \in \Omega_\rho$, by (C_1), (C_2) and (C_3), we obtain
$$\|\Upsilon_\jmath(\theta, \vartheta(.))\| \leq L_{\Upsilon_\jmath}(\theta)\|\vartheta(\theta)\| + \Upsilon_\jmath^0.$$
Hence, for
$$\widetilde{\delta}_2^* = (\alpha_g\rho + g_0^*)\kappa \text{ and } \widetilde{\delta}_1^* = \nabla(\rho + \|\Phi\|_\Bbbk),$$

we obtain

$$\begin{aligned}\|\widehat{\Xi}\varpi(\theta)\| &\leq \Big(L_G\widetilde{\delta_1^*}+G^*\Big)(1+K_{\mathcal{Q}}) \\ &\quad + K_{\mathcal{Q}}\Big[L^*_{\Upsilon_J}\rho+\Upsilon_J^0+p_\Psi\kappa\psi_1(\widetilde{\delta_1^*})+q_\Psi\kappa\psi_2(\widetilde{\delta_2^*})+M_{\mathcal{W}}\kappa^{\frac{1}{2}}\|\vartheta\|_{L^2}\Big] \\ &\leq \rho.\end{aligned}$$

If $\theta\in\mho_J$ and $\varpi\in\Omega_\rho$, then from (C_3), we obtain

$$\begin{aligned}\|\widehat{\Xi}\varpi(\theta)\| &\leq L^*_{\Upsilon_J}\rho+\Upsilon_J^0 \\ &\leq \rho.\end{aligned}$$

Thus,

$$\|\widehat{\Xi}z\|_\Omega\leq\rho.$$

Consequently, $\widehat{\Xi}(\Omega_\rho)\subset\Omega_\rho$ and $\widehat{\Xi}(\Omega_\rho)$ is bounded.

Step 2 : $\widehat{\Xi}$ is continuous.

Let $\{\varpi^n\}_{n\in\mathbb{N}}$ be such that $\varpi_n\to\varpi^*$. Then, for $\theta\in I_0$, we have

$$\begin{aligned}\|(\widehat{\Xi}\varpi^n)(\theta)-(\widehat{\Xi}\varpi^*)(\theta)\| &\leq \|G(\theta,\varpi_\theta^n+x_\theta)-G(\theta,\varpi_\theta^*+x_\theta)\| \\ &\quad + K_{\mathcal{Q}}\int_0^\theta \|\Psi(s,\varpi_s^n+x_s,\mathcal{K}(\varpi^n+x)(s)) \\ &\quad - \Psi(s,(\varpi_s^*+x_s),\mathcal{K}(\varpi^*+x)(s))\|ds.\end{aligned}$$

By the continuity of g and Ψ, we obtain

$$g(\theta,s,(\varpi_s^n+x)(s))\to g(\theta,s,(\varpi^*+x)(s)) \quad as \;\; n\to+\infty,$$

and

$$\|g(\theta,s,(\varpi^n+x)(s))-g(\theta,s,(\varpi^*+x)(s))\|\leq\alpha_g\|\varpi^n-\varpi^*\|_\Omega.$$

By Lebesgue dominated convergence theorem, we obtain

$$\int_0^\theta g(\theta,s,(\varpi^n+x)(s))ds\to\int_0^\theta g(\theta,s,(\varpi^*+x)(s))ds, \quad as \;\; n\to+\infty.$$

Hence, from the continuity of G and Ψ, we obtain

$$\|(\widehat{\Xi}\varpi^n)(t)-(\widehat{\Xi}\varpi^*)(t)\|\to 0, \quad as \;\; n\to+\infty.$$

