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Approximate Identities and Strict Topologies.

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APPROXIMATE IDENTITIES AND
STRICT TOPOLOGIES

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
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Doctor of Philosophy

in

The Department of Mathematics

by
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ABSTRACT

In this paper we investigate certain generalizations of the strict topology and of topological measure theory. Although we are primarily interested in noncommutative C^* -algebras, we do devote some attention to linear spaces of vector-valued functions.

In Chapter I notation and terminology are established.

In Chapter II, we study approximate identities (in Banach algebras) which are interesting in themselves and also provide a useful tool for studying strict topologies and topological measure theory. Our main results in Chapter II have to do with well-behaved and β -totally bounded approximate identities in the function algebra $C_0(S)$. We show that $C_0(S)$ has a β -totally bounded approximate identity if and only if S is paracompact and that if S is connected or locally connected and $C_0(S)$ has a well-behaved approximate identity, then S is paracompact. We also define several other types of approximate identities and give conditions on Banach algebras necessary and/or sufficient that they possess one or more of these types.

Chapter III is the central chapter in our paper. In

the first section we consider the double centralizer algebra $M(A)$ of a C^* -algebra A with its strict topology β and state certain results obtained by R. C. Busby and D. C. Taylor. Among the results in the second section are these: (1) $M(A)_\beta$ is semireflexive if and only if A is dual; (2) $M(A)_\beta$ is nuclear if and only if A is finite-dimensional; (3) $M(A)_\beta$ has a β -compact unit ball if and only if A is a subdirect sum of finite-dimensional C^* -algebras; and (4) $M(A)_\beta$ is (DF) or (WDF) if and only if $\ell^\infty(A)$ is an essential A -module. In Section 3 of Chapter III, we define measure compactness of a C^* -algebra A and show that $*$ -homomorphic images of measure compact C^* -algebras are measure compact, that tensor products of measure compact C^* -algebras are measure compact and that subdirect sums of measure compact C^* -algebras are measure compact. We also show that algebras with countable approximate identities or series approximate identities (plus a cardinality condition) are measure compact and that β weak- $*$ compact subsets of positive linear functionals in the adjoint of $M(A)$, with the strict topology, are β -equicontinuous. We conclude Chapter III with a study of the Stone-Weierstrass theorem for $M(A)$ with the strict topology.

In Chapter IV we define topologies β_0 , β_1 , and β

for $C^*(X:E)$ and σ -additive, τ -additive, and tight linear functionals on $C^*(X:E)$ where X is a completely regular space and E is a normed space. We generalize some of the work of Sentilles by showing that the adjoint spaces of $C^*(X:E)$ with the topologies β_0 , β_1 and β are the spaces of tight, σ -additive and τ -additive linear functionals, respectively. We also give a vector measure characterization of tight linear functionals on $C^*(X:E)$ and show that several different definitions of tightness are equivalent. We conclude Chapter IV by showing that $C^*(X:E)_{\beta_0}$ has the approximation property when E has the metric approximation property and by computing the double centralizer algebras of the algebras $C^*(X:E)$ and $C_0(S:E)$ for E a C^* -algebra, X , a completely regular space, and S , a locally compact Hausdorff space.

INTRODUCTION

The algebra $C(X)$ of continuous functions on a compact Hausdorff space with the norm topology is a mathematical object that has fascinated many mathematicians for several decades. In 1958 R. C. Buck presented in [5] the results of his efforts to find a topology, which he called β , on $C^*(S)$, the bounded complex-valued functions, continuous on a locally compact space S , so that $C^*(S)_\beta$ would enjoy many of the topological vector space properties, as a class, that $C(X)$ does.

Buck's paper on β , the strict topology, was followed by many others on the strict topology and its generalizations. Wells in [66] computed the adjoint space of $C^*(S;E)_\beta$. Conway in [11-13] proved several important results. Perhaps his most interesting contribution was to the problem posed by Buck in [5]: when is $C^*(S)_\beta$ a Mackey space? Conway showed that paracompactness of S is a sufficient condition. This result was also obtained independently in [33].

In [7] Collins studied the space $\ell^\infty(S)_\beta$ and in [8] Collins and Dorroh obtained, among other things, the re-

sults that $C^*(S)_\beta$ has the approximation property and that $C_0(S)$ has a β -totally bounded approximate identity if S is paracompact. Dorroh in [18] studied an important localization property of β .

The strict topology was generalized in a noncommutative direction by R. C. Busby [6] in his study of the double centralizer algebra $M(A)$ of a C^* -algebra A . D. C. Taylor continued this study [57-59] with computations of the dual space of $M(A)$ with its strict topology and extensions of Phillips' Theorem [14, p. 32] and Conway's result on the Mackey problem being only a few of the nice results he obtained.

The strict topology was studied in a Banach module setting in [51,52].

Several authors extended the strict topology to $C^*(X)$ for X completely regular and not necessarily locally compact. Some of this work is to be found in [21,22,27,50,53-56,61,67]. The work of Sentilles is especially important, in our opinion, as he connected the strict topology with topological measure theory as initiated in [62] and subsequently studied in [21,29,30,31,35-37].

The main problem motivating this thesis was to study noncommutative C^* -algebras without identity, particularly in the strict topology context. Approximate identities in

Banach algebras are studied since they seem to be a useful tool for questions about $M(A)_\beta$ and are interesting in their own right.

We are particularly interested in well-behaved approximate identities because of Taylor's result that $M(A)_\beta$ is a strong Mackey space if the C^* -algebra A has a well-behaved approximate identity. We were also interested in the question posed in [8]: does the existence of a β -totally bounded approximate identity for $C_0(S)$ imply that S is paracompact? We answer this question in the affirmative and also show that if S is connected or locally connected and $C_0(S)$ has a well-behaved approximate identity, then S is paracompact.

Returning to the study of $M(A)$, for a C^* -algebra A , we first study topological vector space properties of $M(A)_\beta$ ($M(A)$ with the strict topology). Among our results are the following: $M(A)_\beta$ is semireflexive if and only if A is dual; $M(A)$ has a β -compact unit ball if and only if A is a subdirect sum of finite-dimensional C^* -algebras; $M(A)_\beta$ is nuclear if and only if A is finite-dimensional; and $M(A)$ is (DF) or (WDF) if and only if $\ell^\infty(A)$ is essential (generalizing a result in [54]).

We next generalize topological measure theory to a C^* -algebra context by defining a notion of measure compact-

ness for C^* -algebras. Using this notion, we show that C^* -algebras with countable or series approximate identities (plus a cardinality condition) are measure compact. We also obtain some permanence properties of measure compactness as we show that C^* -tensor products of measure compact C^* -algebras are measure compact and $*$ -homomorphic images of measure compact C^* -algebras are measure compact. Finally, we show that weak- $*$ compact subsets of positive linear functionals in the adjoint space of $M(A)_\beta$ are β -equicontinuous. Chapter III concludes with some partial results on the Stone-Weierstrass Theorem for $M(A)_\beta$.

In Chapter IV the linear space $C^*(X:E)$ is studied with the topologies β_0, β and β_1 which we defined so as to extend definitions in [50]. We define the notions of σ -additivity, τ -additivity, and tightness for bounded linear functionals on $C^*(X:E)$ and compute dual spaces, showing that the adjoint of $C^*(X:E)$ with topology $\beta(\beta_0, \beta_1)$ is the space of τ -additive (tight, σ -additive) functionals. We characterize tight functionals in several ways and give a vector measure representation for tight functionals. Two other results in Chapter IV are a characterization of the double centralizer algebra of $C^*(X:E)$ (for a C^* -algebra E) and a proof that $C^*(X:E)_{\beta_0}$ has the approximation property if E has the metric approximation property.

We list several unsolved problems to conclude the thesis.

We conclude with some remarks on our numbering convention. Chapter I has only one section so items are numbered consecutively, e.g., 1·x denotes item x in Chapter I. In Chapters II, III, and IV items are numbered in the form a·b·c, where item a·b·c is item c in Section b of Chapter a.

CHAPTER I
PRELIMINARIES

The purpose of this chapter is to present some facts and results which we hope will facilitate reading of this thesis. Many of the terms used in the text are defined here and nowhere else in the text. For this reason, familiarity with the contents of Chapter I is helpful.

TOPOLOGY AND CONTINUOUS FUNCTIONS

Our standard references for general topology are [19,28]. Throughout this paper S will always denote a locally compact Hausdorff space and X a completely regular Hausdorff space. A covering of a topological space is locally finite if each point in the space has a neighborhood which meets only finitely many elements of the covering. A Hausdorff space Z is called paracompact if each open cover of Z has an open, locally finite refinement. The next theorem gives a useful characterization of locally compact, paracompact Hausdorff spaces.

1.1 THEOREM [19, p. 107]. The space S is paracompact

iff S is the union of a pairwise disjoint collection of open and closed σ compact subspaces. The space of ordinals less than the first uncountable ordinal, with the order topology, is not paracompact.

Note the use of 'iff' for 'if and only if' in the statement of the theorem. We shall follow this convention in the sequel.

Let $C^*(X)$ denote the space of bounded continuous scalar-valued functions on X . 'Scalar' refers to either the real numbers or the complex numbers and we shall make it clear at each point which meaning we are using. A function f mapping X into the scalar field vanishes at infinity if $\{x \in X : |f(x)| \geq \epsilon\}$ is a subset of a compact subset for each $\epsilon > 0$. A real-valued function f is called upper semicontinuous if $\{x : f(x) < a\}$ is open for each real number a . Let $C_0(X)$ denote the elements of $C^*(X)$ which vanish at infinity and $C_c(X)$ denote the subset of $C^*(X)$ consisting of functions which vanish outside a compact subset of X .

LOCALLY CONVEX SPACES

Our general reference here is [45]. Let E be a vector space (real or complex). A seminorm on E is a

function p from E into the nonnegative reals satisfying $p(0) = 0$, $p(ax) = |a|p(x)$ for scalars a and $x \in E$, and $p(x+y) \leq p(x) + p(y)$ for $x, y \in E$. A locally convex topological vector space (locally convex space, for short) is a vector space E with a topology T , which has a family P of seminorms so that the sets $\{x \in E: p(x) \leq \epsilon\}$ for $p \in P$ and $\epsilon > 0$ form a subbasis for the neighborhood system in T of the origin. The space of continuous linear functionals on E is called the adjoint (adjoint space) of E and denoted E' .

The weak-* topology on E' is that topology determined by the seminorms $f \mapsto |f(x)|$, for $x \in E$ and $f \in E'$. The adjoint of E' with the weak-* topology is E . The weak-* topology on E , thinking of it as the adjoint of E' , is called the weak topology.

We assume familiarity with the Hahn-Banach theorem, the open mapping theorem, and the uniform boundedness principle as presented in [45]. We also assume a knowledge of polar sets. If E is a locally convex space and $A \subseteq E$, the polar of A in E , denote A^0 , is $\{f \in E': |f(a)| \leq 1, \forall a \in A\}$. The strong topology on E' is the topology of uniform convergence on bounded sets of E , i.e., a net $\{f_a\} \subset E'$ converges to 0 in the strong topology iff for every bounded set $B \subseteq E$,

$\{f_a\}$ is eventually in B^0 . The second adjoint of E , denoted E'' , is the adjoint of E' with the strong topology. If for $x \in E$, we define the element F_x in E'' by means of the equation $F_x(f) = f(x)$ for $f \in E'$, the map $x \rightarrow F_x$ is a one-to-one vector space homomorphism of E into E'' . In general, this map is neither onto nor continuous when E has its initial topology and E'' has its strong topology. If the map is onto, E is called semi-reflexive; if it is onto and continuous, E is called reflexive.

1.2 THEOREM. A space E is semi-reflexive iff every bounded weakly closed subset of E is weakly compact.

A set $H \subseteq E'$ is equicontinuous if H^0 is a zero neighborhood in E . Every equicontinuous set has weak-* compact closure, but the converse fails in general.

An important topology for a locally convex space E is the Mackey topology. This is the topology of uniform convergence on weak-* compact convex circled subsets of E' . The Mackey topology is the finest locally convex topology on E for which E' is the adjoint space [45]. We shall call E a Mackey space if its topology is the Mackey topology, or equivalently, if every weak-* compact

convex circled subset of E' is equicontinuous. We shall call E a strong Mackey space if every weak-* compact (but not necessarily convex and circled) subset of E' is equicontinuous. Some authors have used the term 'strong Mackey space' to mean that every weak-* countably compact subset of E' is equicontinuous, e.g., see [12].

MEASURE THEORY

We shall discuss some measure theory concepts for locally compact Hausdorff spaces here; more general measure theory will be summarized in Chapter IV. Let S be a locally compact space and B denote the sigma algebra generated by the closed subsets of S . B is called the Borel sigma algebra. Our reference for measure theory on locally compact Hausdorff spaces is [47].

1.3 THEOREM [47]. Let L be a bounded linear functional on $C_0(S)$. Then there exists a unique Borel measure μ so that $L(f) = \int f d\mu$ for all $f \in C_0(S)$. Furthermore, $\|L\| = \|\mu\|$.

BANACH ALGEBRAS

Our general references for this section are [17,38]. A Banach algebra is a complex Banach space A with a

multiplication satisfying $\|ab\| \leq \|a\| \|b\|$, where $\|x\|$ denotes the norm of the element x in A .

1.4 DEFINITION. Let A be a Banach algebra. A net $\{e_a\} \subset A$ is called an approximate identity for A if: (i) $\|e_a\| \leq 1 \forall a$ and (ii) $\lim_a \|xe_a - x\| = 0 \forall x \in A$.

1.5 DEFINITION. Let A be a Banach algebra. A is said to be a *-algebra or an algebra with involution if there is a linear map $*:A \rightarrow A$ which satisfies the following conditions (where we denote by x^* the image of the element $x \in A$ under the map $*$):

- (1) $\|x^*\| = \|x\|$ (2) $(xy)^* = y^*x^*$ for all $x, y \in A$
 (3) $x^{**} = x \forall x \in A$ (4) $(\lambda a)^* = \bar{\lambda}a^*$ for any complex number λ and $a \in A$ ($\bar{\lambda}$ denotes the complex conjugate of λ).

If A is a Banach algebra, A can be imbedded naturally in a Banach algebra with identity. Let A_1 denote the set of ordered pairs (a, t) where $a \in A$ and t is a complex number. Define a norm on A_1 by $\|(a, t)\|_1 = \|a\| + |t|$. If A is a *-algebra, A_1 can be made into a *-algebra by defining $(a, t)^* = (a^*, \bar{t})$.

Define addition and scalar multiplication coordinatewise.

If (a, t) and $(a_1, t_1) \in A_1$ define the product

$$(a, t)(a_1, t_1) = (aa_1 + ta_1 + t_1a, tt_1).$$

Note that A_1 is a Banach algebra with identity containing A isometrically

and isomorphically and that A_1 is a $*$ -algebra if A is,

with the $*$ -operation on A_1 extending that on A . Let

I denote the identity in A_1 . If $a \in A$, the spectrum

of a , denote $S_p(x)$ is the set of complex numbers λ such that $x - \lambda I$ is not invertible in A_1 .

Let A be a $*$ -algebra and $x \in A$. Then x is called hermetian if $x = x^*$ and positive if it is hermetian and $S_p(x)$ is a subset of the real numbers.

1.6 DEFINITION. Let A be a $*$ -algebra. A is called a C^* -algebra if $\|x\|^2 = \|x^*x\|$ for all $x \in A$. The term B^* -algebra will also be used interchangeably with C^* -algebra in this thesis.

Since this thesis is mainly about C^* -algebras, we shall give several examples and state certain structure theorems we will use. Two examples of C^* -algebras are the algebra $C(X)$ of continuous functions on a compact Hausdorff space X and the algebra $B(H)$ of bounded linear operators on a Hilbert space H .

Let A and B be two $*$ -algebras. A $*$ -homomorphism f is an algebra homomorphism that satisfies $f(a^*) = (f(a))^*$ for all $a \in A$.

The following structure theorems are needed in the sequel [17].

1.7 THEOREM. Let A be a commutative C^* -algebra without identity. Then there is a locally compact Hausdorff space S so that A is isometrically $*$ -isomorphic to $C_0(S)$, i.e., there is a $*$ -homomorphism h mapping A onto $C_0(S)$ such that $\|h(a)\| = \|a\|$ for all $a \in A$.

1.8 THEOREM. Let A be a commutative C^* -algebra with identity. Then A is isometrically $*$ -isomorphic to the algebra $C(M)$ of all continuous functions on some compact Hausdorff space M .

1.9 THEOREM. Let A be a C^* -algebra (not necessarily commutative). Then A is isometrically $*$ -isomorphic with a norm closed self-adjoint subalgebra of $B(H)$, the bounded linear operators on some Hilbert space H .

Let A be a Banach $*$ -algebra. A linear functional f on A is said to be positive if $f(x^*x) \geq 0 \quad \forall x \in A$.

1.10 PROPOSITION (Cauchy-Schwarz Inequality). If f is a positive linear functional on a Banach $*$ -algebra A_1 then $|f(ab)|^2 \leq f(a^*a)f(b^*b) \quad \forall a, b \in A$ [38, p. 187].

We shall need one more theorem about C^* -algebras.

1.11 THEOREM [17]. Every C^* -algebra A has a bounded approximate identity. The adjoint space is algebraically spanned by the positive linear functionals (every positive linear functional on A is continuous).

FACTORIZATION THEOREMS

Let A be a Banach algebra. A Banach space V is called a left A -module if there is a mapping from $A \times V$ into V , whose value at the pair (a, v) in $A \times V$ is denoted $a \cdot v$, satisfying the conditions that $a \cdot v$ is linear in a for fixed v and linear in v for fixed a and $(ab) \cdot v = a \cdot (b \cdot v)$ for $a, b \in A$ and $v \in V$. The left A -module V is said to be isometric if $\|a \cdot v\| \leq \|a\| \|v\|$ for all $a \in A$ and $v \in V$. Suppose that A has a bounded approximate identity $\{e_a\}$. The left A -module V is called essential if $\|e_a \cdot v - v\| \xrightarrow{a} 0$ for all $v \in V$.

The next theorem is very important in our work.

1.12 THEOREM [26] . Let A be a Banach algebra with a bounded approximate identity and V an isometric and essential left A -module. Let $\epsilon > 0$ and $x \in V$. Then $\exists a \in A$ and $y \in V$ such that:

- (i) $x = a \cdot y$
- (ii) $\|y-x\| < \epsilon$
- (iii) $\|a\| \leq 1$
- (iv) y belongs to the closure of $\{a \cdot z \mid a \in A \text{ and } z \in V\}$.

It is pointed out in [26] that if the approximate identity is contained in a closed cone in A , then the element a in 1.12 may be chosen from C . For example, it is known that the positive elements of a C^* -algebra form a closed cone and that a C^* -algebra has an approximate identity consisting of positive elements. Thus if A is a C^* -algebra the element a in 1.12 can be taken to be a positive element of A .

CHAPTER II
APPROXIMATE IDENTITIES AND THE
STRICT TOPOLOGY

The purpose of this thesis, as stated in the abstract, is an attempt to generalize the theory of commutative C^* -algebras to a noncommutative setting. A particular problem along this line is the question of what is to substitute for the maximal ideal space [38] in the noncommutative case. As explained in Section 2 of this chapter, the approximate identity has proven to be a useful device for a large class of problems, particularly those connected with generalizations of the theory of the strict topology (defined in Section 1).

Section 1 of Chapter II contains preliminary material relating to the strict topology. In Sections 2-7 relationships between the existence of certain types of approximate identities on a C^* -algebra and other properties of the algebra are studied. In Section 8, we study a property of C^* -algebras which arose from trying to decide whether every C^* -algebra has a canonical approximate identity. The most interesting results in Chapter II are in Section 3

because they are related to the important Mackey problem posed by Buck [5].

SECTION 1. THE STRICT TOPOLOGY

In this chapter, S will always denote a locally compact Hausdorff space. In this section, we present definitions and theorems pertinent to the strict topology, which was defined and first studied by R. C. Buck [5].

2.1.1 DEFINITION. The strict topology on $C^*(S)$ is that locally convex topology given by the seminorms $f \rightarrow \|f\|_\phi$ for $f \in C^*(S)$ and $\phi \in C_0(S)$. Note that we may consider only nonnegative ϕ and that the sets $V_\phi = \{f \in C^*(S): \|f\|_\phi \leq 1\}$ actually form a neighborhood basis at the origin. In terms of nets, a net $\{f_b\}$ in $C^*(S)$ converges to zero in the strict topology iff $f_b\phi \rightarrow 0$ in the norm topology for each $\phi \in C_0(S)$. The strict topology is denoted β .

2.1.2 DEFINITION. Another important topology for $C^*(S)$ is the compact open topology, denoted $C - Op$. The seminorms are $f \rightarrow \|f\|_K$ for $f \in C^*(S)$ and K a compact subset of S ($\|f\|_K = \sup \{|f(x)|: x \in K\}$). Clearly the seminorms $f \rightarrow \|f\|_\phi$, for $\phi \in C_0(S)$, also determine the compact open topology. Thus β is a finer topology

that $C - Op$ since it has a larger family of seminorms. A net $\{f_b\}$ in $C^*(S)$ converges to zero in the compact open topology (denoted $f_b \rightarrow 0 C - Op$) iff $f_b \rightarrow 0$ uniformly on all compact subsets of S .

The following results on the strict topology are due to Buck [5].

2.1.3 THEOREM. (a) The norm topology and β on $C^*(S)$ agree iff S is compact.

(b) $C^*(S)_\beta$ ($C^*(S)$ with topology β) is complete.

(c) β is metrizable iff S is compact.

(d) A set is β -bounded iff it is norm bounded.

(e) On bounded subsets of $C^*(S)$ $\beta = C - Op$.

(f) $C_c(S)$ is β -dense in $C^*(S)$.

2.1.4 THEOREM. Let L be a β -continuous linear functional on $C^*(S)$. Then there is a unique regular Borel measure μ on S so that $L(f) = \int f d\mu$ for $f \in C^*(S)$. Conversely, if μ is a regular Borel measure on S and $T(f) = \int f d\mu$ for $f \in C^*(S)$, then T is a β -continuous linear functional on $C^*(S)$.

It follows from 2.1.4 that β -continuous complex homo-

morphisms of $C^*(S)$ are given by evaluation at points of S . This fact, 2.1.4, and other results below show that $C^*(S)_\beta$ has many of the same properties as does $C^*(X)$ with its norm topology, where X is a compact Hausdorff space. In [5] Buck posed the question: When is $C^*(S)_\beta$ a Mackey space? A partial answer was given by Conway [12]. Theorem 2.1.5 - 2.1.9 are due to him.

2.1.5 THEOREM. Let $H \subseteq C^*(S)_\beta$. The following statements are equivalent:

(a) H is uniformly bounded and, for every $\epsilon > 0$, there is a compact set $K \subseteq S$ so that $|\mu|(SNK) < \epsilon$ for all $\mu \in H$.

(b) H is β -equicontinuous.

(c) H is uniformly bounded and for every net $\{f_a\}$ in $C_c(S)$ such that $f_a \rightarrow 0$ C -Op and $\|f_a\| \leq 1$ for all a , $f_a \rightarrow 0$ uniformly on H .

2.1.6 THEOREM. Let S be paracompact. If H is a β -weak-* countably compact subset of $M(S)$, then H is β -equicontinuous. Hence $C^*(S)_\beta$ is a strong Mackey space.

A crucial fact used in proving 2.1.6 is this result of Conway:

2.1.7 THEOREM. If S is the space of positive integers with the discrete topology and $H \subseteq \ell^1 = M(S)$, then the following are equivalent:

- (a) H is weakly conditionally compact;
- (b) H is β -weak-* conditionally compact;
- (c) H is norm conditionally compact;
- (d) H is β -equicontinuous.

Conway's next result is a generalization of the classical Ascoli theorem.

2.1.8 THEOREM. If $F \subseteq C^*(S)$ then the following are equivalent:

- (a) F is β conditionally compact;
- (b) F is uniformly bounded and $C - Op$ conditionally compact;
- (c) F is uniformly bounded and for every compact set $K \subseteq S$, $F|_K = \{f|_K : f \in F\}$ is norm conditionally compact in $C^*(K)$ ($f|_K$ denotes the restriction of f to K);
- (d) F is uniformly bounded and an equicontinuous family.

2.1.9 EXAMPLE. Let X denote the space of ordinals less than the first uncountable ordinal with the order topology. $C^*(X)_\beta$ is not a Mackey space.

We shall now mention some of the work of Collins [7] and Collins and Dorroh [8] on the strict topology. The work of several other authors will be considered in Chapters III and IV of this thesis.

2.1.10 THEOREM. The following are equivalent for a locally compact space S :

- (a) S is discrete;
- (b) $C^*(S)_\beta$ is semi-reflexive;
- (c) each bounded subset of $C^*(S)_\beta$ is precompact.

2.1.11 THEOREM. The following conditions are equivalent:

- (a) $C^*(S)_\beta$ is nuclear [49];
- (b) $C^*(S)_\beta$ is semi-reflexive and every unconditionally convergent series in $C^*(S)_\beta$ is absolutely convergent;
- (c) S is finite, i.e., $C^*(S)$ is finite-dimensional.

Results 2.1.10 and 2.1.11 are in [7]. Collins and Dorroh proved 2.1.12 - 2.1.14 in [8].

2.1.12 THEOREM. $C^*(S)_\beta$ has the approximation property [49].

2.1.13 THEOREM. The following are equivalent for locally

compact S :

- (a) S is σ compact;
- (b) $C_0(S)$ has a sequential approximate identity;
- (c) $C_0(S)$ has an approximate identity whose range is a countable subset of $C^*(S)$.

2.1.14 THEOREM. If S is paracompact, $C_0(S)$ has a canonical approximate identity whose range is totally bounded in $C^*(S)_\beta$.

SECTION 2. INTRODUCTION TO APPROXIMATE IDENTITIES.

The problem of extending theorems about commutative B^* algebras to the non-commutative case has received a great deal of attention in recent years. Because many proofs made in the commutative case make use of the spectrum (= maximal ideal space), an obvious question is: what is to replace this device in the case of a non-commutative B^* algebra? Various possible replacements have been sought; e.g., see Akemann [2] and Pedersen [41,42]. Much progress has been made for certain types of problems by means of restrictions on approximate identities for the algebra in question by Taylor [57,59], Akemann [3], and others. The class of problems solved or seemingly susceptible to this technique is rather large. This fact and the paucity of

results for this class of problems obtained by studying Prim A [17] and the space of equivalence classes of irreducible representations suggest that the approximate identity is a useful tool for extending many commutative theorems to a non-abelian setting. A question that arises immediately in the case of a commutative B*-algebra is: what do restrictions on the approximate identity imply about the spectrum of A and vice versa? Along this line, Collins-Dorroh [8] characterize σ compactness of the spectrum and ask for necessary and sufficient conditions on S that $C_0(S)$ (in this chapter, S always denotes a locally compact Hausdorff space) have an approximate identity that is totally bounded in the strict topology. This portion of this thesis answers this question and several related ones, including some in the non-commutative context.

2.2.1 DEFINITION. Let A be a Banach algebra. An approximate identity for A is a net $\{e_\lambda \mid \lambda \in \Lambda\}$ (we generally write simply $\{e_\lambda\}$) with $\lim_{\lambda} \|e_\lambda x - x\| = \lim_{\lambda} \|x e_\lambda - x\| = 0$ for $x \in A$ and $\|e_\lambda\| \leq 1$ for all λ . It is well known that all B*-algebras have approximate identities [17].

2.2.2 DEFINITION. The double centralizer algebra $M(A)$

of a B*-algebra A was studied by R. C. Busby [6] who defined the strict topology as that topology on $M(A)$ generated by the seminorms $x \rightarrow \max \{ \|xy\|, \|yx\| \}$ for $x \in M(A)$ and $y \in A$. Two motivating examples for the double centralizer algebra concept are the algebra $C_0(S)$ of continuous complex functions on S which vanish at infinity (this class is identical with the class of all commutative C*-algebras by the theorem of Gelfand), whose double centralizer algebra was identified by Wang [63] as $C^*(S)$, the algebra of all bounded continuous complex functions on S ; and the algebra of compact operators on a Hilbert space H , whose double centralizer algebra was shown to be the bounded linear operators on H by Busby. For a definition of $M(A)$ and some of its properties, the reader is referred to Chapter III of this paper. By $M(A)_\beta$ we shall mean $M(A)$ endowed with the strict topology β .

2.2.3 DEFINITION. If $f \in C^*(S)$, the support of f , $\text{spt } f$, is the closure in S of $\underline{N(f)} = \{x: f(x) \neq 0\}$.

2.2.4 DEFINITION. S is sham compact if each σ compact subset is relatively compact.

2.2.5 DEFINITION. Let A be a B^* -algebra and $\{e_\lambda\}$ be an approximate identity for A . We shall be interested in the following conditions:

- (a) $\{e_\lambda\}$ is countable, i.e., the range of $\{e_\lambda\}$ is a countable set;
- (b) $\{e_\lambda\}$ is sequential, i.e., Λ is the set of positive integers with the usual order;
- (c) $\{e_\lambda\}$ is canonical, i.e., $e_\lambda \geq 0$ and if $\lambda_1 < \lambda_2$ then $e_{\lambda_1} e_{\lambda_2} = e_{\lambda_1}$;
- (d) $\{e_\lambda\}$ is well-behaved (after Taylor [57]), i.e., $\{e_\lambda\}$ is canonical and if $\lambda \in \Lambda$ and $\{\lambda_n\}$ is a strictly increasing sequence in Λ , there is a positive integer N so that $e_\lambda e_{\lambda_n} = e_\lambda e_{\lambda_m}$ for $n, m > N$;
- (e) $\{e_\lambda\}$ is β totally bounded, i.e., totally bounded in the strict topology;
- (f) $\{e_\lambda\}$ is abelian;
- (g) $\{e_\lambda\}$ is chain totally bounded, i.e., if $\{\lambda_n\}$ is an increasing sequence in Λ , then $\{e_{\lambda_n}\}$ is β totally bounded;
- (h) $\{e_\lambda\}$ is $\sigma(M(A), M(A)_\beta^*)$ relatively compact, where σ denotes the weak topology on $M(A)$ in the pairing with its β dual;
- (i) $\{e_\lambda\}$ is sham compact, i.e., $\{e_\lambda\}$ is canonical and if $\{\lambda_n\}$ is a sequence in Λ , then there is a λ in Λ

so that $\lambda > \lambda_n$ for all integers n .

2.2.6 REMARK. A sequence $\{e_n\}$ in a B*-algebra A which satisfies $\lim_n \|e_n x - x\| = \lim_n \|x e_n - x\| = 0$ is norm bounded by the uniform boundedness principle and the B*-norm property. Thus it is not necessary to require norm boundedness in 2.2.1 for this case.

