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Approximate polyhedra, resolutions of maps and shape fibrations

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Abstract. Shape fibrations between compact metric spaces were introduced by T. B. Rushing and the author in [15]. In this paper one extends the definition so as to apply to maps $p: E \to B$ between arbitrary topological spaces. This is done by considering certain morphisms in pro-Top $\underline{p}: \underline{E} \to \underline{B}$, called resolutions of p. In the compact case resolutions reduce to inverse limit expansions. One requires also that the systems $\underline{E}$ and $\underline{B}$ consist of ANR's, polyhedra or more generally of spaces called approximate polyhedra (AP). A map p is a shape fibration provided it admits an AP-resolution $\underline{p}$, which has a certain approximate homotopy lifting property. Resolutions of spaces are characterized and compared with the inverse limit expansions. Moreover, existence of ANR-resolutions and polyhedral resolutions is demonstrated.

1. Introduction. Shape fibrations $p: E \to B$ between compact metric spaces (more generally, proper shape fibrations between locally compact metric spaces) were introduced and studied by T. B. Rushing and the author in [15], [16], [17]. Further contributions to this theory were made by Z. Čerin, L. S. Husch, M. Jani, J. Keesling, S. Mardešić, A. Matsumoto and T. C. McMillan. For a survey of results on approximate fibrations and shape fibrations see [14] and [22].

This paper originated from an attempt to extend the notion of shape fibration from the rather special case of maps between metric compacta to the general case of maps between arbitrary topological spaces. Results concerning this question are contained in Sections 4 and 8 of this paper.

The main idea consists in considering certain expansions $\underline{p}: \underline{E} \to \underline{B}$ of the map $p: E \to B$, called resolutions of p. They are related to inverse limit expansions of p and appear to be of interest on their own. For resolutions of p one defines the approximate homotopy lifting property (AHLP) as in ([15], § 9). If one allows as members of $\underline{E}$ and $\underline{B}$ only "nice" spaces, then the property AHLP does not depend on the choice of the resolution, but depends only on the map p. Maps which have this property are, by definition, shape fibrations. In § 8 we give a "categorical" definition of shape fibrations.

In § 2 we define and study "nice" spaces under the name of approximate polyhedra. We show that they include ANR's (for metric spaces), CW-complexes and n-dimensional LC^{n-1} paracompacta. In the compact metric case approximate polyhedra coincide with approximate ANR's of M. H. Clapp [6] and with K. Borsuk's NE-sets [4].

In §§ 5 and 6 we first characterize resolutions of spaces. They turn out to be closely related to P. Bacon's complements of inverse systems [2]. Then we compare resolutions with inverse limits.

In § 7 we establish the existence of polyhedral resolutions and ANR-resolutions of spaces and maps. The ANR-resolution, which we construct in § 7, is an essential tool in proving that the categorical definition of shape fibrations coincides with the definition based on resolutions.

In § 9 we show that a resolution of a space induces in the homotopy category an associated system in the sense of K. Morita [19]. Consequently, one can base the development of shape theory also on ANR-resolutions. This can be considered an outgrowth of the original Mardešić–Segal ANR-system approach to shape [18].

2. Approximate polyhedra. Let $\mathscr{V}$ be a covering of Y. We say that the maps $f, g\colon X\to Y$ are *$\mathscr{V}$-near*, and we write $(f, g)\leqslant\mathscr{V}$, provided every $x\in X$ admits a $V\in\mathscr{V}$ such that $f(x), g(x)\in V$. If $\mathscr{W}$ is a star-refinement of $\mathscr{V}$ we write $\mathscr{V}\leqslant^*\mathscr{W}$. We shall often use the fact that $(f, g)\leqslant\mathscr{W}$, $(g, h)\leqslant\mathscr{W}$ and $(h, k)\leqslant\mathscr{W}$ imply $(f, k)\leqslant\mathscr{V}$.

An open covering $\mathscr{V}$ of X is called *normal* if there is a sequence of open coverings $\mathscr{V}_i$, $i = 1, 2, \ldots$, such that $\mathscr{V}_1 = \mathscr{V}$ and $\mathscr{V}_i\leqslant^*\mathscr{V}_{i+1}$. Normal coverings of X coincide with open coverings $\mathscr{V}$, which admit a locally finite partition of unity $(\varphi_V, V\in\mathscr{V})$ subordinated to $\mathscr{V}$ (see e.g. [1], Theorem 10.10 or [13], 3, Theorems 1, 2, 3). Also recall that for paracompact spaces all open coverings are normal (see e.g. [1], Corollary 10.14).

DEFINITION 1. Let $\mathscr{X}$ and $\mathscr{Y}$ be two classes of topological spaces. We say that the class $\mathscr{Y}$ *approximately dominates* the class $\mathscr{X}$, and we write $\mathscr{X}\leqslant_a\mathscr{Y}$, provided for every $X\in\mathscr{X}$ and for every normal covering $\mathscr{U}$ of X there exists a $Y\in\mathscr{Y}$ and there exist maps $f\colon X\to Y$, $g\colon Y\to X$ such that gf and the identity map 1_X are $\mathscr{U}$-near. If $\mathscr{X}\leqslant_a\mathscr{Y}$ and $\mathscr{Y}\leqslant_a\mathscr{X}$ we say that the classes $\mathscr{X}$ and $\mathscr{Y}$ are *approximately equivalent*, and we write $\mathscr{X}\simeq_a\mathscr{Y}$.

Remark 1. The relation $\leqslant_a$ is reflexive and transitive. Indeed, let $\mathscr{X}\leqslant_a\mathscr{Y}$ and $\mathscr{Y}\leqslant_a\mathscr{Z}$. Let $X\in\mathscr{X}$ and let $\mathscr{U}$ be a normal covering of X. Let $\mathscr{V}$ be a normal star-refinement of $\mathscr{U}$. Then there exists a space $Y\in\mathscr{Y}$ and there exist maps $f\colon X\to Y$, $g\colon Y\to X$ such that $(gf, 1_X)\leqslant\mathscr{V}$. Furthermore, there exists a space $Z\in\mathscr{Z}$ and there exist maps $f'\colon Y\to Z$, $g'\colon Z\to Y$ such that $(g'f', 1_Y)\leqslant g^{-1}(\mathscr{V})$. Then

$$(gf, g(g'f')f)\leqslant\mathscr{V}.$$

Hence,

$$(1_X, (gg')(ff'))\leqslant\mathscr{U},$$

which shows that indeed $\mathscr{X}\leqslant_a\mathscr{Z}$.

By a *polyhedron* we mean in this paper the carrier $|K|$ of a simplicial complex K endowed with the CW-topology. We denote by $\mathscr{K}$ the open covering of $|K|$ formed by all the stars $\mathrm{st}(v, K)$ of the vertexes v of K. We call $\mathscr{K}$ the *star-covering* of K. By an ANR we always mean an ANR for metric spaces.

THEOREM 1. *The following four classes of spaces are approximately equivalent: The class $\mathscr{P}$ of polyhedra, the class $\mathscr{SM}$ of simplicial complexes endowed with the metric topology, the class $\mathscr{ANR}$ of absolute neighborhood retracts for metric spaces and the class $\mathscr{CW}$ of CW-complexes.*

Proof. (i) $\mathscr{P}\leqslant_a\mathscr{SM}$. Let X be a polyhedron and $\mathscr{U}$ an open covering of X. Choose a triangulation K of X so fine that the star-covering $\mathscr{K}$ of K refines $\mathscr{U}$. Such triangulations exist (J. H. C. Whitehead [24], Theorem 35; also see ([13] 3, Lemma 11)). Let $Y = |K|$ be the carrier of K endowed with the metric topology. Then the identity map $i\colon X\to Y$ is continuous and admits a homotopy inverse $j\colon Y\to X$ such that ji and 1_X are contiguous with respect to K (see e.g. [7], Corollary A, 2.9 or [12], 2, Theorem 2). Hence $(ji, 1_X)\leqslant\mathscr{K}$.

(ii) $\mathscr{SM}\leqslant_a\mathscr{ANR}$. This assertion is obvious because every simplicial complex with the metric topology is an ANR ([9], III, Theorem 11.3, or [13], 2, Theorem 4).

(iii) $\mathscr{ANR}\leqslant_a\mathscr{CW}$. It is well-known that for every ANR X and for every open covering $\mathscr{U}$ of X there exists a polyhedron P and there exist maps $f\colon X\to P$, $g\colon P\to X$ such that gf and 1_X are even $\mathscr{U}$-homotopic (see e.g. [9], IV, Theorem 6.1).

(iv) $\mathscr{CW}\leqslant_a\mathscr{P}$. Let X be a CW-complex. By results of R. Cauty [5], X can be considered a closed subset of a polyhedron P and a retract of an open neighborhood V of X in P. If $f\colon X\to V$ is the inclusion map and $g\colon V\to X$ is a retraction, then $gf = 1_X$. It thus suffices to notice that an open subset V of a polyhedron is itself a polyhedron. A proof of this fact is outlined in ([23], Exercise 3, p. 149) for the special case of compact polyhedra. However, the same argument is valid in the general case too (cf. [7], footnote on p. 353).

Remark 2. We have actually shown that the classes $\mathscr{P}$, $\mathscr{SM}$, $\mathscr{ANR}$ and $\mathscr{CW}$ are equivalent in the stronger sense that the maps gf and 1_X are $\mathscr{U}$-homotopic and not just $\mathscr{U}$-near.

DEFINITION 2. A space X is called an *approximate polyhedron* provided X is approximately dominated by the class $\mathscr{P}$ of all polyhedra. We denote the class of approximate polyhedra by $\mathscr{AP}$.

An immediate consequence of Theorem 1 is this corollary.

COROLLARY 1. *Polyhedra, simplicial complexes with the metric topology, ANR's for metric spaces and CW-complexes are approximate polyhedra.*

Remark 3. One could define approximate ANR's and approximate CW-complexes in the same way as we defined AP's in Definition 2. However, by Theorem 1, these classes of spaces would coincide with the class $\mathscr{AP}$. One could also introduce a class of homotopy approximate polyhedra using the stronger notion of $\mathscr{U}$-homotopy instead of $\mathscr{U}$-nearness. However, notice that all these spaces would have the homotopy type of polyhedra.

Another class of examples of approximate polyhedra is given by the following theorem.

THEOREM 2. *A paracompact space X, which is* LC^{n-1} *and has covering dimension* $\dim X \leqslant n < \infty$, *is an approximate polyhedron.*

Proof. Let $\mathcal{U}$ be an open covering of X. We choose a star-refinement $\mathcal{U}'$ of $\mathcal{U}$ and a refinement $\mathcal{V}$ of $\mathcal{U}'$ such that for any simplicial complex K with $\dim K \leqslant n$ every partial realization $g_0\colon |L| \to X$ of K relative to $\mathcal{V}$ extends to a full realization $g\colon |K| \to X$ relative to $\mathcal{U}'$. Such a $\mathcal{V}$ exists by ([9], V, Theorem 4.1 and Remark 4.4). Let $\mathcal{W}$ be a star-refinement of $\mathcal{V}$. Since $\dim X \leqslant n$ one can assume that the nerve $N(\mathcal{W})$ has dimension $\leqslant n$. Let $f\colon X \to |N(\mathcal{W})|$ be a canonical map for $\mathcal{W}$, i.e. a map such that $f^{-1}(\mathrm{st}(W, \mathcal{W})) \subseteq W$ for every $W \in \mathcal{W}$. For $K = N(\mathcal{W})$ and for $L \subseteq K$ the 0-skeleton of K we define $g_0\colon |L| \to X$ be choosing for each $W \in \mathcal{W}$ a point $g_0(W) \in W$. If the vertexes $W_0, \ldots, W_n$ of K span a simplex in K, then $W_0 \cap \ldots \cap W_n \neq 0$ and thus $g_0(W_0), \ldots, g_0(W_n) \in W_0 \cup \ldots \cup W_n \subseteq \mathrm{st}(W_0, \mathcal{W}) \subseteq V$ for some $V \in \mathcal{V}$. This shows that g_0 is a partial realization of K relative to $\mathcal{V}$. Consequently, g_0 extends to a full realization $g\colon |N(\mathcal{W})| \to X$ relative to $\mathcal{U}'$.