If $\theta\in I_J$, we obtain

$$\begin{aligned}&\|\widehat{\Xi}(\varpi^n)(\theta)-\widehat{\Xi}(\varpi^*)(\theta)\| \\ &\quad\leq K_{\mathcal{Q}}\|\Upsilon_J(s_J,(\varpi^n)(\theta_J^-))-\Upsilon_J((s_J,(\varpi^*)(\theta_J^-)))\| \\ &\quad\quad + K_{\mathcal{Q}}\|G(s_J,\varpi_{s_J}^n+x_{s_J})-G(s_J,\varpi_{s_J}^*+x_{s_J})\|+\|G(\theta,\varpi_\theta^n+x_\theta)-G(\theta,\varpi_\theta^*+x_\theta)\| \\ &\quad\quad + K_{\mathcal{Q}}\int_{s_J}^\theta \|\Psi(s,(\varpi_s^n+x_s)(s),\mathcal{K}(\varpi^n)(s))-\Psi(s,(\varpi_s^*+x_s),\mathcal{K}(\varpi^*)(s))\|ds.\end{aligned}$$

As in Case 1, since $G,\ h,\ f,\ and\ \Upsilon_J$ are continuous, we obtain

$$\|(\widehat{\Xi}\varpi^n)(t)-(\widehat{\Xi}\varpi^*)(t)\|\to 0, \quad as \;\; n\to+\infty.$$

Now, for $\theta \in \mho_J$, we have

$$\|(\widehat{\Xi}(\varpi^n))(\theta) - \widehat{\Xi}(\varpi^*)(\theta)\| \leq \|\Upsilon_J(\theta, (\varpi^n)(\theta_J^-)) - \Upsilon_J(\theta, (\varpi^*)(\theta_J^-))\|.$$

By the continuity of Υ_J, we obtain

$$\|(\widehat{\Xi}\varpi^n)(t) - (\widehat{\Xi}\varpi^*)(t)\| \to 0, \quad as \ \ n \to +\infty.$$

Thus, $\widehat{\Xi}$ is continuous.

Step 3: $\widehat{\Xi}$ is χ_{PC}-contraction.
For $\Pi \subset \Omega_\rho$, $\varpi \in \Pi$, and $\nu_1, \nu_2 \in I_0$, with $\nu_2 > \nu_1$, we have

$$\begin{aligned}
&\|\widehat{\Xi}\varpi(\nu_1) - \widehat{\Xi}\varpi(\nu_2)\| \\
&\quad \leq \|G(\nu_1, \varpi_{\nu_1} + x_{\nu_1}) - G(\nu_2, \varpi_{\nu_2} + x_{\nu_2})\| \\
&\qquad + \int_0^{\nu_1} \|\mathcal{Q}(\nu_1 - s) - \mathcal{Q}(\nu_2 - s)\|(p_\Psi \psi_1(\delta_1^*) + q_\Psi \psi_2(\delta_2^*) + \|\mathcal{W}\vartheta(s)\|)ds \\
&\qquad + \int_{\nu_1}^{\nu_2} \|\mathcal{Q}(\nu_2 - s)\|(p_\Psi \psi_1(\delta_1^*) + q_\Psi \psi_2(\delta_2^*) + \|\mathcal{W}\vartheta(s)\|)ds \\
&\quad \leq \int_0^{\nu_1} \|\mathcal{Q}(\nu_1 - s) - \mathcal{Q}(\nu_2 - s)\|(\psi_1(\delta_1^*)p_\Psi + \psi_2(\delta_2^*)q_\Psi)ds \\
&\qquad + M_{\mathcal{W}}\left(\int_0^{\theta} \|\mathcal{Q}(\nu_1 - s) - \mathcal{Q}(\nu_2 - s)\|^2\right)^{\frac{1}{2}} ds \|\vartheta\|_{L^2} \\
&\qquad + K_{\mathcal{Q}}(\psi_1(\delta_1^*)p_\Psi + \psi_2(\delta_2^*)q_\Psi)(\nu_2 - \nu_1) + K_{\mathcal{Q}} M_{\mathcal{W}} (\nu_2 - \nu_1)^{\frac{1}{2}} \|\vartheta\|_{L^2}.
\end{aligned}$$

By the strong continuity of $\mathcal{Q}(\cdot)$ and (C_1), we have

$$\|\widehat{\Xi}\varpi(\nu_1) - \widehat{\Xi}\varpi(\nu_2)\| \to 0, \ as \ \nu_1 \to \nu_2.$$

