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Approximate Homogenized Synthesis for Distributed Optimal Control Problem with Superposition Type Cost Functional

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Abstract In this paper, we consider the optimal control problem in the feedback form (synthesis) for a parabolic equation with rapidly oscillating coefficients and not-decomposable quadratic cost functional with superposition type operator. In general, it is not possible to find the exact formula of optimal synthesis for such a problem because the Fourier method can't be directly applied. But transition to the homogenized parameters greatly simplifies the structure of the problem. Assuming that the problem with the homogenized coefficients already admits optimal synthesis form, we ground approximate optimal control in the feedback form for the initial problem. We give an example of superposition operator for specific conditions in this paper.

Keywords The optimal control problem, approximate synthesis, parabolic equation, rapidly oscillating coefficients, homogenization.

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Introduction

In this work, we focus on the finding effective methods of control for complicated infinite-dimensional systems, initiated in the works [1],[2],[3]. Finding control in the feedback form or synthesis plays important role here. In [4] it was proposed and substantiated a procedure for constructing approximate optimal synthesis for a wide class of distributed processes in micro-inhomogeneous medium, investigated earlier in [5]. We use some known facts on G-convergence theory from [6], [7]. In this paper, we consider the optimal control problem in the feedback form for a parabolic equation with rapidly oscillating coefficients and not-decomposable quadratic cost functional with superposition or Nemyckii type operator. In general, to find the exact formula of optimal synthesis is not possible for such a problem because we can not directly apply the Fourier method. But the transition to the homogenized parameters greatly simplifies the structure of the problem. Assuming that the problem with the homogenized coefficients already admits optimal synthesis form, we ground approximate optimal control in the feedback form for the initial problem. We give an example of Nemyckii operator for specific conditions in this paper.

1. Setting of the problem

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $\varepsilon \in (0, 1)$ be a small parameter. In the cylinder $Q = (0, T) \times \Omega$ controlled process $\{y, u\}$ is described by the problem

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$$\begin{cases} \frac{\partial y}{\partial t} = A^\varepsilon y + u(t,x), \\ y|_{\partial\Omega} = 0, \\ y(0,x) = y_0^\varepsilon, \end{cases} \tag{1.1}$$

with a cost functional

$$J(u) = \int_\Omega q_\varepsilon(x, y(T,x))y(T,x)dx + \int_Q u^2(t,x)dtdx \to \inf, \tag{1.2}$$

where

$$A^\varepsilon = div(a^\varepsilon \nabla), \ \ a^\varepsilon(x) = a\Big(\frac{x}{\varepsilon}\Big),$$

a is measurable, symmetric, periodic matrix, satisfying conditions of uniform ellipticity and boundedness: $\exists \nu_1 > 0$, $\nu_2 > 0 \; \forall \eta \in \mathbb{R}^n \; \forall \, x \in \mathbb{R}^n$

$$\nu_1 \sum_{i=1}^n \eta_i^2 \le \sum_{i,j=1}^n a_{i,j}(x)\eta_i\eta_j \le \nu_2 \sum_{i=1}^n \eta_i^2, \tag{1.3}$$

$q_\varepsilon : \Omega \times R \mapsto R$ is a Caratheodory function, and there exist functions $C_1 \in L^2(\Omega)$, $C_2 \in L^1(\Omega)$, and constant $C > 0$, independent of $\varepsilon \in (0,\,1)$, such that for all $\xi \in \mathbb{R}$ and almost all $x \in \Omega$ the following inequalities hold

$$\begin{aligned} |q_\varepsilon(x,\xi)| \le C|\xi| + C_1(x), \\ q_\varepsilon(x,\xi)\xi \ge -C_2(x). \end{aligned} \tag{1.4}$$

Under these conditions [9] Nemyckii's operator $q_\varepsilon(x,\,\cdot)$: $L^2(\Omega) \mapsto L^2(\Omega)$ is continuous. Hence, by conditions (1.3), (1.4) and properties of solutions of the problem (1.1) (see Lemma 2.1) we obtain that the problem (1.1), (1.2) has solution $\{\bar{y}^\varepsilon,\, \bar{u}^\varepsilon\}$ (optimal process) in class $W(0,T) \times L^2(Q)$, where $W(0,T)$ is a class of functions $y \in L^2\left(0,\, T;\, H_0^1(\Omega)\right)$, which have generalized derivatives with respect to t from class $L^2\left(0,\, T;\, H^{-1}(\Omega)\right)$ [1]. In general case, we are not able to find the exact optimal feedback law for the problem (1.1), (1.2). However, in many cases [5] a transition to homogenized parameters simplifies the structure of the problem. We will assume that the problem with homogenized coefficients already admits optimal feedback control of the form $u[t,x,y(t,x)]$.