We claim that $(gf, 1_X) \leqslant \mathcal{U}$. Indeed, if $x \in X$ and $(W_0, \ldots, W_n) \in N(\mathcal{W})$ is the carrier of $f(x)$, then $x \in W_0 \cap \ldots \cap W_n \subseteq W_0 \subseteq U'$ for some $U' \in \mathcal{U}'$. Furthermore, $gf(x) \in g(\overline{W_0, \ldots, W_n}) \subseteq U''$ for some $U'' \in \mathcal{U}'$. Since $\mathcal{U}' \geqslant^* \mathcal{U}$ and

$$g(W_0) \in W_0 \cap g(\overline{W_0, \ldots, W_n}) \subseteq U' \cap U'',$$

we conclude that there is $U \in \mathcal{U}$ such that $x, gf(x) \in U' \cup U'' \subseteq U$.

In order to gain further insight in the class $\mathcal{AP}$, we shall now consider compact metric approximate polyhedra.

In [6] M. H. Clapp has defined a class of metric compacta, called approximative absolute neighborhood retracts (AANR). A metric compactum X is an AANR provided it has the following property:

(C) If X is embedded in a metric space Y, then for every $\varepsilon > 0$ there exist a neighborhood U of X in Y and a map $r\colon U \to X$ such that the distance $d(r(x), x) < \varepsilon$ for all $x \in X$.

In [4] K. Borsuk has studied a class of metric compacta X, called NE-sets, and characterized by the following property:

(B) There exists a compact AR M, which contains X as a subspace and is such that every $\varepsilon > 0$ admits a neighborhood U of X in M and a map $p\colon U \to X$ with $d(p(y), y) < \varepsilon$ for all $y \in U$.

THEOREM 3. *Compact metric approximate polyhedra, Borsuk's* NE-*sets and Clapp's* AANR'*s coincide.*

Proof. (i) Let X be a compact metric approximate polyhedron. We can assume that X is embedded in the Hilbert cube $M = Q$. By Definition 2, for any $\varepsilon > 0$ there exists a polyhedron P and there exist maps $f\colon X \to P$, $g\colon P \to X$ such that $d(gf, 1_X) < \varepsilon$. Since X is compact, one can assume that P is a compact polyhedron and thus an ANR. Therefore, f can be extended to a map $\tilde{f}\colon U \to P$ of a neighborhood U of X in Q. If we choose U small enough, we will still have $d(g\tilde{f}(y), y) < \varepsilon$ for $y \in U$. Hence, $p = g\tilde{f}\colon U \to X$ satisfies condition (B).

(ii) Now let X be an NE-set and let $X \subseteq Y$, where Y is a metric space. Let M be an AR containing X and satisfying (B). For any $\varepsilon > 0$ there exist a neighborhood V of X in M and a map $p\colon V \to X$ such that $d(p(y), y) < \varepsilon$, $y \in V$. Since M is an AR, the identity map $1_X\colon X \to X \subseteq M$ extends to a map $f\colon U \to M$, where U is a neighborhood of X in Y. One can assume that $f(U) \subseteq V$. Then we define $r\colon U \to X$ by $r = pf$. Clearly, $d(r(x), x) = d(p(x), x) < \varepsilon$ for any $x \in X$. We have thus established property (C).

(iii) Now let X be an AANR. One can assume that X is embedded in the Hilbert cube $Q = \prod_{i=1}^{\infty} I_i$, $I_i = [0, 1]$. For a given $\varepsilon > 0$ let $r\colon U \to X$ be a map of a neighborhood U of X in Q such that $d(r(x), x) < \frac{1}{2}\varepsilon$ for $x \in X$. There is no loss of generality in assuming that $d(r(y), y) < \frac{1}{2}\varepsilon$ for $y \in U$. One can also assume that U is of the form $U = P \times \prod_{i>n} I_i$, where P is a compact polyhedron in $\prod_{i \leqslant n} I_i$ and that

$$d((s, t), (s, t')) < \tfrac{1}{2}\varepsilon$$

for any $s \in P$ and $t, t' \in \prod_{i>n} I_i$. Consider the projection $U = P \times \prod_{i>n} I_i \to P$ and let $f\colon X \to P$ be its restriction to X. Let $g\colon P \to X$ be the composition of the embedding $s \mapsto (s, 0) \in P \times \prod_{i>n} I_i = U$, $s \in P$, with the map $r\colon U \to X$. Clearly, if $x = (s, t) \in X$, then $gf(x) = r(s, 0)$ so that

$$d(gf(x), x) \leqslant d(r(s, 0), (s, 0)) + d((s, 0), (s, t)) < \varepsilon.$$

This shows that X is an approximate polyhedron.

Now we shall describe some simple examples.

EXAMPLE 1. Every locally connected continuum in R^2 is an approximate polyhedron ([4], Theorem 5.1). In particular, the compact infinite bouquet of circles is an approximate polyhedron, which does not have the homotopy type of a polyhedron.

EXAMPLE 2. Let $S_1, S_2 \subseteq R^2$ be circles with center 0 and with radii 1 and 2 respectively. Let A be the annulus determined by S_1, S_2 and let $L \subseteq \mathrm{Int} A$ be a homeomorphic copy of R, which spirals to both S_1 and S_2. Then $X = S_1 \cup S_2 \cup L$ is a continuum in R^2, which fails to be an approximate polyhedron.

Indeed, assume that $X \in \mathrm{AP}$ and choose $0 < \varepsilon < 1$. Then there is a polyhedron P and there are maps $f\colon X \to P$, $g\colon P \to X$ such that $d(gf, 1_X) < \varepsilon$. Notice that

$$gf(S_1) \subseteq X \setminus S_2 \quad \text{and} \quad gf(S_2) \subseteq X \setminus S_1.$$

Let Q_i, $i = 1, 2$, be a compact connected polyhedral neighborhood of $f(S_i)$ in P so small that

$$g(Q_1) \subseteq X \setminus S_2 \quad \text{and} \quad g(Q_2) \subseteq X \setminus S_1.$$

Since $Y = Q_1 \cup Q_2 \cup f(X)$ is a pathwise connected continuum and since S_1, S_2, L are the path components of X, it follows that $g(Y)$ must be entirely contained in L. Consequently, $gf(X) \subseteq g(Y)$ is an arc. However, $gf\colon X \to gf(X)$ is an 2ε-mapping. Since ε was arbitrary, we would have the false conclusion that X is an arc-like continuum.

A similar argument shows that the Warsaw circle is not an approximate polyhedron.

EXAMPLE 3. The cone over the continuum X from Example 2 is a contractible continuum, which fails to be an approximate polyhedron. This is an immediate consequence of the fact that a compact metric space is an approximate polyhedron if and only if its cone is an approximate polyhedron ([3], Lemma 3).

3. Resolutions of spaces and maps. We shall consider inverse systems in the category TOP of topological spaces and continuous maps $\underline{E} = (E_\lambda, q_{\lambda\lambda'}, \Lambda)$ indexed by directed sets $(\Lambda, \leqslant)$.

A *map of systems* $\underline{p} = (p_\mu, \pi)\colon \underline{E} \to \underline{B} = (B_\mu, r_{\mu\mu'}, M)$ consists of an increasing function $\pi\colon M \to \Lambda$ and of a collection (p_μ, M) of maps $p_\mu\colon E_{\pi(\mu)} \to B_\mu$ such that for $\mu \leqslant \mu'$ the following diagram commutes

(1)
$$\begin{array}{ccc} E_{\pi(\mu)} & \xleftarrow{\;q_{\pi(\mu)\pi(\mu')}\;} & E_{\pi(\mu')} \\ \downarrow{\scriptstyle p_\mu} & & \downarrow{\scriptstyle p_{\mu'}} \\ B_\mu & \xleftarrow{\;r_{\mu\mu'}\;} & B_{\mu'} \end{array}$$

If $\underline{E} = E$ is a rudimentary system, i.e. Λ consists of a single element, then a map of systems $\underline{p}\colon E \to \underline{B}$ is a collection (p_μ, M) of maps $p_\mu\colon E \to B_\mu$, $\mu \in M$, such that

(2)
$$r_{\mu\mu'} p_{\mu'} = p_\mu, \quad \mu \leqslant \mu' .$$

Maps of systems $\underline{p}'$ and $\underline{p}$ compose by composing π with π' and $p'_{\pi(\mu)}$ with p_μ.

DEFINITION 3. Let E be a space and $\underline{E} = (E_\lambda, q_{\lambda\lambda'}, \Lambda)$ an inverse system of spaces. A *resolution* of E is a map of systems $\underline{q} = (q_\lambda, \Lambda)\colon E \to \underline{E}$, which satisfies the following two conditions:

(R1) Let P be an approximate polyhedron, $\mathscr{V}$ a normal covering of P and $f\colon E \to P$ a map. Then there exist an index $\lambda \in \Lambda$ and a map $f_\lambda\colon E_\lambda \to P$ such that
$$(f_\lambda q_\lambda, f) \leqslant \mathscr{V} .$$

(R2) Let P be an approximate polyhedron and $\mathscr{V}$ a normal covering of P. Then there exists a normal covering $\mathscr{V}'$ of P with the following property:

If $\lambda \in \Lambda$ and $f, f'\colon E_\lambda \to P$ are maps such that $(fq_\lambda, f'q_\lambda) \leqslant \mathscr{V}'$, then there exists a $\lambda' \geqslant \lambda$ such that $(fq_{\lambda\lambda'}, f'q_{\lambda\lambda'}) \leqslant \mathscr{V}$.

An AP-*resolution* (*polyhedral resolution*, ANR-*resolution*) of E is a resolution $\underline{q}\colon E \to \underline{E} = (E_\lambda, q_{\lambda\lambda'}, \Lambda)$ such that all E_λ are approximate polyhedra ($E_\lambda \in \mathscr{P}$, $E_\lambda \in \mathscr{ANR}$).

PROPOSITION 1. *A map of systems $\underline{q}\colon E \to \underline{E}$ is a resolution if and only if the conditions* (R1) *and* (R2) *are fulfilled for all polyhedra P, or equivalently for all* ANR*'s P.*

Proof. Let $P \in \mathscr{AP}$ and let $\mathscr{V}$ be a normal covering of P. Choose a star-refinement $\mathscr{V}'$ of $\mathscr{V}$. There exist a polyhedron P' (an ANRP') and maps $h\colon P \to P'$, $h'\colon P' \to P$ such that $(h'h, 1_P) \leqslant \mathscr{V}'$.

In order to establish (R1) consider for a given map $f\colon E \to P$ the map $hf\colon E \to P'$ and the covering $h'^{-1}(\mathscr{V}')$. By assumption, there exist a $\lambda \in \Lambda$ and a map $g\colon E_\lambda \to P'$ such that $(gq_\lambda, hf) \leqslant h'^{-1}(\mathscr{V}')$ and therefore $(h'gq_\lambda, h'hf) \leqslant \mathscr{V}'$. Since $(h'hf, f) \leqslant \mathscr{V}'$, we conclude that $(h'gq_\lambda, f) \leqslant \mathscr{V}$ and the map $f_\lambda = h'g\colon E_\lambda \to P$ has the desired property.