Now, for $\nu_1, \nu_2 \in I_J$, we obtain

$$\begin{aligned}
&\|\widehat{\Xi}\varpi(\nu_1) - \widehat{\Xi}\varpi(\nu_2)\| \\
&\quad \leq \|\mathcal{Q}(\nu_1 - s_J) - \mathcal{Q}(\nu_2 - s_J)\| \|\Upsilon_J(s_J, (\varpi)(\theta_J^-))\| \\
&\qquad + \int_{s_J}^{\nu_1} \|\mathcal{Q}(\nu_1 - s) - \mathcal{Q}(\nu_2 - s)\| \Big(p_\Psi \psi_1(\widetilde{\delta}_1^*) + q_\Psi \psi_2(\widetilde{\delta}_2^*) + \|\mathcal{W}\vartheta(s)\|\Big) ds \\
&\qquad + \int_{\nu_1}^{\nu_2} \|\mathcal{Q}(\nu_2 - s)\| \Big(p_\Psi \psi_1(\widetilde{\delta}_1^*) + q_\Psi \psi_2(\widetilde{\delta}_2^*) + \|\mathcal{W}\vartheta(s)\|\Big) ds \\
&\quad \leq \|\mathcal{Q}(\nu_1 - s_J) - \mathcal{Q}(\nu_2 - s_J)\|(L_{\Upsilon_J}^* \rho + \Upsilon_J^0) \\
&\qquad + \Big(\psi_1(\widetilde{\delta}_1^*)p_\Psi + \psi_2(\widetilde{\delta}_2^*)q_\Psi\Big) \int_{s_J}^{\nu_1} \|\mathcal{Q}(\nu_1 - s) - \mathcal{Q}(\nu_2 - s)\| ds \\
&\qquad + M_{\mathcal{W}}\left(\int_0^{\theta} \|\mathcal{Q}(\nu_1 - s) - \mathcal{Q}(\nu_2 - s)\|^2\right)^{\frac{1}{2}} ds \|\vartheta\|_{L^2} \\
&\qquad + K_{\mathcal{Q}}(\nu_2 - \nu_1)\Big(\psi_1(\widetilde{\delta}_1^*)p_\Psi + \psi_2(\widetilde{\delta}_2^*)q_\Psi\Big) + K_{\mathcal{Q}} M_{\mathcal{W}} (\nu_2 - \nu_1)^{\frac{1}{2}} \|\vartheta\|_{L^2}.
\end{aligned}$$

By the strong continuity of $\mathcal{Q}(\cdot)$ and assumption (C_1), we obtain

$$\|\widehat{\Xi}\varpi(\nu_1) - \widehat{\Xi}\varpi(\nu_2)\| \to 0, \ as \ \nu_1 \to \nu_2.$$

For $\nu_1, \nu_2 \in \mho_J$, we obtain

$$\|\widehat{\Xi}\varpi(\nu_1) - \widehat{\Xi}\varpi(\nu_2)\| = \|\Upsilon_J(\nu_1, \varpi(\theta_J^-) - \Upsilon_J(\nu_2, \varpi(\theta_J^-)\|.$$

From $(C3)$, the set $\{\Upsilon_J(\theta, \varpi(\theta_J^-))\}_{J=1}^{k_0}$ is equicontinuous, then

$$\|\widehat{\Xi}\varpi(\nu_1) - \widehat{\Xi}\varpi(\nu_2)\| \to 0, \ as\ \nu_1 \to \nu_2.$$

Hence, the set $\widehat{\Xi}(\Pi)$ is equicontinuous, then $\mu_0(\widehat{\Xi}(\Pi)) = 0$.
Now for $\varrho > 0$, there exist $\{\varpi^J\}_{J=0}^{\infty} \subset \Pi$ where for $\theta \in I_0$, and we obtain