2.2.7 REMARK. Taylor [57] introduced the notion of a well-behaved approximate identity and used it to prove many interesting improvements of results of Phillips [14, p. 32], Akemann [3], Bade [4], Collins-Dorroh [8], and Conway [10,12,13].

SECTION 3. A CHARACTERIZATION OF PARACOMPACT SPACES.

Our main result in this section, 2.3.10, answers two questions posed in [8]. Our interest centers exclusively on B*-algebras without identity; for these, we need information about increasing sequences in the directed set of an appropriate identity and about supports. Lemmas 2.3.1 and 2.3.2 provide what we need.

2.3.1 LEMMA. If A is a Banach algebra without identity, $\{e_\lambda\}$ an approximate identity for A , and $\lambda_0 \in \Lambda$, then

$\exists \lambda \in \Lambda$ so that $\lambda > \lambda_0$.

Proof. If the conclusion does not hold, then $\forall \lambda \in \Lambda$, $\lambda \leq \lambda_0$, from which it follows that e_{λ_0} is an identity for A .

2.3.2 LEMMA. Let $\{e_\lambda\}$ be an approximate identity for $C_0(S)$.

(a) If $\{e_\lambda\}$ is canonical, then $\lambda_1 < \lambda_2$ implies $\text{spt } e_{\lambda_1} \subset e_{\lambda_2}^{-1} \{1\} \subset N(e_{\lambda_2})$ and $\lambda \in \Lambda$ implies that the $\text{spt } e_\lambda$ is compact;

(b) If K is a compact subset of S , then $\exists \lambda \in \Lambda$ so that $|e_\lambda| > \frac{3}{4}$ on K .

Proof. This is straightforward.

We are mainly interested (in Section 3) in two types of approximate identities, viz., well-behaved ones, shown to be important by Taylor [57], and β totally bounded ones, the study of which motivated this section.

2.3.3 LEMMA. Let $\{e_\lambda\}$ be an approximate identity for $C_0(S)$ which is either β totally bounded or well-behaved. Then there exists a cover of S by clopen σ compact

sets.

2.3.4 REMARK. All topologies between the compact open and the strict agree on norm bounded sets. Thus 'β totally bounded' may be replaced in 2.3.3 by 'compact open totally bounded.'

Proof. We assume that S is not compact in either case to avoid trivialities. Assume first that $\{e_\lambda\}$ is β totally bounded. Replacing $\{e_\lambda\}$ by $\{|e_\lambda|^2\}$, performing a straight forward computation and using 2.3.4, we may assume that $\{e_\lambda\}$ is compact open totally bounded and $e_\lambda \geq 0$ for each λ . Let $x \in X$ and choose by 2.3.2 (b) $\lambda_1 \in \Lambda$ so that $e_{\lambda_1}(x) > \frac{3}{4}$. Let $K_1 = \{x \in S : e_{\lambda_1}(x) \geq \frac{1}{4}\}$. Suppose that $\{K_j\}$ $j = 1, \dots, n$ and $\{\lambda_j\}$ $j = 1, \dots, n$ have been chosen so that

$$(1) \quad e_{\lambda_j} > \frac{3}{4} \text{ on } K_{j-1} \quad j = 1, \dots, n$$

$$(2) \quad K_j = \{x \in S : e_{\lambda_i}(x) \geq \frac{1}{4^j} \text{ for some } i, 1 \leq i \leq j\}$$

By 2.3.2 (b) again, choose $\lambda_{n+1} \in \Lambda$ so that $e_{\lambda_{n+1}} > \frac{3}{4}$ on K_n and let $K_{n+1} = \{x \in S : e_{\lambda_i}(x) \geq \frac{1}{4^{n+1}} \text{ for some } i, 1 \leq i \leq n+1\}$. By induction we obtain sequences $\{\lambda_n\}$ and $\{K_n\}$ satisfying (1) and (2) above. Let $X = \bigcup_n K_n$.

X is clearly σ compact and contains x . It is open since $K_n \subset \text{interior of } K_{n+1}$. To show that X is closed, take a compact set K . It suffices to show $K \cap X$ is closed [28, p. 231]. The total boundedness condition of $\{e_\lambda\}$ gives the existence of an integer i_0 so that for all positive integers j ,

$$(3) \quad \min_{1 \leq i \leq i_0} \|e_{\lambda_j} - e_{\lambda_i}\|_K < \frac{1}{4}$$

$$(\|f\|_K = \sup_{x \in K} |f(x)| \text{ for } f \in C_b(S)).$$

Let $y \in K_m \cap K$ where $m > i_0$.

By construction $e_{\lambda_{m+1}}(y) > \frac{3}{4}$ so by (3) there is an integer $1 \leq i \leq i_0$ so that $e_{\lambda_i}(y) \geq \frac{1}{2}$ which shows that

$y \in K_i$. Thus $X \cap K = K \cap \bigcup_{i=1}^{i_0} K_i$ so $X \cap K$ is closed.

For the other part of the lemma, let $x \in X$, assume that $\{e_\lambda\}$ is well-behaved, and choose by 2.3.1 and 2.3.2 an increasing sequence $\{\lambda_n\}$ so that $e_{\lambda_1}(x) > 0$. Let $K_n = \text{spt } e_{\lambda_n}$ and note, by 2.3.2, that $K_n \subset \text{interior of } K_{n+1}$. Let $X = \bigcup K_n$ and note that X is open, σ compact and $x \in X$. From 2.3.2 (a) and the definition of well-behaved approximate identity, it follows that $\{e_{\lambda_i}\}$ is totally bounded in the compact open topology and that $y \in X$ implies $e_{\lambda_j}(y) = 1$ for j large enough. With

these observations, the proof that X is closed is the same as in the first part of the lemma.

2.3.5 REMARK. Note that in 3.3, $\bigcup_{n=1}^{\infty} \text{spt } e_{\lambda_n} \subset X$.

2.3.6 COROLLARY. If S is connected and has an approximate identity that is either well-behaved or β totally bounded, then S is σ compact.

2.3.7 PROPOSITION. Let F be a closed subset of S . If $C_0(S)$ has either a well-behaved or a β totally bounded approximate identity, then F contains a σ compact set that is relatively clopen in F .

Proof. Let $\{e_\lambda\}$ be an approximate identity with either of the properties above. For $\lambda \in \Lambda$, let d_λ be the restriction of e_λ to F . Since F is closed, $\{d_\lambda\} \subset C_0(F)$. We claim that $\{d_\lambda\}$ has the same property as $\{e_\lambda\}$ does; i.e., that $\{d_\lambda\}$ is a well-behaved (resp. β totally bounded) approximate identity for $C_0(F)$. To show this, it suffices to show that if $f \in C_0(F)$, then there is an extension g in $C_0(S)$ of f . Let S^* denote the one-point compactification of S and ∞ denote the point at infinity. Let f' be an extension of f to $F \cup \{\infty\}$ ob-

tained by defining $f'(\infty) = 0$. Since $f \in C_0(F)$, f' is continuous and extends to a continuous function p on all of S^* by Tietze's Theorem since $F \cup \{\infty\}$ is closed in S^* . The restriction g of p to S is clearly an extension of f in $C_0(S)$. This concludes the proof of 2.3.7.

2.3.8 COROLLARY. If S is locally connected and $C_0(S)$ has an approximate identity that is either well-behaved or β totally bounded, then S is paracompact.

Proof. By 1.1, it suffices to show that S is a disjoint union of clopen σ compact subspaces. In a locally connected space, the components are clopen and connected and so σ compact by 2.3.7.

2.3.9 LEMMA. Suppose that $C_0(S)$ has a β totally bounded approximate identity and let \mathcal{K} be the family of all clopen σ compact subsets of S constructed by the method of the first part of 2.3.3. If $\mathcal{U} \subset \mathcal{K}$, then

$$\bigcup_{W \in \mathcal{U}} W \text{ is clopen.}$$

Proof. We may assume $e_\lambda \geq 0$ as in 2.3.3. Let $X = \bigcup_{W \in \mathcal{U}} W$ and K be an arbitrary compact subset of S . Since S

is locally compact, it suffices to show that $X \cap K$ is closed. With each W in \mathcal{U} is associated a sequence $\{e_n^W\}$ from the approximate identity such that $\bigcup_{n=1}^{\infty} \text{spt } e_n^W \subset W$ (see 2.3.4) and if $y \in W$ $e_n^W(y) > \frac{3}{4}$ for n large enough. From β total boundedness of $\{e_n^W : W \in \mathcal{U}, n = 1, 2, \dots\}$, we get a set $\{W_i\}_{i=1, \dots, n}$ from \mathcal{U} and associated integers $\{n_i\}_{i=1, \dots, n}$ so that for any V in \mathcal{U} and positive integer p

$$(4) \quad \min_{1 \leq i \leq n} \| e_{n_i}^{W_i} - e_p^V \|_K < \frac{1}{4} .$$

If $y \in X \cap K$, then $y \in K \cap W$ for some $W \in \mathcal{U}$, so choosing p large enough so that $e_p^W(y) > \frac{3}{4}$ we see that $e_{n_i}^{W_i}(y) > 0$ for some $1 \leq i \leq n$ so that $y \in W_i$. We have established that $X \cap K = K \cap \bigcup_{i=1}^n W_i$ so $X \cap K$ is closed. This concludes the proof of 2.3.9.

In [8] Collins and Dorroh show that if S is paracompact then $C_0(S)$ has a β totally bounded approximate identity and ask two questions: (1) Does the existence of a β totally bounded approximate identity imply the existence of a canonical one that is β totally bounded? And (2) does the existence of a β totally bounded approximate identity in $C_0(S)$ imply that S is paracompact? We add to these a third question: Does the existence of a

β totally bounded approximate identity in $C_0(S)$ imply the existence of a well-behaved one? The answer to all these questions is given in 2.3.10.

2.3.10 THEOREM. These are equivalent: (1) S is paracompact; (2) $C_0(S)$ has a canonical approximate identity that is β totally bounded; (3) $C_0(S)$ has an approximate identity that is β totally bounded.

Proof. For the first implication see [8]. Since the second implication is trivial, we prove only that if $\{e_\lambda\}$ is a β totally bounded approximate identity for $C_0(S)$ then S is paracompact. Take \mathcal{W} to be the set in 2.3.9 and well order it. Let W_0 be the first element in \mathcal{W} and $W'_0 = W_0$. If $W \in \mathcal{W}$, and $W \neq W_0$, let

$$W' = W \setminus \bigcup_{\substack{V < W \\ V \in \mathcal{W}}} V.$$

Each set W' is clopen and σ compact by 2.3.3 and 2.3.9.

If $x \in S$ and W is the least element in $\{W: W \in \mathcal{W} \text{ and } x \in W\}$, then x clearly belongs to W' . The collection $\{W': W \in \mathcal{W}\}$ then consists of disjoint sets and so forms a partition of S by clopen σ compact subsets. We apply 1.1 to conclude the proof.

SECTION 4. NON-COMMUTATIVE RESULTS AND EXAMPLES.

Taylor [57] gives the following examples of B^* -algebras with well-behaved approximate identities: algebras with countable approximate identities, algebras with series approximate identities (for a definition, see Ake-mann [3]) such as the compact operators on a Hilbert space, and subdirect sums of algebras having well-behaved approximate identities, such as dual B^* -algebras which are subdirect sums of algebras of compact operators.

In this section, we give examples of algebras with β totally bounded approximate identities using some techniques borrowed from Taylor and some of our own. We also give some partial results, e.g., 2.4.1, relating the existence of approximate identities of one type to existence of another type.

2.4.1 PROPOSITION. Let A be a Banach algebra with a sequential canonical approximate identity $\{e_n\}$. Then $\{e_n\}$ is β totally bounded and well behaved.

The proof requires the following observation whose proof is straightforward:

2.4.2 REMARK. If $\{f_\lambda\}$ is an approximate identity for A ,

then the locally convex topology on $M(A)$ (see Chapter III, Section 1) generated by the seminorms $x \rightarrow \max \{ \|f_\lambda x\|, \|xf_\lambda\| \}$ agrees with the strict topology on norm bounded sets in $M(A)$.

Proof of 2.4.1. Let m and $n_1 < n_2 < \dots$ be positive integers. Choose a positive integer i_0 so that $n_i > m$ for $i \geq i_0$. Then $e_m(e_{n_i} - e_{n_j}) = 0$ for $i, j > i_0$ by the canonical property so $\{e_n\}$ is well-behaved. Total boundedness in the strict topology follows from 2.4.2 and the fact that $\{e_n\}$ is well-behaved. Part (a) of the next result was used by Taylor [57] in his study of well-behaved identities. We shall use it in 2.4.5 to show that algebras with countable approximate identities have ones with other nice properties.

2.4.3 LEMMA. Let A be a Banach algebra. (a) If $\{e_\lambda\}$ is an approximate identity for A and $\{f_p\}$ is an approximate identity for the normed algebra generated by $\{e_\lambda\}$, then $\{f_p\}$ is an approximate identity for A ; (b) If $\{e_\lambda\}$ is a norm bounded net in A and D a dense subset in the Hermitian part of the unit ball of A so that $e_\lambda x \rightarrow x$ and $x e_\lambda \rightarrow x$ for each x in D , then $\{e_\lambda\}$ is an approximate identity for A . In part (b), we assume

A is a B^* -algebra.

Proof. This is a straightforward computation.

Separable B^* algebras have many types of approximate identities as 2.4.4 shows.

2.4.4 LEMMA. Let A be a separable B^* algebra. Then A contains an approximate identity that is canonical, sequential, and abelian (and by 2.4.1, well-behaved and β totally bounded).

Proof. Let $\{x_n\}$ be a countable dense set in the Hermitian part of the unit sphere of A , and let

$x = \sum_{n=1}^{\infty} \frac{1}{2^n} x_n^2$. Since x is a positive element of A ,

the B^* algebra C generated by x is isometrically $*$ -isomorphic to the algebra $C_0(S)$, where S is the maximal ideal space of C . Since $C_0(S)$ is generated by a single function, S is σ compact. We may select from $C(=C_0(S))$ an approximate identity $\{e_k\}$ for C possessing all the properties mentioned in the statement of 2.4.4. It remains only to show that $\{e_k\}$ is an approximate identity for A . Adjoin a unit I to A in the customary manner so that the adjoined algebra is B^* , hence we have that

$\|(I - e_k)x(I - e_k)\|_k \rightarrow 0$. From [17, p. 14] we have that

$$\|(I - e_k) \cdot x_n x_n^* (I - e_k)\| \leq 2^n \|(I - e_k) x (I - e_k)\|$$

so that $\|(I - e_k)x_n\| = \|x_n(I - e_k)\|_k \rightarrow 0$. Thus applying 2.4.3 (b) to $D = \{x_n\}$ and $\{e_k\}$ we see that $\{e_k\}$ is an approximate identity for A .

2.4.5 DEFINITION. Let $\{A_\gamma\}$ be a family of normed algebras. The subdirect sum, $(\Sigma A_\gamma)_0$, of the family $\{A_\gamma\}$ is that subset of $\prod_{\gamma \in \Gamma} A_\gamma$ consisting of all $a = (a_\gamma) \in \prod_{\gamma \in \Gamma} A_\gamma$ so that $\{\gamma \in \Gamma : \|a_\gamma\| \geq \epsilon\}$ is finite for each $\epsilon > 0$. The algebraic operations are pointwise and $\|a\| = \sup \{\|a_\gamma\| : \gamma \in \Gamma\}$.

2.4.6 PROPOSITION. If $A = (\Sigma A_\gamma)_0$ and each A_γ has a β totally bounded approximate identity, then so does A .

Proof. The proof is the same as Proposition 3.2 in [57] where the same result is proved for well-behaved approximate identities.

2.4.7 REMARK. Proposition 2.4.6 is true when "totally bounded" is replaced by any of the types of approximate identities listed in Section 2, except countable and se-

quential. Dual B*-algebras have β totally bounded approximate identities by 2.4.6, and 2.4.5 and 2.4.6 give a proof, different from that in [8], that $C_0(S)$, for S paracompact, has a β totally bounded approximate identity.

2.4.8 CONJECTURE. We conjecture that $C_0(S)$ has a well-behaved approximate identity if and only if S is paracompact. As indicated earlier, our results on this question are incomplete, but we give an example in Section 6 that is perhaps illuminating.

SECTION 5. SHAM COMPACT SPACES AND APPROXIMATE IDENTITIES.

The definition of sham compact space and sham compact approximate identity, given in 2.2.5, is motivated by the space X of ordinals less than the first uncountable ordinal with the order topology, and the algebra $C_0(X)$. For example, let $\Lambda = X$ with the usual order and if $\lambda \in \Lambda$, let f_λ be the characteristic function of the interval $[0, \lambda]$. It is clear that $\{f_\lambda\}$ is a sham compact approximate identity for $C_0(X)$. We note that $C_0(X)$ cannot have a β totally bounded approximate identity since X is not paracompact. Furthermore, it cannot have a well-behaved approximate identity either since it is pseudocom-

pact.

2.5.1 PROPOSITION. Let S be pseudocompact. If $C_0(S)$ has a well-behaved approximate identity, then S is compact.

Proof. Let $\{e_\lambda\}$ be a well-behaved approximate identity for $C_0(S)$, suppose that S is not compact, and choose, by 2.3.1, an increasing sequence $\{\lambda_n\}$ so that $e_{\lambda_i} \not\equiv e_{\lambda_{i+1}}$ for any integer i . Note that $e_{\lambda_1} < e_{\lambda_2} < \dots$, i.e., $\{e_{\lambda_i}\}$ is an increasing sequence. Since the sequence $\{e_\lambda\}$ is Cauchy in the compact open topology and $C^*(S)$ is complete in this topology, there is a function in $C_b(S)$ so that $e_{\lambda_i} \rightarrow f$ uniformly on compact subsets of S . By [24, Theorem 2] $e_{\lambda_i} \rightarrow f$ in norm so f is in $C_0(S)$. By 2.3.2, $f \equiv 1$ on $\bigcup_{i=1}^{\infty} \text{spt } e_{\lambda_i}$ which then is contained in the compact set $K = f^{-1}\{1\}$. Choosing $\lambda \in \Lambda$ so that $e_\lambda \equiv 1$ on K , we obtain a contradiction to the fact that $e_{\lambda_i} \not\equiv e_{\lambda_{i+1}}$ for all i .

2.5.2 REMARK. Proposition 2.5.1 admits the following non-abelian generalization, stated here, without proof, for

completeness: Suppose a B*-algebra A has a well-behaved approximate identity and $M(A)$ satisfies the following condition: whenever $\{a_n\}$ is an increasing sequence in A and $\{a_n\}$ converges in the strict topology to x in $M(A)$, then $\|a_n - x\| \rightarrow 0$. Then A has an identity and $A = M(A)$. (See [24, Proposition 2] to see that this result includes 2.5.1.)

The next proposition relates sham compactness of S , existence of sham compact approximate identities in $C_0(S)$ and the property (DF) of Grothendieck.

2.5.3 DEFINITION. Let E be a locally convex topological vector space with dual E' . The space E is (DF) if there is a countable base for bounded sets in E and if every countable intersection of closed convex circled zero neighborhoods which absorbs bounded sets is a zero neighborhood.

2.5.4 REMARK. The vector space $C^*(S)_\beta$ is complete and the β bounded sets coincide with the norm bounded sets so $C^*(S)_\beta$ is (DF) if each countable intersection of closed convex circled zero neighborhoods which absorbs points of $C^*(S)$ is a zero neighborhood [45, p. 67].

We shall use the following remark in the proof of Theorem 2.5.6.

2.5.5 REMARK. W. H. Summers [54] has recently shown that $C^*(S)_\beta$ is (DF) if $C^*(N; C_0(S))$ is essential, where $C^*(N; C_0(S))$ is the Banach algebra of all norm bounded sequences from $C_0(S)$ with the sup norm topology ($\|\cdot\|_\infty$) and "essential" means that $\|e_\lambda \{f_n\} - \{f_n\}\|_\infty \rightarrow 0$ where $\{e_\lambda\}$ is any approximate identity for $C_0(S)$ and $\{f_n\}$ any element of $C^*(N; C_0(S))$.

2.5.6 THEOREM. These are equivalent: (a) $C^*(S)$ is (DF); (b) S is a sham compact space (c) $C_0(S)$ has a sham compact approximate identity.

Proof. Assume that $C^*(S)_\beta$ is (DF) and X is the union of compact sets K_n , i.e., $X = \bigcup_{n=1}^{\infty} K_n$. For each integer n , let φ_n be a function in $C_0(S)$ so that $0 \leq \varphi_n \leq 1$ and $\varphi_n \equiv 1$ on K_n . Let $V = \{f \in C_b(S) : \|f \varphi_n\| \leq 1, \forall n\}$. V absorbs points of $C^*(S)$; therefore it is a zero neighborhood in the strict topology by (a). It is obvious that the sets $\{f \in C^*(S) : \|f \varphi\| \leq 1\}$ (for $\varphi \geq 0$ in $C_0(S)$) is a base at zero for the strict topology. Thus $\exists \varphi \geq 0$ in $C_0(S)$

so that $\{f \in C^*(S) : \|f\| \leq 1\} \subset V$. This shows that $\varphi(x) \geq 1$ for x in X . For if not, there is an integer n and a point x_0 in K_n so that $\varphi(x_0) < 1$. By a standard Urysohn's lemma argument $\exists f \in C_0(S)$ so that $f(x_0) > 1$ and $\|f\| < 1$. This contradiction establishes our claim, i.e., $X \subset \varphi^{-1}\{1\}$, so X is relatively compact.

Suppose that (b) holds. Let Λ be the set of all pairs (K, O) where $K \subset O \subset S$, K is compact and O is open with compact closure. If $\lambda = (K, O)$ and $\lambda_1 = (K_1, O_1)$, we define $\lambda \geq \lambda_1$ if $\lambda = \lambda_1$ or if $O_1 \subset K$. If $\lambda = (K, O)$ let f_λ be a function in $C_0(S)$ which satisfies: (1) $0 \leq f_\lambda \leq 1$; (2) $f_\lambda \equiv 1$ on K ; and (3) $\text{spt } f_\lambda \subset O$. The net $\{f_\lambda\}$ is by (b) a sham compact approximate identity for $C_0(S)$.

Assume (c), with $\{e_\lambda\}$ a sham compact approximate identity, and let $\{f_n\}$ be a sequence contained in the unit ball of $C_0(S)$, and $\epsilon > 0$. Choose a sequence $\{\lambda_n\}$ from Λ so that $\|e_{\lambda_n} f_n - f_n\| < \epsilon$ for each integer n . Let $\lambda_0 \in \Lambda$ be such that $\lambda_0 > \lambda_n$ for all integers n . Remark 2.5.5 and the following computation finish the proof:

$$\lambda > \lambda_0 \text{ implies } \|e_\lambda f_n - f_n\| =$$

$$\begin{aligned} \|(1 - e_\lambda)f_n\| &= \|(1 - e_{\lambda_n})(1 - e_\lambda)f_n\| \leq \\ &\|(1 - e_{\lambda_n})f_n\| < \epsilon \quad \text{for all } n. \end{aligned}$$

SECTION 6. METACOMPACT SPACES - AN EXAMPLE.

We have been unable to prove our conjecture that S is paracompact if $C_0(S)$ has a well-behaved approximate identity except in special cases (see section 3), but we are able to give an example that shows that metacompactness is not sufficient for existence of a well-behaved approximate identity.

2.6.1 EXAMPLE. Let I be the unit interval with the discrete topology and I^* , the one-point compactification of I , with ∞ denoting the point at infinity. Similarly, let N denote the positive integers with discrete topology, N^* the one-point compactification of N , and w the point at infinity. Let $S = I^* \times N^* \setminus \{(\infty, w)\}$. Being an open set in a compact Hausdorff space, S is locally compact Hausdorff.

To show that X is metacompact, take an open cover \mathcal{U} of X . For each point (∞, n) , there is a finite set F_n of I so that a member of \mathcal{U} contains the open set $U_n = \{(x, n) : x \notin F_n\}$. Similarly, for each point (x, w)

there is a finite set G_x of N with a member of \mathcal{U} containing the open set $W_x = \{(x,n): n \notin G_x\}$. If $(x,y) \in X$ and $x \neq \infty$ and $y \neq w$, (x,y) is discrete. Let $W_{x,y} = \{(x,y)\}$. It is easily checked that the sets $\{W_x\}$, $\{U_n\}$, and $\{W_{x,y}\}$ form a point-finite open refinement of \mathcal{U} . Recalling that a space is metacompact if each open cover has a point-finite open refinement, we see that X is metacompact.

Before we show that $C_0(X)$ has no well-behaved approximate identity, we point out that X is not pseudocompact; thus we cannot simply apply 2.5.1. In our demonstration that $C_0(X)$ does not have a well-behaved approximate identity, we first exhibit a $\sigma(M(X), C^*(X))$ convergent sequence $\{\mu_n\}$ which is not tight, where a subset H of $M(X)$ is tight if it is bounded and for each $\epsilon > 0$ there is a compact set K_ϵ in X so that $|\mu|(X \setminus K_\epsilon) < \epsilon$ for all $\mu \in H$ ($|\mu|$ denotes the total variation of μ). We may then apply Corollary 3.4 in [57] to conclude that $C_0(X)$ does not have a well-behaved approximate identity.

For each positive integer n , let μ_n be the member of $M(X)$ defined by the equation $\mu_n(f) = f((\infty, n)) - f((\infty, n+1))$ for f in $C^*(X)$. Note that the total variation of μ_n satisfies the equation $|\mu_n|(f) =$

$= f((\infty, n)) + f((\infty, n+1))$ for f in $C^*(X)$ and so $\|\mu_n\| \leq 2$ for each integer n . We now show that $\mu_n \rightarrow 0$ in the weak-* topology of $M(X)$. Let $f \in C^*(X)$ and $m \in \mathbb{N}$. Since f is continuous at (∞, n) , for each $\epsilon > 0$, there is a finite subset $I_{\epsilon, n}$ of I so that if $x \notin I_{\epsilon, n}$, $|f(x, n) - f(\infty, n)| < \epsilon$. Thus there is a countable subset I_n of I so that if $x \notin I_n$, $f(x, n) = f(\infty, n)$. If I_f is the union of the sets $\{I_n\}$, we see that it is countable and if $x \notin I_f$, then $f(x, n) = f(\infty, n)$ for all integers n . Choose a point $x_f \notin I_f$. Then the sequence $\{(x_f, n)\}$ converges to the point (x_f, w) so that $f((x_f, n)) \rightarrow f((x_f, w))$. Thus $\lim_n f((\infty, n)) = \lim_n f((x_f, n)) = f((x_f, w))$ so that $\lim_n f((\infty, n)) - f((\infty, n+1)) = 0$, i.e., $\mu_n(f) \rightarrow 0$. Since f is arbitrary, we have shown that $\mu_n \rightarrow 0$ weak-*.

We next see that $\{\mu_n\}$ cannot be tight: Let $\epsilon = \frac{1}{2}$ and note that a compact set in X can contain only finitely many of the points (∞, n) . If K is a compact subset of X and $(\infty, p) \notin K$, we can choose $f \in C^*(X)$ so that $\text{spt } f$ is compact, $f((\infty, p)) = 1$, $f \equiv 0$ on K , and $0 \leq f \leq 1$, i.e., so that $|\mu_p|(X \setminus K) \geq |\mu_p(f)| \geq |f((\infty, p))| = 1$. Applying [57, Corollary 3.4.], we see that $C_0(X)$ does not have a well-behaved approximate identity (note that X is not para-

compact by [57, 3.1 and 3.2].

2.6.2 REMARK. The space $C^*(X)_\beta$ where X is as in 2.6.1 is interesting for several other reasons. First $C^*(X)_\beta$ is not a strong Mackey space. Conway [12] has shown that $C^*(X)_\beta$ is strong Mackey if X is paracompact. The problem of finding topological conditions on X necessary and sufficient for $C^*(X)_\beta$ to be a strong Mackey (or Mackey) space is an intriguing problem. If we let μ_n be the element of $M(X)$ whose value at f in $C^*(X)$ is $f((\infty, n))$, arguments similar to the above show that $\{\mu_n\}$ is weak* Cauchy but has no weak-* limit in $M(X)$, i.e., $M(X)$ is not β weak-* sequentially complete (see [8,5.1]). $C^*(X)_\beta$ is also not sequentially barrelled (see [65]).

SECTION 7. MISCELLANEOUS REMARKS.

2.7.1 REMARK. It is easy to show that if $\{e_\lambda\}$ is a sham compact approximate identity for a (possibly non-abelian) Banach algebra A , then $\{e_\lambda\}$ cannot be well-behaved unless A has an identity. The question one really wants to answer is whether A can have another approximate identity that is well-behaved unless A has an identity element. If A is commutative, the question is answered in

the negative by 2.5.1 and 2.5.6 of this paper. We have the following generalization of Theorem 4.1 in [54] (see Chapter III for a proof).

2.7.2 THEOREM. These are equivalent: (1) $M(A)$ is (DF) ; (2) $M(A)$ is (WDF) ; (3) $\mathcal{L}^\infty(A)$ is both a right and a left essential module ($\mathcal{L}^\infty(A)$ is the set of all bounded sequences in A ; $\mathcal{L}^\infty(A)$ is a right essential module means that if $\{f_\lambda\}$ is any approximate identity for A and $x = \{x_n\} \in \mathcal{L}^\infty(A)$ then $\lim (\sup_\lambda \|x_n f_\lambda - x_n\|) = 0$).

2.7.3 PROPOSITION. Let A have a well-behaved approximate identity and suppose that $\{e_\lambda\}$ is a sham compact approximate identity for A . Then A has an identity.

Proof. Let $x = (x_n) \in \mathcal{L}^\infty(A)$; we can choose, by induction, a sequence $\{\lambda_k\}$ from Λ so that

$$\lim_k \|e_{\lambda_k} x_n - x_n\| = \lim_k \|x_n e_{\lambda_k} - x_n\| = 0$$

for all positive integers n . By the sham compact property, choose e_λ so that $\lambda > \lambda_k$ for all integers k . Thus $e_\lambda e_{\lambda_k} = e_{\lambda_k}$ so that $e_\lambda x_n = x_n$ for all n . Thus $\lim_\lambda (\sup_n \|x_n e_\lambda - x_n\|) = 0$ and $\lim_\lambda (\sup_n \|e_\lambda x_n - x_n\|) = 0$, i.e., $\mathcal{L}^\infty(A)$ is both left and right essential. Suppose

$\{f_\gamma\}$ is a well-behaved approximate identity for A and $\gamma_1 < \gamma_2 < \dots$ is a sequence in Γ so that $0 \neq f_{\gamma_i} \neq f_{\gamma_{i+1}}$ for all integers i . Since $k^\infty(A)$ is essential, there is an element γ_0 in Γ so that

$$\|f_{\gamma_0} f_{\gamma_i} - f_{\gamma_i}\| < \frac{1}{4}$$

for all positive integers i . Since $\{f_\gamma\}$ is well-behaved, there is a positive integer N so that $n, m \geq N$ implies that

$$f_{\gamma_0} (f_{\gamma_n} - f_{\gamma_m}) = 0$$

which further implies that $\|f_{\gamma_n} - f_{\gamma_m}\| < \frac{1}{2}$ for $n, m \geq N$.