Now we establish (R2). Consider P' and its covering $h'^{-1}(\mathscr{V}')$. The assumption yields a covering $\mathscr{W}$ of P'. We claim that the covering $\mathscr{V}'' = h^{-1}(\mathscr{W})$ of P has the desired property. Indeed, let $\lambda \in \Lambda$ and let $f, f'\colon E_\lambda \to P$ be maps such that $(fq_\lambda, f'q_\lambda) \leqslant \mathscr{V}''$. Then $(hfq_\lambda, hf'q_\lambda) \leqslant \mathscr{W}$. By the choice of $\mathscr{W}$ we conclude that there is a $\lambda' \geqslant \lambda$ such that $(h'hfq_{\lambda\lambda'}, h'hf'q_{\lambda\lambda'}) \leqslant \mathscr{V}'$. Since also $(h'hfq_{\lambda\lambda'}, fq_{\lambda\lambda'}) \leqslant \mathscr{V}'$, it follows that indeed $(fq_{\lambda\lambda'}, f'q_{\lambda\lambda'}) \leqslant \mathscr{V}$.

DEFINITION 4. A *resolution of a map* $p\colon E \to B$ consists of a resolution $\underline{q}\colon E \to \underline{E}$ of E, of a resolution $\underline{r}\colon B \to \underline{B}$ of B and of a map of systems $\underline{p}\colon \underline{E} \to \underline{B}$ such that

(3)
$$\underline{p}\underline{q} = \underline{r}p .$$

$(\underline{q}, \underline{r}, \underline{p})$ is an AP-*resolution* (P-*resolution*, ANR-*resolution*) of p provided it is a resolution of p and $\underline{q}$ and $\underline{r}$ are AP-resolutions (P-resolutions, ANR-resolutions) of E and B respectively.

Notice that (3) is equivalent to

(4)
$$p_\mu q_{\pi(\mu)} = r_\mu p, \quad \mu \in M .$$

Remark 4. If B is a completely regular space and $(\underline{q}, \underline{r}, \underline{p})$ is a resolution for both maps $p, p'\colon E \to B$, then $p = p'$.

Indeed, assume that $p(x_0) \neq p'(x_0)$ for some point $x_0 \in E$. Let $f\colon B \to I$ be a map such that

(5)
$$fp(x_0) = 0, \quad fp'(x_0) = 1 .$$

By (R1) there exist a $\mu \in M$ and a map $f_\mu\colon B_\mu \to I$ such that $d(f_\mu r_\mu, f) < 1$. Consequently, we would have

(6)
$$f_\mu r_\mu p(x_0) < 1 ,$$

(7)
$$f_\mu r_\mu p'(x_0) > 1 .$$

However, this is impossible, because by (4), $r_\mu p = p_\mu q_{\pi(\mu)} = r_\mu p'$.

Remark 5. In dealing with resolutions one can always assume that the index sets Λ and M are cofinite, i.e. every element has a finite number of predecessors. This is a consequence of the next proposition.

Proposition 2. *Let* $(\underline{q}, \underline{r}, \underline{p})$ *be a resolution of* $p: E \to B$. *Then there exists a resolution* $(\underline{q}', \underline{r}', \underline{p}')$ *of* p *such that every* $B'_{\mu'}(E'_{\lambda'})$ *is a* $B_\mu(E_\lambda)$, *every* $r'_{\mu'\mu''}(q'_{\lambda'\lambda''})$ *is an* $r_{\mu\mu'}(q_{\lambda\lambda'})$, *every* $p'_{\mu'}(q'_{\lambda'})$ *is a* $p_\mu(q_\lambda)$ *and the index sets* Λ', M' *are cofinite.*

Proof. We define $\underline{B}' = (B'_{\mu'}, r'_{\mu'\mu''}, M')$ and $\underline{r}' = (r'_{\mu'}, M')$ as in the proof of Theorem 7.1 of [11], i.e. we take for M' the set of all finite subsets $\mu' = \{\mu_1, \ldots, \mu_n\}$ of M and we order M' by inclusion. We define an increasing function $m: M' \to M$ such that $m(\{\mu\}) = \mu$ for $\mu \in M$. For $\mu' = \{\mu_1, \ldots, \mu_n\}$ we put $B'_{\mu'} = B_{m(\mu')}$ and for $\mu' \leqslant \mu''$ we put $r'_{\mu'\mu''} = r_{m(\mu')m(\mu'')}$. Finally, we put $r'_{\mu'} = r_{m(\mu')}$ and $\underline{r}' = (r'_{\mu'}, M')$. Conditions (R1) and (R2) are readily verified.

Let $(\Lambda', \leqslant)$ be the set of all finite subsets λ' of Λ, ordered by inclusion. For the index set of $\underline{E}'$ we take the ordered product $(M' \times \Lambda', \leqslant)$. We then define an increasing function $n: M' \times \Lambda' \to \Lambda$ by putting $n(\mu', \lambda') = \pi m(\mu')$. We also define $E'_{(\mu',\lambda')} = E_{n(\mu',\lambda')}$, $q'_{(\mu',\lambda')(\mu'',\lambda'')} = q_{n(\mu',\lambda')n(\mu'',\lambda'')}$, $q'_{(\mu',\lambda')} = q_{n(\mu',\lambda')}$. It is readily seen that $\underline{q}: E \to \underline{E}'$ is also a resolution.

Now we define an increasing function $\pi': M' \to M' \times \Lambda'$ by $\pi'(\mu') = (\mu', \pi(\mu'))$. We also put $p'_{\mu'} = p_{m(\mu')}: E_{\pi m(\mu')} \to B_{m(\mu')}$. This is a well-defined map $p'_{\mu'}: E'_{\pi'(\mu')} \to B'_{\mu'}$ since $\pi'(\mu') = (\mu', \pi(\mu'))$, $E'_{\pi'(\mu')} = E_{n(\mu',\pi(\mu'))} = E_{\pi m(\mu')}$ and $B'_{\mu'} = B_{m(\mu')}$. The maps $p'_{\mu'}$ define a map of systems $\underline{p}': \underline{E}' \to \underline{B}'$, which satisfies $\underline{p}'\underline{q}' = \underline{r}'p$.

4. Shape fibrations of topological spaces. Let $\underline{p} = (p_\mu, \pi): \underline{E} \to \underline{B}$ be a map of systems. We call a pair of indexes $(\lambda, \mu) \in \Lambda \times M$ *admissible* provided $\lambda \geqslant \pi(\mu)$. For every admissible pair we define a map $p_{\mu\lambda}: E_\lambda \to B_\mu$ by

(1) $$p_{\mu\lambda} = p_\mu q_{\pi(\mu)\lambda}.$$

Notice that for admissible pairs $(\lambda, \mu) \leqslant (\lambda', \mu')$ one has

(2) $$p_{\mu\lambda} q_{\lambda\lambda'} = r_{\mu\mu'} p_{\mu'\lambda'}.$$

Definition 5. A map of systems $\underline{p} = (p_\mu, \pi): \underline{E} \to \underline{B}$ has the *homotopy lifting property* (HLP) with respect to a class of topological spaces $\mathscr{X}$ provided the following holds:

Every admissible pair of indexes $(\lambda, \mu) \in \Lambda \times M$ admits an admissible pair of indexes $(\lambda', \mu') \geqslant (\lambda, \mu)$ (called *lifting indexes*) such that for an arbitrary space $X \in \mathscr{X}$ and for arbitrary maps $h: X \to E_{\lambda'}$, $H: X \times I \to B_{\mu'}$, satisfying

(3) $$p_{\mu'\lambda'} h = H_0,$$

there exists a homotopy $\tilde{H}: X \times I \to E_\lambda$ such that

(4) $$\tilde{H}_0 = q_{\lambda\lambda'} h,$$

(5) $$p_\mu \tilde{H} = r_{\mu\mu'} H.$$

Remark 6. If $\underline{B}$ consists of a single space B, then $\underline{p}: \underline{E} \to \underline{B}$ is given by a single map $p: E_{\lambda_0} \to B$, where $\lambda_0 \in \Lambda$ is a fixed index. $\underline{p}$ has the HLP with respect to all spaces if and only if the maps $p_\lambda = pq_{\lambda_0\lambda}$, $\lambda \geqslant \lambda_0$, form an s-fibration in the sense of [10].

Definition 6. A map of systems $\underline{p} = (p_\mu, \pi): \underline{E} \to \underline{B}$ has the *approximate homotopy lifting property* (AHLP) with respect to a class of spaces $\mathscr{X}$ provided the following holds:

For every admissible map $(\lambda, \mu) \in \Lambda \times M$ and for arbitrary normal coverings $\mathscr{V}_\mu$ of B_μ and $\mathscr{U}_\lambda$ of E_λ there exist an admissible pair of indexes $(\lambda', \mu') \geqslant (\lambda, \mu)$ (called *lifting indexes*) and a normal covering $\mathscr{V}_{\mu'}$ of $B_{\mu'}$ (called *lifting mesh*) such that for an arbitrary space $X \in \mathscr{X}$ and for arbitrary maps $h: X \to E_{\lambda'}$, $H: X \times I \to B_{\mu'}$ with

(6) $$(p_{\mu'\lambda'} h, H_0) \leqslant \mathscr{V}_{\mu'},$$

there exists a homotopy $\tilde{H}: X \times I \to E_\lambda$ such that

(7) $$(q_{\lambda\lambda'} h, \tilde{H}_0) \leqslant \mathscr{U}_\lambda,$$

(8) $$(p_{\mu\lambda} \tilde{H}, r_{\mu\mu'} H_\mu) \leqslant \mathscr{V}_\mu.$$

Definitions 5 and 6 are Definitions 8 and 9 of [15].

Remark 7. Let $\underline{p}, \underline{p}': \underline{E} \to \underline{B}$ be maps of systems which determine the same morphism $\underline{E} \to \underline{B}$ in pro-TOP, i.e. let every $\mu \in M$ admit a $\lambda \in \Lambda$ such that $\lambda \geqslant \pi(\mu), \pi'(\mu)$ and $p_{\mu\lambda} = p'_{\mu\lambda}$. It is readily seen that if $\underline{p}$ has the property HLP (AHLP), then so does $\underline{p}'$. Consequently, these properties are actually properties of morphisms in pro-TOP.

We now define shape fibrations.

Definition 7. A map $p: E \to B$ between topological spaces is a *shape fibration* provided there exists an AP-resolution $(\underline{q}, \underline{r}, \underline{p})$ of p such that $\underline{p}: \underline{E} \to \underline{B}$ has the AHLP with respect to all topological spaces.

To assess fully this definition one must bear in mind these two facts:

(i) Every map admits an AP-resolution (see Theorems 11 and 13 of Section 7).

(ii) In order to decide whether a map p is a shape fibration, one can use any AP-resolution of p. More precisely, one has this result.

Theorem 4. *Let* $(\underline{q}, \underline{r}, \underline{p})$ *and* $(\underline{q}', \underline{r}', \underline{p}')$ *be two* AP-*resolutions of the same map* $p: E \to B$. *If* $\underline{p}: \underline{E} \to \underline{B}$ *has the* AHLP *for all spaces* X, *then so does* $\underline{p}': \underline{E}' \to \underline{B}'$.

The proof of this theorem is a straightforward translation of the proof of Theorem 1 of [15] into the present more general setting. Notice that Lemma 1 of [15] is here replaced by Definition 3. The conditions from [15] of the form $\varepsilon \in \Lambda(f, \delta)$ for $\varepsilon > 0$, $\delta > 0$, are now replaced by requirements on normal coverings $\mathscr{U}$ and $\mathscr{V}$ that $\mathscr{U} \geqslant f^{-1}(\mathscr{V})$. One also often uses the fact that a normal covering $\mathscr{U}$ has a normal star-refinement $\mathscr{V} \geqslant^* \mathscr{U}$.