The main goal of this paper is to prove the fact that the form $u[t,x,y(t,x)]$ realizes an approximate feedback control in initial problem (1.1), (1.2), i. e. for any $\eta > 0$

$$|J_\varepsilon(\bar{y}^\varepsilon, \bar{u}^\varepsilon) - J_\varepsilon(y_\varepsilon, u[t,x,y_\varepsilon])| < \eta \tag{1.5}$$

for $\varepsilon > 0$ small enough, where y_ε is a solution of the problem (1.1), (1.2) with control $u[t,x,y_\varepsilon]$.

2. Main results

We shall use $\|\cdot\|$ to denote the norm and $(\,\cdot\,,\,\cdot\,)$ to denote the inner product in $L^2(\Omega)$. Let us assume that there exists a Caratheodory function $q : \Omega \times \mathbb{R} \mapsto \mathbb{R}$, such that

$$\begin{gathered} \forall\, r > 0 \quad q_\varepsilon(x,\xi) \to \ q(x,\xi) \text{ weakly in } L_2(\Omega) \\ \text{uniformly for } |\xi| \le r. \end{gathered} \tag{2.1}$$

We refer to the following problem

$$\begin{cases} \frac{\partial y}{\partial t} = A^0 y + u(t,\,x), \\ y|_{\partial\Omega} = 0, \\ y|_{t=0} = y_0, \end{cases} \tag{2.2}$$

$$J(y,u)=\int_\Omega q(x,y(T,x))y(T,x)dx+\int_Q u^2(t,x)dtdx\to\inf, \tag{2.3}$$

as an homogenized one for the problem (1.1), (1.2). Here a constant matrix a^0 is homogenized for a^ε [6], $A^0=div(a^0\nabla),\ y_0\in L^2(\Omega)$, such that

$$y_0^\varepsilon\to y_0 \text{ weakly in } L^2(\Omega) \text{ as } \varepsilon\to 0. \tag{2.4}$$

In further arguments we will use the following result about convergence of parabolic operators which is the consequence of G-convergence of A^ε to A^0 [6].

Lemma 2.1. *[6],[7] Let $y_0^\varepsilon\to y_0$ weakly in $L^2(\Omega)$, $u_\varepsilon\to u$ weakly in $L^2(Q)$. Then $y^\varepsilon\to y$ in $C\left([\delta,T];L^2(\Omega)\right)$ $\forall\delta>0$, where y^ε is a solution of the problem (1.1) with control u_ε, y is a solution of the problem (2.2) with control u.*

Let us assume that the following conditions hold:

$$\text{the problem (2.2), (2.3) has a unique solution } \{\overline{y},\overline{u}\}; \tag{2.5}$$

$$\text{there exists a measurable map } u:\ [0,T]\times\Omega\times L^2(\Omega)\mapsto L^2(\Omega) \text{ such that}$$
$$\overline{u}(t,x)\equiv u\,[t,x,\,\overline{y}(t,x)]; \tag{2.6}$$

$$\text{there exist constants } D_1>0,\ D_2>0,$$
$$\text{such that for all } t\in[0,T] \text{ and } y,z\in L^2(\Omega) \text{ the inequalities hold}$$

$$\begin{aligned}\|u\,[t,x,y]\|&\le D_1\,(\|y\|+1)\,,\\ \|u\,[t,x,y]-u\,[t,x,z]\|&\le D_2\,\|y-z\|\,.\end{aligned} \tag{2.7}$$

Before we formulate the main result, we give a typical example of the function $q_\varepsilon:\Omega\times\mathbb{R}\mapsto\mathbb{R}$, for which the conditions (1.4), (2.1), (2.5) – (2.7) hold.

Example. Let $q_\varepsilon(x,\xi)=g\left(\frac{x}{\varepsilon}\right)\xi$, where g is measurable, bounded, periodic function with mean value $\langle g\rangle$ [5]. Then conditions (1.4), (2.1) hold for $q(x,\xi)=\langle g\rangle\,\xi$. Moreover, the problem (2.2), (2.3) becomes a classical linear quadratic problem that has a unique solution [1]. Thus, the condition (2.5) holds. Let $\{X_i\},\ \{\lambda_i\}$ be solutions of the spectrum problem

$$\begin{cases} A^0X_i=-\lambda_iX_i,\\ X_i\,|_{\partial\Omega}=0,\end{cases}$$

$\{X_i\}\subset H_0^1(\Omega)$ is an orthonormal basis in $L^2(\Omega)$, $0<\lambda_1\le\lambda_2\le\cdots,\ \lambda_i\to\infty,\ i\to\infty.$
By using Fourier decomposition method, it is easy to obtain optimal control in a feedback form [4]:

$$u[t,x,y]=\sum_{i=1}^{\infty}\beta_i\,(t)\,(y,X_i)\,X_i(x), \tag{2.8}$$

where

$$\beta_i\,(t)=-e^{2\lambda_i(t-T)}\left(\frac{1}{\langle g\rangle}+\frac{1}{2\lambda_i}\left(1-e^{2\lambda_i(t-T)}\right)\right)^{-1}.$$