Let C be the commutative B^* algebra generated by $\{f_{\gamma_n} : \gamma \geq N\}$. We claim that $\text{spt } f_{\gamma_N} \not\subseteq N(f_{\gamma_{N+2}})$. If this is not true, then $\text{spt } f_{\gamma_N} = \text{spt } f_{\gamma_{N+1}} = \text{spt } f_{\gamma_{N+2}}$ and so $f_{\gamma_{N+1}} = f_{\gamma_{N+2}}$ = the characteristic function of the $\text{spt } f_{\gamma_N}$ by 2.3.2, contradicting the choice of $\{f_{\gamma_n}\}_{n=1}^\infty$. Thus $\exists x \in N(f_{\gamma_{N+2}}) \setminus \text{spt } f_{\gamma_N}$ which implies that $\|f_{\gamma_{N+3}}(x) - f_{\gamma_N}(x)\| = 1$. This contradiction concludes the proof that a (nonabelian) B^* algebra A cannot have both a well-behaved and a sham compact approximate identity.

It is easy to give an example of a β totally bounded approximate identity in $C_0(S)$ that is not canonical (and a fortiori, not well-behaved). Our next result points out the rather interesting fact that in an abelian B^* -algebra a canonical chain totally bounded approximate identity is well-behaved.

2.7.4 PROPOSITION. Let $\{e_\lambda\}$ be a canonical chain totally bounded approximate identity for $C_0(S)$. Then $\{e_\lambda\}$ is well-behaved.

Proof. Let $\{\lambda_n\}$ be an increasing sequence in Λ and $F = \bigcup_{n=1}^{\infty} \text{spt } f_{\lambda_n}$. Then F is clopen as in the proof of

2.3.3 and, for any compact subset K of F , $K \subset N(e_{\lambda_N})$ for some integer N , so that $e_{\lambda_n} = 1$ on K for $n > N$. If $\lambda \in \Lambda$, let $K = \text{spt } e_\lambda \cap F$; then $e_\lambda(e_{\lambda_n} - e_{\lambda_m}) = 0$ for n and m large enough by the preceding remarks. Therefore $\{e_\lambda\}$ is well-behaved.

Taylor [57] proves several interesting theorems about $M(A)$ assuming that the B^* -algebra A has a well-behaved approximate identity. From 2.4.3 and 2.7.4 we see that (looking at the algebra generated by the approximate iden-

tity) an abelian, canonical, and chain totally bounded approximate identity for A is a well-behaved approximate identity so Taylor's theorems hold in this case. We conjecture even more, viz., that if A has a canonical chain totally bounded approximate identity, then the theorems in [57] hold. Our reason for believing this is the next proposition, which shows that a canonical chain totally bounded approximate identity is 'almost' well-behaved.

2.7.5 PROPOSITION. If $\{e_\lambda\}$ is a canonical chain totally bounded approximate identity in a Banach algebra A , then $\{e_\lambda\}$ satisfies the following condition: if $\epsilon > 0$, $\{\lambda_n\}$ is an increasing sequence in Λ , and $\lambda \in \Lambda$ there exists a positive integer N so that $n, m > N$ implies

$$\|e_\lambda(e_{\lambda_n} - e_{\lambda_m})\| < \epsilon.$$

Proof. By chain total boundedness of $\{e_{\lambda_n}\}$, there is an integer P so that for all positive integers n

$$\min_{1 \leq p \leq P} \|e_\lambda(e_{\lambda_n} - e_{\lambda_p})\| < \frac{\epsilon}{2}.$$

Choose $N \geq P$ so that if $N < n < p$, $\exists q > p$ so that $\|e_\lambda(e_{\lambda_n} - e_{\lambda_q})\| < \epsilon$. If $n, m > N$ and $n < m$, choose $q > m$ so that $\|e_\lambda(e_{\lambda_n} - e_{\lambda_q})\| < \epsilon$. Then

$$\|e_\lambda(e_{\lambda_n} - e_{\lambda_m})\| = \|e_\lambda(e_{\lambda_n} - e_{\lambda_q})e_{\lambda_m}\| \leq \|e_\lambda(e_{\lambda_n} - e_{\lambda_q})\| < \epsilon.$$

2.7.6 EXAMPLE. We now give an example of an approximate identity that is well-behaved but not β totally bounded.

Let R denote the real line and Λ be the set of pairs (i, j) where i is any positive integer and $j = 0$ or $j = 1$. Order Λ as follows:

- 1) $(i, j) = (i', j')$ if $i = i'$ and $j = j'$;
- 2) $(j, 0) > (i, 1)$ for all integers i and j ;
- 3) $(i, 0) > (j, 0)$ if $i > j$.

If $\lambda = (i, 0)$ let f_λ be in $C_0(R)$ so that $0 \leq f_\lambda \leq 1$ and $f_\lambda \equiv 1$ on $[-i, i]$ and $f_\lambda \equiv 0$ off $[-(i+1), (i+1)]$.

If $\lambda = (i, 1)$, let f_λ again be in $C_0(R)$ so that $0 \leq f_\lambda \leq 1$, $f_\lambda(x_1) = 1$ where $x_1 = \frac{1}{2}(\frac{1}{i+1} + \frac{1}{i})$ and $f_\lambda \equiv 0$ off $[\frac{1}{i+1}, \frac{1}{i}]$. The net $\{f_\lambda\}$ is easily seen to be well-behaved but the infinite sequence $\{f(i, 1)\}$ is clearly not β totally bounded.

2.7.7 EXAMPLE. In 2.3.3, we showed that if $C_0(S)$ has an approximate identity that is well-behaved (or β totally bounded) then S contains a clopen set X so that $C_0(S) = B_1 \oplus B_2$ where $B_1 = \{f \in C_0(S) : f \equiv 0 \text{ on } X\}$ and $B_2 = \{f \in C_0(S) : f \equiv 0 \text{ on } S \setminus X\}$ are 2-sided ideals of $C_0(S)$. Obvious noncommutative generalizations of the

above fail as we now show. Let A be the algebra of compact operators on a Hilbert space H , $\{e_\gamma : \gamma \in \Gamma\}$ an orthonormal basis for H , and Λ the set of finite subsets of Γ ordered by inclusion. If $\lambda \in \Lambda$, let P_λ be the finite-dimensional projection defined by the equation

$$P_\lambda(h) = \sum_{\gamma \in \lambda} \langle h, e_\gamma \rangle e_\gamma \quad \text{for } h \in H.$$

It is easy to show that $\{P_\lambda\}$ is a well-behaved and totally bounded approximate identity for A , but A has no non-trivial decomposition as a direct sum of two-sided ideals [38].

2.7.8 REMARK. It is perhaps worth pointing out that if $C_0(S)$ and $C_0(T)$ have approximate identities with certain properties, so does $C_0(S \times T)$ and the converse is also true. Suppose for example that $C_0(S)$ has a well-behaved approximate identity $\{e_\lambda\}$ and $C_0(T)$ has a well-behaved approximate identity $\{f_\alpha\}$. If f and $g \in C_0(S)$ and $C_0(T)$ respectively let $f \otimes g$ be the function on $S \times T$ defined by $f \otimes g(s, t) = f(s)g(t)$. It is easy to see that $f \otimes g \in C_0(S \times T)$. Because the algebra generated by $\{f \otimes g \mid f \in C_0(S), g \in C_0(T)\}$ is dense in $C_0(S \times T)$ by the Stone-Weierstrass Theorem, the net $\{e_\lambda \otimes f_\alpha\}$ with directed set

all pairs (λ, α) where $(\lambda, \alpha) > (\lambda', \alpha')$ if $\lambda > \lambda'$ and $\alpha > \alpha'$ is an approximate identity for $C_0(\text{SXT})$ which is easily seen to be well-behaved. Conversely, if $\{e_\lambda\}$ is a well-behaved approximate identity for $C_0(\text{SXT})$ and $t_0 \in T$, the net of functions (f_λ) defined by $f_\lambda(s) = e_\lambda(s, t_0)$ is a well-behaved approximate identity for $C_0(S)$.

2.7.9 EXAMPLE. Our investigations of $\sigma(M(A), M(A)_\beta^*)$ relatively compact approximate identities is in the first stages only. We wish to present the following example, however, as it seems interesting. Let $S =$ the ordinals less than first uncountable with the order topology. $C_0(S)$ has no $\sigma(C^*(S), M(S))$ relatively compact approximate identity. For, suppose that $C_0(S)$ has an approximate identity $\{e_\lambda\}$ which is $\sigma(C^*(S), M(S))$ relatively compact. Note that $(|e_\lambda|^2)$ is an approximate identity which is also $\sigma(C^*(S), M(S))$ relatively compact, so we may suppose $e_\lambda \geq 0$. Let $\lambda_1 \in \Lambda$ and $x_1 = \min \{x \in S: y > x \Rightarrow e_{\lambda_1}(y) = 0\}$. Choose $\lambda_2 \in \Lambda$ so that $e_{\lambda_2} > \frac{2}{3}$ on $[0, x_1 + 1]$ and let $x_2 = \min \{x \in S: y > x \Rightarrow e_{\lambda_2}(y) = 0\}$. Note $x_2 \geq x_1 + 1$ so $x_2 > x_1$.

Suppose $\lambda_1, \dots, \lambda_n$ and x_1, \dots, x_n have been chosen so that:

- 1) $e_{\lambda_k} > \frac{k}{k+1}$ on $[0, x_{k-1}+1]$ for $2 \leq k \leq n$
- 2) $x_k = \min \{x \in S: y > x \Rightarrow e_{\lambda_k}(y) = 0\}$
- 3) $x_n > x_{n-1} > \dots > x_2 > x_1$.

By induction we select a sequence (λ_n) in Λ and a sequence (x_n) from X satisfying 1) and 2) and 3). Let $x = \text{lub } \{x_n\}$. By assumption, $\exists f \in C^*(S)$ so that e_{λ_n} clusters $\sigma(C^*(S), M(S))$ to f . If $y > x$, $e_{\lambda_n}(y) = 0$ for all n so that $f(y) = 0$. If $y < x$, then there is an integer N so that $y < x_n$ for $n > N$ so that $e_{\lambda_n}(y)$ clusters to 1; therefore $f(y) = 1$. We now show that f cannot be continuous at x . Since $\{x_n\}$ is strictly increasing, $x_n < x$ for all n so that $e_{\lambda_n}(x) = 0$ for all n and so $f(x) = 0$; on the other hand, $x_n \rightarrow x$, so, if f were continuous, $f(x)$ would be the limit of the constant sequence $f(x_n)$, i.e., 1. This contradiction concludes the proof that $C_0(S)$ has no $\sigma(C^*(S), M(S))$ relatively compact approximate identity.

Our last result 2.7.10 answers only one of a number of questions of the following form: given an algebra A with an approximate identity having property P and another approximate identity $\{e_\lambda\}$, can we select from Λ a subset Λ_0 (cofinal, perhaps) so that $\{e_\lambda: \lambda \in \Lambda_0\}$ has pro-

perty P . Easy examples show that the subset Λ_0 in 2.7.10 need not be cofinal in Λ .

2.7.10 PROPOSITION. If a Banach algebra A has a countable approximate identity $\{f_\lambda\}$ and $\{e_\lambda\}$ is another approximate identity, \exists countable $\Lambda_0 \subseteq \Lambda$ so that $\{e_\lambda : \lambda \in \Lambda_0\}$ is an approximate identity.

Proof. Choose a countable subset Λ_0 of Λ so that $\lim_{\lambda \in \Lambda_0} e_\lambda f_\gamma = \lim_{\lambda \in \Lambda_0} f_\gamma e_\lambda = f_\gamma$ for each $\gamma \in \Gamma$.

SECTION 8. PROPERTY (B) FOR BANACH ALGEBRAS.

The results in this section arose from our attempts to answer the question: Does every C^* -algebra have a canonical approximate identity? This question led to the formulation of Property (B) (defined below) for Banach algebras and an investigation of some of the permanence properties of Property (B). Further study of the canonical approximate identity problem led us to consider group algebras and their group C^* -algebras [17]. This question seems to be very hard. However, Akemann in [1] gave an example of a C^* -algebra without an abelian approximate identity. The C^* -algebra he constructed may not have a canonical approximate identity; we are studying his ex-

ample now.

2.8.1 DEFINITION. Let A be a Banach- $*$ algebra. An element $a \in A$ is called factorizable if $\exists b \in A$, $b = b^*$ so that $ba = ab = a$.

2.8.2 EXAMPLES. If $A = C_0(S)$, then $a \in A$ is factorizable iff $a \in C_c(S)$. If $A = LCH$, the algebra of compact operators on some Hilbert space, the factorizable elements of A are finite rank operators. Note that if A has an identity then every element of A is factorizable. Thus the concept is of interest only in algebras without identity. Also note, however, that the algebra A of operators with separable range on a nonseparable Hilbert space has no identity but every element of A is factorizable. Finally, note that if $x = x^*$ belongs to a C^* -algebra A then x is factorizable iff $\exists a \in A$, $a \geq 0$ and $\|a\| \leq 1$ such that $ax = x$.

In studying approximate identities in C^* -algebras we were led to ask the question: Suppose A is a C^* -algebra and x and y are arbitrary factorizable elements of A . When does $\exists a \in A$ so that $ax = xa = x$ and $ay = ya = y$? This question led us to define Property (B) for Banach al-

gebras.

2.8.3 DEFINITION. Let A be a Banach $*$ -algebra. A is said to have Property (B) if whenever x and y belongs to A and are factorizable then so is $x + y$.

2.8.4 EXAMPLES. Banach algebras with identity have Property (B) trivially. The algebra LCH of compact operators on a Hilbert space H has Property (B) since factorizable elements are finite-dimensional. All commutative C^* -algebras have Property (B); so does the algebra of operators having separable range on a nonseparable Hilbert space.

2.8.5 THEOREM. Let $A = (\sum_{t \in I} A_t)_0$ where each A_t has Property (B). Then A has Property (B).

Proof. This is clear from the definition of a subdirect sum of Banach algebras.

2.8.6 COROLLARY. Dual B^* -algebras [17] have Property (B).

Proof. They are subdirect sums of algebras of compact operators.

Our next result is that a C^* -algebra with Property (B) has a canonical approximate identity. It was this discovery which spurred our interest in Property (B). We need several lemmas first.

2.8.7 LEMMA. Let A be a C^* -algebra. Suppose that whenever $a_1, a_2, x_1, x_2 \in A^+$ (positive part) and $a_i x_i = x_i$, $i = 1, 2$, there exists $a \in A^+$ that $ax_i = x_i$, $i = 1, 2$. Let n be a positive integer and suppose \mathfrak{E} finite sets $\{b_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ contained in A^+ such that $b_i y_i = y_i$, $i = 1, \dots, n$. Then $\exists b, c \in A^+$, so that $\|b\| \leq 1$, $\|c\| \leq 1$, $bc = cb = b$ and $by_i = y_i$, $i = 1, \dots, n$.

Proof. By induction choose d_1 in A^+ so that $d_1 y_i = y_i$ for $1 \leq i \leq n-1$. Hence $d_1 (\sum_{i=1}^{n-1} y_i) = \sum_{i=1}^{n-1} y_i$. By induction again, we select $d \in A^+$ so that $d (\sum_{i=1}^n y_i) = \sum_{i=1}^{n-1} y_i + y_n = \sum_{i=1}^n y_i$. Let C be a commutative C^* -subalgebra of A containing d and $\sum_{i=1}^n y_i$. By 1.7, there is a locally compact Hausdorff space S so that C is isometrically $*$ -isomorphic with $C_0(S)$. Hence $\sum_{i=1}^n y_i \in C_c(S)$, so we may select elements b and c in

A^+ so that $\|b\| \leq 1$, $\|c\| \leq 1$, $bc = c$, $b(\sum_{i=1}^n y_i) = \sum_{i=1}^n y_i$.
 Computing in $M(A)$ (which has an identity, denoted by I), we get that $0 \leq (I-b)y_1(I-b) \leq (I-b)(\sum_{i=1}^n y_i)(I-b) = 0$ so that (using the B^* -norm property) $(I-b)y_1 = 0$, i.e., $by_1 = y_1$ for $1 \leq i \leq n$.

2.8.8 LEMMA. Suppose the C^* -algebra A has Property (B).

Let \mathfrak{F} denote the family of all finite subsets of A^+ . Then families $\{a_F: F \in \mathfrak{F}\}$ and $\{b_F: F \in \mathfrak{F}\}$ can be chosen from A^+ so as to satisfying the following conditions:

- (1) $\|a_F\| \leq 1$ and $\|b_F\| \leq 1$ for every $F \in \mathfrak{F}$;
- (2) If $G \subseteq F$ and $G \neq F$ then $a_F a_G = a_G$;
- (3) $b_F a_F = a_F$;
- (4) If $C(F)$ denotes the cardinality of F , then $\|a_F x - x\| < \frac{1}{C(F)}$ for each $x \in F$.

Proof. If F is a singleton, we may consider the commutative C^* -algebra generated by the element of F and select a_F and b_F using 1.7. We proceed by induction on $C(F)$, the cardinality of the set F . Suppose we have selected a_F and b_F so as to satisfy (1) - (4) for all F such that $C(F) \leq k$. Suppose F belongs to \mathfrak{F} and has $k+1$ (distinct) elements. Let m denote the number of proper, non-void subsets of F ($m = 2^{k+1} - 2$).

Each such subset has two elements of A^+ associated with it satisfying (1) - (4) above. Denote these elements by $\{a_i : 1 \leq i \leq m\}$ and $\{b_i : 1 \leq i \leq m\}$. Next select elements a_{m+1} and b_{m+1} in A^+ so that $a_{m+1}b_{m+1} = a_{m+1}$ and $\|a_{m+1}x - x\| < \frac{1}{k+1}$ for all $x \in F$. Let $z = \sum_{x \in F} \Sigma x$ and

regard, by 1.7, the commutative C^* -algebra generated by z as $C_0(S)$, for some locally compact S . Choose $f, g \in C_0(S)$ so that $0 \leq f \leq 1$, $0 \leq g \leq 1$, $fg = f$ and $\|fz - z\| < \frac{1}{n+1}$. Let a_{m+1} and b_{m+1} be the elements of A whose images are f and g , respectively, under the isomorphism guaranteed by 1.7.

Hence we have the system of equations $b_1 a_1 = a_1$ $i = 1, \dots, m+1$. By 2.8.7, we can choose elements a_F and b_F in A^+ so that (1) and (3) are satisfied and such that $a_F a_i = a_i$, $1 \leq i \leq m+1$. Thus (2) is satisfied also. It remains to be shown that (4) is satisfied. Working in $M(A)$, we have that $\|(I - a_{m+1})x\| < \frac{1}{k+1}$ for every $x \in F$. Thus $\|(I - a_F)x\| = \|(I - a_F)(I - a_{m+1})x\| \leq \|(I - a_{m+1})x\| < \frac{1}{k+1}$ for all $x \in F$, i.e., $\|a_F x - x\| < \frac{1}{k+1}$ for all $x \in F$. This completes the induction step and shows that the families $\{a_F | F \in \mathfrak{F}\}$ and $\{b_F | F \in \mathfrak{F}\}$ can be constructed so as to satisfy (1) - (4).

2.8.9 THEOREM. Let A be a C^* -algebra having Property (B). Then A has a canonical approximate identity.

Proof. The family $\{a_F: F \in \mathfrak{F}\}$ of 2.8.8 becomes a net if \mathfrak{F} is partially ordered by inclusion. This net is a canonical approximate identity for A .

Our next result is that $*$ -homomorphic images of algebras with Property (B) have Property (B).

2.8.10 THEOREM. Suppose A is a C^* -algebra with Property (B) and C is the image of A under a $*$ -homomorphism f . Then C has Property (B).

Proof. Let K_A be the subset of A formed in the following way: take the linear span of the smallest order ideal (cone with the property that if $0 \leq x \leq y$ and y belongs to the cone then so does x) of A containing the factorizable elements of A^+ . Note that Property (B) for A implies that the factorizable elements in A^+ form a cone. Also note if $0 \leq x \leq y$ and $\exists a \in A^+$, with $\|a\| \leq 1$ and $ay = y$ that $0 \leq (I - a)x(I - a) \leq (I - a)y(I - a) = 0$ so that $ax = x$ by familiar arguments. Thus K_A is the linear span of the factorizable

elements of A^+ and, by Property (B) again, if $x \in K_A \ni a \in A^+$ satisfying $ax = xa = x$. In [39,40] it is shown that $f(K_A) = K_C$ where K_C is defined as the analogue in C of K_A . Hence every element of K_C is factorizable. Since K_C is closed under addition, to show that C has Property (B) it suffices to show that if $x \in C$ and x is factorizable then $x \in K_C$. Suppose $c = c^* \in A$ and $cx = xc = x$. Then $cx^* = x^*c = x^*$ so that we may suppose $x = x^*$ (by looking at $\frac{x+x^*}{2}$, $\frac{x-x^*}{2i}$). Let J be the commutative C^* -algebra generated by x and c . Write $x = x^+ - x^-$ where $x^+, x^- \in J^+ \subseteq C^+$ and $x^+x^- = 0$. Note that $cx^+ = x^+c = x^+$ and $cx^-x^-c = x^-$. Clearly x^+ and $x^- \in K_C$ so $x \in K_C$.

We next want to consider algebras of type $C_0(S;E) = \{f|f:T \rightarrow E, f \text{ is continuous, and } s \rightarrow \|f(s)\| \in C_0(S)\}$ where S is locally compact and E is a B^* -algebra. We make $C_0(S;E)$ into a B^* -algebra with pointwise operations and involution and $\|f\| = \sup_{t \in S} \|f(t)\|$. A claim is made in [39] which amounts to the assertion that if E is the algebra of compact operators on a Hilbert space then $C_0(S;E)$ has Property (B) for any locally compact S . To our surprise, we discovered this is not the case. Let T be the one-point compactification of the

positive integers, denoted by N . We will show in 2.8.19 that $C^*(T:LCH) = C_0(T:LCH)$ does not have Property (B) if H is infinite-dimensional.

Before we do this, it is necessary to look at some examples and results about factorizable elements in Banach *-algebras.

2.8.11 EXAMPLE. Let T be an operator on a Hilbert space H . The range projection of T is the smallest projection in the set $\{P \in B(H): P \text{ is a projection and } PT = TP = T\}$. Let A denote the algebra $C^*(N^*:LCH)$, i.e., the algebra of convergent sequences in the compact operators on H . We give a counterexample to a reasonable conjecture, namely that if $g \in A$ and f denotes the sequence whose n -th coordinate is the range projection of $g(n)$ then $f(n)$ converges, i.e., $f \in A$. Let P and Q be finite rank projections in H with $PQ = 0$. Let g be the sequence $(P, \frac{1}{2}Q, \frac{1}{3}P, \frac{1}{4}Q, \dots)$. Then f is the sequence (P, Q, P, Q, \dots) and $f(n)$ clearly does not converge in LCH .

2.8.12 EXAMPLES. An algebra may have a well-behaved approximate identity (and certainly then, a canonical one) without having Property (B). An example is given in [41, p. 135]

of an algebra generated by two elements which does not have Property (B). We thought for a while that the properties of a C*-algebra being liminal or having continuous trace [17] were related to having Property (B). This is not so. The algebra of operators with separable range on a nonseparable Hilbert space has Property (B) but is not liminal and does not have continuous trace. On the other hand, $C^*(N^*;LCH)$ is liminal and has continuous trace but does not have Property (B) as 2.8.19 shows.

2.8.13 QUESTION. Does every C*-algebra have a canonical approximate identity? This seems very hard.

2.8.14 LEMMA. Suppose A and B are compact operators on a Hilbert space H such that $0 \leq A \leq I$, $0 \leq B \leq I$ and $\|A - B\| < \frac{1}{2}$. Let $m(T, \lambda)$ denote the dimension of $\{h \in H: Th = \lambda h\}$ for any operator $T \in B(H)$. Then

$$m(A, 1) \leq \sum_{\lambda \geq \frac{1}{2}} m(B, \lambda).$$

Proof. Suppose the result is false. Let $n = m(A, 1)$ and $p = \sum_{\lambda \geq \frac{1}{2}} m(B, \lambda)$ and suppose $n > p$. Let $\{x_i | 1 \leq i \leq n\}$ be an orthonormal basis for the eigenspace corresponding to the eigenvalue 1 for A . Let H_0 be the subspace

of H spanned by the eigenvectors of B corresponding to eigenvalues $\lambda > \frac{1}{2}$. Note that the dimension of H_0 is p . For each $1 \leq i \leq n$, let $x_i = y_i + z_i$ where $y_i \in H_0$ and z_i belongs to the orthogonal complement of H_0 . Note that, by the Spectral Theorem, if $x \in H_0^\perp$ (the orthogonal complement of H_0) then $\|Bx\| < \frac{1}{2}\|x\|$. Since $p < n$ there are scalars $\{d_i | 1 \leq i \leq n\}$, not all zero,

$$\text{so that } \sum_{i=1}^n d_i y_i = 0. \quad \text{Thus } \sum_{i=1}^n d_i x_i = \sum_{i=1}^n d_i y_i + \sum_{i=1}^n d_i z_i = \\ = \sum_{i=1}^n d_i z_i. \quad \|B(\sum_{i=1}^n d_i x_i)\| = \|B(\sum_{i=1}^n d_i z_i)\| < \frac{1}{2} \sum_{i=1}^n d_i \|x_i\|.$$

$$\text{Hence } \|A - B\| \geq \frac{1}{\|\sum_{i=1}^n d_i x_i\|} \|A(\sum_{i=1}^n d_i x_i) - B(\sum_{i=1}^n d_i x_i)\| = \\ = \frac{1}{\|\sum_{i=1}^n d_i x_i\|} \|\sum_{i=1}^n d_i x_i - B(\sum_{i=1}^n d_i x_i)\| \geq \frac{1}{2}.$$

This contradiction establishes the lemma.

2.8.15 PROPOSITION. Suppose f and g belong to $C^*(N; LCH)$ and $fg = g$ where $0 \leq f$, $0 \leq g$, $\|f\| \leq 1$, $\|g\| \leq 1$. Then $\sup_n \{\text{rank } g(n)\} < +\infty$.

Proof. Choose p in N so that $n \geq p$ implies that

$\|f(n) - f(\infty)\| < \frac{1}{2}$ where ∞ denotes the 'point at infinity' in N^* . By 2.8.14 $m(f(n), 1) \leq \sum_{\lambda \geq \frac{1}{2}} m(f(\infty), \lambda)$

$< +\infty$, since $f(\infty)$ is a compact operator. Thus

$\sup_{n \geq p} m(f(n), 1) < +\infty$ so that $\sup_n m(f(n), 1) < +\infty$.

Hence $\sup_n \text{rank } g(n)$ is finite also.

2.8.16 LEMMA. Suppose T is a compact operator on a Hilbert space and $\{e_n\}$ is an orthonormal sequence. If $\epsilon > 0$ there is an integer p so that $n > p \Rightarrow \|T(e_n) - e_n\| > \epsilon$.

2.8.17 EXAMPLE. Another 'obvious' conjecture is that if $f \in C^*(N^*; LCH)$, then f is factorizable if $\sup_n \text{rank } f(n) < +\infty$. This is false as the following example shows. Let $f(n) = \frac{1}{n}$ times the projection onto e_n where $\{e_n\}_{n=1,2,3,\dots}$ is an orthonormal sequence in an infinite-dimensional Hilbert space H . Note that f is not factorizable by applying 2.8.16.

2.8.18 LEMMA. Suppose T is a completely regular space, K a compact subset of T , $\epsilon > 0$, E a Banach space, and f a continuous function from K into E . Then f has an extension g to T such that $\|g\| \leq \|f\| + \epsilon$.

If T is locally compact, g may be chosen in $C_0(T;E)$.

Proof. First we do a special case. Assume that f is

defined by the formula (for $t \in K$) $f(t) = \sum_{i=1}^n f_i(t)e_i$

where $\{f_i\}_{i=1}^n \subseteq C^*(K)$ and $\{e_i\} \subseteq E$. Extend each func-

tion f_i to $g_i \in C^*(T)$ by Tietze's Theorem and let

$h(t) = \sum_{i=1}^n g_i(t)e_i$ for $t \in T$. Let

$O = \{t \in T: \|g(t)\| < \|f\| + \epsilon\}$ and let $P: T \rightarrow [0,1]$ be continuous and satisfy $p \equiv 1$ on K , $p \equiv 0$ on $T \setminus O$.

Let $g(t) = \sum_{i=1}^n p(t)g_i(t)e_i$ for $t \in T$ and note that

$\|g\| < \|f\| + \epsilon$. Let $C(K) \otimes E$ denote the subspace of

$C^*(K;E)$ consisting of functions representable in the form

assumed for the special case of the theorem done in the

preceding paragraph. Note $C(K) \otimes E$ is a linear subspace

of $C^*(K;E)$.

Let us assume for the time being that a sequence

$\{f_n\}_{n=1}^{\infty}$ can be chosen from $C(K) \otimes E$ so that

$\|f_n - f\| \rightarrow 0$. We will show below that a sequence satis-

fying this condition can be chosen. By extracting a sub-

sequence, if necessary, assume that $\|f_n - f_{n-1}\| < \frac{1}{2^{n-1}}$

for $n = 2, 3, \dots$. Let g_1 be a function in $C^*(T;E)$

which extends f_1 . The special case of this theorem,

done in the first paragraph, guarantees the existence of

g_1 . Let h_1 be an extension of $f_2 - f_1$ in $C^*(T;E)$ which satisfies $\|h_1\| < \frac{1}{2}$. Let $g_2 = h_1 + g_1$. Then $\|g_2 - g_1\| < \frac{1}{2}$ and g_2 extends f_2 . Suppose that g_1, \dots, g_n in $C^*(T;E)$ satisfy: (1) g_i extends f_i $1 \leq i \leq n$; (2) $\|g_i - g_{i-1}\| < \frac{1}{2^{i-1}}$ for $2 \leq i \leq n$.

