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APPROXIMATE METHOD OF ESTIMATION OF EXPONENTIAL TREND PARAMETERS FOR FORECASTING PROCESS PURPOSES

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Abstract

The paper discusses the issue of estimation of exponential trend parameters in terms of its application in the forecast process. Due to the character of a random element, three models were considered: additive, multiplicative, and mixed. For estimating trend parameters, a log transformation method, least squares method, and approximate methods were applied.
As a result of computer simulations, high sensitivity of the log transformation method with regard to the assumed random element model was noticed. This method yields the smallest value of *ex post* error for the multiplicative model but is burdened with a large error for the additive model, where the estimated parameter B takes large values ($B > 0.24$). In the paper, a new approximate method of estimation of exponential trend parameters is proposed. The method is compared with approximate formulas presented in the paper by Purczyński (2008).

Keywords: exponential trend, parameter estimation, least squares, approximate methods

JEL classification: C13, C15, C53

Introduction

An exponential trend has been widely used in economic forecasting (Cieślak, 2001; Zaród, 2017; Zeliaś, Pawełek, Wanat, 2003). Bearing it in mind, the paper discusses the issue of estimation of exponential trend parameters in terms of its application in the forecast process.

Due to the character of a random element, three models will be considered:

– an additive model (Model I)

$$y_t = A \cdot e^{B \cdot t} + \varepsilon_t^a \quad \text{for: t = 1, 2, ..., } n \tag{1}$$

– a multiplicative model (Model II)

$$y_t = A \cdot e^{B \cdot t} \cdot e^{\varepsilon_t^m} \tag{2}$$

– a mixed model (Model III)

$$y_t = A \cdot e^{B \cdot t} \cdot e^{\varepsilon_t^m} + \varepsilon_t^a \tag{3}$$

where:

ε_t^a – a random element with distribution $N(0, \sigma a)$;

ε_t^m – a random element with distribution $N(0, \sigma m)$;

y_t – empirical data.

In the case of Model II, a linear transform method is used (Zeliaś, 1997) as well as the least square method (LSM).

For the additive model (Model I), LSM is recommended with reference to equation (1), as a result of which the set of equations is obtained (Zeliaś, 1997):

$$\sum_{t=1}^{n} y_t e^{BS \cdot t} \cdot \sum_{t=1}^{n} t e^{2BS \cdot t} = \sum_{t=1}^{n} y_t \cdot t e^{BS \cdot t} \cdot \sum_{t=1}^{n} e^{2BS \cdot t} \tag{4a}$$

$$AS = \frac{\sum_{t=1}^{n} y_t e^{BS \cdot t}}{\sum_{t=1}^{n} e^{2BS \cdot t}} \tag{4b}$$

where: *AS* and *BS* refer to the estimation of parameters A and B obtained through LSM.

In order to determine estimation *BS*, equation (4a) should be solved by means of a numerical method. Subsequently, estimation *AS* is calculated from equation (4b).

When discussing the issue of the estimation of nonlinear regression function parameters, including the exponential function, the authors of textbooks and books with problems to solve point out to the limited application of the log transform, which stems from the assumed form of the multiplicative model (equation (2)) (Kmenta, 1990). In Jurkiewicz, Plenikowska-Ślusarz (2001), the authors also highlight two other problems related to the application of the transform: the lack of the transfer of the characteristics of estimator ln(A) onto the estimator of parameter A and the inequivalence of the criteria of LSM used for the curvilinear function (equation (2)). Therefore, Jurkiewicz and Plenikowska-Ślusarz (2001) propose the solution of the set of equations (4) by means of a spreadsheet, e.g. Microsoft Excel. In our paper, approximate methods will be considered which enable the estimation of exponential trend parameters with characteristics similar to the parameter estimations obtained with LSM in relation to Model I (equation (1)). The proposed method will be compared with the formulas presented in Purczyński (2008). The aim of this paper is to propose the approximate method which yields a forecast error not larger than the error of the methods described in the paper by Purczyński (2008).