Corollary 2. *Let* E *and* B *be metric compacta. A map* $p: E \to B$ *is a shape fibration in the sense of Definition* 7 *if and only if* p *is a shape fibration in the sense of* [15] (*or equivalently, in the sense of* [17]).

Proof. Let $\underline{p}: \underline{E} \to \underline{B}$ be a level map of inverse sequences of compact ANR's with $\lim \underline{p} = p$. Then $E = \lim \underline{E}$, $B = \lim \underline{B}$ and the projections $q_i: E \to E_i$, $r_i: B \to B_i$ form maps of systems $\underline{q}: E \to \underline{E}$, $\underline{r}: B \to \underline{B}$. It follows from Theorem 8 of Section 6 that $\underline{q}$ and $\underline{r}$ are ANR-resolutions of E and B respectively. Since $\underline{pq} = \underline{rp}$, it follows that $(\underline{q}, \underline{r}, \underline{p})$ is an ANR-resolution of p.

If p is a shape fibration in the sense of [15], then by Theorem 1 of [15], $\underline{p}$ has the AHLP for all spaces and thus p is a shape fibration in the sense of Definition 7.

Conversely, if p is a shape fibration in the sense of Definition 7, then by Theorem 4, $\underline{p}$ must have the property AHLP for all spaces. Consequently, p is a shape fibration in the sense of [15].

COROLLARY 3. *Let E and B be approximate polyhedra and let $p: E \to B$ be a Hurewicz fibration. Then p is a shape fibration. In particular, this is true if E and B are* ANR's, *CW-complexes or paracompact* LC^{n-1} *spaces of dimension* $\leqslant n < \infty$ (see Corollary 1 and Theorem 2).

More generally, we have this result.

COROLLARY 4. *Let E and B be approximate polyhedra. A map $p: E \to B$ is a shape fibration if it has this property:*

For arbitrary normal coverings $\mathscr{V}$ of B and $\mathscr{U}$ of E there exists a normal covering $\mathscr{V}'$ of B such that for an arbitrary space X and for arbitrary maps $h: X \to E$, $H: X \times I \to B$ with $(ph, H_0) \leqslant \mathscr{V}'$, there exists a homotopy $\tilde{H}: X \times I \to E$ such that $(h, \tilde{H}_0) \leqslant \mathscr{U}$ and $(p\tilde{H}, H) \leqslant \mathscr{V}$.

Proof. $p: E \to B$ can be viewed as a rudimentary AP-resolution of p. For this resolution the property (AHLP) coincides with the above property of p.

EXAMPLE 4. Let $E = S_1 \cup S_2 \cup L$ be the plane continuum described in Example 2. Let $B = S_2$ and let $p: E \to B$ be the radial projection. The map p is a fibre bundle and thus a Hurewicz fibration. Nevertheless, p fails to be a shape fibration (see [22]).

The simplest way to see this is to consider the pro-homotopy exact sequence of a shape fibration [16]:

$$\text{pro-}\pi_1(F, *) \to \text{pro-}\pi_1(E, *) \to \text{pro-}\pi_1(B, *)\,.$$

In our case the fibre F is a totally disconnected compactum, so that $\text{pro-}\pi_1(F, *) = 0$. Since B is a circle, $\text{pro-}\pi_1(B, *) = \pi_1(B, *) = Z$. Finally, E decomposes the plane in three regions and thus has the shape of the figure 8. Therefore, $\text{pro-}\pi_1(E, *) = \pi_1(E, *) = Z * Z$. Consequently, the exact sequence assumes the form

$$0 \to Z * Z \to Z\,,$$

which is obviously a contradiction.

The explanation for the existence of such an example lies in the fact that E fails to be an approximate polyhedron, as it was shown directly in Example 2.

5. Characterizing resolutions of spaces. In this section we shall first give sufficient conditions for a map of systems $\underline{q}: E \to \underline{E}$ to be a resolution of the space E.

THEOREM 5. *A map of systems $\underline{q}: E \to \underline{E}$ is a resolution of E if it satisfies the following two conditions:*

(B1) *For every normal covering $\mathscr{U}$ of E there exist a $\lambda \in \Lambda$ and a normal covering $\mathscr{U}_\lambda$ of E_λ such that $q_\lambda^{-1}(\mathscr{U}_\lambda)$ refines $\mathscr{U}$.*

(B2) *For every $\lambda \in \Lambda$ and for every open set U in E_λ, which contains $\overline{q_\lambda(E)}$, there exists a $\lambda' \geqslant \lambda$ such that $q_{\lambda\lambda'}(E_{\lambda'}) \subseteq U$.*

Proof. According to Proposition 1, it suffices to prove that conditions (R1) and (R2) hold for polyhedra P.

Proof of (R1). Consider a polyhedron P, an open covering $\mathscr{V}$ of P and a map $f: E \to P$. Let K be a triangulation of P such that the star-covering $\mathscr{K}$ of K refines $\mathscr{V}$. Notice that $f^{-1}(\mathscr{K})$ is a normal covering of E. Therefore, by (B1), there exists a $\lambda \in \Lambda$ and there exists a normal covering $\mathscr{U}_\lambda$ of E_λ such that $q_\lambda^{-1}(\mathscr{U}_\lambda)$ refines $f^{-1}(\mathscr{K})$.

Now consider the pair of spaces (E_λ, E_λ'), where $E_\lambda' = q_\lambda(E)$, and consider the coverings $\mathscr{U}_\lambda$ of E_λ and $\mathscr{U}_\lambda' = \mathscr{U}_\lambda | E_\lambda'$ of E_λ'. Let N and N' be the nerves of $\mathscr{U}_\lambda$ and $\mathscr{U}_\lambda'$ respectively. For each $U \in \mathscr{U}_\lambda$, for which $U' = U \cap q_\lambda(E) \neq \emptyset$, identify the vertex U' of N' with the vertex U of N. Then N' becomes a subcomplex of N.

Now choose for each vertex U' of N' a vertex $h(U')$ of K such that

$$q_\lambda^{-1}(U) \subseteq f^{-1}\big(\operatorname{st}(h(U'), K)\big)\,. \tag{1}$$

$h: N' \to K$ is a simplicial map. Indeed, if the vertexes $U_0', \ldots, U_n'$ of N' span a simplex in N', then $q_\lambda(E) \cap U_0 \cap \ldots \cap U_n \neq \emptyset$ and therefore, by (1),

$$f^{-1}\big(\operatorname{st}(h(U_0'), K)\big) \cap \ldots \cap f^{-1}\big(\operatorname{st}(h(U_n'), K)\big) \neq \emptyset\,, \tag{2}$$

i.e. the vertexes $h(U_0'), \ldots, h(U_n')$ span a simplex in K.

Since $\mathscr{U}_\lambda$ is a normal covering, there exists a canonical map $g: E_\lambda \to |N|$, i.e. a map g such that

$$g^{-1}(\operatorname{st}(U, N)) \subseteq U, \qquad U \in \mathscr{U}_\lambda\,. \tag{3}$$

Notice that

$$g(E_\lambda') \subseteq |N'|\,. \tag{4}$$

Indeed, if $x \in E$ and if $(U_0, \ldots, U_n)$ is the simplex of N, which carries $gq_\lambda(x)$, then

$$q_\lambda(x) \in g^{-1}(\operatorname{st}(U_0, N) \cap \ldots \cap \operatorname{st}(U_n, N)) \subseteq U_0 \cap \ldots \cap U_n\,. \tag{5}$$

Consequently, each U_i meets E_λ' and is therefore a U_i', and the vertexes $U_0', \ldots, U_n'$ span a simplex in N'. Since $h: N' \to K$ is a simplicial map, the vertexes $h(U_0'), \ldots, h(U_n')$ of K span a simplex in K, which contains $hgq_\lambda(x)$ in its interior. Therefore,

$$hgq_\lambda(x) \in \operatorname{st}(h(U_0'), K)\,. \tag{6}$$

On the other hand, (5) and (1) imply

$$x \in q_\lambda^{-1}(U_0) \subseteq f^{-1}\big(\operatorname{st}(h(U_0'), K)\big)\,, \tag{7}$$

so that both points $hgq_\lambda(x)$ and $f(x)$ belong to $\operatorname{st}(h(U_0'), K) \in \mathscr{K}$ and thus also to some $V \in \mathscr{V}$. This shows that the maps hgq_λ and f are $\mathscr{V}$-near.

Now notice that $|N'|$ is a neighborhood retract of $|N|$ (see e.g. [9], Lemma 10.1, p. 101). Therefore, there is an open neighborhood G of $|N'|$ in $|N|$ and there is a retraction $r\colon G \to |N'|$. This enables us to extend the map $h\colon |N'| \to |K|$ to a map $\tilde{h}\colon G \to |K|$ by putting $\tilde{h} = hr$.

Since $|N'|$ is closed in $|N|$ and $g^{-1}(G) \supseteq g^{-1}(|N'|) \supseteq E'_\lambda$ we find by (B2) an index $\lambda' \geqslant \lambda$ such that

$$q_{\lambda\lambda'}(E_{\lambda'}) \subseteq g^{-1}(G)\,. \tag{8}$$

We claim that λ' has all the desired properties required by (R1). Indeed, let $f_{\lambda'}\colon E_{\lambda'} \to P$ be defined by

$$f_{\lambda'} = \tilde{h} r g q_{\lambda\lambda'}\,. \tag{9}$$

This is a well-defined map because of (8). Furthermore,

$$f_{\lambda'} q_{\lambda'} = \tilde{h} r g q_\lambda = h g q_\lambda\,, \tag{10}$$

because $gq_\lambda(E) = g(E'_\lambda) \subseteq |N'|$ and $r\,||N'| = 1_{|N'|}$. Since we have already proved that hgq_λ and f are $\mathscr{V}$-near maps, the same is true of $f_{\lambda'}q_{\lambda'}$ and f. This establishes (R1).

Proof of (R2). Let $\lambda \in \Lambda$, let P be a polyhedron and let $\mathscr{V}$ be an open covering of P. Let $\mathscr{V}'$ be a star-refinement of $\mathscr{V}$. We shall show that $(fq_\lambda, f'q_\lambda) \leqslant \mathscr{V}'$ implies $(fq_{\lambda\lambda'}, f'q_{\lambda\lambda'}) \leqslant \mathscr{V}$ for some $\lambda' \geqslant \lambda$.

We first prove that every $y \in \overline{E'_\lambda}$ admits a $V \in \mathscr{V}$ such that

$$f(y), f'(y) \in V\,. \tag{11}$$

Indeed, let $V'_1, V_2 \in \mathscr{V}'$ be such that $f(y) \in V'_1, f'(y) \in V'_2$. Let U be a neighborhood of y in E_λ so small that

$$f(U) \subseteq V'_1\,,\quad f'(U) \subseteq V'_2\,. \tag{12}$$

Since $y \in \overline{E'_\lambda}$, there exists a point $x \in E$ such that $q_\lambda(x) \in U$. Consequently,

$$fq_\lambda(x) \in V'_1\,,\quad f'q_\lambda(x) \in V'_2\,. \tag{13}$$

Furthermore, by assumption, there exists a $V' \in \mathscr{V}'$ such that

$$fq_\lambda(x),\quad f'q_\lambda(x) \in V'\,. \tag{14}$$

We see thus that $V'_1 \cap V' \neq \emptyset$ and $V'_2 \cap V' \neq \emptyset$. Consequently,

$$V'_1 \cup V'_2 \subseteq \mathrm{st}(V', \mathscr{V}')\,. \tag{15}$$

Since $\mathscr{V}'$ is a star-refinement of $\mathscr{V}$, there exists a $V \in \mathscr{V}$ such that $\mathrm{st}(V', \mathscr{V}') \subseteq V$ and therefore

$$\{f(y), f'(y)\} \subseteq V'_1 \cup V'_2 \subseteq V\,. \tag{16}$$

By (11), every $y \in \overline{E'_\lambda}$ admits an open neighborhood $U(y)$ such that $f(z), f'(z) \in V$ for $z \in U(y)$. Consequently, there is an open set $U \supseteq \overline{E'_\lambda}$ such that $f|U$ and $f'|U$ are $\mathscr{V}$-near. Now we conclude, by (B2), that there is a $\lambda' \geqslant \lambda$ such that

$$q_{\lambda\lambda'}(E_{\lambda'}) \subseteq U\,. \tag{17}$$

Consequently, the maps $fq_{\lambda\lambda'}$ and $f'q_{\lambda\lambda'}$ are $\mathscr{V}$-near. This completes the proof of Theorem 5.