$$\begin{aligned}
&\chi(\widehat{\Xi}(\Pi)(\theta)) \\
&\quad \le \chi\left(\left\{G(\theta, \varpi_\theta + x_\theta) + \int_0^\theta \mathcal{Q}(t-s)\Psi(s, \varpi_s + x_s, \mathcal{K}(\varpi + x)(s))ds\ ;\ \varpi \in \Pi\right\}\right) \\
&\quad \le L_G^* \chi(\Pi(\theta)) + 2\chi\left(\left\{\int_0^\theta \mathcal{Q}(t-s)\Psi(s, \varpi_s^J + x_s, \mathcal{K}(\varpi^J + x)(s))ds\ ;\ \varpi \in \Pi\right\}\right) + \varrho \\
&\quad \le L_G^* \chi(\Pi(\theta)) + 4\int_0^\theta K_{\mathcal{Q}} l_\Psi \chi(\{\Pi(s)\})ds + \varrho \\
&\quad \le L_G^* \chi(\Pi(\theta)) + \int_0^\theta e^{\varepsilon\Sigma(s)} e^{-\varepsilon\Sigma(s)} \Sigma'(s)\chi(\Pi(s))ds + \varrho \\
&\quad \le L_G^* \chi(\Pi(\theta)) + \int_0^\theta \Sigma'(s) e^{\varepsilon\Sigma(s)} \sup_{s\in[0,t]} e^{-\varepsilon\Sigma(s)} \chi(\Pi(s))ds + \varrho \\
&\quad \le \frac{1}{\varepsilon}\chi(\Pi(\theta)) + \chi_{PC}(\Pi)\int_0^\theta \left(\frac{e^{\varepsilon\Sigma(s)}}{\varepsilon}\right)' ds + \varrho \\
&\quad \le \frac{2e^{\varepsilon\Sigma(\theta)}}{\varepsilon}\chi_{PC}(\Pi) + \varrho.
\end{aligned}$$

Since ϱ is arbitrary, we obtain

$$\chi(\widehat{\Xi}(\Pi)(\theta)) \le \frac{2e^{\varepsilon\Sigma(\theta)}}{\varepsilon}\chi_{PC}(\Pi).$$

Thus,

$$\chi_{PC}(\widehat{\Xi}(\Pi)) \le \frac{2}{\varepsilon}\chi_{PC}(\Pi).$$

Now, if $\theta \in I_J$, we obtain

$$\begin{aligned}
\chi(\widehat{\Xi}(\Pi)(\theta)) &\le K_{\mathcal{Q}}\, \chi\left(\{\Upsilon_J(s, \varpi(\theta_J^-)); \varpi \in \Pi\}\right) + (K_{\mathcal{Q}} + 1)\chi(\{G(\theta, \varpi_\theta + x_\theta)\ ;\ \varpi \in \Pi\}) \\
&\quad + \chi\left(\left\{\int_0^\theta \mathcal{Q}(t-s)\Psi(s, \varpi_s + x_s, \mathcal{K}(w)(s))ds\ ;\ w \in \Pi\right\}\right) \\
&\le \frac{2}{\varepsilon}\chi(\Pi(\theta)) + 4\int_0^\theta K_{\mathcal{Q}} l_\Psi \chi(\{\Pi(s)\})ds + \varrho \\
&\le \frac{3e^{\varepsilon\Sigma(\theta)}}{\varepsilon}\chi_{PC}(\Pi) + \varrho.
\end{aligned}$$

Therefore,

$$\chi_{PC}\left(\widehat{\Xi}(\Pi)\right) \le \frac{3}{\varepsilon}\chi_{PC}(\Pi).$$

If $\theta \in \mho_\jmath$, by $(C3)$, we obtain

$$\begin{aligned}\chi\left(\widehat{\Xi}(\Pi)(\theta)\right) &= \chi\left(\left\{\Upsilon_\jmath(\theta, \varpi(\theta_\jmath^-)); \varpi \in \Pi\right\}\right) \\ &\leq \frac{1}{K_{\mathcal{Q}}\varepsilon}\chi(\Pi(\theta)) \\ &\leq \frac{e^{\varepsilon\Sigma(\theta)}}{\varepsilon\, K_{\mathcal{Q}}}\chi_{PC}(\Pi).\end{aligned}$$

Then,

$$\chi_{PC}\left(\widehat{\Xi}(\Pi)\right) \leq \frac{1}{\varepsilon\, K_{\mathcal{Q}}}\chi_{PC}(\Pi).$$

Theorem 2 implies that $\widehat{\Xi}$ has at least one fixed point w^*. Consequently, $x^* = w^* + x$ is a fixed point of Ξ, which represents a mild solution of (2). □