1. Simplified methods of the estimation of exponential trend parameters

Let us assume that for the additive model (1) the following theoretical form was determined $\hat{y}_t$:

$$\hat{y}_t = \hat{A} \cdot e^{\hat{B} \cdot t} \tag{5}$$

where: $\hat{A}$ and $\hat{B}$ refer to the estimation of trend parameters.

In the paper by Purczyński (2008), the following form of parameter $\hat{B}$ estimation of model (5) was proposed:

$$\hat{B}_j = ArshE_j = ln\left(E_j + \sqrt{1 + E_j^{\,2}}\right) \quad ; \quad j = 1{,}2 \tag{6a}$$

where: E_j is determined by the formulas:

$$E_1 = \frac{\sum_{t=2}^{n-1}(y_{t+1} - y_{t-1})}{2\sum_{t=2}^{n-1} y_t}; \; E_2 = \frac{\sum_{t=2}^{n-1}(y_{t+1} - y_{t-1})y_t}{2\sum_{t=2}^{n-1} y_t^{\,2}} \tag{6b}$$

The estimation of parameter A is obtained from equation (4b):

$$\hat{A}_j = \frac{\sum_t y_t e^{\hat{B} j\, t}}{\sum_t e^{2\hat{B} j\, t}} \quad ; \quad j = 1,2 \tag{7}$$

where $\hat{B}_j$ is described by equation (6).

The method which refers to the case $j = 1$ in equations (6) and (7) will be referred to as Approximate Method I, and the case of $j = 2$ refers to Approximate Method II.

It should be noticed that expression E_1 (equation (6b)) was determined in the following way: function $y(t) = A \cdot e^{B \cdot t}$ fulfils a differential equation:

$$y'(t) = B \cdot y(t). \tag{8}$$

By applying a symmetric equation to a derivative:

$$y_t' = \frac{y_{t+1} - y_{t-1}}{2 \cdot h},$$

a recurrence equation is obtained:

$$y_{t+1} - y_{t-1} = 2 \cdot h \cdot B \cdot y_t \tag{9}$$

By performing the summation on the left and right side of equation (9), expression E_1 is obtained.

In the paper, the derivative is determined as:

$$y_t' = \frac{y_{t+1} - y_t}{h}.$$

The derivative defined in this way refers to a half of interval [t; t + 1], which means that in equation (8), $y(t)$ should be replaced by $y\left(t + \frac{1}{2}\right)$. As the value of $y\left(t + \frac{1}{2}\right)$, the arithmetic mean of value y_t and y_{t+1} can be assumed. Equation (8) takes the form:

$$\frac{y_{t+1} - y_t}{h} = B \cdot \frac{1}{2}\left[y_t + y_{t+1}\right] \tag{10}$$

By finding the sums and assuming $h = 1$, we obtain:

$$B_3 = \frac{2 \cdot \sum_{t=1}^{t=n-1} \left[y_{t+1} - y_t \right]}{\sum_{t=1}^{t=n-1} \left[y_{t+1} + y_t \right]} \tag{11}$$

The estimation of parameter A is derived from equation (7), where $j = 3$. The proposed method will be referred to as Approximate Method III.

By analogy to equation (10), for elements y_{t-1} and y_t, the following relation applies:

$$\frac{y_t - y_{t-1}}{h} = B \cdot \frac{1}{2} \left[y_{t-1} + y_t \right] \tag{12}$$

By adding the sides of equations (10) and (12), we obtain:

$$\frac{y_{t+1} - y_{t-1}}{h} = B \cdot \frac{1}{2} \left[y_{t-1} + 2y_t + y_{t+1} \right].$$

By finding the sums and assuming $h = 1$, we obtain:

$$B_4 = \frac{2 \cdot \sum_{t=2}^{t=n-1} \left[y_{t+1} - y_{t-1} \right]}{\sum_{t=1}^{t=n-1} \left[y_{t+1} + 2y_t + y_{t-1} \right]} \tag{13}$$

The estimation of parameter A is derived from equation (7), where $j = 4$ – Approximate Method IV.