The last formula yields

$$\exists \beta > 0 \; \forall i \geq 1 \quad \sup_{t \in [0,T]} |\beta_i(t)| \leq \beta,$$

thus, conditions (2.6), (2.7) hold with $D_1 = D_2 = \beta$.

Further, using feedback law (2.6), we consider the problem

$$\begin{cases} \frac{\partial y}{\partial t} = A^{\varepsilon} y + u[t,\, x, y], \\ \quad y\,|_{\partial\Omega} = 0, \\ \quad y\,|_{t=0} = y_0^{\varepsilon}. \end{cases} \tag{2.9}$$

Under conditions (2.7) the problem (2.8) has a unique solution y_ε in the class $W(0,T)$ [8].
The main result of this article is the following theorem.

Theorem 2.1. *Let conditions (1.3), (1.4), (2.1), (2.5) - (2.7) hold and, moreover, there exists a positive function $l \in L^{\infty}(\Omega)$, such that for all $\varepsilon \in (0,1)$*

$$|q_\varepsilon(x,\xi_1) \; - \; q_\varepsilon(x,\xi_2)| \leq l(x)|\xi_1 - \xi_2|. \tag{2.10}$$

Then for arbitrary $\eta > 0$ there exists $\bar{\varepsilon} \in (0,1)$, such that $\forall \varepsilon \in (0, \bar{\varepsilon})$

$$|J_\varepsilon(\bar{y}^{\varepsilon}, \bar{u}^{\varepsilon}) - J_\varepsilon(y_\varepsilon, u\,[t,x,y_\varepsilon(t,x)])| < \eta,$$

where $\{\bar{y}^{\varepsilon},\, \bar{u}^{\varepsilon}\}$ is an optimal process for the problem (1.1), (1.2), y_ε is a solution of the problem (2.9), the control $u\,[t,x,y_\varepsilon(t,x)]$ is defined from (2.6).

Proof
At the beginning we show that as $\varepsilon \to 0$ both the solution y_ε of the problem (2.9) and the solution $\bar{y}^{\varepsilon}$ of the problem (1.1), (1.2) tend to $\overline{y}$ in some sense, where $\{\overline{y}, \overline{u}\}$ is the optimal process in the problem (2.2), (2.3). We consider first the problem (2.9). For almost all (a.a.) $t \in (0,T)$, the following estimate holds for the solution y_ε

$$\frac{d}{dt}\,\|y_\varepsilon(t)\|^2 + 2v_1\,\|y_\varepsilon(t)\|^2_{H_0^1} \leq (2D_1+1)\,\|y_\varepsilon(t)\|^2 + D_1^2. \tag{2.11}$$

Using Gronwall's Lemma, from (2.11) we obtain that the sequence $\{y_\varepsilon\}$ is bounded in $W(0,T)$. Then, by Compactness Lemma [8] there exists a function $z \in W(0,T)$, such that along subsequence

$$\begin{gathered} y_\varepsilon \to z \text{ in } L^2(Q) \text{ and almost everywhere in } Q, \\ y_\varepsilon(t) \to z(t) \text{ in } L^2(\Omega) \text{ for almost all } t \in (0,T), \\ y_\varepsilon(t) \to z(t) \text{ in } L^2(\Omega) \text{ weakly } \forall\, t \in [0,T], \\ y_\varepsilon \to z \text{ weakly in } L^2\left(0,T; H_0^1(\Omega)\right). \end{gathered} \tag{2.12}$$

From this and from (2.7) we derive that

$$u[t,x,y_\varepsilon] \to u[t,x,z] \text{ in } L^2(Q). \tag{2.13}$$

From Lemma 2.1 we obtain that z is a solution of the problem (2.9) with operator A^0 and initial data y^0, and as $\varepsilon \to 0$

$$y_\varepsilon \to z \text{ in } C\left([\delta,T]; L^2(\Omega)\right) \;\forall \delta > 0. \tag{2.14}$$

Since the optimal control problem (2.2), (2.3) has a unique solution $\{\overline{y}, \overline{u}\}$ and formula $\overline{u}(t,x) = u[t,x,\overline{y}(t,x)]$ is valid for control $\overline{u}$, then $\overline{y}$ is a solution of the problem (2.9) with operator A^0 and initial data y^0. However, this problem also has a unique solution, so $\overline{y} \equiv z$, and moreover, the convergences (2.12) – (2.14) hold as $\varepsilon \to 0$ (not only along subsequence).