4.2. Approximate Controllability

In this section we investigate the approximate controllability for System (2). For any given $\eta^{\theta_{\jmath+1}} \in \digamma$ and $\lambda \in (0,1]$, we take the control function $\vartheta^\lambda(\theta)$ as follows:

$$\vartheta^\lambda(\theta) = \mathcal{W}^*\mathcal{Q}^*(\theta_{\jmath+1} - s)R\left(\lambda, \Gamma_{s_\jmath}^{\theta_{\jmath+1}}\right)\widetilde{\Delta}(\delta^{\theta_{\jmath+1}}, \theta);\ \jmath = 0, \ldots, m,$$

where

$$\widetilde{\Delta}(\eta^{\theta_{\jmath+1}}, \theta) = \eta^{\theta_{\jmath+1}} - \widetilde{\Delta}_\jmath(\theta) - G(\theta, x_\theta) - \int_{s_\jmath}^{\theta}\mathcal{Q}(\theta - s)\Psi(s, x_s, (\mathcal{K}x)(s))ds,$$

and

$$\widetilde{\Delta}_\jmath(\theta) = \begin{cases}\mathcal{Q}(\theta)\ (\Phi(0) - G(0, \Phi));\ \text{if } \jmath = 0, \\ \mathcal{Q}(\theta - s_\jmath)\left[\Upsilon_\jmath(s_\jmath, x(\theta_\jmath^-)) - G(s_\jmath, x_{s_\jmath})\right];\ \text{if } \jmath \in N_1^m.\end{cases}$$

Theorem 7. *Assume that the hypotheses $(C_0) - (C_4)$ are satisfied and in addition, the function f is uniformly bounded. Then, Equation (2) is approximately controllable on $[0, \kappa]$.*

Proof. According to Theorem 6, we can know that system (2) has at least one mild solution $\vartheta^\lambda \in \Omega_\rho$. Then, we obtain

$$\begin{aligned}\vartheta^\lambda(\theta_{\jmath+1}) &= \widetilde{\Delta}_\jmath(\theta_{\jmath+1}) + G(\theta_{\jmath+1}, x_{\theta_{\jmath+1}}) \\ &\quad + \int_{s_\jmath}^{\theta_{\jmath+1}}\mathcal{Q}(\theta_{\jmath+1} - s)(\Psi(s, x_s, (\mathcal{K}x)(s)) + \mathcal{W}\vartheta(s))ds \\ &= \widetilde{\Delta}_\jmath(\theta_{\jmath+1}) + G(\theta_{\jmath+1}, x_{\theta_{\jmath+1}}) + \int_{s_\jmath}^{\theta_{\jmath+1}}\mathcal{Q}(\theta_{\jmath+1} - s)(\Psi(s, x_s, (\mathcal{K}x)(s)))ds \\ &\quad + \int_{s_\jmath}^{\theta_{\jmath+1}}\mathcal{Q}(\theta_{\jmath+1} - s)\left(\mathcal{W}^*\mathcal{Q}^*(\theta_{\jmath+1} - s)R\left(\lambda, \Gamma_{s_\jmath}^{\theta_{\jmath+1}}\right)\Delta(\eta^{\theta_{\jmath+1}}, \theta_{\jmath+1})\right)ds \\ &= \eta^{\theta_{\jmath+1}} + (\Gamma_{s_\jmath}^{\theta_{\jmath+1}}R\left(\lambda, \Gamma_{s_\jmath}^{\theta_{\jmath+1}}\right) - I)\Delta(\eta^{\theta_{\jmath+1}}, \theta_{\jmath+1}) \\ &= \eta^{\theta_{\jmath+1}} + \lambda R\left(\lambda, \Gamma_{s_\jmath}^{\theta_{\jmath+1}}\right)\Delta(\eta^{\theta_{\jmath+1}}, \theta_{\jmath+1}).\end{aligned}$$