In the most important cases the converse of Theorem 5 holds too. More precisely, we have this theorem.

THEOREM 6. *Let $q\colon E \to \underline{E} = (E_\lambda, q_{\lambda\lambda'}, \Lambda)$ be a resolution of E. If all E_λ are normal spaces, then q has properties* (B1) *and* (B2).

Proof of (B1). Let $\mathscr{U}$ be a normal covering of E. Let N be the nerve of $\mathscr{U}$ and let $f\colon E \to |N|$ be a canonical map. Then

$$f^{-1}(\mathrm{st}(U, N)) \subseteq U\,,\quad U \in \mathscr{U}\,. \tag{18}$$

Let $\mathscr{K}$ be the star-covering of N and let $\mathscr{V}$ be a star-refinement of $\mathscr{K}$. By (R1), there exist a $\lambda \in \Lambda$ and a map $f_\lambda\colon E_\lambda \to |N|$ such that $(f_\lambda q_\lambda, f) \leqslant \mathscr{V}$. Clearly, $f_\lambda^{-1}(\mathscr{V})$ is a normal covering of E_λ. We claim that $q_\lambda^{-1} f_\lambda^{-1}(\mathscr{V})$ refines $\mathscr{U}$. More precisely, we claim that

$$\mathrm{st}(V, \mathscr{V}) \subseteq \mathrm{st}(U, N) \Rightarrow q_\lambda^{-1} f_\lambda^{-1}(V) \subseteq U\,,\quad V \in \mathscr{V},\ U \in \mathscr{U}\,. \tag{19}$$

Indeed, if $x \in q_\lambda^{-1} f_\lambda^{-1}(V)$, then $f_\lambda q_\lambda(x) \in V$. Since $(f_\lambda q_\lambda, f) \leqslant \mathscr{V}$, there exists a $V' \in \mathscr{V}$ such that $f_\lambda q_\lambda(x), f(x) \in V'$. Hence, $V \cap V' \neq \emptyset$ and

$$f(x) \in V' \subseteq \mathrm{st}(V, \mathscr{V}) \subseteq \mathrm{st}(U, N)\,.$$

Now (18) implies

$$x \in f^{-1}(\mathrm{st}(U, N)) \subseteq U\,.$$

Proof of (B2). Let $\lambda \in \Lambda$ and let $U \subseteq E_\lambda$ be an open neighborhood of $\overline{E'_\lambda} = \overline{q_\lambda(E)}$. Since E_λ is normal, there exists a map $f\colon E_\lambda \to I$ such that

$$f\,|\,E_\lambda \setminus U = 0\,,\quad f\,|\,\overline{E'_\lambda} = 1\,. \tag{20}$$

We also consider the constant map $f'\colon E_\lambda \to I$ such that $f'|\overline{E'_\lambda} = 1$. Notice that

$$fq_\lambda = 1 = f'q_\lambda\,. \tag{21}$$

Consequently, $(fq_\lambda, f'q_\lambda) \leqslant \mathscr{V}'$ for any open covering $\mathscr{V}'$ of I.

Now let $\mathscr{V} = ([0, 1), (0, 1])$. By (R2), there is a $\lambda' \geqslant \lambda$ such that the maps $fq_{\lambda\lambda'}$ and $f'q_{\lambda\lambda'} = 1$ are $\mathscr{V}$-near. Since $f\,|\,E_\lambda \setminus U = 0$ and $1 \notin [0, 1)$, it follows that $fq_{\lambda\lambda'}(E_{\lambda'}) \subseteq (0, 1]$. Finally, since $f\,|\,E_\lambda \setminus U = 0$, we must have

$$q_{\lambda\lambda'}(E_{\lambda'}) \subseteq U\,. \tag{22}$$

This completes the proof of Theorem 6.

Remark 8. In [2] P. Bacon has defined the complement of an inverse system $\underline{E}$ as a map of systems $q\colon E \to \underline{E}$, which satisfies condition (B1) and this stronger form of condition (B2):

(B2′) For every $\lambda \in \Lambda$ and for every open set U in E_λ, which contains $q_\lambda(E)$, there exists a $\lambda' \geqslant \lambda$ such that $q_{\lambda\lambda'}(E_{\lambda'}) \subseteq U$.

Consequently, a complement $q\colon E \to \underline{E}$ in the sense of Bacon is also a resolution in our sense.

6. Resolutions and inverse limits. In this section we compare resolutions $q: E \to \underline{E}$ with inverse limits of $\underline{E}$.

It is well-known that an inverse limit of topologically complete (Dieudonné complete) spaces is topologically complete, because this property is preserved under direct products and closed subsets. Since paracompact spaces are topologically complete, it follows that an inverse limit of a system of polyhedra is necessarily a topologically complete space. However, every topological space E admits a polyhedral resolution $q: E \to \underline{E}$ (see Theorem 10 in Section 7). Hence, a space, which fails to be topologically complete, yields a resolution, which fails to be an inverse limit. An example of such a space is the space of all countable ordinals.

On the other hand it is easy to find inverse systems $\underline{E}$ whose inverse limit $q: E \to \underline{E}$ is not a resolution. One can even achieve that E and all E_λ be metric.

EXAMPLE 5. Let $E_n = \{0\} \cup [n, \infty) \subseteq R$, $n \in N$, and let $q_{nn'}: E_{n'} \to E_n$ be inclusions, $n \leqslant n'$. Then $E = \lim \underline{E} = \{0\}$ and the inclusions $q_n: E \to E_n$ fail to satisfy (B2). Hence, by Theorem 6, q is not a resolution.

The next theorem was suggested by Theorem 3.3 of [21].

THEOREM 7. *Let $q: E \to \underline{E}$ be a resolution. If E is a topologically complete space and if all E_λ are normal spaces, then q is the inverse limit of $\underline{E}$.*

Proof. It suffices to prove that for every map of systems $q': E' \to \underline{E}$ there is a unique map $h: E' \to E$ such that

$$qh = q'. \tag{1}$$

We shall first prove that every $x' \in E'$ admits a point $x \in E$ such that

$$q'_\lambda(x') = q_\lambda(x), \quad \lambda \in \Lambda. \tag{2}$$

We begin by noticing that for every $\lambda \in \Lambda$ and for every open set $U_\lambda \subseteq E_\lambda$ containing $q'_\lambda(x')$ one has

$$q_\lambda^{-1}(U_\lambda) \neq \emptyset. \tag{3}$$

Indeed, assume that for some $\lambda \in \Lambda$ and for some U_λ one has $q_\lambda^{-1}(U_\lambda) = \emptyset$, i.e.

$$q_\lambda(E) \subseteq E_\lambda \setminus U_\lambda \subseteq E_\lambda \setminus \{q'_\lambda(x')\}. \tag{4}$$

Then, by (B2), one can find a $\lambda' \geqslant \lambda$ such that

$$q_{\lambda\lambda'}(E_{\lambda'}) \subseteq E_\lambda \setminus \{q'_\lambda(x')\}. \tag{5}$$

However, this is a contradiction, because $q_{\lambda\lambda'} q'_{\lambda'}(x') = q'_\lambda(x')$.

Now we consider the collection $\mathscr{C} = (q_\lambda^{-1}(U_\lambda))$, where $\lambda \in \Lambda$ and U_λ ranges over closed neighborhoods of $q'_\lambda(x')$ in E_λ. The collection $\mathscr{C}$ is centered. Indeed, if we are given $\lambda_1, \ldots, \lambda_n$ and $U_{\lambda_1}, \ldots, U_{\lambda_n}$, then we can find a $\lambda \geqslant \lambda_1, \ldots, \lambda_n$ and a U_λ such that

$$q'_\lambda(x') \in U_\lambda \subseteq \bigcap_{i=1}^{n} (q_{\lambda_i\lambda})^{-1}(U_{\lambda_i}). \tag{6}$$

Clearly,

$$\bigcap_{i=1}^{n} q_{\lambda_i}^{-1}(U_{\lambda_i}) = q_\lambda^{-1}\Big(\bigcap_{i=1}^{n} (q_{\lambda_i\lambda})^{-1}(U_{\lambda_i})\Big) \supseteq q_\lambda^{-1}(U_\lambda) \neq \emptyset. \tag{7}$$

We claim that the centered collection $\mathscr{C}$ is a Cauchy collection for the fine (maximal) uniformity of E. Indeed, this uniformity is generated by normal coverings $\mathscr{U}$ of E. By (B1), there is a $\lambda \in \Lambda$ and a normal covering $\mathscr{U}_\lambda$ of E_λ such that $q_\lambda^{-1}(\mathscr{U}_\lambda)$ refines $\mathscr{U}$. Let U_λ be a closed neighborhood of $q'_\lambda(x')$ contained in an element of $\mathscr{U}_\lambda$. Then $q_\lambda^{-1}(U_\lambda)$ is an element of the centered collection $\mathscr{C}$ and it is contained in some element of $\mathscr{U}$. Since E is topologically complete, the fine uniformity of E is complete and therefore, the intersection of the members of $\mathscr{C}$ is a single point $x \in E$. Hence, for every $\lambda \in \Lambda$ and for every closed neighborhood U_λ of $q'_\lambda(x')$, one has $q_\lambda(x) \in U_\lambda$. This proves that $q_\lambda(x) = q'_\lambda(x')$.

Now we define h by putting $h(x') = x$. Clearly, h satisfies (1) and is unique. It remains to show that h is continuous. Let $x' \in E'$ and let V be an open neighborhood of $h(x') = x$. Then there is a normal covering $\mathscr{U}$ of E such that $\mathrm{st}(x, \mathscr{U}) \subseteq V$. Let $\mathscr{U}'$ be a normal star-refinement of $\mathscr{U}$. If $x \in U' \in \mathscr{U}'$, then $\mathrm{st}(U', \mathscr{U}') \subseteq \mathrm{st}(x, \mathscr{U}) \subseteq V$. By the definition of h, there exist a $\lambda \in \Lambda$ and a closed neighborhood U_λ of $q'_\lambda(x')$ such that $q_\lambda^{-1}(U_\lambda)$ is contained in a member U' of $\mathscr{U}'$. Hence,

$$q_\lambda^{-1}(U_\lambda) \subseteq \mathrm{st}(U', \mathscr{U}') \subseteq V.$$

Since $q'_\lambda: E' \to E_\lambda$ is continuous, there is a neighborhood V' of x' in E' such that $q'_\lambda(V') \subseteq U_\lambda$. Consequently, $q_\lambda h = q'_\lambda$ implies $h(V') \subseteq q_\lambda^{-1}(U_\lambda) \subseteq V$.

Now we shall exhibit some simple sufficient conditions for an inverse limit to be a resolution.

THEOREM 8. *Let $\underline{E}$ be an inverse system of compact Hausdorff spaces. Then the inverse limit $q: E \to \underline{E}$ is a resolution.*

Proof. By Theorem 5, it suffices to verify properties (B1) and (B2'). However, these are well-known facts (see e.g. [8], VIII, Theorem 3.7 and X, Lemma 3.7).