2. Results of computer simulations

In order to assess the usefulness of the proposed approximate formulas, a number of computer simulations were performed for various values of parameters A and B, and a number of observations n. The results for parameters $A = 20$, $B = 0.12, 0.24, 0.36$ and the number of observations n = 8 with forecast period $T = 9, 10, 11$, and also $n = 12$, $T = 13, 14, 15$ will be given below.

A similar numerical experiment was described in the paper by Purczyński (2003), where six approximate methods of the estimation of exponential trend parameters were verified. In the experiment, the quality of an econometric model was evaluated by means of the changeability coefficient and indeterminate coefficient (divergence coefficient).

For a given level of the random element, M = 20,000 simulations were performed using a random number generator with the normal distribution, calculating, according to equations (1–3), the values of observations $y_{m,t}$. For each simulation, the value of forecast $YP_{m,T}$ was determined:

$$YP_{m,T} = \hat{A}_m \cdot \exp\left(\hat{B}_m T\right) \tag{14}$$

where:

$m = 1, 2, ..., M$ – the number of a subsequent simulation,

$\hat{A}_m$ and $\hat{B}_m$ – estimations of trend parameters for particular simulations,

T – a forecast period.

By analogy, the value of the realization of the predicted variable $YR_{m,T}$ was determined, substituting $t = T$ in equations (1–3). In order to evaluate the quality of the forecast, *ex post* relative forecast error b_T was applied:

$$b_T = \frac{\sqrt{\frac{1}{M}\sum_{m=1}^{M}\left(YP_{m,T} - YR_{m,T}\right)^2}}{\overline{YR_T}} \cdot 100 \tag{15}$$

where $\overline{YR_T} = \frac{1}{M}\sum_{m=1}^{M} YR_{m,T}$.

Assuming the consistency of the error variance caused by the presence of a random element for Models I and II, in the paper by Purczyński (2003), the relation between the value of standard deviation σ_a and σ_m was determined:

$$\sigma_a = \left[E\left(yd^2\right) \cdot \left(e^{2\cdot(\sigma_m)^2} - e^{(\sigma_m)^2}\right) + V(yd) \cdot \left(e^{\frac{1}{2}\cdot(\sigma_m)^2} - 1\right)^2 \right]^{\frac{1}{2}} \tag{16}$$

where:

$yd_t = A \cdot e^{Bt}$,

$E(yd^2)$ mean value $(yd_t)^2$,

$V(yd)$ variance yd_t.

For small values of σ_m, the following approximate formula is applicable:

$$\sigma_m = \frac{\sigma_a}{\sqrt{E\left(yd^2\right)}} \tag{17}$$

where $E\left(yd^2\right) = \frac{A^2}{n} \cdot \frac{\exp(2Bn) - 1}{1 - \exp(-2B)}$.

For the random element level analyzed in this paper, the application of equation (17) in place of (16) yields the relative error not exceeding 0.7%.

During the simulations, the random number generator of distribution $N(0, \sigma_a)$ was activated, and then, using equation (1), the values of y_t were calculated for Model I. During the realization of Model II, an equivalent value of standard deviation σ_m was determined based on equation (16). The mixed model (III) was realized as the arithmetic mean of series $y_{m,t}$, which were generated for the additive and multiplicative models.

For a given level of the random element (σ_a), the value of the *ex post* forecast error depends on the values of trend parameters A and B. In order to minimize the influence of the values of the exponential function parameters, normalization was carried out introducing the relative value of the random element level σ_v:

$$\sigma_v = \frac{\sigma_a}{\bar{y}} \tag{18}$$

where $\bar{y} = \frac{1}{n}\sum_{t=1}^{n} y_t$.

As a result of the normalization, for all computer simulations, parameter σv belonged to the same interval [0; 0.1].