Thus,

$$\|\vartheta^\lambda(\theta_{J+1}) - \eta^{\theta_{J+1}}\| \leq \left\| R\left(\lambda, \Gamma_{s_J}^{\theta_{J+1}}\right) \left[\eta^{\theta_{J+1}} - \widetilde{\Delta}_J(\theta_{J+1}) - G(\theta_{J+1}, x_{\theta_{J+1}}) \right] \right\| + \left\| R\left(\lambda, \Gamma_{s_J}^{\theta_{J+1}}\right) \left[\int_{s_J}^{\theta_{J+1}} \mathcal{Q}(\theta_{J+1} - \varkappa) \Psi(\varkappa, x_\varkappa, (\mathcal{K}x)(\varkappa)) d\varkappa \right] \right\|.$$

From the uniform boundedness of $\Psi(\cdot, \cdot, \cdot)$ and similar to the proof of Theorem 5, we obtain

$$\int_0^\kappa \|\Psi(s, x_s^\lambda, (\mathcal{K}x^\lambda)(s)) - \widetilde{\Psi}(s)\| ds \xrightarrow[\lambda \to 0]{} 0.$$

Then,

$$\begin{aligned} &\|\vartheta^\lambda(\theta_{J+1}) - \eta^{\theta_{J+1}}\| \\ &\leq \left\| R\left(\lambda, \Gamma_{s_J}^{\theta_{J+1}}\right) \left[\eta^{\theta_{J+1}} - \widetilde{\Delta}_J(\theta_{J+1}) - G(\theta_{J+1}, x_{\theta_{J+1}}) \right] \right\| \\ &\quad + \left\| R\left(\lambda, \Gamma_{s_J}^{\theta_{J+1}}\right) \left[\int_{s_J}^{\theta_{J+1}} \mathcal{Q}(\theta_{J+1} - \varkappa) \Big(\Psi(s, x_s, (\mathcal{K}x)(s)) - \widetilde{\Psi}(s) \Big) d\varkappa \right] \right\| \\ &\quad + \left\| R\left(\lambda, \Gamma_{s_J}^{\theta_{J+1}}\right) \left[\int_{s_J}^{\theta_{J+1}} \mathcal{Q}(\theta_{J+1} - \varkappa) \widetilde{\Psi}(s) ds \right] \right\| \xrightarrow[\lambda \to 0]{} 0. \end{aligned}$$

Thus, $\vartheta^\lambda(\theta_{J+1}) \to \eta^{\theta_{J+1}}$ holds. Therefore, we obtain the approximate controllability of system (2), and the proof is complete. □

5. An Example

Let

$$\Im := L^2(0, \pi) = \left\{ \vartheta : (0, \pi) \longrightarrow \mathbb{R} : \int_0^\pi |\vartheta(x)|^2 dx < \infty \right\},$$

be the Hilbert space with the scalar product $\langle \vartheta, \varkappa \rangle = \int_0^\pi \vartheta\,(x) \varkappa(x) dx$, and the norm

$$\|\vartheta\|_2 = \left(\int_0^\pi |\vartheta(x)|^2 dx \right)^{1/2}.$$

The phase space $\Bbbk = BUC(\mathbb{R}^-, \Im)$ is the space of bounded uniformly continuous functions endowed with the following norm: $\|\psi\|_\Bbbk = \sup_{-\infty < \varepsilon \leq 0} \|\psi(\varepsilon)\|_{L^2}, \psi \in \Bbbk$. It is well known that $\Bbbk$ satisfies the axioms (A_1) and (A_2) with $K = 1$ and $\widetilde{\wp}(\theta) = \widehat{\wp}(\theta) = 1$ (see [33]).
Let $\aleph$ be induced on $\Im$ as follows:

$$\aleph z = z'', \text{ and } D(\aleph) = H^2(0, \pi) \cap H_0^1(0, \pi).$$

Additionally, we have

$$\aleph z = \sum_{n=1}^\infty n^2 \langle z, z_n \rangle z_n, \quad z \in D(\aleph),$$

where $z_n(s) = -\sqrt{\frac{2}{\pi}} \sin ns; n = 1, 2, \ldots$, is the orthogonal set of eigenvectors of $\aleph$. It is well known that $\aleph$ is the infinitesimal generator of a C_0 semigroup $\{T(\theta), \theta \geq 0\}$ in the Hilbert space $\Im$ and

$$T(\theta)z = \sum_{n=1}^\infty \exp\Big(-n^2\theta\Big) \langle z, z_n \rangle z_n, \quad z \in \Im.$$