Lemma 2.2. *Let functions $q_n := q_{\varepsilon_n}$ satisfy conditions (1.4), (2.1), (2.10) for $\varepsilon_n \to 0$, and $y_n \to y$ in $L^2(\Omega)$. Then*

$$q_n(x, y_n) \to q(x, y) \text{ weakly in } L^2(\Omega).$$

Proof
For any $\phi \in L^2(\Omega)$ we consider

$$I_n := \int_\Omega (q_n(x, y_n(x)) - q(x, y(x)))\phi(x)dx =$$

$$= \int_\Omega (q_n(x, y_n(x)) - q_n(x, y(x)))\phi(x)dx +$$

$$+ \int_{\Omega_1(r)} (q_n(x, y(x)) - q(x, y(x)))\phi(x)dx +$$

$$+ \int_{\Omega \setminus \Omega_1(r)} (q_n(x, y(x)) - q(x, y(x)))\phi(x)dx =$$

$$= I_n^{(1)} + I_n^{(2)}(r) + I_n^{(3)}(r),$$

where $\Omega_1(r) = \{x \in \Omega \mid |y(x)| \le r\}$. Let us prove that

$$\forall \eta > 0 \exists N \; \forall n \ge N \; I_n < \eta.$$

From condition (2.10) we deduce that

$$\exists N_1 \; \forall n \ge N_1 \; I_n^{(1)} < \eta/3.$$

From condition (1.4) we deduce that

$$\exists r > 0 \; \forall n \ge 1 \; I_n^{(3)}(r) < \eta/3.$$

From condition (2.1) we deduce that

$$\exists N(r) > 0 \; \forall n \ge N(r) \; I_n^{(2)}(r) < \eta/3.$$

Choosing $N = \max\{N_1, N(r)\}$, we obtain the desired weak convergence. Lemma is proven. □

From Lemma 2.2 and convergences (2.12), (2.13), we derive

$$J_\varepsilon(y_\varepsilon, u[t, x, y_\varepsilon]) \to J(\overline{y}, \overline{u}), \; \varepsilon \to 0. \tag{2.15}$$

Now we consider the optimal process $\{\bar{y}^\varepsilon, \; \bar{u}^\varepsilon\}$ of the problem (1.1), (1.2). We have the inequality

$$-\int_\Omega C_2(x)dx + \int_Q (\bar{u}^\varepsilon)^2(t, x)dtdx \le$$

$$\le J_\varepsilon(\bar{y}^\varepsilon, \bar{u}^\varepsilon) \le \int_\Omega z_\varepsilon^2(T, x)dx,$$

where z_ε is a solution of the problem (1.1) with control $u \equiv 0$. For z_ε the following estimate holds

$$\frac{d}{dt}\|z_\varepsilon(t)\|^2 + 2v_1\|z_\varepsilon(t)\|^2_{H_0^1} \le 0,$$

so, the sequence $\{\bar{u}^\varepsilon\}$ is bounded in $L^2(Q)$. Then, there exists $v \in L^2(Q)$, such that along some subsequence

$$\bar{u}^{\varepsilon} \rightarrow v \text{ weakly in } L^2(Q), \;\; \varepsilon \rightarrow 0.$$

By the boundedness of $\{\bar{u}^{\varepsilon}\}$ in $L^2(Q)$ and estimate

$$\frac{d}{dt}\|\bar{y}^{\varepsilon}(t)\|^2 + 2v_1 \|\bar{y}^{\varepsilon}(t)\|^2_{H_0^1} \leq 2\left|(\bar{y}^{\varepsilon}(t), \bar{u}^{\varepsilon}(t)\right|, \tag{2.16}$$

and by Gronwall's Lemma, we deduce the boundedness of the sequence $\{\bar{y}^{\varepsilon}\}$ in $W(0,T)$ and along subsequence it tends to some function $y \in W(0,T)$ as $\varepsilon \rightarrow 0$ within the meaning of (2.12). Using Lemma 2.1, we obtain that y is a solution of the problem (2.2) with control v, and with $\varepsilon \rightarrow 0$

$$\bar{y}^{\varepsilon} \rightarrow y \text{ in } C\left([\delta, T]; L^2(\Omega)\right) \; \forall \delta > 0. \tag{2.17}$$