Figure 1 illustrates equation (18). The particular lines in Figure 1 correspond to the respective values of parameter B, i.e. 0.12; 0.24; and 0.36.

Since two forms of the approximate formula of parameter B estimation were proposed in the paper, (equations (11) and (13)), as the first step, it had to be decided which form yields a smaller value of the *ex post* error. Table 1 presents the ratio of the values of the *ex post* error b3/b4 determined for $sv = 0.1$ and $T = 9$, where b3 indicates the error of the Approximate Method III and b4 – the error of the Approximate Method IV. Table 1 proves that for all the considered cases, the ratio is larger than one, which means that Method IV yields a smaller error than Method III. Therefore, the rest of the paper will concentrate on Method IV only (equation (13)).

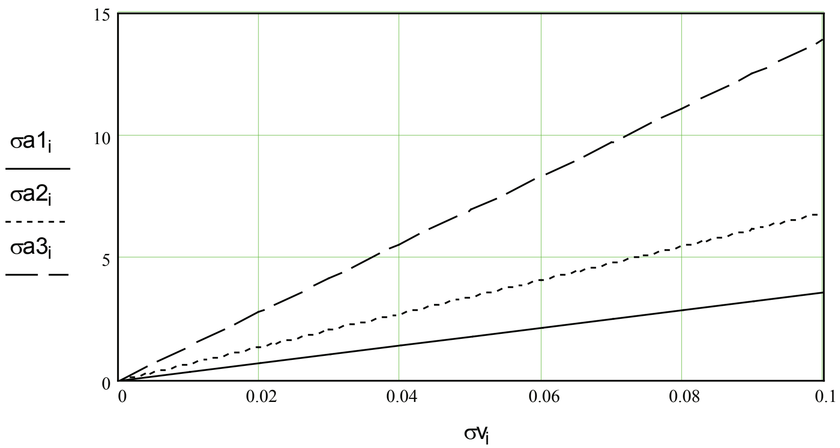

Figure 1. Values of the standard deviation of the random element σ_a as a function of the relative value of the level of the random element σv_i. Solid line $\sigma_{1,i}$ corresponds to parameter $B = 0.12$, dotted line $\sigma_{2,i}$ corresponds to $B = 0.24$, dashed line $\sigma_{3,i}$ corresponds to $B = 0.36$

Source: author's own study.

Table 1. The ratio of the values of the *ex post* error for Methods III and IV (b3/b4)

	B = 0.12	B = 0.24	B = 0.36
Model I	1.070	1.062	1.041
Model II	1.034	1.029	1.028
Model III	1.044	1.035	1.029

Source: author's own study.

Figure 2 presents the results of the simulations made for the additive model of the random element – equation (1): $A = 20$, $B = 0.24$, $n = 8$ – forecast period $T = 9$.

Figure 2 presents the values of *ex post* relative error (expressed in percentages) as a function of the level of random element σ_v. In Figure 2, it can be observed that the values of the errors for both LSM and approximate methods are similar. In order to differentiate between the effectiveness of particular methods, Table 2 was created. Table 2 includes e*x post* relative forecast error (equation (15)), expressed in percentages, for the number of observations $n = 8$, and the forecast period $T = 9$, $T = 10$, $T = 11$. The presented results represent parameter $B = 0.24$ and $B = 0.36$ – the results for parameter $B = 0.12$ were not included in the table because of the limitations imposed on the volume of the paper. On the basis of the results presented in Table 2, it can be concluded that the smallest value of the *ex post* forecast error is provided by LSM, and the largest value of the error is provided by the log transform method. The forecasts obtained for the approximate methods demonstrate similar values of the *ex post* error, where the smallest error (b4) is demonstrated by Method IV and the largest by Method II.