The next theorem is a variation of ([21], Theorem 5.1).

THEOREM 9. *Let E be a subspace of a T_1-space F. Let $\underline{E}$ be an inclusion system, whose members E_λ form a basis of neighborhoods of E in F, and let $q_\lambda: E \to E_\lambda$ also be inclusions. The inverse limit $q: E \to \underline{E}$, defined by the maps q_λ, is a resolution of E if either of the following two assumptions holds:*

(i) *E is P-embedded in F.*

(ii) *The neighborhoods E_λ are paracompact.*

Proof. By Theorem 5, it suffices to verify properties (B1) and (B2').

Proof of (B1). Let $\mathscr{U}$ be a normal covering of E. We must produce a $\lambda \in \Lambda$ and a normal covering $\mathscr{U}_\lambda$ of E_λ such that $q_\lambda^{-1}(\mathscr{U}_\lambda) \geqslant \mathscr{U}$.

Case (i). Since E is P-embedded in F, there exists a normal covering $\mathscr{U}'$ of F such that the restriction $\mathscr{U}'|E$ refines $\mathscr{U}$. Hence, for any $\lambda \in \Lambda$ the covering $\mathscr{U}_\lambda = \mathscr{U}'|E_\lambda$ of E_λ has the desired property.

Case (ii). For every $U \in \mathcal{U}$ there exists an open set $U' \subseteq F$ such that $U' \cap E = U$. Moreover, there exists a $\lambda \in \Lambda$ such that E_λ is contained in $\bigcup U'$, $U \in \mathcal{U}$. Clearly, $\mathcal{U}_\lambda = (U' \cap E_\lambda \colon U \in \mathcal{U})$ is an open covering of E_λ such that $q_\lambda^{-1}(\mathcal{U}_\lambda) = \mathcal{U}_\lambda | E \geqslant \mathcal{U}$. Since E_λ is paracompact, $\mathcal{U}_\lambda$ is normal.

Proof of (B2'). Let $U \subseteq E_\lambda$ be an open set containing $q_\lambda(E) = E$. Since $\underline{E}$ is a basis of neighborhoods, there is a $\lambda' \geqslant \lambda$ such that $q_{\lambda'}(E_{\lambda'}) = E_{\lambda'} \subseteq U$.

7. Existence of polyhedral and ANR-resolutions. The purpose of this section is to establish several existence theorems for $\mathcal{P}$-resolutions and $\mathcal{ANR}$-resolutions.

THEOREM 10. *Every space B admits a polyhedral resolution $\underline{r}\colon B \to \underline{B}$.*

According to Remark 8, every complement $\underline{r}\colon B \to \underline{B}$ is a resolution. Therefore, Theorem 10 is a consequence of this result of P. Bacon ([2], Theorem 3.2).

THEOREM 10' (P. Bacon). *Every space B admits a complement $\underline{r}\colon B \to \underline{B}$, where $\underline{B}$ is an inverse system of polyhedra.*

THEOREM 11. *Every map $p\colon E \to B$ admits a polyhedral resolution $(\underline{q}, \underline{r}, \underline{p})$.*

Notice that Theorem 10 is obtained from Theorem 11 by putting $p = 1_B\colon B \to B$.

The proof of Theorem 11, which we shall now exhibit, uses techniques that Bacon used in his proof of Theorem 10'.

Let Γ denote the set of all normal coverings γ of B. For every $\gamma \in \Gamma$ we choose a locally finite partition of unity $(\psi_V, V \in \gamma)$ subordinated to γ. Let $N(\gamma)$ be the nerve of γ and let $B_\gamma = |N(\gamma)|$. The partition of unity $(\psi_V, V \in \gamma)$ determines a map $r_\gamma\colon B \to B_\gamma$, which sends the point $y \in B$ into the point $r_\gamma(y)$, whose barycentric coordinate with respect to the vertex $V \in \gamma$ equals $\psi_V(y)$. We call such a map the canonical map of the partition $(\psi_V, V \in \gamma)$.

Now assign to every $\gamma \in \Gamma$ the normal covering $p^{-1}(\gamma)$ of E and put $E_\gamma = |N(p^{-1}(\gamma))|$. Let $(\varphi_V, V \in \gamma)$ be the partition of unity on E given by $\varphi_V = \psi_V p$. Clearly $(\varphi_V, V \in \gamma)$ is a locally finite partition of unity subordinated to $p^{-1}(\gamma) = (p^{-1}(V), V \in \gamma)$, and it determines a canonical map $q_\gamma\colon E \to E_\gamma$. We also define a simplicial map $p_\gamma\colon E_\gamma \to B_\gamma$ by sending each vertex $V \in \gamma$ of $N(p^{-1}(\gamma))$, $p^{-1}(V) \neq \emptyset$, to the vertex V of $N(\gamma)$. It is readily seen that

$$p_\gamma q_\gamma = r_\gamma p, \quad \gamma \in \Gamma. \tag{1}$$

We denote by Λ' the set of all normal coverings of E, which are not of the form $p^{-1}(\gamma)$, $\gamma \in \Gamma$. For every $\alpha \in \Lambda'$ we choose a locally finite partition of unity $(\varphi_U, U \in \alpha)$ on E. It determines a canonical map $q_\alpha\colon E \to E_\alpha = |N(\alpha)|$. Now we put $\Lambda = \Lambda' \cup \Gamma$ and define $\pi\colon \Gamma \to \Lambda$ as the inclusion map.

Let M be the set of all finite subsets of Γ ordered by inclusion. Notice that M is directed and cofinite. If $\mu = \{\gamma_1, \ldots, \gamma_n\}$ we take for B_μ the nerve of the covering

$$\gamma_1 \wedge \ldots \wedge \gamma_n = (V_1 \cap \ldots \cap V_n\colon (V_1, \ldots, V_n) \in \gamma_1 \times \ldots \times \gamma_n).$$

If $\mu \leqslant \mu' = \{\gamma_1, \ldots, \gamma_n, \ldots, \gamma_m\}$, let $r_{\mu\mu'}\colon B_{\mu'} \to B_\mu$ be the simplicial map, which takes

the vertex $(V_1, \ldots, V_m)$ of $N(\gamma_1 \wedge \ldots \wedge \gamma_m)$, $\bigcap_{i=1}^m V_i \neq \emptyset$, to the vertex $(V_1, \ldots, V_n)$ of $N(\gamma_1 \wedge \ldots \wedge \gamma_n)$. It is readily seen that

$$r_{\mu\mu'} r_{\mu'\mu''} = r_{\mu\mu''}, \quad \mu \leqslant \mu' \leqslant \mu''. \tag{2}$$

Next we define $r_\mu\colon B \to B_\mu$ for $\mu = \{\gamma_1, \ldots, \gamma_n\} \in M$ by using the partition of unity $(\psi_{(V_1,\ldots,V_n)}, (V_1, \ldots, V_n) \in \gamma_1 \times \ldots \times \gamma_n)$, where $\psi_{(V_1,\ldots,V_n)} = \psi_{V_1} \cdot \ldots \cdot \psi_{V_n}$. It is readily seen that

$$r_{\mu\mu'} r_{\mu'} = r_\mu, \quad \mu \leqslant \mu'. \tag{3}$$

Similarly, we define the set A of all finite subsets of Λ and we order it by inclusion. If $\lambda = \{\alpha_1, \ldots, \alpha_n\} \in A$, we take for E_λ the nerve of $\alpha_1 \wedge \ldots \wedge \alpha_n$ and we define $q_{\lambda\lambda'}\colon E_{\lambda'} \to E_\lambda$ in the same way as we have defined $r_{\mu\mu'}$.

Now we extend $\pi\colon \Gamma \to \Lambda$ to an increasing function $\pi\colon M \to A$ by putting $\pi(\{\gamma_1, \ldots, \gamma_n\}) = \{\gamma_1, \ldots, \gamma_n\}$. For each $\mu = \{\gamma_1, \ldots, \gamma_n\}$ we define a simplicial map $p_\mu\colon E_{\pi(\mu)} \to B_\mu$ by sending the vertex $(V_1, \ldots, V_n)$, $p^{-1}(V_1 \cap \ldots \cap V_n) \neq \emptyset$, of the nerve of $p^{-1}(\gamma_1 \wedge \ldots \wedge \gamma_n) = p^{-1}(\gamma_1) \wedge \ldots \wedge p^{-1}(\gamma_n)$ to the vertex $(V_1, \ldots, V_n)$ of $N(\gamma_1 \wedge \ldots \wedge \gamma_n)$. Notice that

$$p_\mu q_{\mu\mu'} = r_{\mu\mu'} p_{\mu'}, \quad \mu \leqslant \mu'. \tag{4}$$

Finally, we define for each $\mu = \{\gamma_1, \ldots, \gamma_n\} \in M$, $n > 1$, a map $p_\mu\colon E \to E_\mu$ using the partitions $\varphi_{(V_1,\ldots,V_n)} = \varphi_{V_1} \cdot \ldots \cdot \varphi_{V_n}$, where $\varphi_{V_i} = \psi_{V_i} p$, $i = 1, \ldots, n$. One easily checks that

$$p_\mu q_\mu = r_\mu p, \quad p \in M. \tag{5}$$

We have thus obtained two inverse systems of polyhedra $\underline{E} = (E_\lambda, q_{\lambda\lambda'}, A)$, $\underline{B} = (B_\mu, r_{\mu\mu'}, M)$ and three maps of systems $\underline{p}\colon \underline{E} \to \underline{B}$, $\underline{q}\colon E \to \underline{E}$, $\underline{r}\colon B \to \underline{B}$ such that

$$\underline{p}\underline{q} = \underline{r}p. \tag{6}$$

Now one can easily verify property (B1) for $\underline{r}$ and $\underline{q}$. Indeed, let γ be a normal covering of B. Then $\gamma \in \Gamma$ and $r_\gamma\colon B \to B_\gamma = |N(\gamma)|$ is a canonical map for γ. Consequently, if $\mathcal{K}$ denotes the star-covering of $N(\gamma)$, then $r_\gamma^{-1}(\mathcal{K})$ refines γ. Hence, $\underline{r}$ has property (B1). The same argument shows that also $\underline{q}$ has property (B1).

In order to obtain also property (B2'), we shall replace $\underline{B}$ and $\underline{E}$ by certain larger systems $\underline{B}' = (B'_\nu, r'_{\nu\nu'}, M')$ and $\underline{E}' = (E'_\varrho, q'_{\varrho\varrho'}, A')$, which contain beside the members B_μ of $\underline{B}$ and E_λ of $\underline{E}$ also open neighborhoods of $r_\mu(B)$ in B_μ and of $q_\lambda(E)$ in E_λ respectively. Here is the precise description of this construction.

Let M' be the set of all pairs $\nu = (\mu, V)$, where $\mu \in M$ and V is an open neighborhood of $r_\mu(B)$ in B_μ. We order M' by putting $\nu = (\mu, V) \leqslant (\mu', V') = \nu'$ whenever $\mu \leqslant \mu'$ and $r_{\mu\mu'}(V') \subseteq V$.

For $\nu = (\mu, V) \in M'$ we put $B'_\nu = V$ and $r'_\nu = r_\mu\colon B \to V$, and for $\nu \leqslant \nu'$ we put $r'_{\nu\nu'} = r_{\mu\mu'} | V'\colon V' \to V$. Clearly, $\underline{B}' = (B'_\nu, r'_{\nu\nu'}, M')$ is a polyhedral inverse system and $\underline{r}'\colon B \to \underline{B}'$ is a map of systems.