Let h_n be an extension in $C^*(T;E)$ of $f_{n+1} - f_n$ satisfying $\|h_n\| < \frac{1}{2^n}$ and let $g_{n+1} = h_n + g_n$. Note that g_{n+1} extends f_{n+1} and $\|g_{n+1} - g_n\| < \frac{1}{2^n}$. By induction, we have a sequence $\{g_n\}_{n=1}^\infty \subseteq C^*(T;E)$ satisfying the following conditions: (1) g_n extends f_n and (2) $\|g_{n+1} - g_n\| < \frac{1}{2^n}$. Clearly then $\{g_n\}$ is Cauchy in $C^*(T;E)$. If g is the limit of $\{g_n\}_{n=1}^\infty$ then $g \in C^*(T;E)$ and g extends f . By multiplying g by a suitably chosen function p (chosen by Urysohn's Lemma) that has value one on K , we may assure that $\|g\| < \|f\| + \epsilon$. Note that p can be chosen in $C_c(T)$ if T is locally compact, so that $g \in C_0(T;E)$.

All that remains to be shown is that f is the limit of a sequence $\{f_n\}$ in $C(K) \otimes E$. Let $\epsilon > 0$. Since $f(K)$ is bounded, K has a finite open cover $\{O_i\}_{i=1}^n$ so that if $x, y \in O_i$ $|f(x) - f(y)| < \epsilon$. Let $\{\varphi_i\}_{i=1}^n$ be a partition of unity [19] on K subordinate to the cover $\{O_i\}_{i=1}^n$. Then for each $1 \leq i \leq n$ $0 \leq \varphi_i \leq 1$,

$\varphi_i \in C^*(K)$ and $\sum_{i=1}^n \varphi_i = 1$. Choose a point $x_i \in O_i$ and let h in $C(K) \otimes E$ be the function defined by the equation $h(x) = \sum_{i=1}^n \varphi_i(x)f(x_i)$ for $x \in K$. Note that $\|h - f\| < \epsilon$. Hence it is clear that the sequence $\{f_n\}_{n=1}^{\infty}$ can be chosen as assumed.

2.8.19 THEOREM. Suppose T is a locally compact Hausdorff space and T contains a point p which has a discrete sequence $\{p_n\}_{n=1}^{\infty} \subseteq S$ such that $p_n \rightarrow p$. Then if H is an infinite-dimensional Hilbert space $C_0(T;LCH)$ does not have Property (B), where LCH denotes the algebra of compact operators on H .

Proof. Let $K = \{p_n\} \cup \{p\}$. Clearly K is compact and homeomorphic to N^* , the one-point compactification of N , the space of positive integers with the discrete topology. From 2.8.10 and 2.8.18, we see that $C_0(T;LCH)$ cannot have Property (B) unless the algebra $A = C^*(N^*;LCH)$ has Property (B). We shall show that A does not have Property (B). Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal sequence in H . For each $n \in N^*$, let $f(n)$ denote the projection of H onto the span of e_1 . Let f denote the constant function $\{f(n)\}_{n \in N^*}$ in A . Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of positive real numbers satisfying:

$a_n^2 + b_n^2 = 1$ for all n , and a_n converges upwards to 1. Let $y_n = a_n e_1 + b_n e_{n+1}$. Note that $\|y_n\| = 1$ for all positive integers n and $y_n \rightarrow e_1$. For $n \in \mathbb{N}$, let $g(n)$ denote the projection of H onto y_n . Let g be the function in A with co-ordinates $g(n)$ for $n \in \mathbb{N}$ and $g(\infty) = f(1)$. (Note that $y_n \rightarrow e_1$ implies that $g(n) \rightarrow f(1)$.) Since $f^2 = f$ and $g^2 = g$, f and g are clearly factorizable. We will show that $f + g$ is not. Suppose $\exists a \in A$ so that $a(f + g) = (f + g)a = f + g$. We may assume that $a \geq 0$ and $\|a\| \leq 1$. By the familiar C^* -norm argument, we may then assume that $af = f$ and $ag = g$. Hence $a(n)(e_n) = e_n$ so that for n large enough $\|a(\infty)(e_n) - e_n\| < \frac{1}{2}$. This contradicts the fact that $a(\infty) \in LCH$, by 2.8.16.

2.8.20 EXAMPLE. An interesting question is whether every C^* -algebra has an abelian approximate identity. An example is given in [1] to show this is not true.

In looking for examples of C^* -algebras, in our study of approximate identities, we were led to group algebras and the group C^* -algebras of a locally compact topological group [17]. Let $L^1(G)$ denote the group algebra of the locally compact group G and m denote Haar measure on

G . We will use the symbol $C^*(G)$ for the group C^* -algebra of G , for the rest of Section 8. There is no problem of confusing this symbol with our previous use of it to represent the set of bounded continuous functions, since we shall not use it this way for the rest of Chapter II.

2.8.21 DEFINITION. Let G be a group and $U \subseteq G$. U is said to be invariant if $xUx^{-1} \subseteq U$ for each $x \in G$. U is said to be symmetric if $x \in U$ implies $x^{-1} \in U$.

2.8.22 LEMMA. Let G be a compact group and W an open neighborhood of the identity e . Then \exists an invariant open set U containing e so that $U \subseteq W$.

Proof. Suppose not. Then $\exists W$, a neighborhood of the origin, so that for each neighborhood U of the identity there are elements $x_U \in G$ and $y_U \in U$ so that $x_U y_U x_U^{-1} \notin W$. Regard $\{x_U\}$ and $\{y_U\}$ as nets. Since $\bigcap U = e$, $y_U \rightarrow e$. By taking subnets and using compactness, assume that $x_U \rightarrow x_0 \in G$. Hence $x_U y_U x_U^{-1} \rightarrow e$. This contradicts the fact that W is a neighborhood of e and concludes the proof of the lemma.

2.8.23 LEMMA. Let G be a unimodular [38] locally compact group with a neighborhood basis at the identity consisting of invariant, symmetric sets. Then $L^1(G)$ has an abelian approximate identity.

Proof. Let \mathcal{U} denote this neighborhood basis of symmetric, invariant sets. For $U \in \mathcal{U}$ let f_U be the function $(m(U))^{-1}$ times the characteristic function of U , where m denotes Haar measure on G . It is easy to see that the net $\{f_U\}$, with the U 's ordered by reverse inclusion, is a bounded approximate identity for $L^1(G)$. It is also clear that $f_U(xy) = f_U(yx)$ for every pair x and y in G . We shall show that $\{f_U\}$ is contained in the center of $L^1(G)$. Let $*$ denote convolution in $L^1(G)$. Then if $x \in G$ and $f \in L^1(G)$, $(f * f_U)(x) = \int_G f(y) f_U(y^{-1}x) dy = \int_G f(y) f_U(xy^{-1}) dy = \int_G f(y^{-1}x) f_U(y) dy = \int_G f_U(y) f(y^{-1}x) dy = (f_U * f)(x)$. Hence $f * f_U = f_U * f$, for every $f \in L^1(G)$, i.e., $\{f_U\}$ is contained in the center of $L^1(G)$.

2.8.24 COROLLARY. If G is a compact group, $L^1(G)$ has an abelian approximate identity.

We are interested in studying approximate identities

in the group C^* -algebras $C^*(G)$ [17, 13.9] of a locally compact group. Our next result concerns the existence of abelian approximate identities in $C^*(G)$.

2.8.25 THEOREM. Suppose the locally compact group G is either (1) separable, (2) abelian, (3) compact, (4) a group with an invariant neighborhood basis, or (5) first countable. Then $C^*(G)$ has an abelian approximate identity.

Proof. Recall that $C^*(G)$ is the completion of $L^1(G)$ under a smaller norm than the given norm on $L^1(G)$. If G is separable, then so is $L^1(G)$ [17, 13.2.4]. Hence $C^*(G)$ is separable and so has an abelian approximate identity by the results in Section 4 of this chapter. If G is abelian, so is $C^*(G)$. If G is compact or has an invariant neighborhood basis, $L^1(G)$ has an abelian approximate identity by 2.8.23 and 2.8.24. This approximate identity is clearly an approximate identity for $C^*(G)$. Finally, if G is first countable then $L^1(G)$ has a countable approximate identity; hence, $C^*(G)$ has a countable approximate identity and we apply the results of Section 4 again.

We have three more results in this chapter. The first is stated without proof, since its proof is similar to that of 2.8.23. The other two results have to do with Property (B) and canonical approximate identities for $L'(G)$ for G abelian.

2.8.26 PROPOSITION. Let G be a locally compact group. Then $L'(G)$ has an approximate identity $\{f_U\}$ based on a neighborhood basis \mathcal{U} such that if $U_1, U_2 \in \mathcal{U}$ and $U_1 \subset U_2$, then $f_{U_1} * f_{U_2} = f_{U_2} * f_{U_1}$.

2.8.27 PROPOSITION. Let G be an abelian locally compact group. Then $L'(G)$ has Property (B).

Proof. For $f \in L'(G)$, let \hat{f} denote the Fourier transform of f [46,38]. Also let \hat{G} denote the character group of G . Note that $\hat{f} \in C_0(\hat{G})$ so that if f is factorizable in $L'(G)$, $\hat{f} \in C_c(\hat{G})$. Since $f \rightarrow \hat{f}$ is a linear mapping, the fact that if $K \subseteq \hat{G}$ is compact then $\exists h \in L'(G)$ such that $0 \leq \hat{h}$ and $\hat{h} = 1$ on K [46, 2.6.8] shows that $L'(G)$ has Property (B).

2.8.28 PROPOSITION. Let G be a locally compact abelian group. Then $L'(G)$ contains an approximate identity

$\{e_\lambda \mid \lambda \in \Lambda\}$ satisfying:

$$(1) \quad \|e_\lambda\| \leq 2 \quad \forall \lambda \in \Lambda$$

$$(2) \quad \lambda_1 < \lambda_2 \Rightarrow e_{\lambda_1} e_{\lambda_2} = e_{\lambda_1} .$$

Proof. Let $\Lambda = \{f \in L'(G) : 0 \leq \hat{f} \leq 1, \hat{f} \in C_c(\hat{G}), \|f\| \leq 2\}$. Order V as follows $f_1 \geq f_2$ iff $f_1 = f_2$ or $f_1 * f_2 = f_2$. This is clearly a partial ordering. We must show that Λ is directed under this ordering. This follows from [46, 2.6.8] which asserts if K is a compact subset of \hat{G} , there is an element g of $L'(G)$ such that $0 \leq \hat{g} \leq 1$, $\|g\| \leq 2$, and $\hat{g} = 1$ on K . We now show that $\{f\}_{f \in \Lambda}$ is an approximate identity for $L'(G)$. From [46, Section 2.6] we have that $\{g \in L'(G) : g \text{ has compact support}\}$ is dense in $L'(G)$. It suffices to show that if $g \in L'(G)$ such that \hat{g} has compact support, then $\exists f \in V$ so that $\hat{f} = 1$ on the support of \hat{g} (hence $f * g = g$ since $\hat{f} \hat{g} = \hat{g}$ and $L'(G)$ is semisimple [38]). This is guaranteed by [46, 2.8.3]. This concludes the proof.

CHAPTER III
DOUBLE CENTRALIZER ALGEBRAS

In this chapter we extend certain results of Buck and others concerning the strict topology. We have several interesting results including characterizations of nuclearity and semireflexivity for the double centralizer algebra of a C^* -algebra, endowed with the strict topology. We then develop a generalization to C^* -algebras of topological measure theory and obtain many interesting theorems. Finally, we give some partial results concerning a general Stone-Weierstrass Theorem for double centralizer algebras; although our results on this topic are incomplete, we feel that techniques developed in our study of the question may lead us to a solution.

In the first section of this chapter, the double centralizer algebra of a C^* -algebra is defined and some of its important properties listed. Our references are [6,59].

SECTION 1. PRELIMINARIES.

3.1.1 DEFINITION. Let A be a C^* -algebra. By a double

centralizer on A , we mean a pair (T', T''') of functions from A to A such that $xT'(y) = T'''(x)y$ for x, y in A . We denote the set of all double centralizers of A by $M(A)$.

3.1.2 THEOREM. Let $(T', T''') \in M(A)$. Then

- (1) T' and T''' are bounded linear operators from A to A .
- (2) $T'(xy) = T'(x)y \quad \forall x, y \in A$
- (3) $T'''(xy) = xT'''(y) \quad \forall x, y \in A$

3.1.3 LEMMA. If $(T', T''') \in M(A)$, $\|T'\| = \|T'''\|$.

3.1.4 DEFINITION. Let L be a function from A to A .

Let L^* be the function from A to A satisfying

$$L^*(x) = (L(x^*))^* \quad \forall x \in A.$$

3.1.5 LEMMA. Let $(T', T''') \in M(A)$. Then (T''', T') $\in M(A)$.

3.1.6 LEMMA. Let (T', T''') and (S', S''') belong to $M(A)$. Then $(T'S', S''T''') \in M(A)$.

3.1.7 DEFINITION. Let (T', T''') and (S', S''') belong to $M(A)$ and λ be a complex number. Define $*$ -algebra and

norm structures on $M(A)$ as follows:

- (1) $(T', T'') + (S', S'') = (T' + S', T'' + S'')$
- (2) $\lambda(T', T'') = (\lambda T', \lambda T'')$
- (3) $(T', T'')(S', S'') = (T'S', S'T'')$
- (4) $(T', T'')^* = (T''^*, T'^*)$
- (5) $\|(T', T'')\| = \|T'\| = \|T''\|$.

3.1.8 THEOREM. $M(A)$, provided with the operations and norm defined in 3.1.7, is a C^* -algebra with identity.

3.1.9 THEOREM. $\forall a \in A$, let L_a be the operator in A defined by $L_a(b) = ab$, $\forall b \in A$, and let R_a be the operator in A defined by $R_a(b) = ba$, $\forall b \in A$. Then $(L_a, R_a) \in M(A)$. Let $\mu_0: A \rightarrow M(A)$ be the map $\mu_0(a) = (L_a, R_a)$ $\forall a \in A$. Then μ_0 is an injective $*$ -homomorphism and $\mu_0(A)$ is a closed two-sided ideal in $M(A)$. The map μ_0 is surjective iff A has an identity. $M(A)$ is commutative if and only if A is commutative.

3.1.10 DEFINITION. The strict topology is that locally convex topology on $M(A)$ generated by the seminorms $x \rightarrow \|yx\|$ and $x \rightarrow \|xy\|$ for $y \in A$ and $x \in M(A)$ (regarding A as contained in $M(A)$ by virtue of 3.1.9). Let β denote the strict topology on $M(A)$ and $M(A)_\beta$ de-

note $M(A)$ endowed with β .

3.1.11 PROPOSITION. A is β -dense in $M(A)$ and $M(A)$ is β -complete.

3.1.12 EXAMPLES. Wang [63] showed that $M(C_0(S)) = C^*(S)$. Reid in [43] shows that if $A \subseteq B(H)$, then $M(A) = \{t \in B(H) : tA \subseteq A \text{ and } At \subseteq A, te = et = t\}$ where e is the projection of the Hilbert space H onto $A \cdot H$.

3.1.13 THEOREM. Suppose B is a C^* -algebra containing the C^* -algebra A as a two-sided ideal. Then there is a unique $*$ -homomorphism $\mu : B \rightarrow M(A)$ with the property that $\mu(x) = \mu_0(x)$ for $x \in A$. Let $A^0 = \{x \in B \mid xA = \{0\}\}$. Then A^0 is the kernel of μ . Let β_A denote the strict topology on B generated by A . Then $\mu : (B, \beta_A) \rightarrow M(A)_\beta$ is a continuous vector space homomorphism open onto its image. Suppose that $A^0 = \{0\}$ and that B is β_A complete. Then μ is an imbedding of B in $M(A)$ and is onto in this case, i.e., $B = M(A)$.

The next results, 3.1.14 - 3.1.16, are due to D. C. Taylor [59].

3.1.14 THEOREM. If A is a C^* -algebra, then

$M(A)_{\beta}' = \{a \cdot f : a \in A, f \in M(A)'\} = \{f \cdot a : a \in A \text{ and } f \in M(A)'\}$, where $a \cdot f(x) = f(xa)$ and $f \cdot a(x) = f(ax)$ $\forall x \in M(A)$. Under the strong topology, $M(A)_{\beta}'$ is a Banach space that is isomorphic with A' .

3.1.15 THEOREM. Let A be a C^* -algebra and $\{e_{\lambda} : \lambda \in \Lambda\}$ an approximate identity for A . If $H \subseteq M(A)_{\beta}'$ the following are equivalent:

- (1) H is β -equicontinuous;
- (2) H is uniformly bounded and $e_{\lambda} \cdot f + f \cdot e_{\lambda} - e_{\lambda} \cdot f \cdot e_{\lambda} \rightarrow f$ uniformly on H (see 3.1.14).

Using 3.1.15, Taylor shows that $\beta = \beta'$, where β' is the finest locally convex linear topology on $M(A)$ agreeing with β on bounded sets of $M(A)$. He also shows that if A has a well-behaved approximate identity, then $M(A)_{\beta}$ is a strong Mackey space, i.e., every β -weak- $*$ -compact subset of $M(A)_{\beta}'$ is β -equicontinuous. Taylor also proves the following result which we will need later in Chapter III.

3.1.16 THEOREM. Let B be a C^* -algebra and A be a closed two-sided ideal of B . If $f \in B'$, $Ef_1 \in (B, \beta_A)'$

and $f_2 \in A^\perp = \{g \in B: g \equiv 0 \text{ on } A\}$ so that $f = f_1 + f_2$.
 Furthermore if f is a positive linear functional on B
 then so are f_1 and f_2 .

SECTION 2. BASIC RESULTS.

We begin our study of $M(A)$ by proving some facts,
 3.2.1 - 3.2.5, which may be known but do not appear ex-
 plicitly in the literature, as far as we know.

3.2.1 PROPOSITION. Let S denote the unit sphere in $M(A)$.
 Then S is closed in the strict topology.

Proof. Suppose (T_α, T'_α) is a net in $M(A)$ and
 $(T'_\alpha, T_\alpha) \rightarrow (T, T')$ strictly. Regard A as contained in
 $M(A)$ and note that the strict topology on $M(A)$ is a type
 of pointwise convergence on A , i.e., $(S'_\alpha, S_\alpha) \rightarrow 0$
 strictly iff $S_\alpha(x) \rightarrow 0$ in norm and $S'_\alpha(x) \rightarrow 0$ in norm
 for each $x \in A$. Thus $T'_\alpha(x) \rightarrow T(x)$ for each $x \in A$.
 Hence $\|T(x)\| \leq \|x\|$ for all $x \in A$, i.e., $\|T\| \leq 1$.
 Hence $\|(T, T')\| \leq 1$ by 3.1.3.

Our next result is a partial generalization of Conway's
 "Ascoli" Theorem for $C^*(S)_\beta$ [11, 12].

3.2.2 PROPOSITION. Let $F \subseteq M(A)$. The following are equivalent:

- (1) F is totally bounded in the strict topology;
- (2) F is norm bounded and totally bounded in the kappa topology (see 3.2.3) of some bounded approximate identity for A ;
- (3) F is norm bounded and totally bounded in the kappa topology of all bounded approximate identities for A .

3.2.3 DEFINITION. Let A be a C^* -algebra and $\{e_\lambda\}$ a bounded approximate identity for A . The kappa topology associated with $\{e_\lambda\}$ is that locally convex topology on $M(A)$ generated by the seminorms $x \rightarrow \|xe_\lambda\|$ and $x \rightarrow \|e_\lambda x\|$ for $x \in M(A)$ [52]. For example, if $A = C_0(S)$ and a bounded approximate identity is chosen from $C_0(S)$, the corresponding kappa topology is the compact-open topology. This example is, in large, the motivation for the definition of kappa topology as given in [52].

Proof of 3.2.2. This proposition follows from the fact that all kappa topologies agree with the strict topology on norm bounded subsets of $M(A)$. It is clear from the definition that β is finer than any kappa topology. Let $\{e_\lambda\}$ be an approximate identity in A and suppose $\{f_\nu\}$

is a net in the ball of $M(A)$ such that $f_b \rightarrow 0$ in the kappa topology associated with $\{e_\lambda\}$. Let $a \in A$ and $\epsilon > 0$. Choose λ_0 so that $\lambda \geq \lambda_0$ implies $\|a - e_\lambda a\| < \frac{\epsilon}{2}$ and $\|a - ae_\lambda\| < \frac{\epsilon}{2}$. Choose b_0 so that $b \geq b_0$ implies $\|f_b e_{\lambda_0}\| < \frac{\epsilon}{2\|a\|}$ and $\|e_{\lambda_0} f_b\| < \frac{\epsilon}{2\|a\|}$. Then for $b \geq b_0$, $\|f_b a\| \leq \|f_b e_{\lambda_0} a - f_b a\| + \|f_b e_{\lambda_0} a\| < \epsilon$. Similarly $\|af_b\| < \epsilon$ for $b \geq b_0$, i.e., $f_b \rightarrow 0\beta$.

The next result 3.2.4 is a generalization of 2.1.3 (a).

3.2.4 PROPOSITION. Let A be a C^* -algebra. The β and norm topologies on $M(A)$ coincide iff A has an identity, i.e., iff $A = M(A)$.

Proof. Clearly, we need only show that if the norm and β topologies coincide on $M(A)$ then A has an identity. If this is the case, there exists $a \geq 0$ in A so that $\|x\| \leq \max\{\|ax\|, \|xa\|\}$ for all $x \in M(A)$. Let C be the commutative C^* -algebra generated by a . Then C may be regarded as the algebra $C_0(S)$, where the locally compact space S is the maximal ideal space of C . The inequality $\|x\| \leq \|ax\|$ for $x \in C$ implies that $\|a(s)\| \geq 1$ for all

$s \in S$, by Urysohn's lemma. Thus S is compact, i.e.,
 $\exists e \in C$ so that $ea = ae = a$. Let $y \in A$. Then if I
denotes the identity in $M(A)$, $a[(I-e)yy^*(I-e)] = 0 =$
 $[(I-e)yy^*(I-e)]a$. Hence $(I-e)yy^*(I-e) = 0$ and thus
 $ey = y = ye$ from the C^* -norm property.

3.2.5 PROPOSITION. Let A be a C^* -algebra. Then $M(A)_\beta$
is metrizable iff A has an identity.

Proof. If $M(A)_\beta$ is metrizable, the open mapping theorem
for Fréchet spaces [45] applied to the identity mapping from
 $(M(A), \text{norm})$ to $M(A)_\beta$ yields that the β and norm topo-
logies agree. Apply 3.2.4.

The next result is used in the proofs of some of the
principal results of this chapter. Let H be a Hilbert
space and LCH denote the algebra of compact operators and
 $B(H)$, the algebra of bounded linear operators on H .
 $M(LCH)$ can be canonically identified with $B(H)$ (see
3.1.13), so the expression strict-LCH topology on $B(H)$
makes sense.

3.2.6 PROPOSITION. Let H be a Hilbert space. The unit
ball in $B(H)$ is compact in the strict-LCH topology iff

H is finite-dimensional.

Proof. If H is finite-dimensional, $LCH = B(H)$ and the strict topology is the norm topology. It is well known that the bounded closed sets in a finite-dimensional normed space are compact.

For the converse, note that β is finer than the topology of pointwise convergence on H . Hence the unit ball of $B(H)$ is compact under the topology of pointwise convergence on H . Let $x \in H$ and $\|x\| = 1$. There are finite-rank operators T_1, \dots, T_n in the ball of $B(H)$ so that if T is any finite-rank operator in the ball of $B(H)$, $\|T(x) - T_i(x)\| < 1$ holds for at least one $1 \leq i \leq n$. If H is not finite-dimensional, choose $y \in H$ so that $\|y\| = 1$ and y is orthogonal to the set $\{T_i(x) | 1 \leq i \leq n\}$. Let $T(z) = \langle z, x \rangle y$ for $z \in H$, where ' $\langle \rangle$ ' denotes the inner product in H . T is a finite-rank operator in the unit ball of $B(H)$. But $\|T(x) - T_i(x)\|^2 = \|y - T_i(x)\|^2 = \|y\|^2 + \|T_i(x)\|^2 \geq 1$ for all $1 \leq i \leq n$. This contradicts the assumption that H is infinite-dimensional.

The following question is motivated by Reid's characterization of $M(A)$ for subalgebras A of $B(H)$. Suppose

$H_1 = A \cdot H$ and $R: B(H) \rightarrow B(H_1)$ is the map taking $a \in A$ to ea restricted to H_1 , where e is the projection of H onto H_1 . Is $R(M(A)) = M(R(A))$? Note that H_1 is a Hilbert space by 1.12.

3.2.7. Let A, H, H_1 , and R be as above. Then R , restricted to $M(A)$, defines an isometric $*$ -isomorphism of $M(A)$ into $B(H_1)$ (here we are taking $M(A) \subseteq B(H)$ by Reid's Theorem). Furthermore $R(M(A)) = M(R(A))$.

Proof. By Reid's theorem, $M(A) = \{t \in B(H): ete=t \text{ and } tA + At \subseteq A\}$. Thus e commutes with everything in $M(A)$ so R is a $*$ -homomorphism. If $x \in M(A)$ and $R(x) = 0$ then $xe = 0$ so $x = exe = 0$. Thus R is one-to-one and so an isometry [17]. Finally, if $t \in M(R(A))$, let $s = ete$. Then $R(s) = t$. This concludes the proof.

Our next result relates the double centralizer algebra of a C^* -algebra A with the double centralizer algebra of the closed C^* -subalgebra generated by an approximate identity for A .

3.2.8 PROPOSITION. Let A be a C^* -algebra and A_0 the

C^* -subalgebra of A generated by some approximate identity for A . Then $M(A_0)$ is the idealizer of A_0 in $M(A)$, i.e., $M(A_0) = \{x \in M(A) : xA_0 + A_0x \subseteq A_0\}$.

Proof. Let I denote the idealizer of A_0 in $M(A)$ and let $\{e_\lambda\}$ be the approximate identity for A which generates A_0 . We shall apply 3.1.3. Clearly A_0 is an ideal in I . Suppose $x \in I$ and $xA_0 = \{0\}$. Then $xe_\lambda = 0$ for all λ , so $x = 0$. Let β_0 denote the topology on $M(A)$ generated by the seminorms $x \rightarrow \|xy\|$ and $x \rightarrow \|yx\|$ for $x \in M(A)$ and $y \in A_0$. From [51, Cor. 2.3] we have that $\beta = \beta_0$. Since I is β -closed, it is β -complete. Thus I is β_0 -complete. Hence, by 3.1.3, I is canonically isomorphic with $M(A_0)$.

The question of necessary and sufficient conditions on a C^* -algebra A in order that the unit ball of $M(A)$ be β -compact is an interesting one, raised by Collins [7] in the commutative case. We answer the question in general in the next theorem.

3.2.9 THEOREM. Let A be a C^* -algebra. Then $M(A)$ has a β -compact unit ball iff $A = (\Sigma A_\lambda)_0$ where each A_λ is a finite-dimensional C^* -algebra.

Proof. First, suppose that $A = (\Sigma A_\lambda)_0$ where each A_λ is a finite-dimensional C^* -algebra. Since the unit ball in $M(A)$ is β -complete, it suffices to show it is β -totally bounded. Let $a = (a_\lambda) \in A$ and assume, without loss of generality, that $\|a\| \leq 1$. Since each A_λ is finite-dimensional, it has an identity, i.e., $M(A_\lambda) = A_\lambda$. Hence $M(A) = \Sigma A_\lambda = \{a \in \prod A_\lambda : a = (a_\lambda) \text{ and } \sup \|a_\lambda\| < +\infty\}$, with pointwise operations [59]. Let $\epsilon > 0$ and choose a finite set $\lambda_1, \dots, \lambda_n$ so that $\sup_{\lambda \notin \{\lambda_i\}_{i=1}^n} \|a_\lambda\| < \epsilon$. For each $1 \leq i \leq n$, let S_i be a finite set in the unit ball of A_{λ_i} so that if $x \in A_{\lambda_i}$ and $\|x\| \leq 1$, then $\min_{s \in S_i} \|s-x\| < \epsilon$. Suppose $x = (x_\lambda) \in M(A)$ and $\|x\| \leq 1$. Let $y_i \in S_i$ so that $\|x_{\lambda_i} - y_i\| < \epsilon$. Let $y = (y_\lambda) \in M(A)$ where $y_{\lambda_i} = y_i$ and $y_\lambda = 0$ if $\lambda \notin \{\lambda_i\}_{i=1}^n$. Note that $\|(y-x)a\| < \epsilon$ and $\|a(y-x)\| < \epsilon$. Since the set of all such elements y is a finite set depending only on a and ϵ , the unit ball of $M(A)$ is β -totally bounded.

Conversely, suppose the unit ball of $M(A)$ is β -totally bounded. Let S denote the unit ball in A . Then S is β -totally bounded so that aS and Sa are norm totally bounded (hence, norm relatively compact) for each $a \in A$. By [17, p. 99] A is dual, i.e., $A = (\Sigma A_\lambda)_0$

where each A_λ is the algebra LCH_λ of compact operators on the Hilbert space H_λ . We shall show that H_λ is finite-dimensional for each λ . From 3.1.12, $M(A_\lambda) = B(H_\lambda)$. Since $M(A) = \Sigma B(H_\lambda)$, it is clear that the unit ball in $B(H_\lambda)$ is compact in the strict topology defined by $A_\lambda = LCH_\lambda$, for each λ . By 3.2.6 we get that H_λ is finite-dimensional for each λ ; hence $A_\lambda = LCH_\lambda$ is finite-dimensional also.

3.2.10 DEFINITION. Let E be a locally convex space and E' its adjoint. Recall that E is said to be (DF) (the terminology is due to Grothendieck) if it has a countable base for bounded sets and the union of a countable number of equicontinuous sets in E' is again equicontinuous, provided it is bounded in the strong topology on E' . An equivalent formulation is that E has a countable base for bounded sets and any countable intersection of closed absolutely convex zero neighborhoods in E which absorbs bounded sets of E is a zero neighborhood. If E is complete, we need only check that this intersection absorbs points [45, p. 67]. Since $M(A)_\beta$ has a countable base of bounded sets and is complete, it is (DF) if each countable intersection of closed absolutely convex zero neighborhoods which absorbs points of $M(A)$ is a zero neigh-

borhood.

For the case $C_0(S)$, Collins and Dorroh [8] say that $C^*(S)_\beta$ has (WDF) if each uniformly bounded real sequence in $C_0(S)$ has an upper bound in $C_0(S)$. They point out that this is equivalent to requiring that each countable intersection of zero neighborhoods of type $V_a = \{f \in C^*(S) : \|fa\| \leq 1\}$, for some $a \in C_0(S)$ which absorbs points of $C^*(S)$, be a β zero neighborhood. We use this idea to define property (WDF) for $M(A)_\beta$.

3.2.11 DEFINITION. Let A be a C^* -algebra. $M(A)_\beta$ has (WDF) if, for each Hermitian sequence $\{a_n\}_{n=1}^\infty$ in A , $V = \bigcap_{n=1}^\infty V_n$ absorbs points of $M(A)$ implies V is a β zero neighborhood in $M(A)$, where $V_n = \{x \in M(A) : \|xa_n\| \leq 1 \text{ and } \|a_n x\| \leq 1\}$.