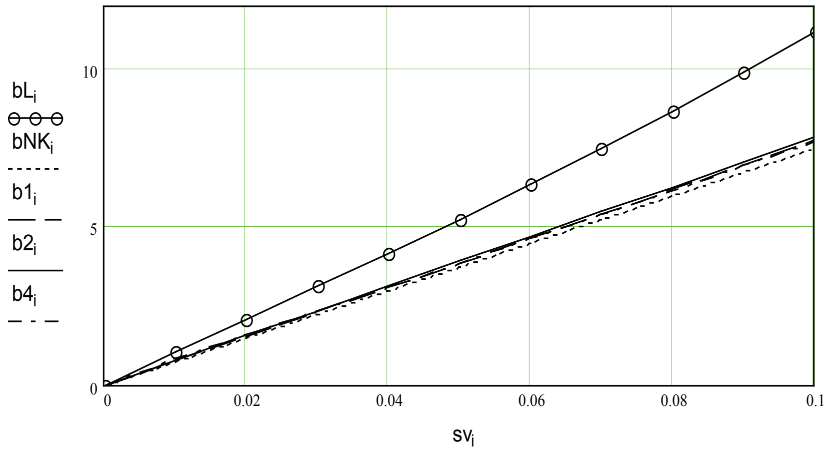

Figure 2. Values of the *ex post* relative error (in percentages) for the additive model of the random element as a function of the level of random element σ_{vi}; parameter $B = 0.24$, forecast period $T = 9$. The following labelling was applied: solid line with circles bL – log transform, dotted line bNK – LSM, dashed line $b1_i$ – Approximate Method I, solid line $b2_i$ – Approximate Method II, dot-dashed line $b3_i$ – Approximate Method III

Source: author's own study.

Table 2. Ex *post* relative forecast error (equation (15)) expressed in percentages, for the random element level $\sigma_v = 0$ and the number of observations $n = 8$

Random element model	Parameter B	Forecast period	bNK	bL	b1	b2	b4
1	2	3	4	5	6	7	8
Additive	0.24	T = 9	6.864	12.066	7.122	7.165	7.106
	0.24	T = 10	7.797	17.589	8.254	8.335	8.203
	0.24	T = 11	9.123	28.112	9.754	9.921	9.666
	0.36	T = 9	5.793	79.794	6.101	6.095	6.093
	0.36	T = 10	7.081	163.29	7.606	7.673	7.535
	0.36	T = 11	8.732	325.508	9.434	9.633	9.281
Multiplicative	0.24	T = 9	8.945	7.434	8.641	9.984	8.461
	0.24	T = 10	10.466	8.204	9.989	12.111	9.696
	0.24	T = 11	12.351	9.263	11.681	14.617	11.270
	0.36	T = 9	8.878	6.230	8.147	9.824	7.803
	0.36	T = 10	11.154	7.039	9.901	12.714	9.312
	0.36	T = 11	13.756	8.103	11.941	15.930	11.086

1	2	3	4	5	6	7	8
Mixed	0.24	T = 9	9.455	10.104	9.413	10.034	9.345
	0.24	T = 10	10.482	11.448	10.438	11.462	10.309
	0.24	T = 11	11.833	13.028	11.764	13.278	11.566
	0.36	T = 9	8.711	14.944	8.492	9.246	8.376
	0.36	T = 10	10.153	24.255	9.757	11.141	9.514
	0.36	T = 11	11.981	43.446	11.353	13.461	10.958

Source: author's own study.

Table 3 was created for the number of observations n = 12 and the forecast period T = 13, T = 14, T = 15. The values of the errors for LSM and Approximates Methods in Table 3 are smaller than in Table 2. However, for the log transform method, the values of errors in Table 3 are larger than in Table 2. The following regularities can be observed in both tables: in the case of the additive model, the smallest error values are provided by LSM, and the largest – by the log transform method.

The other errors can be ordered b4 < b1 < b2. For the multiplicative model, the smallest error values are provided by the log transform method. The other methods can be ordered b4 < b1 < bMNK < b2. For the mixed model b4 < b1 < bMNK < b2 < bL, inequality is fulfilled.