An analogous procedure, applied to $q\colon E\to \underline{E}$, yields $\underline{q}'\colon E\to \underline{E}'$. Now we define $\underline{p}'\colon \underline{E}'\to \underline{B}'$ as follows. If $\nu=(\mu, V)\in M'$, then $\pi'(\nu)=\big(\pi(\mu), p_\mu^{-1}(V)\big)\in A'$. Clearly, $\pi'\colon M'\to A'$ is an increasing function. For $p'_\nu\colon E'_{\pi'(\nu)}\to B'_\nu$ we take $p'_\nu=p_\mu|p_\mu^{-1}(V)\colon p_\mu^{-1}(V)\to V$. It is readily seen that $\underline{p}'=(p'_\nu, \pi')$ is a map of systems $\underline{E}'\to \underline{B}'$ and that

$$\underline{p}\underline{q}'=\underline{r}'\underline{p}\,. \tag{7}$$

The following assertion will complete the proof of Theorem 11:

$$\underline{r}'\colon B\to \underline{B}' \quad\text{and}\quad \underline{q}'\colon E\to \underline{E}' \text{ are resolutions}\,.$$

It suffices to verify conditions (B1) and (B2′).

(B1) is fulfilled for $\underline{q}'$ because it is fulfilled for $\underline{q}$ and for $\nu=(\mu, B_\mu)\in M'$ we have $B'_\nu=B_\mu$ and $r'_\nu=r_\mu$.

Now we verify (B2′). Let $\nu=(\mu, V)\in M'$ and let U be an open neighborhood of $r'_\nu(B)=r_\mu(B)$ in $B'_\nu=V\subseteq B_\mu$. Then (μ, U) is an element ν' of M'. Clearly, $\nu\leqslant\nu'$ and $r'_{\nu\nu'}=r_{\mu\mu}|U$ is the inclusion map $U\to V$. Since $B'_{\nu'}=U$, we see that $r'_{\nu\nu'}(B'_{\nu'})=r'_{\nu\nu'}(U)=U$, which establishes (B2′).

The proof that $\underline{q}'\colon E\to \underline{E}'$ is a resolution is analogous.

Remark 9. By a slight modification in the above construction one can achieve that A' and M' be cofinite. First note that A and M are cofinite. Instead of taking in M' all pairs (μ, V), $\mu\in M$, V open, $r_\mu(B)\subseteq V\subseteq B_\mu$, one can associate with each μ an indexed basis of open neighborhoods V_σ of $q_\mu(B)$ in B_μ, where σ ranges over a cofinite directed set and $\sigma\leqslant\sigma'$ implies $V_\sigma\subseteq V_{\sigma'}$. An analogous procedure applies also to A'.

THEOREM 12. *Every space B admits an* ANR-*resolution* $\underline{r}\colon B\to \underline{B}$.

THEOREM 13. *Every map $p\colon E\to B$ admits an* ANR-*resolution* $(\underline{q}, \underline{r}, \underline{p})$.

One possible proof consists in repeating the proof of Theorem 11 always endowing nerves of coverings with the metric topology (which makes them ANR's).

We shall give here another proof, which has some merits on its own, and will be used essentially in Section 8. In this proof we need the following simple lemma.

LEMMA 1. *Let E be a topological space and let $f\colon E\to Y$ be a map into an* ANR *Y. Then there exists an* ANR *X, with density $s(X)\leqslant s(E)$, and there exist maps $g\colon E\to X$, $h\colon X\to Y$ such that $f=hg$.*

Proof. By the Kuratowski–Wojdysławski embedding theorem (see e.g. [9], III, Theorem 2.1), one can assume that the metric space $f(E)$ is contained in a normed vector space L and that $f(E)$ is closed in its convex hull $K\subseteq L$. It is easy to see that $s(K)=s\big(f(E)\big)$ and thus $s(K)\leqslant s(E)$. Since $Y\in$ ANR, the inclusion map $f(E)\to Y$ extends to a map $h\colon U\to Y$, where U is an open neighborhood of $f(E)$ in K. Let $g\colon E\to U$ be the composition of $f\colon E\to f(E)$ and of the inclusion map $f(E)\to U$. Then $hg=f$, $X=U$ is an ANR and $s(U)\leqslant s(K)\leqslant s(E)$.

Proof of Theorem 13. We say that two maps $r\colon B\to P$, $r'\colon B\to P'$ are equivalent if there is a homeomorphism $h\colon P\to P'$ such that $hr=r'$. Let Γ be the set of all equivalence classes of maps of B into ANR's of density $\leqslant s(B)$. For every $\gamma\in\Gamma$ let $r_\gamma\colon B\to B_\gamma$ be a map from the class γ. Let M be the set of all finite subsets of Γ ordered by inclusion. Let $B_{\{\gamma_1,\dots,\gamma_n\}}=B_{\gamma_1}\times\dots\times B_{\gamma_n}$. If $\mu=\{\gamma_1,\dots,\gamma_n\}\leqslant\mu'=\{\gamma_1,\dots,\gamma_n,\dots,\gamma_m\}$, we define $r_{\mu\mu'}\colon B_{\mu'}\to B_\mu$ as the projection

$$B_{\gamma_1}\times\dots\times B_{\gamma_n}\times\dots\times B_{\gamma_m}\to B_{\gamma_1}\times\dots\times B_{\gamma_n}\,.$$

We also define $r_\mu\colon B\to B_\mu$ as the map $r_\mu=r_{\mu_1}\times\dots\times r_{\mu_n}\colon B\to B_{\gamma_1}\times\dots\times B_{\gamma_n}$. Notice that $B_\mu\in$ ANR, $s(B_\mu)\leqslant\max\{s(B), \aleph_0\}$, and (2) and (3) hold, so that $\underline{B}=(B_\mu, r_{\mu\mu'}, M)$ is an ANR-system and $\underline{r}=(r_\mu, M)\colon B\to \underline{B}$ is a map of systems.

For every $\gamma\in\Gamma$ we now apply Lemma 1 to the map $r_\gamma p\colon E\to B_\gamma$. We obtain an ANR E_γ, $s(E_\gamma)\leqslant s(E)$, and maps $q_\gamma\colon E\to E_\gamma$ and $p_\gamma\colon E_\gamma\to B_\gamma$ such that (1) holds.

Let A' be the set of equivalence classes of maps of E into ANR's of density $\leqslant s(E)$, which do not contain any of the maps q_γ, $\gamma\in\Gamma$. For each $\alpha\in A'$ we choose a map $q_\alpha\colon E\to E_\alpha$ from the class α. Now we consider $A=A'\cup\Gamma$ and define $\pi\colon\Gamma\to A$ as the inclusion map. Next we define Λ as the set of all finite subsets of A and we order it by inclusion. We put $E_{\{\alpha_1,\dots,\alpha_n\}}=E_{\alpha_1}\times\dots\times E_{\alpha_n}$ and we take for $q_{\lambda\lambda'}\colon E_{\lambda'}\to E_\lambda$, $\lambda\leqslant\lambda'$, $\lambda,\lambda'\in\Lambda$, the corresponding projections. We also define $q_\lambda\colon E\to E_\lambda$ by $q_{\{\alpha_1,\dots,\alpha_n\}}=q_{\alpha_1}\times\dots\times q_{\alpha_n}$. Since $q_{\lambda\lambda'}q_{\lambda'}=q_\lambda$, $\lambda\leqslant\lambda'$, $\underline{E}=(E_\lambda, q_{\lambda\lambda'}, \Lambda)$ is an ANR-system and $\underline{q}=(q_\lambda, \Lambda)\colon E\to \underline{E}$ is a map of systems.

We define $\underline{p}\colon \underline{E}\to \underline{B}$ by taking for $p_\mu\colon E_{\pi(\mu)}\to B_\mu$ the map

$$p_{\gamma_1}\times\dots\times p_{\gamma_n}\colon E_{\gamma_1}\times\dots\times E_{\gamma_n}\to B_{\gamma_1}\times\dots\times B_{\gamma_n},\qquad \mu=\{\gamma_1,\dots,\gamma_n\}\,.$$

Obviously, (4) holds. Finally, (5) holds because of (1), so that we have also (7).

Now we verify that $\underline{q}$ and $\underline{r}$ satisfy condition (R1) for ANR's. They actually satisfy the following stronger condition for ANR's:

(R1′) If P is an ANR and $f\colon E\to P$ is a map, then there exists a $\lambda\in\Lambda$ and there exists a map $f_\lambda\colon E_\lambda\to P$ such that

$$f_\lambda q_\lambda=f\,. \tag{8}$$

Proof. By Lemma 1 f admits a factorization $f=hg$, where $g\colon E\to X$, $h\colon X\to P$, and $X\in\mathscr{ANR}$, $s(X)\leqslant s(E)$. Hence, there exists an $\alpha\in A\subseteq\Lambda$ such that g is equivalent with a map $q_\alpha\colon E\to E_\alpha$. Consequently, there is a homeomorphism $h'\colon E_\alpha\to X$ such that $g=h'q_\alpha$. Now we put $f_\alpha=hh'$. The same argument establishes (R1′) for $\underline{r}$.

In order to complete the proof, we apply to $(\underline{q}, \underline{r}, \underline{p})$ the construction described in the last part of the proof of Theorem 11. We obtain thus new ANR-systems $\underline{E}'$, $\underline{B}'$ and maps of systems $\underline{q}'$, $\underline{r}'$, $\underline{p}'$ satisfying $\underline{p}'\underline{q}'=\underline{r}'\underline{p}$.

The map of systems $\underline{r}'$ has property (R1′) because we have just extended $\underline{B}$ and $\underline{r}$. It has also the following property (R2′), which is stronger than property (R2):

(R2′) Let P be an ANR and $\mathscr{V}$ an open covering of P. Let $\nu\in M'$ and let $f, f'\colon B'_\nu\to P$ be maps such that $(fr'_\nu, f'r'_\nu)\leqslant\mathscr{V}$. Then there exists a $\nu'\geqslant\nu$ such that $(fr'_{\nu\nu'}, f'r'_{\nu\nu'})\leqslant\mathscr{V}$.

Indeed, for every $x \in B$ there is a $V \in \mathscr{V}$ such that $fr'_v(x), f'r'_v(x) \in V$. By continuity of f and f', there is an open neighborhood U_x of $r'_v(x)$ in B'_v such that

(9) $$f(y), f'(y) \in V, \quad y \in U_x .$$

Let $U = \bigcup_{x \in E} U_x$. Then U is an open neighborhood of $r'_v(B)$ in B'_v and

(10) $$(f|U, f'|U) \leqslant \mathscr{V} .$$

Since $v = (\mu, V) \in M'$, $B'_v = V$, $r'_v(B) = r_\mu(B) \subseteq V$, it follows that U is an open neighborhood of $r_\mu(B)$ in V. Therefore, $v' = (\mu, U) \in M'$ and $B'_{v'} = U$. Since $r_{\mu\mu}(U) = U \subseteq V$, we have $v \leqslant v'$ and $r'_{vv'}(B'_{v'}) = r_{\mu\mu}(U) = U \subseteq V = B'_v$. Therefore, (10) implies

(11) $$(fr'_{vv'}, f'r'_{vv'}) \leqslant \mathscr{V} .$$

The proof for q' is analogous.

We call the ANR-resolution $(\underline{q}, \underline{r}, \underline{p})$ of p constructed in this proof the *standard* ANR-*resolution of* p.

8. A characterization of shape fibrations. In this section we give a relatively simple characterization of shape fibrations, which does not involve neither resolutions nor inverse systems.