Collins and Dorroh asked whether the properties (DF) and (WDF) are equivalent for $C^*(S)_\beta$ after first showing that (DF) was at least as strong as (WDF). Summers [54] answered the question in the affirmative for $C^*(S)_\beta$ and we have the generalization of his result to the noncommutative case.

3.2.12 THEOREM. Let A be a C^* -algebra. Then the following are equivalent:

- (1) $M(A)$ is (DF) ;
- (2) $M(A)$ is (WDF) ;
- (3) $\ell^\infty(A) = C^*(N:A)$ is an essential A -module under both left and right action.

Proof. (1) implies (2) is clear. To show (2) \rightarrow (3), let $\{e_\lambda\}$ be an approximate identity in A . Suppose $a = \{a_n\}_{n=1}^\infty \in \ell^\infty(A)$, i.e., $\{a_n\}_{n=1}^\infty$ is a uniformly bounded sequence in A . Let $V_n = \{x \in M(A) : \|xa_n\| \leq 1 \text{ and } \|a_n x\| \leq 1\}$ and $V = \bigcap_{n=1}^\infty V_n$. Note that V is closed absolutely convex and absorbs points of $M(A)$. By the assumption that A has property (WDF), we get that V is a β zero neighborhood. Note that $e_\lambda \xrightarrow{\lambda} I$ in the strict topology, where I denotes the identity in $M(A)$. Hence $\exists \lambda_0$ so that $\lambda \geq \lambda_0$ implies $e_\lambda - I \in V$, i.e., $\|(e_\lambda - I)a_n\| \leq 1$ and $\|a_n(e_\lambda - I)\| \leq 1$ for all n . Hence for $\lambda \geq \lambda_0$ $\|e_\lambda a_n - a_n\| \leq 1$ and $\|a_n e_\lambda - a_n\| \leq 1$ for all n . Thus $\ell^\infty(A)$ is an essential left A -module and an essential right A -module since a was an arbitrary sequence in $\ell^\infty(A)$.

To prove (3) \rightarrow (1), let us assume that $\ell^\infty(A)$ is essential. Let $H = \bigcup_{n=1}^\infty H_n$ be a subset of the unit sphere in

$M(A)'_{\beta}$ and each H_n be equicontinuous. We must show that

H is equicontinuous. By examining D. C. Taylor's proof

[59] of his criterion for β -equicontinuity, we see that

$\exists a = \{a_n\}_{n=1}^{\infty} \in \ell^{\infty}(A)$, $\|a\| \leq \delta$, so that

$\{x \in M(A): \|xa_n\| \leq 1 \text{ and } \|a_n x\| \leq 1\} \subset H_n^0$. Thus

$H \subseteq \bigcup_{n=1}^{\infty} H_n^{00} \subseteq \bigcup_{n=1}^{\infty} \{x \in M(A): \|xa_n\| \leq 1 \text{ and } \|a_n x\| \leq 1\}^0$.

From [45, p. 36] we have that $\{x \in M(A): \|xa_n\| \leq 1 \text{ and}$

$\|a_n x\| \leq 1\}^0 =$ the β -weak-* closure of the set sum

$\{x \in M(A): \|xa_n\| \leq 1\}^0 + \{x \in M(A): \|a_n x\| \leq 1\}^0$. Let β_1

and β_2 denote the topologies on $M(A)$ given by right and left multiplications, respectively, by elements of A .

Taylor [59] has shown that $M(A)'_{\beta_1} = M(A)'_{\beta_2} = M(A)'_{\beta}$. By

Alaoglu's Theorem [45] applied to $M(A)_{\beta_1}$,

$\{x \in M(A): \|xa_n\| \leq 1\}^0$ is β -weak-* compact in $M(A)'_{\beta}$.

By Alaoglu's Theorem applied to $M(A)_{\beta_2}$,

$\{x \in M(A): \|a_n x\| \leq 1\}^0$ is β -weak-* compact in $M(A)'_{\beta}$.

Thus the set sum $\{x \in M(A): \|xa_n\| \leq 1\}^0 + \{x \in M(A):$

$\|a_n x\| \leq 1\}^0$ is β -weak-* compact and hence β -weak-* closed

as a subset of $M(A)'_{\beta}$. Our next step is to calculate

$\{x \in M(A): \|a_n x\| \leq 1\}^0$ and $\{x \in M(A): \|xa_n\| \leq 1\}^0$. For

$f \in M(A)'$ and $b \in M(A)$ let $f \cdot b(x) = f(bx)$ and $b \cdot f(x) =$

$f(xb)$ for $x \in M(A)$ and note that $f \cdot b$ and $b \cdot f \in M(A)'$

also. Also note that if $b \in A$ $f \cdot b$ and $b \cdot f \in M(A)'_{\beta}$.

We see that $\{f \cdot a_n : f \in M(A)', \text{ and } \|f\| \leq 1\} \subseteq \{x \in M(A) : \|a_n x\| \leq 1\}^0$. Suppose $f \in \{x \in M(A) : \|a_n x\| \leq 1\}^0$. Then $|f(x)| \leq \|a_n x\|$ for all $x \in M(A)$. Define a linear functional g on $a_n M(A)$ by $g(a_n x) = f(x)$. Then g is well defined and $\|g\| \leq 1$, so g extends to $h \in M(A)'$ such that $\|h\| \leq 1$. If $x \in M(A)$, $h \cdot a_n(x) = h(a_n x) = g(a_n x) = f(x)$. Hence $h \cdot a_n = f$ so that $\{f \cdot a_n : f \in M(A)', \|f\| \leq 1\} = \{x \in M(A) : \|a_n x\| \leq 1\}^0$. Similarly, $\{a_n \cdot f : f \in M(A)', \|f\| \leq 1\} = \{x \in M(A) : \|x a_n\| \leq 1\}^0$. Thus $H \subseteq \{a_n \cdot f + g \cdot a_n : \|f\| \leq 1, \|g\| \leq 1, n = 1, 2, 3, \dots\}$. We now show that H is β -equicontinuous. Since $\mathcal{L}^\infty(A)$ is essential, by assumption, we can write $a_n = ac_n$ and $a_n = b_n d$ with $\|d\| \leq 1$, $\|a\| \leq 1$, $\|c_n\| \leq 10$ for all n , and $\|b_n\| \leq 10$ for all n . By Taylor's criterion [59] we must show that $\|(I - e_\lambda) \cdot h \cdot (I - e_\lambda)\| \rightarrow 0$ uniformly for $h \in H$ (where $\{e_\lambda\}$ is an approximate identity for A and I denotes the identity in $M(A)$). Since a typical $h \in H$ is of form $a c_n \cdot f + g \cdot b_n d$, where $\|f\| \leq 1$, $\|g\| \leq 1$, $\|c_n\| \leq 10$, $\|b_n\| \leq 10$, this is clear. This concludes the proof of 3.2.12.

Our next results concern separability of $M(A)_\beta$.

3.2.13 PROPOSITION. If A is norm separable, then $M(A)_\beta$

is separable.

Proof. A is β dense in $M(A)$ and β is weaker than the norm topology on $M(A)$.

Next we specialize to the case that $A = C_0(S)$. Todd [60] shows that if S is σ compact then $C^*(S)_\beta$ is separable iff S is metrizable. He makes the following conjecture:

3.2.14 CONJECTURE (TODD). $C^*(S)_\beta$ is separable iff S is separable and metrizable.

We give a counterexample to this conjecture by exhibiting an uncountable discrete space S such that $\ell^\infty(S)_\beta$ is separable. Noting that $C_0(S)$ is not separable for this space S , we see that the notions of norm separability of A and β separability of $M(A)$ are not equivalent, even in the commutative case.

3.2.15 THEOREM. Let I denote the unit interval with the discrete topology. Then $\ell^\infty(I)_\beta$ is separable.

Proof. Let $f \in \ell^\infty(I)$, $\epsilon > 0$, and $g \in C_0(I)$, $g \geq 0$.

Choose a finite set $0 = i_0 < i_1 < \dots < i_n = 1$ in I so that $\{x \in I: g(x) \geq \frac{\epsilon}{2\|f\| + \frac{\epsilon}{\|g\|}}\} \subseteq \{i_0, \dots, i_n\}$.

Let $0 = r_0 < r_1 < \dots < r_n = 1$ be rational numbers so that $r_j \leq i_j < r_{j+1}$ for $1 \leq j \leq n-1$. Choose rational numbers $\{q_k\}_{k=0}^n$ so that $|q_k - f(i_k)| < \frac{\epsilon}{\|g\|}$, $0 \leq k \leq n$.

Define a function h on I by the formula

$$h(x) = \begin{cases} q_k & \text{if } r_k \leq x < r_{k+1}, 0 \leq k \leq n-1 \\ q_n & \text{if } x = 1 \end{cases}$$

Check that $\|g(h-f)\| < \epsilon$. Since the set of all such h is countable, $\mathcal{L}^\infty(I)_\beta$ is separable.

3.2.15 PROPOSITION. If $C^*(S)_\beta$ is separable, each open σ compact subset is separable and metrizable.

Proof. Let $X \subseteq S$ be open and σ compact and $\{f_i\}$ a countable β -dense subset, which we may assume to be closed under complex conjugation. Let ϕ be a nonnegative function in $C_0(S)$ such that $\{x \in S: \phi(x) > 0\} = X$. For $f \in C^*(S)$, let $R(f)$ denote the restriction of f to X . Consider $\{R(\phi f_i)\}_{i=1}^\infty$ and note that $R(\phi f_i) \in C_0(X)$ for each $i \geq 1$. This set separates points of $C_0(X)$ so the algebra it generates is dense in $C_0(X)$ by the Stone-

Weierstrass Theorem. Hence $C_0(X)$ is norm separable and so $C(X^*)$ (X^* = the one-point compactification of X) is separable. By a well-known result for compact Hausdorff space, X^* is metrizable and second countable. Hence X is metrizable and separable.

3.2.17 PROPOSITION. Let S be paracompact and locally compact. $C^*(S)_\beta$ is separable iff S is metrizable and the cardinality of S is less than or equal to that of the continuum.

Proof. Collins has shown that $C^*(S)_\beta$ is separable only if the cardinality of S is less than or equal that of the continuum. Since S is paracompact, if $C^*(S)_\beta$ is separable, by 1.1 and 3.2.16, S is a topological sum of metrizable spaces. Hence S is metrizable.

Conversely, suppose that S is metrizable and the cardinality condition (above) on S is satisfied. Then S has a decomposition into $\leq C$ σ compact subspaces (C = the cardinality of the continuum). Using the fact that $C_0(X)$ is separable if X is separable and metrizable, we can do a construction suggested by the $\ell^\infty(I)_\beta$ result to show that $C^*(S)_\beta$ is separable. We leave the details to the reader.

Results 3.2.16 - 3.2.18 are probably known, for similar results appear in a preprint by W. H. Summers [56]. Results 3.2.16 - 3.2.17 were obtained independently by the author. The next result was obtained by the author and Dr. Robert Wheeler during a conversation.

3.2.18 THEOREM. For a locally compact space S , the following are equivalent:

- (1) $C^*(S)_\beta$ is separable;
- (2) the topology on S is stronger than a separable metrizable topology on S ;
- (3) $(C(S), C - Op)$ is separable where $C(S)$ is the algebra of all continuous functions on S .

Proof. We need only show (1) \Leftrightarrow (3) as Warner in [64] showed (2) \Leftrightarrow (3). Suppose that $C^*(S)_\beta$ is separable and let $\{f_i\}_{i=1}^\infty$ be a countable β -dense subset of $C^*(S)$. Let $\epsilon > 0$, K a compact subset of S , and $g \geq 0 \in C(S)$. Let $M = \sup_{x \in K} g(x)$ and $h(x) = \begin{matrix} f(x) & \text{if } x \in K \\ M & \text{otherwise} \end{matrix}$.

Then $h \in C^*(S)$ and $g(x) = h(x)$ for $x \in K$. Let $\phi \in C_0(S)$ such that $\phi = 1$ on K . Then for some $i \geq 1$, $\|(f_i - h)\phi\| < \epsilon$, i.e., $\sup_{x \in K} \|f_i(x) - f(x)\| < \epsilon$. Hence $\{f_i\}_{i=1}^\infty$ is $C - Op$ dense in $C(S)$, i.e., $C(S)$ is separable.

rable for the $C - Op$ topology.

Conversely, suppose that $\{f_i\}_{i=1}^{\infty}$ is a sequence of nonnegative functions dense in $C(S)^+ = \{f \in C(S) : f \geq 0 \text{ on } S\}$ with the compact-open topology. We show that $C^*(S)^+$ is β separable. For a nonnegative real number a and a real function f in $C(S)$, let $f \wedge a$ be the function defined by the equation $f \wedge a(x) = f(x)$ if $f(x) \leq a$ and $f \wedge a(x) = a$ otherwise. Then it is clear that the doubly-indexed sequence $\{f_n \wedge a_m\}_{n=1}^{\infty} \{m=1}^{\infty}$ is a countable β -dense subset of $C^*(S)$.

Our next result is a generalization of a result of Collins [7] concerning nuclearity [25,49] of $M(A)_{\beta}$.

3.2.19 THEOREM. $M(A)_{\beta}$ is nuclear iff A is finite-dimensional.

Proof. If A is finite-dimensional, $M(A) = A$ and β is the norm topology. Since finite-dimensional spaces are well-known to be nuclear, there is no problem with this half of the theorem.

Suppose, on the other hand, that $M(A)_{\beta}$ is nuclear. Then the unit ball in $M(A)$ is β compact since closed and bounded sets in nuclear spaces are compact. By 3.2.9, $A = \left(\sum_{\lambda \in \Lambda} A_{\lambda} \right)_0$, where each A_{λ} is a finite-dimensional

C^* -algebra. Thus $M(A) = \sum_{\lambda \in \Lambda} A_\lambda$ (confer 3.2.9). We use another property of nuclear spaces to show that Λ is finite. In nuclear spaces unconditionally convergent sequences are absolutely convergent [49,25]. Suppose that Λ is infinite and choose an infinite sequence $\{\lambda_i\}_{i=1}^{\infty}$ of distinct elements from Λ . Let e_n denote the identity element in A_{λ_n} and let e'_n be the element of $M(A) = \sum_{\lambda \in \Lambda} A_\lambda$ which has value e_n in the λ_n -th co-ordinate and zero in the other co-ordinates. Let $x \in A = (\sum_{\lambda} A_\lambda)_0$ be such that $x(\lambda_n) = \frac{1}{n} e_n x(\lambda) = 0$ if $\lambda \notin \{\lambda_n\}_{n=1}^{\infty}$. Then $\sum_{n=1}^{\infty} \|x e'_n\| = \sum_{n=1}^{\infty} \frac{1}{n} = +\infty$ so that $\{e'_n\}_{n=1}^{\infty}$ is not an absolutely summable sequence in $M(A)_\beta$. We will contradict nuclearity of $M(A)_\beta$ by showing that $\{e'_n\}$ is unconditionally convergent, i.e., that every rearrangement converges. Let $x = (x_\lambda) \in A$, $\epsilon > 0$, and $\Gamma = \{\lambda \in \Lambda : \|x_\lambda\| \geq \epsilon\}$. Let $F = \{n : n \text{ is an integer and } \lambda_n \in \Gamma\}$. If F_0 is a finite subset of the set of positive integers and $F \cap F_0 = \{\emptyset\}$, then

$$\|x \sum_{n \in F_0} e'_n\| = \sup_{n \in F_0} \|x_{\lambda_n} e_n\| \leq \sup_{n \in F_0} \|x_{\lambda_n}\| < \epsilon.$$

Hence $\{e'_n\}_{n=1}^{\infty}$ is unconditionally convergent. Since $M(A)_\beta$ is nuclear, Λ must be finite and therefore A is finite-dimensional.

The following theorem relates the existence of a countable approximate identity for A with metrizability of a kappa topology on A .

3.2.20 THEOREM. Let $\{e_\lambda\}_{\lambda \in \Lambda}$ be a canonical bounded approximate identity for a C^* -algebra A . If A with the corresponding kappa topology is metrizable, then there is a countable subset Λ_0 of Λ such that $\{e_\lambda: \lambda \in \Lambda_0\}$ is an approximate identity for A . Hence by 2.7.10, every approximate identity for A has a countable subset which is also an approximate identity for A .

Proof. Suppose that the kappa topology corresponding to $\{e_\lambda\}$ is metrizable. Since $\{e_\lambda\}$ is canonical and the kappa topology has a countable base at zero, \exists a sequence $\lambda_1 < \lambda_2 < \dots$ in Λ so that the sets $V_n = \{x \in M(A): \|xe_{\lambda_n}\| \leq \frac{1}{n} \text{ and } \|e_{\lambda_n}x\| \leq \frac{1}{n}\}$, for $n = 1, 2, 3, \dots$, form a decreasing base at zero for the kappa topology. We shall show that $\{e_{\lambda_n}\}_{n=1}^\infty$ is an approximate identity for $M(A)$. It suffices to show that the sequence $e_{\lambda_n} - I \xrightarrow{n} 0$ in the kappa topology, where I is the identity in $M(A)$. Fix a positive integer n . If $m > n$ then $\lambda_m > \lambda_n$ so that $(e_{\lambda_m} - I)e_{\lambda_n} = 0 =$

$= e_{\lambda_n} (e_{\lambda_m} - I)$. Hence $e_{\lambda_m} - I \in V_n$ for $m > n$. Since $\{V_n\}_{n=1}^{\infty}$ forms a neighborhood base at zero for the kappa topology on $M(A)$, $e_{\lambda_n} - I \xrightarrow{\kappa} 0$ in the kappa topology.

The next few remarks are only a beginning of our attempts to answer the interesting question: For a locally compact S , when does every regular, bounded Borel measure have compact support? This is the same as asking when is $(C^*(S)_{\beta})' = (C^*(S), C - Op)'$, since $(C^*(S), C - Op)' = \{\mu \in M(S) : \text{the support of } \mu \text{ is compact}\}$.

3.2.21 REMARK. If $(C^*(S)_{\beta})' = (C^*(S), C - Op)'$, then S is countably compact. For if $\{x_n\}_{n=1}^{\infty} \subseteq S$,

$\mu = \sum_{n=1}^{\infty} \frac{1}{2^n} \delta(x_n) \in M(S)$, and so by assumption it has

compact support (if $x \in S$ $\delta(x)$ denotes the point mass at the point x). Since $\{x_n\}_{n=1}^{\infty}$ is contained in the support of μ ,

$\{x_n\}_{n=1}^{\infty}$ is relatively compact. Thus if S is realcompact [23] or metacompact [19,28] and

$M(S) = (C^*(S), C - Op)'$, then S is compact. As another example, assume that $M(S) = (C^*(S), C - Op)'$ and S has a base for σ compact subsets consisting of open and closed σ compact subsets. Then S is sham compact for each

σ compact subset is contained in a clopen σ compact subset. Each of these clopen subsets of S , being a closed subset of a countably compact space, is countably compact and so compact. Also if $C_0(S)$ has a well-behaved bounded approximate identity and $M(S) = (C^*(S), C - Op)'$, then S is compact by 2.5.1.

3.2.22 PROPOSITION. Suppose S is a locally separable locally compact topological space (each point has a separable neighborhood). Then $M(S) = (C^*(S), C - Op)'$ iff S is sham compact.

Proof. If S is sham compact, β and the compact-open topologies agree on $C^*(S)$ [63], so that the equality $(C^*(S), C - Op)'$ is clear.

Conversely, suppose that $M(S) = (C^*(S), C - Op)'$ and that K is a compact subset of S . The local separability condition on S implies that K is separable. Let $\{x_n\}_{n=1}^{\infty} \subseteq S$ be a countable dense subset and let $\mu = \sum \delta(x_n)$ (see 3.2.21). Then $\mu \in M(S)$ and K is contained in the support of μ . Hence if $X \subseteq S$ is σ compact, and $X = \bigcup_{n=1}^{\infty} K_n$ where $K_n \subseteq S$ is compact and $\mu_n \in M(S)$ has K_n contained in its support, then \mathbb{E} is a sequence of positive numbers $\{c_n\}_{n=1}^{\infty}$ so that $\mu = \sum_{n=1}^{\infty} c_n \mu_n \in M(S)$

and has support containing X . Hence X is relatively compact and S is therefore sham compact.

Our next result is a generalization of a result in [10] which states that if there is a bounded projection from $C^*(S)$ onto $C_0(S)$, then S must be pseudocompact. A result similar to ours has been obtained by A. J. Lazar and D. C. Taylor [32].

3.2.23 DEFINITION. Let A be a C^* -algebra. Then A has property Q if there does not exist a sequence

$\{a_n\}_{n=1}^{\infty} \subseteq A$ so that:

(1) $a_n = a_n^*$, $\|a_n\| = 1$ for $n \geq 1$;

(2) $a_n a_m = 0$ for $n \neq m$;

(3) The partial sums of the "series" $\sum_{n=1}^{\infty} \lambda_n a_n$ form a

Cauchy sequence in the strict topology for each

$\lambda = (\lambda_n) \in \ell^{\infty}$.

3.2.24 THEOREM [24]. Let S be a locally compact space. Then S is pseudocompact iff $C_0(S)$ has property Q .

3.2.25 THEOREM. Suppose A is a C^* -algebra and there is a bounded projection of $M(A)$ onto A . Then A has property Q .

Proof. Let P denote the bounded projection of $M(A)$ onto A . Suppose that A does not have property Q and let $\{a_n\} \subseteq A$ be a sequence satisfying (1) - (3) of 3.2.23. For $\lambda = (\lambda_n) \in \ell^\infty$, let $T(\lambda) = \sum_{n=1}^{\infty} \lambda_n a_n$ (the β -limit in $M(A)$). Then $T: \ell^\infty \rightarrow M(A)$. Note that $\|T(\lambda)\| \leq \|\lambda\|$. For $\|\sum_{n=1}^p \lambda_n a_n\| = \max_{1 \leq n \leq p} |\lambda_n| \leq \|\lambda\|$, and the unit ball in $M(A)$ is β -closed.

By considering the commutative C^* -algebra generated by $\{a_n\}_{n=1}^{\infty}$ and using the Hahn-Banach theorem, construct a sequence $\{\phi_n\}_{n=1}^{\infty} \subseteq A'$ so that $\|\phi_n\| = 1$ and $\phi_n(a_n^2) = 1$ for $u \geq 1$. For $x \in A$, let $S(x)$ denote the sequence whose n -th co-ordinate is $\phi_n(x a_n)$. Since $\|x a_n\| \rightarrow 0$ $S(x) \in c_0$. Note that $\|S\| \leq 1$ and that S is linear. Finally, note that $S \circ P \circ T$ projects ℓ^∞ onto c_0 , in violation of a well-known theorem of Phillips. This contradiction shows that A must have property Q .

Our next result is motivated by a result of Rotman and Finney [20].

3.2.26 DEFINITION. Let A be a ring and X , an A -module. X is said to be projective if, whenever Y and Z are A -modules, $f: X \rightarrow Y$ an A -module homomorphism, and $g: Z \rightarrow Y$ a surjective A -module homomorphism, there exists $h: X \rightarrow Z$

so that $f = g \cdot h$.

Finney and Rotman show that $C_c(S)$ is a projective $C(S)$ -module ($C(S)$ is the set of all continuous functions on S) iff S is paracompact. We asked the analogous question for $C^*(S)$ and $C_0(S)$.

3.2.27 LEMMA. Suppose that $C_0(S)$ is a projective $C^*(S)$ module. Then S is sham compact.

Proof. From Proposition 3.1, page 132, of Homological Algebra by Henri Cartan and Samuel Eilenberg, we have that there is a set B and $C^*(S)$ -module homomorphisms $\varphi_\beta: C_0(S) \rightarrow C^*(S) \quad \forall \beta \in B$ and a family $\{f_\beta: \beta \in B\} \subseteq C_0(S)$ so that for $g \in C_0(S)$

- (1) $\varphi_\beta(g) = 0$ for all but finitely many $\beta \in B$; and
 (2) $g = \sum_{\beta \in B} \varphi_\beta(g) f_\beta$.

Since $C_0(S) = C_0(S) \cdot C_0(S)$ by 1.12, each φ_β actually maps into $C_0(S)$. Let $X \subseteq S$ be σ compact and $g \geq 0$ in $C_0(S)$ so that g is strictly positive on X . Let $B_0 \subseteq B$ be finite so that $\beta \notin B_0 \Rightarrow \varphi_\beta(g) = 0$. Then

$$g = \sum_{\beta \in B_0} \varphi_\beta(g) f_\beta = \sum_{\beta \in B_0} f_\beta \varphi_\beta(g) = \sum_{\beta \in B_0} \varphi_\beta(f_\beta g) = \sum_{\beta \in B_0} g \varphi_\beta(f_\beta).$$

Hence $\sum_{\beta \in B_0} \varphi_\beta(f_\beta) \equiv 1$ on X . But $\sum_{\beta \in B_0} \varphi_\beta(f_\beta) \in C_0(S)$,

so X is relatively compact. Therefore S is sham compact.

3.2.28 THEOREM. $C_0(S)$ is a projective $C^*(S)$ module iff S is compact.

Proof. By 3.2.27, S is sham compact. Hence $C^*(S) = C(S)$ and $C_c(S) = C_0(S)$ [63]. Thus $C_c(S)$ is a projective $C(S)$ module and so S is paracompact by the Rotman-Finney result. A sham compact paracompact space is compact.

3.2.29 THEOREM. $M(A)_\beta$ is semireflexive iff A is a dual C^* -algebra [17, p. 99].

We need a lemma from general topology.

3.2.30 LEMMA. Let X be a regular (and Hausdorff) topological space and F a dense subspace of X . X is compact if every net in F has a cluster point in X .

Proof. Suppose X is not compact and let \mathcal{U} be an open cover of X without a finite subcover. By regularity, construct another \mathcal{V} of X so that the closures of the sets in \mathcal{V} refines \mathcal{U} . Note that no finite subset of \mathcal{V}

covers F . Let H denote the set whose elements are finite unions of sets in \mathcal{N} . For each set h in H , let $x_h \in F \setminus h$. Then the net $\{x_h : h \in H\}$, with H ordered by set inclusion, cannot have a cluster point in X . This contradicts the supposition that X is not compact.

Proof of 3.2.29. First we need some notation. Let $\{B_\lambda\}$ be a family of normed algebras or vector spaces. By $(\Sigma B_\lambda)_1$ we denote the set $\{b = (b_\lambda) \in \prod B_\lambda : \sum \|b_\lambda\| < +\infty\}$ with pointwise operations. By ΣB_λ we denote the set $\{b = (b_\lambda) \in \prod B_\lambda : \sup_\lambda \|b_\lambda\| < +\infty\}$ with pointwise operations.

If A is dual, $A = (\Sigma A_\lambda)_0$ where A_λ is the algebra of compact operators on the Hilbert space H_λ . Then $M(A) = \Sigma B(H_\lambda)$ and $M(A)'_\beta = (\Sigma A'_\lambda)_1$ and $(M(A)'_\beta)' = ((\Sigma A'_\lambda)_1)' = \Sigma A''_\lambda = \Sigma B(H_\lambda) = M(A)$ since the second dual (bidual) of the algebra of compact operators on H_λ is $B(H_\lambda)$ [16,17]. If we note that the strong topology on $M(A)'_\beta$ equals the norm topology on $(\Sigma A'_\lambda)_1$, the argument for the first assertion of 3.2.29 is complete.

Conversely, assume that $M(A)_\beta$ is semireflexive. To show that A is dual, it suffices [17, p. 99] to show that aS and Sa are relatively weakly compact for each $a \in A$, where S denotes the unit ball of A . By 3.2.30, it is

enough to prove that each net in aS (or Sa) has a weak cluster point in A . Let $\{x_\alpha\}$ be a net in S and $a \in A$. Since $M(A)_\beta$ is semireflexive, the unit ball in $M(A)$ is $\sigma(M(A), M(A)_\beta')$ compact. Hence $\exists x_0 \in M(A)$ so that $\{x_\alpha\}$ clusters $\sigma(M(A)_\beta, M(A)_\beta')$ to x_0 . We shall prove that $\{ax_\alpha\}$ clusters weakly to ax_0 . Let $f \in A'$ and g be a Hahn-Banach extension of f to $M(A)$. Then $g \cdot a \in M(A)_\beta'$ so that $\{g \cdot a(x_\alpha - x_0)\}$ clusters to zero, i.e., $\{g(ax_\alpha - ax_0)\}$ clusters to zero. Hence $\{f(ax_\alpha - ax_0)\}$ clusters to zero. Thus $\{ax_\alpha\}$ clusters weakly to ax_0 as claimed and aS (and similarly Sa) are relatively weakly compact. This concludes the proof of 3.2.29.

E. McCharen [34] has shown that if $I(A) = \{x \in A'' : xA + Ax \subseteq A\}$, then $I(A) = A''$ iff A is dual. Also she has shown that $M(A)$ and $I(A)$ are canonically isomorphic. Since $M(A)_\beta'$ is isomorphic to A'' , there is a natural imbedding of $M(A)$ into A'' . By checking out Arens Multiplication [34] on A'' , we see that the image of $M(A)$ under this imbedding is contained in $I(A)$. If $M(A)_\beta$ is semireflexive, then the image of $M(A)$ in A'' is all of A'' . Since $I(A)$ contains the image of $M(A)$, as noted above, $I(A) = A''$

so that A is dual by her result. Hence her theorem implies ours. On the other hand, our theorem implies hers. For if A is dual, by our theorem, the image of $M(A)$ (under the canonical imbedding into A'') = A'' . Hence $I(A) = A''$ since it is squeezed in-between. For the converse, we need to see that $I(A)$ is contained in the canonical image of $M(A)$ in A'' . Suppose $F \in I(A)$. Then if $a \in A$ and F_a denotes the image of a under the canonical imbedding into A'' , write $FF_a = F_{T(a)}$ and $F_a F = F_{S(a)}$. Note that T and S are maps from A into A . We show that $(T,S) \in M(A)$. Let $a, b \in A$. Then $F_a T(b) = F_a F_{T(b)} = F_a F F_b = F_{S(a)} F_b = F_{S(a)b}$. Hence $aT(b) = S(a)b$ since the map $a \rightarrow F_a$ is one-to-one on A . Hence $(T,S) \in M(A)$.

We need to check that $F_{(T,S)} = F$, i.e., that $f((T,S)) = F(f)$ for all $f \in A'$ (which is isometrically isomorphic to $M(A)_\beta'$). It suffices to show that $F F_a = F_{(T,S)} F_a$ for all $a \in A$. For then $F F_a(f) = F_{(T,S)} F_a(f) \quad \forall f \in A'$, hence $F(a \cdot f) = F_{(T,S)}(a \cdot f)$ for all $f \in A'$. But $A' = \{a \cdot f: f \in A', a \in A\}$ [59] so $F = F_{(T,S)}$.