Table 3. Ex *post* relative forecast error (equation (15)), expressed in percentages, for the random element level σ_v = 0.1 and the number of observations n = 12

Random element model	Parameter B	Forecast period	bNK	bL	b1	b2	b4
Additive	0.24	T = 13	4.870	45.343	5.166	5.303	5.159
	0.24	T = 14	5.304	66.756	5.772	6.065	5.747
	0.24	T = 15	6.057	96.542	6.686	7.167	6.640
	0.36	T = 13	3.961	231.838	4.174	4.274	4.262
	0.36	T = 14	7.743	405.713	5.023	5.377	5.156
	0.36	T = 15	5.797	714.833	6.109	6.751	6.275
Multiplicative	0.24	T = 13	7.317	5.246	7.137	8.878	6.995
	0.24	T = 14	8.425	5.473	8.149	10.807	7.923
	0.24	T = 15	9.791	5.919	9.418	13.003	9.110
	0.36	T = 13	7.356	4.163	6.716	8.429	6.460
	0.36	T = 14	9.188	4.440	8.113	10.940	7.681
	0.36	T = 15	11.222	4.893	9.692	13.658	9.075
Mixed	0.24	T = 13	7.672	11.035	7.687	8.491	7.461
	0.24	T = 14	8.207	13.390	8.223	9.566	8.135
	0.24	T = 15	9.112	16.551	9.129	11.065	8.998
	0.36	T = 13	6.904	94.723	6.728	7.482	6.666
	0.36	T = 14	7.887	146.042	7.527	8.953	7.393
	0.36	T = 15	9.235	222.832	8.666	10.809	8.447

Source: author's own study.

Figure 3 is also related to Model I, yet here parameter B = 0.36. It should be noticed that the log transform method (bL) yields ten times larger value of the *ex post* error than the other methods.

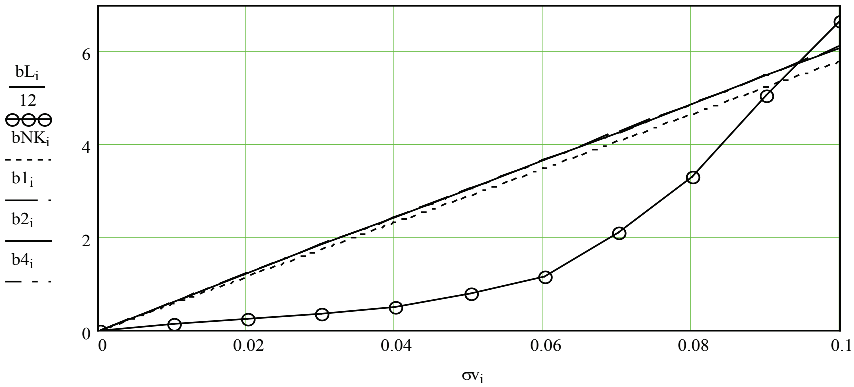

Figure 3. Values of *ex post* relative error (in percentages) for the additive model of the random element as a function of the level of random element σv – parameter B = 0.36, T = 9. The labelling used is identical to Figure 2.

Source: author's own study.

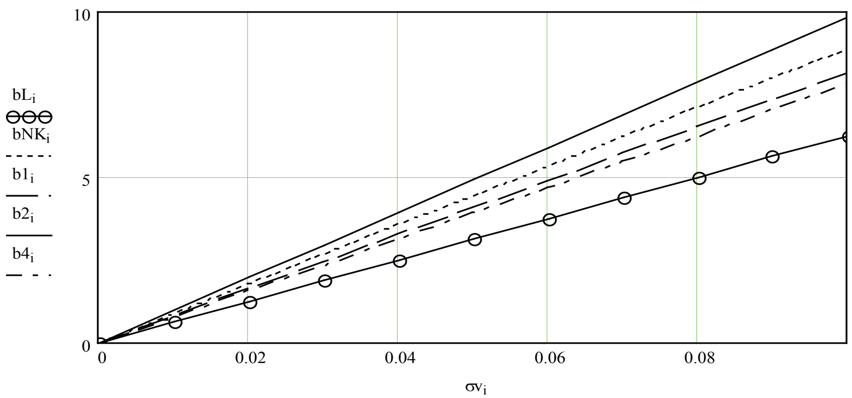

Figure 4. Values of *ex post* relative error (in percentages) for the multiplicative model of the random element as a function of the level of random element σv_i – parameter B = 0.36, forecast period T = 9. The labelling used is identical to Figure 2.