We define the operators $\Lambda(\theta) : D(\aleph) \subset \Im \mapsto \Im$ as follows:

$$\Lambda(\theta)z = \Gamma(\theta)\aleph z, \ for\ \theta \geq 0,\ z \in D(\aleph).$$

More appropriate conditions on operator Λ, (C_4) holds. Furthermore, the above C_0 semi-group $\{T(\theta), \theta \geq 0\}$ is compact for $\theta > 0$ and thus it is operator-norm continuous for $\theta > 0$, and so the resolvent operator for $\theta > 0$. More details about these facts can be seen from the monograph [34].

Consider the following system:

$$\begin{cases} \frac{\partial \nu(\theta,\zeta_\theta)(x)}{\partial \theta} = \frac{\partial^2 \nu(\theta,\zeta_\theta)(x)}{\partial^2 x} - \int_0^\theta \Gamma(\theta - s)\frac{\partial^2 \nu(s,\zeta_s)(x)}{\partial^2 x} ds + \int_{-\infty}^{-\theta} \frac{e^\varepsilon \|\zeta(\theta+\varepsilon,x)\|_2}{77((\theta+\varepsilon)^2+1)} d\varepsilon \\ \qquad - \frac{e^{\frac{\pi}{2}}\sqrt{\sinh(\pi)}}{77(\theta^2+1)} + \int_0^1 \frac{\sin(\theta)\ln(1+e^{-\theta^2})(1+\zeta(s,x))}{133(1+2\theta^2+s^2)e^\theta} ds + \mathcal{L}(\theta,x), \\ \qquad \text{if } \theta \in I_1 \cup I_2 \cup I_3 \text{ and } x \in (0,\pi), \\ \zeta(\theta,0) = \zeta(\theta,\pi) = 0, \quad \text{for } \theta \in [0,1], \\ \zeta(\theta,x) = \frac{1}{63+\theta^6}\|(\zeta(\jmath^-,x))\|_2, \text{ if } \theta \in \mho_1 \cup \mho_2, \ x \in (0,\pi), \\ \zeta(\theta,x) = \Phi(\theta,x), \text{ if } \theta \in \mathbb{R}_- \text{ and } x \in (0,\pi), \end{cases} \tag{4}$$

where $k_1 = \frac{1}{16}$, $k_2 = \frac{1}{9}$, $k_3 = \frac{1}{8}$, $k_4 = \frac{1}{4}$, $I_1 = (0, k_1]$, $I_2 = (k_2, k_3]$, $I_3 = (k_4, 1]$, $\mho_1 = (k_1, k_2]$, $\mho_2 = (k_3, k_4]$, $\mathcal{L} : [0,1] \times [0,\pi] \to [0,\pi]$.

The function ν is defined by

$$\nu(\theta,\zeta_\theta)(x) = \zeta(\theta,x) - \int_{-\infty}^0 \frac{a_0(\theta)a_1(s)}{63}\|\zeta(s,x)\|_2 ds,$$

where $a_0 \in C([0,1],\mathbb{R}^+)$ and $a_1 \in L^1(\mathbb{R}^-,\mathbb{R}^+)$ such that $\|a_0\|\|a_1\|_{L^1} < \frac{34}{5}$.

Let $\mathcal{W} : U \to \Im$ be defined by $\mathcal{W}\vartheta(\theta)(x) = \mathcal{L}(\theta,x)$, $x \in [0,\pi]$, $\vartheta \in U$, where $\mathcal{L} : [0,1] \times [0,\pi] \to \Im$ is continuous.