Let us show that the process $\{y, v\}$ is optimal in the problem (2.2), (2.3). From the optimality of $\{\bar{y}^{\varepsilon}, \bar{u}^{\varepsilon}\}$ for arbitrary $u \in L^2(Q)$ the following inequality holds

$$J_{\varepsilon}(\bar{y}^{\varepsilon}, \bar{u}^{\varepsilon}) \leq J_{\varepsilon}(p^{\varepsilon}, u), \tag{2.18}$$

where p_{ε} is a solution of the problem (1.1) with control u. Hence, replacing u^{ε} by u, the estimate (2.16) holds for p_{ε}. Thus, $\{p^{\varepsilon}\}$ is bounded in $W(0,T)$. With the above thinking we obtain that p_{ε} converges to some function $p \in W(0,T)$ as $\varepsilon \rightarrow 0$ in the meaning of (2.12). Moreover, p is a solution of the problem (2.2) with control u and p_{ε} converges to p in the meaning of (2.17).

Then Lemma 2.2 implies

$$\begin{gathered} J_{\varepsilon}(p^{\varepsilon}, u) \rightarrow J(p, u), \\ \varliminf_{\varepsilon \rightarrow 0} J_{\varepsilon}(\bar{y}^{\varepsilon}, \bar{u}^{\varepsilon}) \geq J(y, v). \end{gathered} \tag{2.19}$$

From (2.18), (2.19) we ultimately derive inequality

$$J(y, v) \leq J(p, u),$$

which means that $\{y, v\}$ is an optimal process in the problem (2.2), (2.3). By the uniqueness it follows $\{y, v\} \equiv \{\bar{y}, \bar{u}\}$.

Now we will prove that

$$J_{\varepsilon}(\bar{y}^{\varepsilon}, \bar{u}^{\varepsilon}) \rightarrow J(\bar{y}, \bar{u}), \;\; \varepsilon \rightarrow 0. \tag{2.20}$$

Fix the control $u = \bar{u}$ in the problem (1.1) for this. Let y^{ε} be a solution of this problem.

Using the reasoning above, we have the following relations

$$J_{\varepsilon}(\bar{y}^{\varepsilon}, \bar{u}^{\varepsilon}) \leq J_{\varepsilon}(y^{\varepsilon}, \bar{u}),$$

$$J_{\varepsilon}(y^{\varepsilon}, \bar{u}) \rightarrow J(\bar{y}, \bar{u}), \;\; \varepsilon \rightarrow 0. \tag{2.21}$$

From (2.19), (2.21) we obtain inequalities

$$J(\bar{y}, \bar{u}) \leq \varliminf_{\varepsilon \rightarrow 0} J_{\varepsilon}(\bar{y}^{\varepsilon}, \bar{u}^{\varepsilon}),$$

$$\varlimsup_{\varepsilon \rightarrow 0} J_{\varepsilon}(\bar{y}^{\varepsilon}, \bar{u}^{\varepsilon}) \leq J(\bar{y}, \bar{u}),$$

which mean the fulfillment of (2.20). Besides, from (2.17) and (2.20) it follows that

$$\int_Q (\bar{u}^{\varepsilon})^2(t,x)dtdx \;\rightarrow \int_Q \bar{u}^2(t,x)dtdx,$$

and together with weak convergence this provides a strong convergence

$$\bar{u}^{\varepsilon} \to \ \overline{u} \text{ in } L^2(Q),\ \varepsilon \to 0. \tag{2.22}$$

Finally, from (2.15) and (2.20) we get the statement of the theorem.

□

Remark 1. The convergences (2.13) and (2.18) provide closeness not only for quality criteria but also for controls and phase variables in the following way:

$$\bar{u}^{\varepsilon} - \ u[t,x,y_{\varepsilon}] \ \to \ 0 \text{ in } L^2(Q), \ \ \varepsilon \to 0,$$
$$\bar{y}^{\varepsilon} - \ y_{\varepsilon} \to \ \ 0 \,\text{in}\, C\left([\delta,T]; L^2\left(\Omega\right)\right) \ \forall \delta > 0.$$

3. Conclusion

In this paper, we investigated the optimal control problem for a parabolic equation with rapidly oscillating coefficients and a special type cost functional with superposition or, another words, Nemyckii type operator. For good understanding we give an example of Nemyckii operator under specific conditions in this paper. We describe a problem with the homogenized coefficients, corresponding to the initial optimal control problem. Under some known facts on G-convergence theory and assuming that it already admits optimal synthesis form, we ground approximate optimal control in the feedback form for the initial problem.

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