Let $f \in A'$. Then $(F_{(T,S)} F_a)(f) = F_{(T,S)}(a \cdot f) = f((T,S)(L_a, R_a)) = f((L_{T(a)}, R_{T(a)})) = f(T(a)) = F_{T(a)}(f) = (F F_a)(f)$. We have, of course, made the identifications

resulting from imbedding A into $M(A)$. Hence $F F_a = F_{(T,S)} F_a$, so $F = F_{(T,S)}$ and the proof that E. McCharen's result and our 3.2.29 are equivalent is complete.

3.2.31 DEFINITION. Let A be a locally convex space. Then A is said to have the approximation property if, for any zero neighborhood V and totally bounded set B in A , there is a continuous linear operator S , of finite rank, on A so that $S(x) - x \in V$ for all $x \in B$ [25]. If A is a normed space and S can be chosen so that $\|S\| \leq 1$, then A is said to have the metric approximation property. For more information, see [49].

In 3.2.32 below, β_1 and β_2 denote the topologies on $M(A)$ generated by left and right multiplications, respectively, by elements of A .

3.2.32 THEOREM. Suppose the C^* -algebra A has the metric approximation property. Then $M(A)_{\beta_1}$ and $M(A)_{\beta_2}$ have the approximation property.

Proof. We prove the result for β_1 ; the other argument is similar. Let K be a β_1 totally bounded subset of

$M(A)$. By the uniform boundedness principle, K is norm bounded. Let $d = \sup_{x \in K} \|x\|$ and $\epsilon > 0$. Suppose that $a \geq 0 \in A$ and $\|a\| \leq 1$. Choose $0 \leq b \leq 1$, $0 \leq c \leq 1$ in A so that $\|a-b\| < \frac{\epsilon}{4d}$ and $cb = bc = b$. Since the set cK is a norm totally bounded subset of A , there exists a finite-rank operator T on A such that $\|T\| \leq 1$ and $\|T(cx) - cx\| < \frac{\epsilon}{2}$ for $x \in K$. Let $S: M(A) \rightarrow M(A)$ be the operator defined by the equation $S(x) = T(cx)$ for $x \in M(A)$. Note that $\|S\| \leq 1$ and that $\|b(S(x) - x)\| = \|b(S(x) - cx)\| = \|b(T(cx) - cx)\| < \frac{\epsilon}{2}$ for $x \in K$. Hence $\|a(S(x) - x)\| \leq \|(a-b)(S(x) - x)\| + \|b(S(x) - x)\| < (\frac{\epsilon}{4d})(2d) + \frac{\epsilon}{2} = \epsilon$.

Since S is clearly of finite rank, we need only verify that S is continuous on $M(A)$ with the β_1 topology. To this end, let $x_\alpha \rightarrow 0_{\beta_1}$, with $\{x_\alpha\} \subseteq M(A)$. Then $cx_\alpha \rightarrow 0$ in norm so $\|S(x_\alpha)\| = \|T(cx_\alpha)\| \rightarrow 0$ since T is norm continuous. Thus $S(x_\alpha) \rightarrow 0_{\beta_1}$ since the norm topology is finer than β_1 on $M(A)$.

3.2.33 COROLLARY (Collins-Dorroh [8]). Suppose that S is locally compact. Then $C^*(S)_\beta$ has the approximation property.

Proof. It is well-known that $C^*(X)$ for X a compact

Hausdorff space has the metric approximation property [15,25]. $C_0(S)$ is an ideal in $C_0(S^*)$ (S^* = the one-point compactification of S) so $C_0(S)$ has the metric approximation property by 3.2.34 below. Then apply 3.2.32.

3.2.34 LEMMA. Let A be a Banach algebra and I an ideal in A such that I has a bounded approximate identity. If A has the metric approximation property, then so does I .

Proof. Let $\epsilon > 0$ and $K \subseteq I$ such that K is totally bounded. Choose $a \in I$, $\|a\| \leq 1$ so that $\|ax-x\| < \frac{\epsilon}{2}$ for $x \in K$ [9]. Let $T:A \rightarrow A$ be of finite rank such that $\|T\| \leq 1$ and $\|T(x) - x\| < \frac{\epsilon}{2}$ for $x \in K$. Define $S:I \rightarrow I$ by $S(x) = aT(x)$. Hence $\|S(x) - x\| \leq \|S(x) - ax\| + \|ax-x\| < \epsilon$ for $x \in K$. Therefore I has the metric approximation property.

SECTION 3. TOPOLOGICAL MEASURE THEORY.

The next topic we take up is our generalization of topological measure theory. This is an important subject that was initiated in a classic paper by Varadarajan [62]. The problems have to do with studying certain classes of linear functionals on $C^*(X)$ where X is a completely re-

gular space. Many of these results are still very interesting in the case that X is locally compact. Topological measure theory is intimately connected with the important Mackey problem of Buck [5], mentioned earlier. We have generalized many of the topological measure theory results for locally compact spaces to the C^* -algebra setting.

The relevant definitions and theorems on topological measure theory, in the classical case, have been placed in the preliminary sections of Chapter IV. The reader should read those sections before he reads any further in Chapter III, in order to have a perspective for reading our theorems.

3.3.1 DEFINITION. Let A be a C^* -algebra. Suppose $\{f_\alpha\}$ is a net in $M(A)$. Write $f_\alpha \downarrow 0$ if the following three conditions are satisfied:

- (1) $0 \leq f_\alpha \leq I$ for all α ;
- (2) $f_\alpha \leq f_\beta$ for $\beta \leq \alpha$;
- (3) $f_\alpha \rightarrow 0$ in the strict topology.

3.3.2 REMARK. If $A = C_0(S)$, we can replace (3) in 3.3.1 by the assumption that $f_\alpha \rightarrow 0$ pointwise on S , by Dini's theorem.

3.3.3 DEFINITION. Let Φ be a positive linear functional on $M(A)$. Then Φ is said to be σ -additive if for any sequence $\{f_n\} \subseteq M(A)$ such that $f_n \downarrow 0$, $\Phi(f_n) \rightarrow 0$. Φ is called τ -additive if for any net $\{f_\alpha\} \subseteq M(A)$ such that $f_\alpha \downarrow 0$, $\Phi(f_\alpha) \rightarrow 0$. Finally, Φ is called tight if Φ is β continuous. For any arbitrary linear functional $\Phi \in M(A)'$ write $\Phi = \Phi_1 - \Phi_2 + i\Phi_3 - i\Phi_4$, where Φ_1, Φ_2, Φ_3 , and Φ_4 are positive linear functionals on $M(A)$. This decomposition is unique [17]. Φ is said to be σ -additive (τ -additive, tight) if Φ_1, Φ_2, Φ_3 and Φ_4 are σ -additive (τ -additive, tight). Let $M_\sigma(A)$, $M_\tau(A)$ and $M_t(A)$ denote the σ -additive, τ -additive, and tight linear functionals in $M(A)'$.

3.3.4 PROPOSITION. $M_t(A) = M_\tau(A) \subseteq M_\sigma(A)$.

Proof. All we need to prove is that if Φ is a positive linear functional in $M_\tau(A)$, then $\Phi \in M_t(A)$. Choose an increasing bounded approximate identity $\{e_\lambda\}$ for A satisfying $0 \leq e_\lambda \leq I \ \forall \lambda$. Note that $I - e_\lambda \downarrow 0$ so that $\Phi(I - e_\lambda) \rightarrow 0$. Let $(I - e_\lambda)^{1/2}$ denote the positive square root of $I - e_\lambda$, for each λ . Then

$$\|\Phi \cdot (I - e_\lambda)^{1/2}\| = \sup_{\|x\| \leq 1} \|\Phi((I - e_\lambda)^{1/2}x)\| \leq$$

$$\leq \sup_{\|x\| \leq 1} \bar{\psi}(I - e_\lambda)^{1/2} \bar{\psi}(x^*x)^{1/2} \leq \|\bar{\psi}\|^{1/2} \bar{\psi}(I - e_\lambda)^{1/2} \xrightarrow{\lambda} 0$$

Thus $\|\bar{\psi} \cdot (I - e_\lambda)\| = \|[\bar{\psi} \cdot (I - e_\lambda)^{1/2}] \cdot (I - e_\lambda)^{1/2}\| \leq \|\bar{\psi} \cdot (I - e_\lambda)^{1/2}\| \xrightarrow{\lambda} 0$. Hence $\bar{\psi}$ is β continuous, i.e., $\bar{\psi} \in M_t(A)$.

In the classical case of topological measure theory (see Chapter IV), the support of a positive measure μ on a completely regular space is defined as the $\bigcap \{Z: Z \subseteq X \text{ is a zero set and } \mu(Z) = \mu(X)\}$. If the support of μ is the empty set, μ is said to be entirely without support [30]. The next two theorems develop the analogue of measures entirely without support.

3.3.5 THEOREM [58]. Let $f \in M(A)'$. Then $f = f_1 + f_2$ where $f_1 \in M(A)'$ and $f_2 \in A^\perp = \{g \in M(A)': g \equiv 0 \text{ on } A\}$. The decomposition is unique. Furthermore, if f is a positive linear functional, then so are f_1 and f_2 .

3.3.6 DEFINITION. Let A be a C^* -algebra. We say that A is measure compact if $M_0(A) = M_t(A)$. This definition coincides with that given in Chapter IV for the case $A = C_0(S)$.

The next result shows that if A is not measure com-

compact, then the situation is the worst possible, i.e., there is an element of $M_\sigma(A)$ which annihilates A .

3.3.7 THEOREM. Suppose that A is not measure compact. Then there is a nonzero positive linear functional in $M_\sigma \cap A^\perp$ (see 3.3.5).

Proof. By hypothesis, there is a positive linear functional g in $M_\sigma(A)$ but not in $M_t(A)$. By 3.3.5, write $g = g_1 + g_2$ with g_1 and g_2 positive linear functionals on $M(A)$ such that $g_1 \in M(A)' = M_t(A)$ and $g_2 \in A^\perp$. Then $g_2 \in M_\sigma(A)$ since $g_2 = g - g_1$ and $M_t(A) \subseteq M_\sigma(A)$.

The next result is the analogue of the result that a Lindelöf (σ compact, in the locally compact case) space is measure compact.

3.3.8 THEOREM. Suppose A has a countable approximate identity. Then A is measure compact.

Proof. We may assume that A has a bounded approximate identity $\{e_n\}$ satisfying $e_n \leq e_m$ for $n \leq m$ and $0 \leq e_n \leq I$ for all n . If A is not measure compact, there is a nonzero positive linear functional g in

$M_G \cap A^\perp$ by 3.3.7. Hence $g(e_n) = 0$ for all n . But $I - e_n \downarrow 0$ so $g(e_n) \rightarrow g(I) = \|g\| \neq 0$. This contradiction establishes the result.

A set X is said to have measure compact cardinal if the set X with the discrete topology is measure compact, i.e., if each countably additive measure defined on all subsets of X which vanishes on points of X is identically zero. The next result is motivated by the result [30] that a paracompact locally compact space with measure compact cardinal is measure compact (see 1.1).

3.3.9 THEOREM. Suppose $A = \left(\sum_{\lambda \in \Lambda} A_\lambda \right)$ where Λ is a set having measure compact cardinal and each A_λ is measure compact. Then A is measure compact.

Proof. Note that $M(A) = \sum_{\lambda \in \Lambda} M(A_\lambda)$. Suppose A is not measure compact and that g is a nonzero positive linear functional in $M_G(A) \cap A^\perp$. Note that $g \equiv 0$ on $M(A_\lambda)$ (i.e., the canonical image of $M(A_\lambda)$ in $M(A)$). Let I_λ denote the identity in $M(A_\lambda)$. Let b^λ denote the element of $M(A)$ whose λ -th coordinate is I_λ and whose α -th coordinate is zero, for $\alpha \neq \lambda$. For any subset Λ_0 of Λ , let $\mu(\Lambda_0) = g\left(\sum_{\lambda \in \Lambda_0} b^\lambda\right)$. Note that μ vanishes

on points of Λ . A straightforward computation shows that μ is σ -additive as a measure on Λ , using the fact that $g \in M_{\sigma}(A)$. Also $\mu(\Lambda) = \mathbb{1}(I) \neq 0$. The existence of μ contradicts the assumption that Λ has measure compact cardinal. Hence A is measure compact.

3.3.10 COROLLARY. Let S be locally compact and paracompact. Then S is measure compact.

Proof. We must show that $C_0(S)$ is measure compact (see Chapter IV). By 1.1, S is a topological sum of σ compact spaces S_{α} so $C_0(S) = (\Sigma C_0(S_{\alpha}))_0$. Apply 3.3.9 and 3.3.8.

The concept of well-behaved approximate identity was discussed before (Chapter II). A reasonable conjecture is that if A has a well-behaved approximate identity, then A is measure compact. We prove a special case of this theorem, for algebras having a series approximate identity [3]. A series approximate identity for a C^* -algebra is a family $\{f_{\alpha}\}_{\alpha \in \Gamma}$ of projections in A such that, for each $a \in A$, $\|a(\sum_{\alpha \in F} f_{\alpha})a - a\| \rightarrow 0$ as F runs through the finite subsets of Γ . It is easy to see that the net $\{\sum_{\alpha \in F} f_{\alpha} : F \text{ finite } \subseteq \Gamma\}$ is a well-behaved approximate identity for A , where

the F 's are ordered by inclusion.

3.3.11 THEOREM. Suppose A has a series approximate identity $\{e_\lambda : \lambda \in \Lambda\}$ with Λ having measure compact cardinal. Then A is measure compact.

Proof. We assume Λ is infinite; otherwise A has an identity, $A = M(A)$ and β is the norm topology so that A is trivially measure compact. Let F be a nonzero positive linear functional in $M_\sigma(A) \cap A^\perp$. For $\Lambda_0 \subseteq \Lambda$, define $\mu(\Lambda_0) = F(\sum_{\lambda \in \Lambda_0} e_\lambda)$. We will show that μ is a nonzero σ -additive measure defined for all subsets of Λ which vanishes on points of Λ . Let $\{\Lambda_n\}_{n=1}^\infty$ be a decreasing collection of subsets of Λ such that $\bigcap_{n=1}^\infty \Lambda_n = \{\emptyset\}$. Let $x_n = \sum_{\lambda \in \Lambda_n} e_\lambda$. The sequence $\{x_n\}_{n=1}^\infty$ in $M(A)$ is bounded. We claim that $x_n \rightarrow 0$ β ; it suffices to show that $\|e_\gamma x_n\| \rightarrow 0$ and $\|x_n e_\gamma\| \rightarrow 0$ for any $\gamma \in \Lambda$. This is clear since $\{e_\lambda\}$ consists of orthogonal projections and $\bigcap_{n=1}^\infty \Lambda_n$ is empty. Thus $\mu(\Lambda_n) = F(x_n) \rightarrow 0$ since $F \in M_\sigma(A)$ and $\{x_n\}$ is clearly decreasing. Since $\mu(\Lambda) = F(I) \neq 0$, the existence of μ contradicts the assumption that Λ has measure compact cardinal.

3.3.12 LEMMA [6]. Suppose A and B are C^* -algebras and $\tilde{\varphi}:A \rightarrow B$ is a surjective (onto) $*$ -homomorphism. Then $\exists F:M(A) \rightarrow M(B)$ such that F is a $*$ -homomorphism extending $\tilde{\varphi}$ and such that F maps the identity of $M(A)$ onto the identity of $M(B)$. Furthermore, F is continuous when $M(A)$ and $M(B)$ are given their respective strict topologies.

3.3.13 THEOREM. Suppose $\tilde{\varphi}:A \rightarrow B$ is a surjective $*$ -homomorphism and A is measure compact. Then B is measure compact.

Proof. Suppose g is a positive linear functional in $M_{\sigma}(B) \cap B^+$. Let $F:M(A) \rightarrow M(B)$ be the extension of $\tilde{\varphi}$ whose existence is guaranteed by 3.3.12. Let $F':M(B)' \rightarrow M(A)'$ denote the adjoint map. Let $h = F'(g)$. If $a \in A$, $h(a) = (F'(g))(a) = g(F(a)) = g(\tilde{\varphi}(a)) = 0$ since $g \equiv 0$ on B . Suppose $\{a_n\} \subseteq A$ and $a_n \downarrow 0$. Then $h(a_n) = g(F(a_n)) \rightarrow 0$ since F preserves order, is norm decreasing and β continuous. Thus $h \in M_{\sigma}(A)$ and $h \equiv 0$ on A . Since A is measure compact, $h \equiv 0$. Let I_A and I_B denote the identity elements in $M(A)$ and $M(B)$

respectively. Then $0 = h(I_A) = (F'(g))(I_A) = g(F(I_A)) = g(I_B) = \|g\|$. Hence $g = 0$ so B is measure compact since g was an arbitrary positive linear functional in $M_C(B)$ which annihilates B .

3.3.14 COROLLARY. If A and B are *-isomorphic C^* -algebras then A is measure compact if and only if B is.

3.3.15 COROLLARY. Suppose X and Y are homeomorphic locally compact topological spaces. Then X is measure compact iff Y is.

3.3.16 COROLLARY. Suppose S is a measure compact topological space and F a closed subset of S . Then F is measure compact.

Proof. Let $R: C_0(S) \rightarrow C_0(F)$ denote the restriction map. By 2.3.7, R is surjective. Hence $C_0(F)$ is measure compact.

An important problem in the commutative topological measure theory is the determination of the Prohorov spaces. A completely regular space X is called Prohorov (see Chapter IV) if every weak-* compact subset of the positive

tight measures on X is uniformly tight [62]. It is known that locally compact spaces are Prohorov. Our next result is a generalization of this commutative result.

3.3.17 THEOREM. Suppose $H \subseteq M(A)'_{\beta}$ is β weak-* compact and consists of positive linear functionals. Then H is β equicontinuous.

Proof. We use the criterion for β equicontinuity in [59]. Note that H is norm bounded. Suppose that $\{e_{\lambda} : \lambda \in \Lambda\}$ is an approximate identity for A such that $0 \leq e_{\lambda} \leq I$ $\forall \lambda \in \Lambda$ and $\{e_{\lambda}\}$ is increasing. Let $\epsilon > 0$. Since $I - e_{\lambda} \rightarrow 0_{\beta}$, the sets $\{F \in H : F(I - e_{\lambda}) < \epsilon\}$ form a β -weak-* open cover of H . The compactness of H implies that we can choose a finite set $\{\lambda_i\}_{i=1}^n$ from Λ so that $H \subseteq \bigcup_{i=1}^n \{F \in H : F(I - e_{\lambda_i}) < \epsilon\}$. Choose λ_0 in Λ such that $\lambda_0 \geq \lambda_i$, $i = 1, \dots, n$. Then $\lambda \geq \lambda_0$ implies $0 \leq I - e_{\lambda} \leq I - e_{\lambda_i}$ for $1 \leq i \leq n$ so that $F(I - e_{\lambda}) < \epsilon$ for all $F \in H$. Hence, letting d_{λ} denote the positive square root of $I - e_{\lambda}$, $\|d_{\lambda} \cdot F \cdot d_{\lambda}\| = d_{\lambda} \cdot F \cdot d_{\lambda}(I) = F(I - e_{\lambda}) < \epsilon$ for $\lambda \geq \lambda_0$. Thus $\|(I - e_{\lambda}) \cdot F \cdot (I - e_{\lambda})\| = \|d_{\lambda}^2 \cdot F \cdot d_{\lambda}^2\| \leq \|d_{\lambda}\|^2 \|d_{\lambda} \cdot F \cdot d_{\lambda}\| < \epsilon$ for $\lambda \geq \lambda_0$. Thus H is β -equicontinuous by Taylor's criterion [59].

3.3.18 QUESTION. Our definitions of σ -additivity, tightness, and τ -additivity for elements of $M(A)'$ were made in terms of positive functionals. One natural question that arises is this: suppose $F \in M(A)'$ and, for every sequence $\{a_n\} \subseteq A$ such that $a_n \downarrow 0$, $F(a_n) \rightarrow 0$. Is F σ -additive, i.e., if we write $F = F_1 - F_2 + iF_3 - iF_4$ where $\{F_i\}_{i=1}^n$ is a set of positive linear functionals on $M(A)$, is $\{F_i\}_{i=1}^n \subseteq M_\sigma(A)$. The question is answered affirmatively in the commutative case using lattice structure. A similar question may be asked about τ -additive and tight functionals. It is clear that we need consider only Hermitian linear functionals, i.e., functionals which are real-valued on Hermitian elements. The question for tight functionals has been answered by D. C. Taylor.

3.3.19 THEOREM.[59]. Suppose A is a C^* -algebra. Then $M(A)'_\beta$ is a Banach space which is isometrically order isomorphic to A' .

Our next result generalizes the idea of a measure entirely without support which we discussed in the paragraph preceding 3.3.5. If B is a von Neumann algebra [16,17] and $f \in B'$, there is a smallest projection E_f in the set $\{P: P \text{ is a projection in } B \text{ and } f \cdot P = f\}$. E_f is

called the support of f . Suppose we regard A as a subset of $M(A)$ in the canonical way. $M(A)''$ is a von Neumann algebra and any positive linear functional on $M(A)$ has an extension to $M(A)''$ that is continuous with respect to the $\sigma(M(A)'', M(A)')$ topology.

If $A = C_0(S)$, functionals $F \in C_0(S)^* = M(S)$ such that $F \equiv 0$ on A correspond under the mapping defined in 2.1.14 to measures entirely without support. Let βS denote the Stone-Cech compactification of S [28]. Borel measures on S can be thought of as Borel measures on βS and Borel measures entirely without support are measures whose support is contained in $\beta S \setminus S$.

Our next result is an analogue of this idea in the noncommutative case.

3.3.20 THEOREM. Let F be a positive linear functional on $M(A)$ and let H denote the $\sigma(M(A)'', M(A)')$ -continuous extension of F to $M(A)''$. Let E_H denote the support of H in $M(A)''$ and I_A denote the principal identity of A which is the identity of the $\sigma(M(A)'', M(A)')$ closure of A in $M(A)''$. Then $F \equiv 0$ on A iff $I_A E_H = 0$.

Proof. Suppose that $F \equiv 0$ on A . Let $\{e_\lambda\}$ be an increasing, positive approximate identity for A . Then

$e_\lambda \rightarrow I_A$ $\sigma(M(A)'', M(A)')$ [48]. Since H is $\sigma(M(A)'', M(A)')$ continuous, $H(I_A) = \lim_{\lambda} F(e_\lambda) = 0$. Let I denote the identity in $M(A)''$. Then $H \cdot (I - I_A) = H$ so that $I - I_A \geq E_H$, i.e., $E_H I_A = 0$.

Conversely, suppose that $I_A E_H = 0$. Then $H(I_A) = H \cdot E_H(I_A) = H(E_H I_A) = 0$. If $a \in A$ such that $0 \leq a$ and $\|a\| \leq 1$, then $a \leq I_A$ so that $F(a) \leq H(I_A) = 0$, i.e., $F \equiv 0$ on A .

There are many questions suggested by our work on generalized topological measure theory. What we have presented here, together with the next result, is only a beginning. As a corollary to the theorem on tensor products below, we have that finite products of locally compact measure compact spaces are measure compact.

For this result, we need the concept of the C^* -tensor product of C^* -algebras. We sketch the construction and refer the reader to [48] for a more detailed treatment.

3.3.21 DEFINITION. Let A and B be C^* -algebras and $A \otimes B$ denote the algebraic tensor product of A and B , regarded as a set of bilinear forms on $A' \times B'$. Define involution for "elementary tensors" $a \otimes b$ by $(a \otimes b)^* = a^* \otimes b^*$ and extend linearly. If $(f, g) \in A' \times B'$, let

$f \otimes g \left(\sum_{i=1}^n a_i \otimes b_i \right) = \sum_{i=1}^n f(a_i)g(b_i)$. If f and g are positive, $f \otimes g$ is positive [48]. The C*-tensor product of A and B is the completion of $A \otimes B$ under the norm α_0 , defined for $x \in A \otimes B$ by $\alpha_0(x) = \sup \left\{ \sqrt{\frac{f \otimes g(y^*x^*xy)}{f \otimes g(y^*y)}} \right\}$: f and g are positive linear functionals of norm ≤ 1 on A and B respectively, and $y \in A \otimes B$ such that $f \otimes g(y^*y) \neq 0$. From a slightly different viewpoint, if we let $\pi_{f \otimes g}$ denote the representation of $A \otimes B$ associated with the positive functional $f \otimes g$ [17, Par. 2], then $\alpha_0(x) = \sup_{f \otimes g} \|\pi_{f \otimes g}(x)\|$, where f and g are positive linear functionals of norm ≤ 1 on A and B respectively.

We shall let $A \otimes_{\alpha_0} B$ denote the C*-tensor product of A and B .

3.3.22 THEOREM. Suppose A and B are measure compact. Then so is $A \otimes_{\alpha_0} B$.

Proof. Regard A as a subset of $M(A)$ and B as a subset of $M(B)$ in the natural way. Suppose that $\tilde{\nu}$ is a σ -additive positive linear functional on $M(A \otimes_{\alpha_0} B)$ and $\tilde{\nu} \neq 0$. We shall show that $\tilde{\nu} \neq 0$ on $A \otimes B$. First, we

show that $M(A \underset{\alpha_0}{\otimes} B)$ contains naturally imbedded copies of $M(A)$ and $M(B)$.

Let $a' \in M(A)$ and $\Pi_1(a') = (La', Ra')$ where La' is the linear map on $A \otimes B$ defined by $La'(\sum_{i=1}^n a_i \otimes b_i) = \sum_{i=1}^n a' a_i \otimes b_i$, for $\sum_{i=1}^n a_i \otimes b_i \in A \otimes B$, and $Ra'(\sum_{i=1}^n a_i \otimes b_i) = \sum_{i=1}^n a_i a' \otimes b_i$. Ra' and La' are well-defined. For if $\sum_{i=1}^n a_i \otimes b_i = 0$, i.e., represents the zero form so that $\sum_{i=1}^n \varphi(a_i) \psi(b_i) = 0$ for all $\varphi \in A'$, $\psi \in B'$ then $\sum a' a_i \otimes b_i = 0$ also since if $\varphi \in A'$ and $\psi \in B'$, then $\sum_{i=1}^n \varphi(a' a_i) \psi(b_i) = \sum_{i=1}^n p(a_i) \psi(b_i)$ where $p = \varphi \cdot a'$. Thus La' is well-defined; similarly, Ra' is well-defined. We now show that La' and Ra' are bounded operators on $A \otimes B$ so that they may be extended to all of $A \underset{\alpha_0}{\otimes} B$.

First, let us assume that $a' \in A$. Let φ be a positive linear functional on B . We want to show that

$$\|\Pi_{\varphi \otimes \psi}(\sum a' a_i \otimes b_i)\| \leq \|a'\| \|\Pi_{\varphi \otimes \psi}(\sum a_i \otimes b_i)\|.$$

Let H_φ , H_ψ , and $H_{\varphi \otimes \psi}$ be the Hilbert spaces constructed from A, B and $A \otimes B$ by means of the positive func-

tionals φ, ψ , and $\varphi \otimes \psi$. Sakai notes [48, p. 61] that $H_{\varphi \otimes \psi}$ is unitarily equivalent to $H_{\varphi} \tilde{\otimes} H_{\psi}$ (where ' $\tilde{\otimes}$ ' denotes the tensor product of Hilbert spaces defined in Dixmier [16, p. 21]). By making the appropriate identifications, we are able to restrict our attention to the pre-Hilbert space $A_{\varphi} \otimes B_{\psi}$ where A_{φ} denotes the pre-Hilbert space constructed from φ and A and B_{ψ} denotes the pre-Hilbert space constructed from ψ and B . Let I_{ψ} and I_{φ} denote the identity operator on H_{ψ} and H_{φ} respectively. Then $\Pi_{\varphi} \otimes \Pi_{\psi}(\Sigma a'_i \otimes b_i) = (\Pi_{\varphi}(a') \otimes I_{\psi}) \circ (\Pi_{\varphi} \otimes \Pi_{\psi}(\Sigma a_i \otimes b_i))$. Hence $\|\Pi_{\varphi \otimes \psi}(\Sigma a'_i \otimes b_i)\| = \|\Pi_{\varphi} \otimes \Pi_{\psi}(\Sigma a_i \otimes b_i)\| \leq \|\Pi_{\varphi}(a') \otimes I_{\psi}\| \|\Pi_{\varphi} \otimes \Pi_{\psi}(\Sigma a_i \otimes b_i)\| = \|\Pi_{\varphi}(a') \otimes I_{\psi}\| \|\Pi_{\varphi \otimes \psi}(\Sigma a_i \otimes b_i)\|$. The first equality and the last follow because $\Pi_{\varphi \otimes \psi}$ and $\Pi_{\varphi} \otimes \Pi_{\psi}$ are unitarily equivalent, as noted above.

Our problem then reduces to computing the norm of the operator $\Pi_{\varphi}(a') \otimes I_{\psi}$ on $A_{\varphi} \otimes B_{\psi}$. By using 3.2.23 (following this theorem) plus the fact that Π_{φ} is norm decreasing, we see that

$$\|\Pi_{\varphi}(a') \otimes I_{\psi}\| \leq \|\Pi_{\varphi}(a')\| \|I_{\psi}\| \leq \|a'\|.$$

Thus

$$\|\Pi_{\varphi \otimes \psi}(\Sigma a'_i \otimes b_i)\| \leq \|a'\| \|\Pi_{\varphi \otimes \psi}(\Sigma a_i \otimes b_i)\|$$

so that $L_{a'}$ is a bounded linear operator on $A \otimes B$ of norm $\leq \|a'\|$ and so extends to $A \otimes_{\alpha_0} B$. Similarly, $R_{a'}$ extends to all of $A \otimes_{\alpha_0} B$ with norm $\leq \|a'\|$.