We put $\zeta(\theta)(x) = \zeta(\theta,x)$, for $\theta \in [0,1]$, and define

$$\begin{aligned} \Psi(\theta,x_1,x_2)(x) &= \int_{-\infty}^{-\theta} \frac{e^\varepsilon \|x_1(\theta+\varepsilon,x)\|_2}{77((\theta+\varepsilon)^2+1)} d\varepsilon - \frac{e^{\frac{\pi}{2}}\sqrt{\sinh(\pi)}}{77(\theta^2+1)} + \frac{\sin(\theta)x_2(\theta)(x)}{e^\theta}, \\ x_2(\theta)(x) = \mathcal{K}(x_1)(x) &= \int_0^1 \frac{\ln(1+e^{-\theta^2})(1+x_1(s,x))}{133(1+2\theta^2+s^2)} ds, \\ \Upsilon_\jmath(\theta,x(\theta,x)) &= \frac{1}{63+\theta^6}\|(x(\jmath^-,x))\|_2, \\ G(\theta,\hat{x}(\theta,x)) &= \int_{-\infty}^0 \frac{a_0(\theta)a_1(s)}{63}\|\hat{x}(s,x)\|_2 ds. \end{aligned}$$

These definitions allow us to depict the system (4) in the abstract form (2).

Now, for $\theta \in [0,1]$, we have

$$|\Psi(\theta,\varkappa_{1(\theta)},\varkappa_2(\theta))| \leq \frac{e^{\frac{\pi}{2}}\sqrt{\sinh(\pi)}}{77(\theta^2+1)}(1+\|\varkappa_1\|_{\Bbbk}) + \sin(\theta)e^{-\theta}(|\varkappa_2(\theta)|).$$

So, $\psi_{i+1}(\theta) = t + i$; $i = 0,1$ are continuous nondecreasing functions, and we have

$$p_\Psi = \frac{e^{\frac{\pi}{2}}}{77\sqrt{2}}\sqrt{\left(\frac{\pi}{1+\pi^2} + \tan^{-1}(\pi)\right)\sinh(\pi)}, \text{ and } q_\Psi = \sqrt{8^{-1}(1-e^{-2\pi})}.$$

Additionally, for any bounded set $\Pi \subset \Im$, and $\Pi_\theta \in \Bbbk$, we obtain

$$\chi(\Psi(\theta,\Pi_\theta,\mathcal{K}(\Pi(\theta)))) \leq (p_\Psi + q_\Psi)\chi(\Pi).$$

Now, regarding g and Υ_J, we obtain

$$\begin{aligned} \|g(\theta,s,\varkappa_1)-g(\theta,s,\varkappa_2)\|_2 &\leq \frac{\ln(2)}{133}\|\varkappa_1-\varkappa_2\|_2, \\ \|\Upsilon_J(\theta,\varkappa_1)-\Upsilon_J(\theta,\varkappa_2)\|_2 &\leq \frac{1}{63}\|\varkappa_1-\varkappa_2\|_2, \\ \|G(\theta,\widetilde{\varkappa_1})-G(\theta,\widetilde{\varkappa_2})\|_2 &\leq \frac{\|a_0\|\|a_1\|_{L^1}}{63}\|\varkappa_1-\varkappa_2\|_{\Bbbk}. \end{aligned}$$

Then $\varepsilon K_Q(L^*_{\Upsilon_J}+L_G)<1$, so we can take $\varepsilon=7$, and for $p_3=\|\varkappa_1\|_{\Bbbk}$, $p_4=\|\varkappa_2\|_2$, we obtain

$$\|\Psi(\cdot,\varkappa_{1(\cdot)},\varkappa_2(\cdot))\|_2\leq e^{\frac{\pi}{2}}\sqrt{\sinh(\pi)}(1+p_3+p_4),\ \text{ for all } \varkappa_1\in\Bbbk,\ \varkappa_2\in\Im.$$

For reasoning similar to that in [8], the linear control system corresponding to system (4) is approximately controllable on $[0,1]$. Hence, (C_0) holds. After verifying all the requirements of Theorems 6 and 7, we can conclude that (4) is approximately controllable.

6. Conclusions

In this paper, we have presented an analysis of the approximate controllability for a class of abstract neutral integro-differential equations with non-instantaneous impulsions and state-dependent delay. Our approach utilizes resolvent operators and Darbo's type fixed point theorem to obtain the results. Through our analysis, we have shown that the system is approximately controllable under certain conditions. Furthermore, we have illustrated the practical applications of our results through a specific example. We hope that our analysis can inspire further research in this area and contribute to the development of more complex systems. In our future work, we aim to study the approximate and exact controllability, attractivity and Ulam stability for second-order impulsive differential evolution equations with different types of delay.

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