Source: author's own study.

In the case of the multiplicative model (Figure 4), it can be observed that the largest error (b2) is yielded by Approximate Method II, and the smallest error (bL) – by the log transform method. The errors of the other methods fulfil inequality: b4 < b1 < bNK.

In the case of the mixed model (Figure 5), the three methods yield similar values of the *ex post* error. Therefore, the interval of the variability of the relative level of random element $\sigma v_i \in [0.06; 0.1]$ was limited.

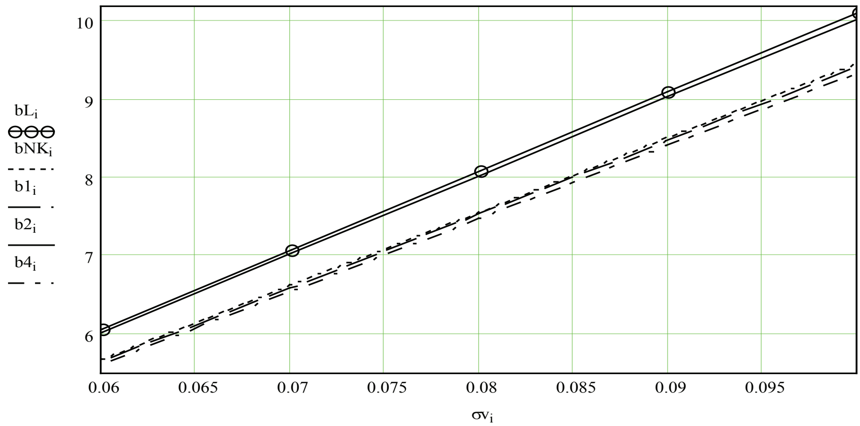

Figure 5. Values of *ex post* relative error (in percentages) for the mixed model of the random element as a function of the level of random element σv_i – parameter $B = 0.24$. The labelling used is identical to Figure 2

Source: author's own study

On the basis of the results presented in Figure 5, it can be observed that the largest error (bL) is yielded by the log transform method and the smallest error (b4) by Approximate Method IV. The errors of the other methods fulfil inequality: b1 < bN < b2.

Conclusions

The paper discusses certain aspects of the application of the exponential trend in the forecast process.

The computer simulations described in Part 2 proved high sensitivity of the log transform method to the assumed model of the random element. It is a well-known fact that the method works well for the multiplicative model and – to a lesser extent – for the mixed model. However, this method should not be used for the additive model if the estimated parameter B takes large values (B > 0.24), since it leads to a very large value of the *ex post* error. In Figure 3, created for parameter B = 0.36, the relative error amounted to 79.8% (Table 2). It confirmed the reservations concerning limitations of the log transformation method expressed in papers by Kmenta (1990) and Jurkiewicz and Plenikowska-Ślusarz (2001).

In terms of the evaluation of the approximate methods, for all the three models of the random element, the smallest value of the *ex post* error was obtained for Approximate Method IV (b4). The largest error was obtained for Approximate Method II (b2). By drawing these conclusions, the author has reached the aim of the paper, showing that the proposed Approximate Method IV yields the forecast error lower than the errors obtained for the approximate methods presented in the paper by Purczyński (2008).

It should be noticed that LSM provides the smallest error only for the additive model. However, for the multiplicative and mixed models, the proposed method yields smaller error than LSM.

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