In the above discussion, we assumed that $a' \in A$. Now suppose that $a' \in M(A)$ and that $\{a'_\alpha\}$ is a net in A such that $a'_\alpha \rightarrow a'$ in the strict topology. Assume further that $\|a'_\alpha\| \leq \|a'\|$ for all α . Then

$$\alpha_0(L_{a'}, (\Sigma a_i \otimes b_i)) = \alpha_0(\Sigma a'_i a_i \otimes b_i) = \sup \left\{ \sqrt{\frac{\varphi \otimes \psi (y^* (\Sigma (a'_i a_i)^* \otimes b_i^*) (\Sigma a'_i a_i \otimes b_i)) y}{\varphi \otimes \psi (y^* y)}} : 0 \leq \varphi \in A^*, \right.$$

$\|\varphi\| \leq 1, 0 \leq \psi \in A^*, \|\psi\| \leq 1, \varphi \otimes \psi (y^* y) \neq 0, y \in A \otimes B$. Suppose $y = \Sigma c_j \otimes d_j$. Since we may regard φ as an element of $M(A)_\beta'$, inserting $\Sigma c_i \otimes d_i$ for y in the above expression and evaluating $\varphi \otimes \psi$, we get that $\alpha_0(\Sigma a'_i a_i \otimes b_i) = \lim_\alpha \alpha_0(\Sigma a'_\alpha a_i \otimes b_i)$. This last fact follows from the β -continuity of φ . Hence $\alpha_0(L_{a'}, \Sigma a_i \otimes b_i) \leq \lim_\alpha \|a'_\alpha\| \alpha_0(\Sigma a_i \otimes b_i) \leq \|a'\| \alpha_0(\Sigma a_i \otimes b_i)$. Thus $\|L_{a'}\| \leq \|a'\|$ for arbitrary $a' \in M(A)$ and a similar result holds for $R_{a'}$. By using the fact that α_0 is a cross norm ($\alpha_0(x \otimes y) = \|x\| \|y\|$) we see that $\|R_{a'}\| = \|L_{a'}\| = \|a'\|$. It is also clear that if $x, y \in A \otimes B$, then $x L_{a'}(y) = R_{a'}(x)y$ so that by continuity of $L_{a'}$ and $R_{a'}$, we

get that $(L_{a'}, R_{a'}) \in M(A \otimes_{\alpha_0} B)$.

We may summarize the preceding arguments by saying that $\Pi_1: M(A) \rightarrow M(A \otimes_{\alpha_0} B)$ defined by $\Pi_1(a') = (L_{a'}, R_{a'})$ is a *-isomorphism into. Similarly we may define a *-isomorphism $\Pi_2: M(B) \rightarrow M(A \otimes_{\alpha_0} B)$ with $\Pi_2(b') = (L_{b'}, R_{b'})$ for $b' \in M(B)$ where $L_{b'}, (\Sigma a_i \otimes b_i) = \Sigma a_i \otimes b' b_i$ and $R_{b'}, (\Sigma a_i \otimes b_i) = \Sigma a_i \otimes b_i b'$.

We are now ready to prove the theorem. First note the following fact: if $\{a'_n\} \subseteq M(A)$ and $a'_n \rightarrow 0\beta$ then $\Pi_1(a'_n) \rightarrow 0\beta$, i.e., in the strict topology on $M(A \otimes_{\alpha_0} B)$ defined by $A \otimes_{\alpha_0} B$. For note that by the uniform boundedness theorem $\{\|a'_n\|\}_{n=1}^\infty$ is bounded; thus it suffices to verify that $L_{a'_n}$ and $R_{a'_n}$ converge to zero pointwise as operators on $A \otimes B$. This is clear. Similarly if $\{b'_n\} \subseteq M(B)$ and $b'_n \rightarrow 0\beta$ then $\Pi_2(b'_n) \rightarrow 0$ in the strict topology of $M(A \otimes_{\alpha_0} B)$ defined by $A \otimes B$.

With these remarks in mind, one sees that the linear functional $\tilde{\phi}_1$ on $M(A)$ defined by the equation $\tilde{\phi}_1(x) = \tilde{\phi}(\Pi_1(x))$, for $x \in M(A)$, is a non-zero σ -additive positive linear functional. By measure compactness of A , $\exists a \in A^+$ so that $\tilde{\phi}(\Pi_1(a^2)) > 0$. Define a linear functional $\tilde{\phi}_2$ on $M(B)$ by the equation $\tilde{\phi}_2(y) = \tilde{\phi}(\Pi_1(a)\Pi_2(y)\Pi_1(a))$ and note that $\tilde{\phi}_2$ is nonzero,

σ -additive, and positive. Measure compactness of B then yields the existence of $b \in B^+$ so that

$\mathfrak{f}(\Pi_1(a)\Pi_2(b)\Pi_1(a)) > 0$. Since it is easily checked that $\Pi_1(a)\Pi_2(b)\Pi_1(a) = a^2 \otimes b$, we have that $\mathfrak{f}(a^2 \otimes b) > 0$, i.e., \mathfrak{f} does not annihilate $A \otimes B$. Since \mathfrak{f} is an arbitrary σ -additive linear functional on $M(A \otimes_{\alpha_0} B)$, it follows that $A \otimes_{\alpha_0} B$ is measure compact.

We now state and prove the lemma used in this proof.

3.3.23 LEMMA. Let H_1 and H_2 be 2 Hilbert spaces and $H_1 \tilde{\otimes} H_2$ be the completion of $H_1 \otimes H_2$ with the inner product defined for elementary tensors by

$\langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle = \langle x_1, y_1 \rangle \langle x_2, y_2 \rangle$ with $x_1, y_1 \in H_1$ and $x_2 \in H_2$. Suppose $T \in B(H_1)$ and I_2 is the identity operator on H_2 . Let $T \otimes I_2$ be the operator on $H_1 \otimes H_2$ defined by $T \otimes I_2(\Sigma x_i \otimes g_i) = \Sigma T(x_i) \otimes y_i$. Then $T \otimes I_2 \in B(H_1 \otimes H_2)$ and $\|T \otimes I_2\| \leq \|T\|$ so that T extends to all of $H_1 \tilde{\otimes} H_2$.

Proof. Let $\Sigma x_i \otimes y_i$ be an element of $H_1 \otimes H_2$. We may suppose that $\{y_i\}$ is an orthonormal set. Then

$$\begin{aligned} \|T \otimes I_2(\Sigma x_i \otimes y_i)\|^2 &= \|\Sigma T(x_i) \otimes y_i\|^2 = \Sigma \|T(x_i)\|^2 \leq \\ \|T\|^2 \Sigma \|x_i\|^2 &= \|T\|^2 \|\Sigma x_i \otimes y_i\|^2. \end{aligned}$$

Hence $\|T \otimes I_2\| \leq \|T\|$ as

claimed.

SECTION 4. STONE-WEIERSTRASS THEOREMS.

The final section of this chapter has to do with a general Stone-Weierstrass Theorem for the double centralizer algebra of a C^* -algebra. Our results in Section 4 are only partial ones; we have not finished working out the details of better and more general results.

The first proposition is about LCH , the algebra of compact operators on the Hilbert space H and its double centralizer algebra $B(H)$.

3.4.1 PROPOSITION. Let $A = LCH$ and B be a β closed $*$ -subalgebra of $B(H)$ which separates the set consisting of the pure states of A together with the zero functional on $M(A)$. We will think of the pure states of A as being pure states of $M(A)$ which are β continuous [59,41,42]. Then $B = B(H)$.

Proof. From the result A12 in [17], we have that the closure of B in the ultrastrong operator topology [16] is the double commutant [38] of B if we can show that $B \cdot H = \{b(x) : b \in B, x \in H\}$ is dense in H . Suppose that $B \cdot H$ is not dense in H . Then $\exists h \in H, \|h\| = 1$, so

that h is orthogonal to $B \cdot H$. Hence $\langle b(h), h \rangle = 0$
 $\forall b \in B$, where ' $\langle \rangle$ ' denotes the inner product in
 H . Thus B does not separate the pure state of $B(H)$
 $T \mapsto \langle T(h), h \rangle$, for $T \in B(H)$, from the zero functional.
 Since this pure state is β -continuous on $B(H)$, we have
 arrived at a contradiction and conclude that $B \cdot H$ is dense
 in H .

Next we show that the double commutant of B is
 $B(H)$ by showing that the commutant of B consists only
 of scalar multiples of the identity operator, i.e., that
 the identity representation of B is irreducible. If
 not, there are orthogonal invariant subspaces H_1 and H_2 .
 Let $h_1 \in H_1$ and $h_2 \in H_2$ so that $\|h_1\| = \|h_2\| = 1$.
 If $T \in B$, $\langle T(h_1 - h_2), h_1 - h_2 \rangle = \langle T(h_1 + h_2), h_1 + h_2 \rangle$
 so the pure states of $B(H)$ determined by $h_1 - h_2$ and
 $h_1 + h_2$ are equal on $B(H)$, since they are equal on B .
 Thus $h_1 + h_2$ is a multiple of $h_1 - h_2$. This is nonsense
 and hence we conclude that the double commutant of B is
 $B(H)$ as claimed.

Summarizing the above arguments, we have shown that
 B is dense in $B(H)$ with respect to the ultrastrong operator
 topology. We now show that $(B(H)_\beta)^\beta \subseteq (B(H), \text{ultrastrong operator topology})^\beta$
 which will prove that $B = B(H)$
 by means of the separation theorem for locally convex

spaces. Let F be a positive linear functional continuous on $B(H)$ in the strict topology. Using [17, p. 83-84] we have the existence of sequences $\{x_i\}_{i=1}^{\infty}$ from H and $\{\lambda_i\}_{i=1}^{\infty}$ from the positive reals so that $\sum \lambda_i < +\infty$, $\|x_i\| = 1$ for $i \geq 1$ and $F(T) = \sum_{i=1}^{\infty} \lambda_i \langle Tx_i, x_i \rangle$ for $T \in LCH$. It is easy to see that this formula for F actually holds for all $T \in B(H)$. By using the fact that β is the finest locally convex topology agreeing with itself on norm bounded sets [59] of $M(A)$, we see that the expression for $F(T)$ defines a β continuous linear functional on norm bounded sets of $B(H)$, hence a β continuous linear functional on $B(H)$. Recall that LCH is β dense in $B(H)$ and our claim follows.

Since the ultrastrong topology is defined by seminorms $A \rightarrow \sum_{n=1}^{\infty} \langle Ax_i, x_i \rangle$ where $\sum \|x_i\|^2 < +\infty$, F is clearly ultrastrongly continuous on $B(H)$.

Our next result is related to Glimm's work on the Stone-Weierstrass Theorem [17]. Since A is an ideal in $M(A)$, pure states of A extend to β continuous pure states of $M(A)$ [59, 41, 42]. These are, in fact, exactly the β continuous pure states of $M(A)$. States of A also extend, uniquely, to states of $M(A)$ which are β continuous. When we speak of pure states and states of A as elements

of $M(A)_{\beta}'$, are speaking of these functionals extended to $M(A)$ so as to preserve β continuity and/or pureness.

3.4.2 LEMMA. Suppose B is a β -closed self-adjoint subalgebra of $M(A)$ containing the identity of A and separating the states of A . Then $B = M(A)$.

Proof. Let B_1 and $M(A)_1$ denote the self-adjoint elements in B and $M(A)$, respectively. If $B \neq M(A)$, then $B_1 \neq M(A)_1$. Noting that B_1 is β -closed, we have, by duality theory, a real linear functional f which is β continuous on $M(A)_1$ so that $f \equiv 0$ on B_1 , but f is not identically zero. Extend f to all of $M(A)$ in the obvious way and note that the extension, which we shall denote by g , is a β continuous hermitian (real valued on hermitian elements of $M(A)$) functional. Since A is a C^* -algebra, we may write $f = h_1 - h_2$ where h_1 and h_2 are positive linear functionals on A . Extending h_1 and h_2 (uniquely) to β continuous positive linear functionals p_1 and p_2 , respectively, on $M(A)$, we have that $g = p_1 - p_2$. Since $g(I) = 0$, $\|p_1\| = \|p_2\| = p_1(I) = p_2(I)$. Thus the states $\frac{1}{\|p_1\|} p_1$ and $\frac{1}{\|p_2\|} p_2$ agree on B , but

not on $M(A)$. This is a contradiction to the assumption that $B \neq M(A)$.

3.4.3 DEFINITION. Let A be a C^* -algebra. A is called liminal if the image of A under every irreducible representation is a subset of the compact operators on the Hilbert space of the representation. A is called anti-liminal if A contains no nonzero two-sided closed liminal ideal.

3.4.4 LEMMA. Suppose A is a C^* -algebra such that every state of A is a weak- $*$ limit of a net of pure states on A . If f is a state in $M(A)_{\beta}'$, then f is a β weak- $*$ limit of a net of β continuous pure states of $M(A)$.

Proof. Using D. C. Taylor's methods [59, Th. 2.1] and [17, 1.6.10], we may factor $f = a \cdot g \cdot a$ where g is a β continuous state of $M(A)$ and $a \in A^+$. Let $\{f_{\alpha}\}$ be a net of β continuous pure states of $M(A)$ so that $f_{\alpha}(x) \rightarrow g(x)$ for $x \in A$. Then $a \cdot f_{\alpha} \cdot a \rightarrow a \cdot g \cdot a = f$ β weak- $*$ on $M(A)$. We may assume that $f_{\alpha}(a^2) = \|a \cdot f_{\alpha} \cdot a\| \neq 0$ for any α . Let $C_{\alpha} = \frac{1}{f_{\alpha}(a^2)}$. Then $\frac{1}{C_{\alpha}} = f_{\alpha}(a^2) \rightarrow g(a^2) = f(I) = 1$. Thus $C_{\alpha}(a \cdot f_{\alpha} \cdot a) \rightarrow f$

β weak-* on $M(A)$. Note that $C_\alpha(a \cdot f_\alpha \cdot a)$ is a β continuous state of $M(A)$ for all α . We need to check that it is a pure state of $M(A)$. This is handled by the next lemma.

3.4.5 LEMMA. Suppose f is a pure state of $M(A)$ and $a \in A^+$ such that $f(a^2) = 1$. Then $a \cdot f \cdot a$ is a pure state of $M(A)$ also.

Proof. Let I denote the identity in $M(A)$. Let Π be the canonical irreducible representation defined by f such that $f(a) = \langle \Pi(b), h_0, h_0 \rangle$ where h_0 is the image of I in the Hilbert space defined by f and ' $\langle \cdot, \cdot \rangle$ ' denotes the inner product in this Hilbert space [17, Section 2]. Then $a \cdot f \cdot a(x) = \langle \Pi(x)(\Pi_a(h_0)), (\Pi(a)h_0) \rangle$. From [17, 2.5.1], we obtain that every positive functional g dominated by $a \cdot f \cdot a$ is of the type $g(x) = \langle \Pi_x T(\Pi_a(h_0)), T(\Pi_a(h_0)) \rangle$ where T is a positive operator, of norm ≤ 1 , on the Hilbert space of the representation and T commutes with all operators in $\Pi(M(A))$. Since Π is irreducible, T is a multiple of the identity operator. Hence g is a multiple of $a \cdot f \cdot a$, i.e., $a \cdot f \cdot a$ is a pure state [17, Section 2].

3.4.6 THEOREM. Let A be an antiliminal C^* -algebra such that any two nonzero two-sided closed ideals of A have nonzero intersection. Let Q denote the β weak- $*$ closure of the pure states of A (i.e., the β continuous pure states of $M(A)$). If B is a β closed self-adjoint subalgebra of $M(A)$ containing the identity of $M(A)$ such that B separates $Q \cap M(A)'_{\beta}$, then $B = M(A)$.

Proof. From [17, 11.2.4], we have that every state of A is a $\sigma(A', A)$ -limit of a net of pure states on A . Hence, by 3.4.4, every state in $M(A)'_{\beta}$ is a β weak- $*$ limit of a net of pure states in $M(A)'_{\beta}$, i.e., $Q \cap M(A)'_{\beta}$ contains the states in $M(A)'_{\beta}$. By 3.4.2, $B = M(A)$.

3.4.7 REMARK. If A has the properties of 3.4.6, then so does $M(A)$. For if I is an ideal in $M(A)$ and $I \neq 0$, then $I \cap A \neq \{0\}$.

3.4.8 EXAMPLE. For a separable Hilbert space H , $A = B(H)/LCH$ has the properties in the hypothesis of 3.4.6.

CHAPTER IV
VECTOR-VALUED FUNCTIONS

In this chapter we discuss the basic theory of topological measure theory along with the recent work of Sentilles which helps make clear the connection between topological measure theory and the strict topology. We then present our contributions to the theory in Sections 2 and 3. In Section 4, we compute the double centralizer algebras of two algebras of vector-valued functions. The most interesting results, in our opinion, are 4.2.3, 4.3.2, 4.3.4, 4.3.7, 4.3.12, and 4.3.13.

SECTION 1. PRELIMINARIES.

We first need to develop some additional measure theory. A good reference for this is [62]. Let X denote a completely regular topological space. The Baire algebra of X , denoted $B_a^*(X)$ is the smallest algebra of subsets of X containing the zero-sets of functions in $C^*(X)$. We use $B_a(X)$ to denote the smallest σ -algebra containing the zero-sets. In this chapter, $C^*(X)$ always means real-valued continuous functions and all linear spa-

ces considered are real linear spaces. A positive Baire measure μ on X is a non-negative, finite, positive real-valued, finitely-additive set function on $B_a^*(X)$ so that $A \in B_a^*(X) \Rightarrow \mu(A) = \sup \{\mu(Z) : Z \subseteq A, Z \text{ a zero set of } X\}$. A Baire measure is the different of two positive Baire measures. The collection of all Baire measures and positive Baire measures are denoted $M(X)$ and $M^+(X)$ respectively. If m is a Baire measure, the set functions $m^+(A) = \sup \{m(B) : B \subseteq A, B \in B_a^*(X)\}$, for $A \in B_a^*(X)$, and $m^-(A) = - \inf \{m(B) : B \in B_a^*(X) \text{ and } B \subseteq A\}$, for $A \in B_a^*(X)$, are elements of $M^+(X)$ and $m = m^+ - m^-$. Let $|m| = m^+ + m^-$. Then $|m| \in M^+(X)$ and is called the absolute value of the Baire measure m . $M(X)$ with the norm $\|m\| = m^+(X) + m^-(X)$ is a Banach space. There is an equivalent definition of $M(X)$ that is sometimes useful. Let m be a finitely-additive set function on $B_a^*(X)$. Then $m \in M(X)$ iff (1) $|m(A)| \leq C$ for some $C > 0$ and all $A \in B_a^*(X)$ and (2) for any $A \in B_a^*(X)$ and $\epsilon > 0$, there is a zero-set $Z \subseteq A$ so that $|m(B)| < \epsilon$ for all $B \subseteq A \setminus Z$.

The adjoint of $C^*(X)$ can be identified with $M(X)$. If $\mathfrak{f} \in C^*(X)'$, there is a unique Baire measure $m \in M(X)$ such that $\mathfrak{f}(f) = \int f dm$ for $f \in C^*(X)$. Conversely, if \mathfrak{f} is defined by the preceding formula for $m \in M(X)$,

then $\mathfrak{z} \in M(X)$. Furthermore, $\|\mathfrak{z}\| = \|m\|$. The correspondence is a vector space homomorphism and preserves order, that is, \mathfrak{z} is a positive linear functional ($\mathfrak{z}(f) \geq 0$ for $f \geq 0$ in $C^*(X)$) iff $m \in M^+(X)$ [62, Theorem 6].

We shall be particularly interested in three classes of measures on X . A Baire measure m is said to be σ -additive if $m(Z_n) \rightarrow 0$ for every sequence $\{Z_n\}_{n=1}^{\infty}$ of zero-sets of X such that $Z_{n+1} \subseteq Z_n \forall n$ and $\bigcap_{n=1}^{\infty} Z_n = \{\emptyset\}$ (we denote this by $Z_n \downarrow \emptyset$). A measure $m \in M(X)$ is called τ -additive if $m(Z_\alpha) \rightarrow 0$ for every net $\{Z_\alpha\}$ of zero-sets of X such that $Z_\alpha \subseteq Z_\beta$ for $\alpha \geq \beta$ and $\bigcap_{\alpha} Z_\alpha = \{\emptyset\}$ (we denote this by $Z_\alpha \downarrow \emptyset$). The measure $m \in M(X)$ is called tight if for every $\epsilon > 0$, \exists a compact set $K_\epsilon \subseteq X$ so that $|m|_*(X \setminus K_\epsilon) < \epsilon$, where for $E \subseteq X$ $|m|_*(E) = \sup \{|m|(Z) : Z \text{ is a zero-set of } X \text{ and } Z \subseteq E\}$.

If $\mathfrak{z} \in C^*(X)'$, \mathfrak{z} is called σ -additive if $\mathfrak{z}(f_n) \rightarrow 0$ for every sequence $\{f_n\}_{n=1}^{\infty}$ in $C^*(X)$ such that $f_{n+1} \leq f_n \forall n$ and $f_n \rightarrow 0$ pointwise on X (we denote this by $f_n \downarrow 0$). The functional $\mathfrak{z} \in C^*(X)'$ is called τ -additive if $\mathfrak{z}(f_\alpha) \rightarrow 0$ for every net $\{f_\alpha\} \subseteq C^*(X)$ such that $f_\alpha \leq f_\beta$ for $\alpha \geq \beta$ and $f_\alpha \rightarrow 0$ pointwise on X . Finally, $\mathfrak{z} \in C^*(X)'$ is called tight if $\mathfrak{z}(f_\alpha) \rightarrow 0$

for every net $\{f_\alpha\}$ contained in the unit ball of $C^*(X)$ such that $f_\alpha \rightarrow 0$ $C - Op$.

In [62], it is shown that if $\tilde{\mu} \in C^*(X)'$ and $m \in M(X)$ such that $\tilde{\mu}(f) = \int f dm$, then $\tilde{\mu}$ is σ -additive (τ -additive, tight) iff m is σ -additive (τ -additive, tight). Identifying functionals and the corresponding Baire measures, we denote the class of σ -additive, τ -additive, and tight functionals (σ -additive, τ -additive, and tight Baire measures) by $M_\sigma(X)$, $M_\tau(X)$, and $M_t(X)$, respectively. Note that $M_t(X) \subset M_\tau(X) \subset M_\sigma(X)$.

One of the big problems of topological measure theory is to determine when $M_\tau(X) = M_\sigma(X)$. This problem is first mentioned in [62] and has been studied by many authors. Some good references for the interested reader are [21, 29, 30, 31, 35-37, 50, 62, 67]. A space X satisfying $M_\sigma(X) = M_\tau(X)$ is called measure compact. If X is a set, not necessarily with a topology, X is said to have measure compact cardinal if X , equipped with the discrete topology, is measure compact. The question of whether every set has measure compact cardinal is a deep question of set theory which is related to the existence of Ulam measures [23] and the question of whether every discrete space is realcompact [23]. For a locally compact space, $M_t(X) = M_\tau(X)$ [35]. Measure compactness is a to-

topological property, i.e., if X and Z are homeomorphic topological space X is measure compact iff Z is .

In Chapter III we spoke of measure compact C^* -algebras (see 3.3.6). If X is a locally compact space, X is measure compact iff $M_\sigma(X) = M_t(X)$. If A is a commutative C^* -algebra then A is isometrically $*$ -isomorphic with the algebra $C_0(S)$ for some locally compact Hausdorff space; it is clear that A is measure compact in the sense of Chapter III, 3.3.6, iff S is measure compact, i.e., iff $M_\sigma(S) = M_r(S) = M_t(S)$. Closed subsets of measure compact spaces are measure compact. Countably infinite products of locally compact measure compact spaces are measure compact [29]. If X is Lindelöf, then X is measure compact [29]. If X has measure compact cardinal and X is paracompact, then X is measure compact.

In Section 3 of Chapter III, we prove theorems which extend most of these results for locally compact X to the non-commutative C^* -algebra setting. See, e.g., 3.3.4, 3.3.7, 3.3.8, 3.3.9, 3.3.13, 3.3.22.

F. D. Sentiiles and other have extended Buck's strict topology β to $C^*(X)$ for X completely regular (instead of the more restrictive requirement that X be locally compact); see [21, 22, 27, 50, 55, 61]. In doing so, connections were established between these new "strict"

topologies and some aspects of topological measure theory. In our opinion, the best work so far is that in [21] and [50]. We shall describe the work of Sentilles.

The topology β_0 on $C^*(X)$ is defined to be the finest locally convex linear topology agreeing with the compact-open topology on norm bounded sets. Let βX denote the Stone-Cech compactification of X [19] and if $f \in C^*(X)$, let \bar{f} denote the unique continuous extension of f to βX . For each compact set $Q \subseteq \beta X \setminus X$, let $C_Q(X) = \{f \in C^*(X) : \bar{f} \equiv 0 \text{ on } Q\}$. Let β_Q be the topology on $C^*(X)$ defined by the seminorms $f \rightarrow \|fh\|$ for $f \in C^*(X)$ and $h \in C_Q(X)$. Let β be the intersection of the topologies β_Q , where Q varies through all compact sets in $\beta X \setminus X$. If we instead allow Q to vary through the zero-sets (of continuous functions defined on βX) contained in $\beta X \setminus X$, the topology is called β_1 . Let p denote the topology of pointwise convergence on X and $C - Op$ that of uniform convergence on compact subsets of X and $\| \cdot \|$ the norm topology on $C^*(X)$.

Sentilles shows that $p \leq C - Op \leq \beta_0 \leq \beta \leq \beta_1 \leq \| \cdot \|$ and that all these topologies are locally convex and Hausdorff. He also shows that $C^*(X)_{\beta_Q}$ is topologically and isometrically isomorphic with $C^*(\beta X \setminus Q)$ with the strict topology defined by $C_0(\beta X \setminus Q)$ (in Buck's sense). It is

known that $\beta = \beta_0$ if X is locally compact [18,51].

Sentilles [50] makes a very important contribution when he calculates the adjoint spaces of $C^*(X)$ endowed with the topologies β_0 , β , and β_1 . It is this result which allows him to use the interplay between topological measure theory techniques and functional analytic techniques to obtain a deeper understanding of both topological measure theory and his strict topologies. Sentilles shows that $C^*(X)'_{\beta_0} = M_t(X)$, $C^*(X)'_{\beta_1} = M_\sigma(X)$ and $C^*(X)'_{\beta} = M_r(X)$ and that X is measure compact iff $\beta_1 = \beta$. He also proves many other interesting results which we will list as we need.

In the rest of Chapter IV, except for Section 4, we extend many of the above mentioned results to vector-valued functions.

In what follows E will always denote a real normed linear space (in most of the results, if not all of them, E could be any locally convex space, but we feel that notation is made simpler by restricting ourselves to this case). Let X denote a completely regular topological space and let $C^*(X;E)$ denote the set of all bounded continuous functions from X to E . $C^*(X;E)$ is a real linear space.

We define the topology β_0 on $C^*(X;E)$ to be the finest locally convex linear topology agreeing with the

compact-open topology on norm bounded sets. For Q a compact subset of $\beta X \setminus X$, the topology β_Q on $C^*(X;E)$ is that topology defined by the seminorms $f \rightarrow \|hf\|$, where $h \in C_Q(X)$ and $f \in C^*(X;E)$. Then β_1 and β are defined as the intersection of topologies β_Q , exactly as in the scalar case.

Note that we may restrict ourselves to nonnegative functions in C_Q in defining the β_Q seminorms and the resulting topology is β_Q . Also note that if X is locally compact, then $\beta = \beta_0$ is the topology defined for $C^*(X;E)$ by Buck in [5] and studied in [66]. That β_0 coincides with Buck's topology in the case X is locally compact follows from [51].

In [50], β_0 , as defined by Sentilles, is compared with several other 'strict topologies' in the literature. We now define these topologies for $C^*(X;E)$ and state the generalization of Sentilles' theorem to $C^*(X;E)$ without proof, since the proof in [50] goes through, with obvious modifications. Let w_1 denote the topology on $C^*(X;E)$ defined by the seminorms $f \rightarrow \|hf\|$, for $f \in C^*(X;E)$, where h is a bounded real-valued function on X such that $\{x: |h(x)| \geq \epsilon\}$ is compact for each $\epsilon > 0$ and let w_2 denote the topology on $C^*(X;E)$ defined by the seminorms $f \rightarrow \|hf\|$ where h is a nonnegative real-valued

function on X which is bounded and satisfies the condition that $\{x \in X: h(x) \geq \epsilon\}$ is compact for each $\epsilon > 0$. Summers studied w_2 in the scalar case in [53,55]. Finally, let m denote the mixed topology $\gamma[C - Op, \|\cdot\|]$ on $C^*(X;E)$ as defined by Wiweger [68].

4.1.1 THEOREM. (a) $\beta_0 = m$ and β_0 has a base of neighborhoods of the form $W(K_1, a_1) = \bigcap_{i=1}^{\infty} \{f: \|f\|_{K_1} \leq a_1\}$ where $0 < a_1 \rightarrow \infty$ and K_1 is a compact subset of X , for each i .

(b) $\beta_0 = w_1 = w_2$.

If S is locally compact, most of the results for $C^*(S)_\beta$ go through for $C^*(S;E)_\beta$, perhaps with the additional assumption that E is complete ($C^*(S;E)_\beta$ is complete iff E is complete). For example, β and $C - Op$ coincide iff S is sham compact. The norm topology is β iff X is compact.

Conway in [11] proves an interesting characterization of $C^*(S)_\beta$ as a projective limit of Banach spaces. His proof extends easily to the case $C^*(S;E)$. In [11], Conway also asks about the Mackey problem for $C^*(S;E)_\beta$. Using the main technique in [57] we have the following theorem.

4.1.2 THEOREM. Suppose S is a locally compact Hausdorff space and $C_0(S)$ has a well-behaved approximate identity and E is a Banach space. Then $C^*(S;E)$ is a strong Mackey space.

4.1.3 COROLLARY. If S is paracompact and E is a Banach space, then $C^*(S;E)$ is a strong Mackey space.

The following analogue of Sentilles' theorem relating the topology of pointwise convergence denoted p , the compact-open topology denoted $C - Op$, and the norm and strict topologies holds. The proof which is similar to that of Sentilles is omitted.

4.1.4 THEOREM. (a) $p \leq C - Op \leq \beta_0 \leq \beta_1 \leq \beta_2 \leq \| \|$ on $C^*(X;E)$. (b) The topologies $C - Op$ through $\| \|$ are all equal iff X is compact. (c) If X is locally compact, $\beta = \beta_0$. (d) X is compact iff β is barrelled [45] iff β is bornological [45] iff β is metrizable iff β is normable. (e) X is pseudocompact iff β_1 is barrelled iff β_1 is bornological iff β_1 is metrizable iff β_1 is normable.

SECTION 2. THE STRICT TOPOLOGIES β AND β_1 FOR VECTOR-VALUED FUNCTIONS.

As in Section 1, let E be a real normed linear space, X a completely regular space, and $C^*(X;E)$ denote the real linear space of bounded continuous functions from X to E . When no other topology is explicitly mentioned, $C^*(X;E)$ is to be assumed given the norm topology.

4.2.1 DEFINITION. Let $\phi \in C^*(X;E)'$. Then ϕ is said to be σ -additive if for every sequence $\{f_n\} \subseteq C^*(X)$ such that $f_n \downarrow 0$ $\phi(f_n g) \rightarrow_n 0$ uniformly for g in $C^*(X;E)$ of norm ≤ 1 . Similarly, ϕ is said to be τ -additive if, whenever $\{f_\alpha\}$ is a net in $C^*(X)$ such that $f_\alpha \downarrow 0$, then $\phi(f_\alpha g) \rightarrow_\alpha 0$ uniformly for g in $C^*(X;E)$ of norm ≤ 1 .

4.2.2 REMARK. The definitions in 4.2.1 generalize the usual ones. In order to see this, we need only show that if $\phi \in C^*(X)'$ and ϕ is a positive linear functional, then ϕ is σ -additive (τ -additive) in the sense of [62] implies it is σ -additive (τ -additive) in the sense of 4.2.1. This follows immediately from 1.10.

4.2.3 THEOREM. Let $\phi \in C^*(X;E)'$. Then (a) ϕ is

σ -additive iff ϕ is β_1 continuous on $C^*(X:E)$;

(b) ϕ is τ -additive iff ϕ is β continuous on $C^*(X:E)$.

Proof. (a) Suppose that ϕ is σ -additive. We wish to show that $\phi \in (C^*(X:E)_{\beta_1})'$. It clearly suffices to show that $\phi \in (C^*(X:E)_{\beta_Q})'$ for an arbitrary zero-set Q contained in $\beta X \setminus X$. Let Q be a zero-set of βX such that $Q \subseteq \beta X \setminus X$. Since Q is a compact G_δ , $C_Q(X)$ has a countable approximate identity $\{h_n\}_{n=1}^\infty$ satisfying $0 \leq h_n \leq 1 \forall n$ and $1 - h_n \downarrow 0$ on X . Thus $\phi((1 - e_n)g) \rightarrow 0$ uniformly for g in $C^*(X:E)'$ in the natural way, i.e., if $g \in C_Q(X)$ and $\phi \in C^*(X:E)'$ $g \cdot \phi(h) = \phi(gh) \forall h \in C^*(X:E)$. With this notation we have shown that $\|e_n \cdot \phi - \phi\| \xrightarrow{n} 0$. Thus $\phi \in W = \{p: p \in C^*(X:E)'$ and $\|e_n \cdot p - p\| \rightarrow 0\}$. Clearly W is a Banach space and an essential left $C_Q(X)$ -module in the language of Chapter I. By 1.12, if $p \in W$, $p = a \cdot q$ where $a \in C_Q(X)$ and $q \in W$. Clearly then $W \subseteq (C^*(X:E)_{\beta_Q})'$. Thus $\phi \in (C^*(X:E)_{\beta_Q})'$ for each compact zero-set $Q \subseteq \beta X \setminus X$; hence $\phi \in (C^*(X:E)_{\beta_1})'$.

Conversely, suppose that ϕ is β_1 continuous,

$\|\phi\| \leq 1$, $\epsilon > 0$, and $\{f_n\} \subseteq C^*(X)$ such that

$\|f_n\| \leq 1 \forall n$ and $f_n \downarrow 0$ on X . For $f \in C^*(X)$, let \mathbb{F}

denote the unique continuous extension of f to βX .

Let $K = \bigcap_{n=1}^{\infty} \{t \in \beta X: \bar{F}_n(t) \geq \frac{\epsilon}{2}\}$. Then K is a compact nonempty subset of $\beta X \setminus X$. Since $K =$

$\bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \{t \in \beta X: \bar{F}_n(t) > \frac{\epsilon}{2} - \frac{1}{m}\}$ K is a compact G_δ and

hence a zero-set of βX . Since $\emptyset \in C^*(X:E)_{\beta_K}'$ there are

are functions $0 \leq h \leq 1$ in $C_K(X)$ and $\psi \in C^*(X:E)'$ with $\|\psi\| \leq 2$ so that $\emptyset = h \cdot \psi$. Thus $|\emptyset(f)| = |\psi(hf)| \leq 2 \|hf\|$ for all $f \in C^*(X:E)$. Let $O = \{t \in \beta X: \bar{h}(f) < \frac{\epsilon}{2}\}$.

Then O is open, $K \subseteq O$, and $\beta X \setminus O$ is compact. Since $O \supseteq K$, there is an integer N so that $\{t \in \beta X: \bar{F}_n(t) \geq \frac{\epsilon}{2}\} \subseteq O$ for $n \geq N$. Then if $g \in C^*(X:E)$ and $\|g\| \leq 1$, $|\emptyset(f_n g)| \leq 2 \|hf_n g\| \leq 2 \|hf_n\| < \epsilon$ for $n > N$. Hence $\emptyset(f_n g) \rightarrow 0$ uniformly for g of norm ≤ 1 in $C^*(X:E)$, i.e., \emptyset is σ -additive.

(b) The proof of this equivalence is similar to that given in (a) and so is omitted.

SECTION 3. THE TOPOLOGY β_0 ON $C^*(X:E)$.

In this section we characterize the dual space $C^*(X:E)_{\beta_0}'$, show that $C^*(X:E)_{\beta_0}$ has the approximation property if E has the metric approximation property, and give a vector-valued measure representation for elements of $C^*(X:E)_{\beta_0}'$, generalizing the work in [66]. We also

extend 4.2.3.

4.3.1 DEFINITION. Let $F \in C^*(X:E)'$. Then F is said to be tight if $F(g_\alpha) \rightarrow 0$ for every net $\{g_\alpha\} \subset C^*(X:E)$ such that $\|g_\alpha\| \leq 1 \quad \forall \alpha$ and $g_\alpha \rightarrow 0$ uniformly on compact subsets of X .

4.3.2 THEOREM. Let $F \in C^*(X:E)'$. The following statements are equivalent:

- (1) $F \in C^*(X:E)_{\beta_0}'$;
- (2) F is tight;
- (3) The real linear functional T on $C^*(X)$ defined for $f \geq 0$ in $C^*(X)$ by the equation $T(f) = \sup \{F(g) : \|g(x)\| \leq f(x), \forall x \in X, g \in C^*(X:E)\}$ and extended by linearity to all of $C^*(X)$ is tight;
- (4) if $\epsilon > 0$, \exists compact $K_\epsilon \subseteq X$ so that if $f \in C^*(X:E)$ and $\|f\| \leq 1$, then $f \equiv 0$ on K_ϵ implies that $|F(f)| < \epsilon$;
- (5) $F(f_\alpha g) \rightarrow 0$ uniformly for $g \in C^*(X:E)$ of norm ≤ 1 , for every net $\{f_\alpha\} \subset C^*(X)$ such that $\|f_\alpha\| \leq 1 \quad \forall \alpha$ and $f_\alpha \rightarrow 0$ C -Op.

Proof. (1) \Rightarrow (2). Suppose F is β_0 continuous and $g_\alpha \rightarrow 0$ uniformly on compact subsets of X . Since β_0

agrees with the compact-open topology on norm bounded subsets of $C^*(X;E)$, $g_\alpha \rightarrow 0$ β_0 ; hence $F(g_\alpha) \rightarrow 0$.

(2) \Rightarrow (1). Since β_0 is defined as the finest locally convex linear topology agreeing with the compact-open topology on norm bounded sets a linear functional F on $C^*(X;E)$ is β_0 continuous iff its restriction to norm bounded subsets of $C^*(X;E)$ is continuous in the compact-open topology, i.e., iff F is tight.

(5) \Rightarrow (3). For $f \geq 0$ in $C^*(X)$ define $T(f) = \sup \{F(g) : \|g(x)\| \leq f(x), \forall x \in X\}$. We first want to show that we can extend T to a real linear functional on $C^*(X)$. In order to establish this, all we need to show is that $T(f+g) = T(f) + T(g)$, for $f, g \geq 0$ in $C^*(X)$.

Let $h \in C^*(X;E)$ so that $\|h\| \leq f+g$. For $x \in X$ such that $f(x) + g(x) > 0$, define $h_1(x) = \frac{f(x)h(x)}{f(x)+g(x)}$ and $h_2(x) = \frac{g(x)h(x)}{f(x)+g(x)}$. If $f(x) + g(x) = 0$, let $h_1(x) = h_2(x) = 0$. Note that h_1 and $h_2 \in C^*(X;E)$ and $\|h_1\| \leq f$ and $\|h_2\| \leq g$. Thus $F(h) = F(h_1) + F(h_2) \leq T(f) + T(g)$. Taking the supremum over all such functions h , we get $T(f+g) \leq T(f) + T(g)$. For the other inequality, let $\epsilon > 0$ and $h_1, h_2 \in C^*(X;E)$ with $\|h_1\| \leq f$, $\|h_2\| \leq g$ and $0 \leq F(h_1) \leq T(f) \leq F(h_1) + \frac{\epsilon}{2}$ and $0 \leq F(h_2) \leq T(g) \leq F(h_2) + \frac{\epsilon}{2}$. Then $T(f) + T(g) \leq F(h_1) + F(h_2) + \epsilon = F(h_1 + h_2) + \epsilon \leq T(f+g) + \epsilon$. Since $\epsilon > 0$ is

arbitrary we get $\mathbb{T}(f) + \mathbb{T}(g) \leq \mathbb{T}(f+g)$. Hence $\mathbb{T}(f+g) =$
 $\mathbb{T}(f) + \mathbb{T}(g)$ and so \mathbb{T} extends to a linear functional
 on $C^*(X)$ which is bounded.
 We now show \mathbb{T} is tight. Suppose $\{f^\alpha\} \subseteq C^*(X)$,
 $\|f^\alpha\| \leq 1$ and $f^\alpha \rightarrow 0$ $C - Op$. We need to show
 $\mathbb{T}(f^\alpha) \rightarrow 0$ so we may assume that $f^\alpha \geq 0$ or $f^\alpha \leq 0$.
 We first show that it suffices to show that $\mathbb{T}(f^\alpha) \rightarrow 0$
 for all positive $f \geq 0$. Assume, for simplicity, that
 $\|f\| \leq 1$. Then $0 \leq \mathbb{T}(f) \leq \|f\|$ for all $f \geq 0$ in $C^*(X)$.
 From elementary calculus, we get that $\sup_{0 \leq t \leq 1} t - \frac{t^2}{2} = \frac{1}{2}$
 for $t > 0$. Hence $0 \leq \mathbb{T}(f^\alpha) - f^\alpha \leq \|f^\alpha\| \leq 1 - \frac{f^\alpha}{2}$
 for $f^\alpha > 0$, and if $\mathbb{T}(f^\alpha) \rightarrow 0$ for each $f^\alpha \geq 0$, then
 $\mathbb{T}(f^\alpha) \rightarrow 0$.
 It remains to be shown that $\mathbb{T}(f^\alpha) \rightarrow 0$ for each real
 number $r > 1$. Fix $r > 1$, let $\epsilon > 0$ and $g \in C^*(X; \mathbb{R})$ if
 so that $\|g\| \leq f^\alpha$ for some α . Define $h(x) = \frac{f^\alpha(x)}{g(x)}$ if
 $f^\alpha(x) \neq 0$ and $h(x) = 0$ otherwise. Then $h \in C^*(X; \mathbb{R})$,
 $\|h\| \leq r$, and $g = f^\alpha h$. Thus $\mathbb{T}(g) = \mathbb{T}(f^\alpha h)$. Choose
 α_0 so that $\alpha \geq \alpha_0$ implies $|\mathbb{T}(f^\alpha h)| \leq \epsilon$ for all
 $h \in C^*(X; \mathbb{R})$ such that $\|h\| \leq r$. Then if $\alpha \geq \alpha_0$,
 $|\mathbb{T}(g)| \leq \epsilon$. Taking the supremum over all possible $g \leq f^\alpha$,
 we get $\mathbb{T}(f^\alpha) \leq \epsilon$ for $\alpha \geq \alpha_0$. Thus $\mathbb{T}(f^\alpha) \rightarrow 0$ for each
 $f^\alpha \geq 0$.

(1) \Rightarrow (4). Suppose F is β_0 continuous and $\epsilon > 0$. By 4.1.1, there is a bounded nonnegative upper semicontinuous function g which vanishes at infinity such that $|F(f)| \leq \|gf\|$ for all $f \in C^*(X;E)$. If $\|f\| \leq 1$ and $f \equiv 0$ on $K_\epsilon = \{x \in X: g(x) \geq \epsilon\}$, then $|F(f)| < \epsilon$.

(4) \Rightarrow (5). Suppose that (4) holds and that $\{f_\alpha\} \subseteq C^*(X)$, $\|f_\alpha\| \leq 1$ and $f_\alpha \rightarrow 0$ $C - Op$. We want to show that $F(f_\alpha g) \xrightarrow{\alpha} 0$ uniformly for g in $C^*(X;E)$ of norm ≤ 1 . Clearly, we may assume that $f_\alpha \geq 0 \forall \alpha$ and that $\|F\| \leq 1$. Let $\epsilon > 0$ and let K_ϵ be the compact subset of X given by (4). Choose α_0 so that $\alpha \geq \alpha_0$ implies $\|f_\alpha\|_{K_\epsilon} < \epsilon$. Let $h_\alpha = \min \{f_\alpha, \epsilon\}$. Then if $g \in C^*(X;E)$ and $\|g\| \leq 1$, $|F(f_\alpha g - h_\alpha g)| < \epsilon$ for $\alpha \geq \alpha_0$ since $f_\alpha - h_\alpha = 0$ on K_ϵ . Thus, for $\alpha \geq \alpha_0$, $|F(f_\alpha g)| \leq |F(f_\alpha g) - F(h_\alpha g)| + |F(h_\alpha g)| < 2\epsilon$. Hence $F(f_\alpha g) \xrightarrow{\alpha} 0$ uniformly for g in the unit ball of $C^*(X;E)$.

(3) \Rightarrow (1). Suppose that T is tight. Then, from Santilles' result $M_t = C^*(X)_{\beta_0}^t$ and 4.1.1, \exists a bounded nonnegative upper semicontinuous function h vanishing at infinity such that $\|T(g)\| \leq \|hg\|$ for all $g \in C^*(X)$. Let $f \in C^*(X;E)$. Then $|F(f)| \leq T(\|f\|) \leq \| \|f\| h \| = \|hf\|$; therefore F is β_0 continuous by 4.1.1 again. We have shown (2) \Rightarrow (1) \Rightarrow (2) and (1) \Rightarrow (4) \Rightarrow (5) \Rightarrow (3) \Rightarrow (1), so the proof of 4.3.2 is complete.

4.3.3 REMARK. We have the following improvement of 4.2.3, whose proof is clear if we look at 4.2.3 along with the proof of (5) \Rightarrow (3) in 4.3.2 and make the observation that (3) \Rightarrow (5) in 4.3.2 is trivial (although we did not prove 4.3.2 this way).

4.3.4 THEOREM. Let $F \in C^*(X;E)'$.

- (a) The following are equivalent: (1) F is σ -additive; (2) F is β_1 continuous; (3) if T is defined in terms of F as in 4.3.2, T is σ -additive.
- (b) The following are equivalent: (1) F is τ -additive; (2) F is β continuous; (3) if T is defined in terms of F as in 4.3.2, T is τ -additive.

The next topic we take up is the approximation problem in $C^*(X;E)_{\beta_0}$ (see 3.2.31), where we generalize a result in [21]. The following lemma is well-known.

4.3.5 LEMMA. Let X be a completely regular space, C a compact subset of X , K a compact subset of $C^*(X)_{\beta_0}$ and $\epsilon > 0$. Then \exists a finite partition of unity (see 4.3.6) $\{g_i\}_{i=1}^n$ on X and points $\{c_i \mid 1 \leq i \leq n\}$ in C so that if P is the linear operator on $C^*(X)$ defined by the equation $Pf(x) = \sum_{i=1}^n g_i(x)f(c_i)$, then P is

β_0 continuous, $\|P\| \leq 1$, P is of finite rank and
 $\|Pf - f\|_C < \epsilon$ for $f \in K$.

4.3.6 DEFINITION. Let X be a completely regular space and $\{f_\alpha\} \subseteq C^*(X)$ such that $0 \leq f_\alpha \leq 1$ for each α . The family $\{f_\alpha\}$ is called a partition of unity on X if the sets $\text{spt } f_\alpha$ (see 2.2.3) form a locally finite cover of X and $\sum f_\alpha = 1$ on X .

If there is a covering \mathcal{A} of X so that $\text{spt } f_\alpha \subseteq \alpha$ for each $\alpha \in \mathcal{A}$, then $\{f_\alpha\}$ is called a partition of unity subordinate to \mathcal{A} .

4.3.7 THEOREM. Let E be a normed linear space with the metric approximation property and X a completely regular Hausdorff space. Then $C^*(X;E)_{\beta_0}$ has the approximation property.

Proof. Let $\epsilon > 0$, Q a β_0 totally bounded subset of $C^*(X;E)$ and h a nonnegative bounded upper semicontinuous function on X which vanishes at infinity such that $\|h\| \leq 1$. Since Q is norm bounded, let us assume that Q is a subset of the unit ball in $C^*(X;E)$.

Let $C = \{x: h(x) \geq \frac{\epsilon}{2}\}$. Then C is compact. Note that $D = \{f(x): f \in Q, x \in C\}$ is a totally bounded subset

of E . Hence there is a finite rank operator T on E , with $\|T\| \leq 1$, such that $\|T(d) - d\| < \frac{\epsilon}{2} \quad \forall d \in D$. Since T is finite-rank, there is a finite set $\{\varphi_i: 1 \leq i \leq n\} \subseteq E'$ and a finite set $\{e_i: 1 \leq i \leq n\} \subseteq E$ so that

$$T(e) = \sum_{i=1}^n \varphi_i(e) e_i \quad \text{for } e \in E. \quad \text{Thus } T(f(x)) =$$

$$= \sum_{i=1}^n \varphi_i \circ f(x) e_i \quad \text{for } f \in C^*(X; E). \quad \text{Since the set}$$

$\{\varphi_i \circ f: 1 \leq i \leq n, f \in Q\}$ is a β_0 totally bounded subset of $C^*(X)$, there is, by 4.3.5, a finite-rank operator P on $C^*(X)$ with $\|P\| \leq 1$, such that P is continuous for the β_0 topology on $C^*(X)$ and such that

$$\|P(\varphi_i \circ f) - (\varphi_i \circ f)\|_C < \frac{\epsilon}{2} \quad \text{for } f \in Q \text{ and } 1 \leq i \leq n.$$

Furthermore, we may assume P given by a formula such as that in 4.3.5. Let S be the linear operator on $C^*(X; E)$ defined for $f \in C^*(X; E)$ by the equation $Sf(x) =$

$$\sum_{i=1}^n P(\varphi_i \circ f)(x) e_i \quad \text{for all } x \in X. \quad \text{Note that } S \text{ is } \beta_0 \text{ con-}$$

tinuous and of finite rank. In order to compute $\|S\|$, we write P more explicitly. As in 4.3.5, let $\{g_j | 1 \leq j \leq m\}$

be a partition of unity on X and $\{c_j: 1 \leq j \leq m\} \subseteq C$ so that $Pf(x) = \sum_{j=1}^m g_j(x) f(c_j) \quad \forall f \in C^*(X)$. If $f \in C^*(X; E)$

$$\begin{aligned} \text{and } x \in X, \text{ then } Sf(x) &= \sum_{i=1}^n P(\varphi_i \circ f)(x) e_i = \\ &= \sum_{i=1}^n \sum_{j=1}^m g_j(x) (\varphi_i \circ f(c_j)) e_i = \sum_{j=1}^m g_j(x) \left(\sum_{i=1}^n \varphi_i \circ f(c_j) e_i \right) = \\ &= \sum_{j=1}^m g_j(x) T(f(c_j)). \quad \text{Thus } \|Sf(x)\| \leq \max_{1 \leq j \leq m} \|T(f(c_j))\| \leq \|f\| \end{aligned}$$

and so $\|S\| \leq 1$.

What remains to be shown is that $\|h(Sf - f)\| < \epsilon$ for $f \in Q$. If $x \in X \setminus C$, $h(x) < \frac{\epsilon}{2}$ so that $\|h(x)(Sf(x) - f(x))\| < (\frac{\epsilon}{2})(2\|f\|) < \epsilon$. If $x \in C$, then $\|Sf(x) - f(x)\| \leq \left\| \sum_{i=1}^n P(\varphi_i \circ f)(x)e_i - \sum_{i=1}^n \varphi_i \circ f(x)e_i \right\| + \|T(f(x)) - f(x)\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Thus if $x \in C$, $\|h(x)(Sf - f)(x)\| < \epsilon$ since $\|h\| \leq 1$. Hence $\|h(Sf - f)\| < \epsilon \quad \forall f \in Q$ and the proof is complete.

Our last results in Section 3 have to do with a vector measure representation for tight linear functionals on $C^*(X; E)$.

4.3.8 DEFINITION. Let X be a completely regular space and E a normed linear space. By $M(X; E')$ we denote the set of all set functions m defined on $B_a^*(X)$, with range in E' , which satisfy the following two conditions:

(a) the measure $m(\cdot)e$ defined for $e \in E$ by $m(\cdot)e(A) = m(A)(e)$, $A \in B_a^*(X)$, belongs to $M(X)$;

(b) $\exists C > 0$ so that $\sum_{i=1}^n \|m(A_i)\| < C$ for every partition of X into sets $X_i \in B_a^*(X)$. Let $M_\sigma(X; E')$, $M_t(X; E')$ and $M_\tau(X; E')$ denote the set of $m \in M(X; E')$ so that for each $e \in E$ $m(\cdot)e \in M_\sigma(X)$, $M_t(X)$, and $M_\tau(X)$, respectively.

4.3.9 PROPOSITION. Let $m \in M(X;E')$ and $A \in B_a^*(X)$. Let $|m|(A) = \sup \left\{ \sum_{i=1}^n \|m(A_i)\| : \{A_i\} \subseteq B_a^*(X) \text{ is a partition of } A \right\}$. Then $|m| \in M(X)$. If $m \in M_\sigma(X;E')$ ($M_t(X;E')$), then $|m| \in M_\sigma(X)$ ($M_t(X)$).

Proof. The proof of the first assertion and the proof of the assertion that $m \in M_t(X;E')$ implies $|m| \in M_t(X)$ are straightforward.

Suppose that $m \in M_\sigma(X;E')$, $A \in B_a(X)$ and $e \in E$. From [62, TH. 18] there is a unique countably additive (regular) measure m_e on $B_a(X)$ which extends $m(\cdot)e$. Let $m'(A) = m_e(A)$ for $A \in B_a(X)$. By regularity and uniqueness of extension $m' \in M(X;E')$ and $|m| = |m'|$ on $B_a^*(X)$. Also m' is countably additive in norm, i.e., if $\{A_n\}_{n=1}^\infty$ is a disjoint collection in $B_a(X)$, $\|m'(\bigcup_{n=1}^p A_n) - \sum_{n=1}^p m'(A_n)\| \rightarrow 0$. Hence by modifying standard arguments such as [47, TH. 6.2], $|m'|$ is countably additive. Hence $|m|$ is σ -additive.

4.3.10 DEFINITION. Let $m \in M(X;E')$ and $f \in C^*(X;E)$. The integral of f with respect to m , denoted $\int_X f dm$, is the real number r if for $\epsilon > 0$ there is a finite partition $P(\epsilon)$ of X into elements of $B_a^*(X)$ so that

$|\sum_{i=1}^n m(A_i)(f(x_i)) - n| < \epsilon$ if $\{A_i\}_{i=1}^n \subseteq B_a^*(X)$ is any partition of X refining $P(\epsilon)$ and $\{x_i\}_{i=1}^n$ is any choice of points such that $x_i \in A_i$ for $1 \leq i \leq n$.

4.3.11 LEMMA. Let $f \in C^*(X:E)$ and $m \in M(X:E')$. Then

$$\int_X f dm \text{ exists and } \left| \int_X f dm \right| \leq \int_X \|f\| d|m|.$$

4.3.12 PROPOSITION. Let $m \in M(X:E')$ and $F(f) = \int_X f dm$ for $f \in C^*(X:E)$. Then $F \in C^*(X:E)'$ and $\|F\| = |m|(X)$. If $m \in M_\sigma(X:E')$ or $m \in M_t(X:E')$, then F is σ -additive or tight, respectively.

Proof. Apply 4.3.11 and 4.3.9 plus Sentilles' results [50], for all assertions but the equality $\|F\| = |m|(X)$. By 4.3.11, it is clear that $\|F\| \leq |m|(X)$.

For the reverse inequality, it suffices to show that

$\sum_{i=1}^n m(Z_i)(e_i) \leq \|F\| + \epsilon$ for every $\epsilon > 0$, finite set $\{e_i\}_{i=1}^n$ contained in the unit ball of E , and disjoint collection $\{Z_i\}_{i=1}^n$ of zero-sets such that $m(Z_i)(e_i) \geq 0$ for $1 \leq i \leq n$.

Suppose that $\{Z_i\}_{i=1}^n$ and $\{e_i\}_{i=1}^n$ are sets as above and $\epsilon > 0$. For $\mu \in M(X)$, let $|\mu|$ denote the total variation of μ [47]. Choose disjoint cozero-sets $\{D_i\}$,

$1 \leq i \leq n$, so that $Z_i \subseteq D_i$ and $|m(\cdot)e_i|(D_i \setminus Z_i) < \frac{\epsilon}{n}$,
 and functions $\{f_i: 1 \leq i \leq n\} \subseteq C^*(X)$ such that
 $0 \leq f_i \leq 1$, $1 \leq i \leq n$, $f_i \equiv 1$ on Z_i and $f_i \equiv 0$ on
 $X \setminus D_i$. For $f \in C^*(X)$ and $e \in E$, let $f \otimes e(x) = f(x)e$,
 $\forall x \in X$. Note that $f \otimes e \in C^*(X; E)$. Then $\sum_{i=1}^n m(Z_i)(e_i) =$
 $\sum_{i=1}^n \int_{Z_i} f_i \otimes e_i \, dm \leq \left| \sum_{i=1}^n \int_{D_i} f_i \otimes e_i \, dm \right| + \epsilon \leq |F(\sum_{i=1}^n f_i \otimes e_i)| +$
 $+ \epsilon \leq \|F\| + \epsilon$. Hence $|m|(X) \leq \|F\|$ as claimed.

4.3.13 THEOREM. Suppose F is a tight linear functional
 on $C^*(X; E)$. Then $\exists m \in M_t(X; E')$ so that $F(f) = \int_X f \, dm$
 $\forall f \in C^*(X; E)$.

Proof. For $g \in C^*(X)$ and $e \in E$, let $g \otimes e(x) = g(x)e$
 $\forall x \in X$. Let $C^*(X) \otimes E$ denote the linear subspace of
 $C^*(X; E)$ spanned by all functions $g \otimes e$ for $g \in C^*(X)$ and
 $e \in E$. By using partitions of unity, we see that $C^*(X) \otimes E$
 is β_0 -dense in $C^*(X; E)$.

For $e \in E$, let $F_e(f) = F(f \otimes e)$ for all $f \in C^*(X)$.
 Since $F \in C^*(X; E)_{\beta_0}'$, $F_e \in C^*(X)_{\beta_0}'$ so by Sentiilles re-
 sults [50], there is a unique $m_e \in M_t(X)$ so that $F_e(f) =$
 $= \int_X f \, dm_e \quad \forall f \in C^*(X)$. Note that $\|m_e\| = \|F_e\| \leq \|F\| \|e\|$
 [62, TH. 6]. For $A \in B_a^*(X)$, let $m(A)(e) = m_e(A)$.
 Note that $m(A) \in E'$ and $m(\cdot)e \in M_t(X) \quad \forall A \in B_a^*(X)$ and

$\forall e \in E$. We show that $\sum_{n=1}^p \|m(A_n)\| \leq \|F\|$, for every partition $\{A_n\}_{n=1}^p$ of X into Baire sets, exactly as in the last part of 4.3.12. Hence $m \in M_t(X:E')$. Note that $F(f) = \int_X f dm \quad \forall f \in C^*(X) \otimes E$. By 4.3.12 and β_0 -denseness of $C^*(X) \otimes E$, $F(f) = \int_X f dm$ holds $\forall f \in C^*(X:E)$.

SECTION 4. EXAMPLES.

Let E be a C^* -algebra, X , a completely regular k -space [19], and S , a locally compact Hausdorff space. We shall compute the double centralizer algebras $M(C^*(X:E))$ and $M(C_0(S:E))$. In 4.4.1, $C^*(X:M(E)_\beta)$ denotes the set of all functions f , continuous from X into $M(E)$ with the strict topology, such that the range of f is a β -bounded (hence, norm bounded) set in $M(E)$. $C^*(X:M(E)_\beta)$ is a C^* -algebra, with $\|f\| = \sup_{x \in X} \|f(x)\| \quad \forall f \in C^*(X:M(E)_\beta)$, since if $f: X \rightarrow M(E)$ then f is in $C^*(X:M(E)_\beta)$ iff $fe \in C^*(X:E) \quad \forall e \in E$.

4.4.1 THEOREM. $M(C^*(X:E)) = C^*(X:M(E)_\beta)$ and $M(C_0(S:E)) = C^*(S:M(E)_\beta)$.

Proof. We prove only the first assertion, as the second has a similar proof.

We shall use the criterion in 3.1.13. First, note

that $C^*(X:E)$ is an ideal in $C^*(X:M(E)_\beta)$. For example, if $g \in C^*(X:M(E)_\beta)$, $f \in C^*(X:E)$, and K compact $\subseteq X$, choose, by 1.12 and 2.2.18, $e \in E$ and $h \in C^*(X:E)$ so that $f = eh$ on K . Then $gf = geh$ on K . Since $ge \in C^*(X:E)$, gf is continuous on K . Hence $gf \in C^*(X:E)$ since X is a k -space.

Using the uniform boundedness principle and the fact that, if $f: X \rightarrow M(E)$, $f \in C^*(X:M(E)_\beta)$ iff $fe \in C^*(X:E) \forall e \in E$, we see that $C^*(X:M(E)_\beta)$ is complete in the strict topology defined by $C^*(X:E)$. Since $g \in C^*(X:M(E)_\beta)$ satisfies $gf = 0 \forall f \in C^*(X:E)$ only if $g = 0$, we have that $C^*(X:M(E)_\beta) = M(C^*(X:E))$ by 3.1.13.

PROBLEMS

1. Determine conditions on S necessary and sufficient for $C_0(S)$ to have a well-behaved approximate identity. As a start consider metacompact and normal S . It is possible that the existence of a metacompact, normal, non-paracompact space S is independent of the axioms of set theory. This problem seems very difficult.
2. What does a well-behaved approximate identity for A (Chapter III) have to do with A being measure compact?
3. Finish the vector-measure representative of τ -additive and tight linear functionals in Chapter IV.
4. Does every C^* -algebra have a canonical approximate identity?
5. Consider Question 3.3.18.

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Major Field: Mathematics

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Date of Examination:

May 1, 1972