For a fixed map $p: E \to B$ we shall often have to consider commutative diagrams of the form

$$\begin{array}{ccc} E' & \overset{q'}{\leftarrow} & E \\ \downarrow p' & & \downarrow p \\ B' & \underset{r'}{\leftarrow} & B \end{array}$$

We shall denote such a diagram simply by $D' = (q', r', p')$. If E', B' are ANR's, we shall speak of an ANR-diagram D'. If $D'' = (q'', r'', p'')$, we define a map of diagrams $D'' \to D'$ as a pair (q, r) of maps $q: E'' \to E'$, $r: B'' \to B'$, such that

(1) $$q' = qq'' ,$$

(2) $$r' = rr'' .$$

Clearly, we obtain in this way a category $D(p)$. If there exists a map $D'' \to D'$, we write for short $D' \leqslant D''$. This is a reflexive and transitive relation.

DEFINITION 8. A map $p: E \to B$ has the property SFP (*shape fibration property*) provided the following holds:

Let $D' = (q', r', p')$ be an ANR-diagram and let $\mathscr{U}$, $\mathscr{V}$ be open coverings of E' and B' respectively. Then there exist an ANR-diagram $D'' = (q'', r'', p'')$, an open covering $\mathscr{V}'$ of B'' and a map of diagrams $(q, r): D'' \to D'$ with the following property. If X is an arbitrary space and $h: X \to E''$, $H: X \times I \to B''$ are maps such that

(3) $$(p''h, H_0) \leqslant \mathscr{V}' ,$$

then there exists a homotopy $\tilde{H}: X \times I \to E'$ such that

(4) $$(\tilde{H}_0, qh) \leqslant \mathscr{U} ,$$

(5) $$(p'\tilde{H}, rH) \leqslant \mathscr{V} .$$

THEOREM 14. *A map* $p: E \to B$ *is a shape fibration if and only if it has the property* SFP.

LEMMA 2. *Every* ANR-*diagram* D' *admits an* ANR-*diagram* D'' *such that* $D' \leqslant D''$ *and* $s(E'') \leqslant s(E)$, $s(B'') \leqslant s(B)$.

Proof. We first apply Lemma 1 to the map $r': B \to B'$. We obtain an ANR B'', $s(B'') \leqslant s(B)$, and a factorization $r' = rr''$ through B''. Now we apply Lemma 1 to the map $q' \times r''p: E \to E' \times B''$. We obtain an ANR E'', $s(E'') \leqslant s(E)$, and a factorization $q' \times r''p = (q \times p'')q''$ through E''. Clearly, (q'', r'', p'') is an ANR-diagram D'' and since $q' = qq''$, (q, r) is a map $D'' \to D'$ so that $D' \leqslant D''$.

LEMMA 3. *Let* $D' = (q', r', p')$ *be an* ANR-*diagram and let* $(\underline{q}, \underline{r}, \underline{p})$ *be the standard* ANR-*resolution of* p (see the proof of Theorem 13). *Then there exists an admissible pair* (λ, μ) *such that* $D'' = (q_\lambda, r_\mu, p_{\mu\lambda})$ *is an* ANR-*diagram and* $D' \leqslant D''$.

Proof. By Lemma 2, one can assume that $s(E') \leqslant s(E)$ and $s(B') \leqslant s(B)$. By the definition of $\Gamma \subseteq M$ (see the proof of Theorem 13), there exist a $\gamma \in \Gamma$ and a homeomorphism $h: B_\gamma \to B'$ such that

(6) $$r' = hr_\gamma .$$

Similarly, there is an $\alpha \in A \subseteq \Lambda$ and a homeomorphism $h': E_\alpha \to E'$ such that

(7) $$q' = h'q_\alpha .$$

Notice that the maps $q_\gamma: E \to E_\gamma$, $p_\gamma: E_\gamma \to B_\gamma$ satisfy

(8) $$r_\gamma p = p_\gamma q_\gamma .$$

Hence, if $\alpha = \gamma$, $D'' = (q_\gamma, r_\gamma, p_\gamma)$ is an ANR-diagram and $(h', h): D'' \to D'$ is a map of diagrams, which proves that $D' \leqslant D''$.

If $\alpha \neq \gamma$, we consider $\lambda = \{\alpha, \gamma\} \in \Lambda$. Since $\gamma \leqslant \lambda$, $\gamma = \pi(\gamma)$, (λ, γ) is an admissible pair and we have

(9) $$p_{\gamma\lambda} q_\lambda = p_\gamma q_{\gamma\lambda} q_\lambda = p_\gamma q_\gamma = r_\gamma p .$$

This proves that $D'' = (q_\lambda, r_\gamma, p_{\gamma\lambda})$ is an ANR-diagram. We also have maps $h: B_\gamma \to B'$ and $h'q_{\alpha\lambda}: E_\lambda \to E'$. They satisfy

(10) $$hr_\gamma = r' ,$$

(11) $$h'q_{\alpha\lambda} q_\lambda = q' ,$$

which proves that $D' \leqslant D''$.

Proof of Theorem 14. (i) The condition SFP is necessary. Let $p: E \to B$ be a shape fibration, let $D' = (q', r', p')$ be an ANR-diagram and let $\mathscr{U}$ and $\mathscr{V}$ be open coverings of E' and B' respectively. Let $(\underline{q}, \underline{r}, \underline{p})$ be the standard ANR-resolution of p. By Lemma 3, there is an admissible pair (λ, μ) and a map (q, r): $D'' \to D'$, where $D'' = (q_\lambda, r_\mu, p_{\mu\lambda})$. Let $\mathscr{V}_1$ be a star-refinement of $\mathscr{V}$. Notice that

$$p'qq_\lambda = p'q' = r'p = rr_\mu p = rp_{\mu\lambda}q_\lambda .$$

Therefore, property (R2) applied to B', $\mathscr{V}_1$, $p'q$ and $rp_{\mu\lambda}$ yields a $\lambda' \geq \lambda$ such that

$$(p'qq_{\lambda\lambda'}, rp_{\mu\lambda}q_{\lambda\lambda'}) \leq \mathscr{V}_1 .$$

This shows that there is no loss of generality in assuming that

(12) $$(p'q, rp_{\mu\lambda}) \leq \mathscr{V}_1 .$$

Now we apply the AHLP to (λ, μ), $r^{-1}(\mathscr{V}_1)$ and $q^{-1}(\mathscr{U})$, and we obtain a pair of lifting indexes $(\lambda', \mu') \geq (\lambda, \mu)$ and a lifting mesh $\mathscr{V}'$ on $B_{\mu'}$. We claim that the diagram $D^* = (q_{\lambda'}, r_{\mu'}, p_{\mu'\lambda'})$, the covering $\mathscr{V}'$ and the map $(qq_{\lambda\lambda'}, rr_{\mu\mu'})$: $D^* \to D'$ have all the properties required by SFP.

Indeed, let h: $X \to E_{\lambda'}$, H: $X \times I \to B_{\mu'}$ be such that

(13) $$(p_{\mu'\lambda'}h, H_0) \leq \mathscr{V}' .$$

Then we obtain a homotopy H': $X \times I \to E_\lambda$ such that

(14) $$(q_{\lambda\lambda'}h, H'_0) \leq q^{-1}(\mathscr{U}) ,$$

(15) $$(r_{\mu\mu'}H, p_{\mu\lambda}H') \leq r^{-1}(\mathscr{V}_1) .$$

Let $\tilde{H} = qH'$: $X \times I \to E'$. Then, by (14),

(16) $$(\tilde{H}_0, qq_{\lambda\lambda'}h) \leq \mathscr{U} ,$$

and by (15)

$$(rr_{\mu\mu'}H, rp_{\mu\lambda}H') \leq \mathscr{V}_1 .$$

Since (12) implies

(17) $$(p'\tilde{H}, rp_{\mu\lambda}H') \leq \mathscr{V}_1 ,$$

we conclude that also

(18) $$(p'\tilde{H}, rr_{\mu\mu'}H) \leq \mathscr{V} .$$

(ii) The condition SFP is sufficient. Let p: $E \to B$ be a map with the property SFP. Let $(\underline{q}, \underline{r}, \underline{p})$ be the standard ANR-resolution of p. Let (λ, μ) be an admissible pair of indexes and let $\mathscr{U}$ and $\mathscr{V}$ be open coverings of E_λ and B_μ respectively. Let $\mathscr{U}_1 \geq^* \mathscr{U}$ and $\mathscr{V}_1 \geq^* \mathscr{V}$. Applying the property SFP to $D = (q_\lambda, r_\mu, p_{\mu\lambda})$, $\mathscr{U}_1$ and $\mathscr{V}_1$, we obtain an ANR-diagram $D' = (q', r', p')$, a map (q, r): $D' \to D$ and a covering $\mathscr{V}'$ of B'. By Lemma 3, there exist an admissible pair (λ', μ') and a map (q'', r''): $D'' \to D'$, where $D'' = (q_{\lambda'}, r_{\mu'}, p_{\mu'\lambda'})$. There is no loss of generality in assuming that

(19) $$(p'q'', r''p_{\mu'\lambda'}) \leq \mathscr{V}'_1 ,$$

where $\mathscr{V}'_1 \geq^* \mathscr{V}'$. One can also achieve that $(\lambda, \mu) \leq (\lambda', \mu')$. Moreover, since $r_{\mu\mu'}r_{\mu'} = r_\mu = rr''r_{\mu'}$, and $q_{\lambda\lambda'}q_{\lambda'} = q_\lambda = qq''q_{\lambda'}$, one can achieve that

(20) $$(rr'', r_{\mu\mu'}) \leq \mathscr{V}_1 ,$$

(21) $$(qq'', q_{\lambda\lambda'}) \leq \mathscr{U}_1 .$$

We claim that (λ', μ') is a pair of lifting indexes and that $r''^{-1}(\mathscr{V}'_1)$ is a lifting mesh for (λ, μ), $\mathscr{U}$, $\mathscr{V}$ and p. Indeed, let h: $X \to E_{\lambda'}$, H: $X \times I \to B_{\mu'}$ be maps satisfying

(22) $$(p_{\mu'\lambda'}h, H_0) \leq r''^{-1}(\mathscr{V}'_1) .$$

Then

(23) $$(r''p_{\mu'\lambda'}h, r''H_0) \leq \mathscr{V}'_1 .$$

Since, by (19), also

(24) $$(p'q''h, r''p_{\mu'\lambda'}h) \leq \mathscr{V}'_1 .$$

we obtain

(25) $$(p'q''h, r''H_0) \leq \mathscr{V}' .$$

By the choice of D', $\mathscr{V}'$ and (q, r), we infer that there exists a homotopy $\tilde{H}$: $X \times I \to E_\lambda$ such that

(26) $$(\tilde{H}_0, qq''h) \leq \mathscr{U}_1 ,$$

(27) $$(p_{\mu\lambda}\tilde{H}, rr''H) \leq \mathscr{V}_1 .$$

Combining (26) and (27) with (21) and (20), we finally obtain

(28) $$(\tilde{H}_0, q_{\lambda\lambda'}h) \leq \mathscr{U} ,$$

(29) $$(p_{\mu\lambda}\tilde{H}, r_{\mu\mu'}H) \leq \mathscr{V} ,$$

which completes the proof of the theorem.

9. Resolutions and associated systems. In this section we establish the relationship between resolutions of spaces and associated systems in the sense of K. Morita [19].

The homotopy functor [ ]: TOP → HTOP converts every inverse system $\underline{E}$ in TOP into an inverse system $[\underline{E}]$ in the homotopy category HTOP. By definition, if $\underline{E} = (E_\lambda, q_{\lambda\lambda'}, \Lambda)$, then $[\underline{E}] = (E_\lambda, [q_{\lambda\lambda'}], \Lambda)$. Similarly, [ ] takes a map of systems $\underline{q}$: $E \to \underline{E}$ into a morphism of pro-HTOP $[\underline{q}]$: $E \to [\underline{E}]$. Recall that $[\underline{E}]$ is *associated with E via* $[\underline{q}]$ provided the following conditions are satisfied:

(M1) If P is a polyhedron and f: $E \to P$ is a map, then there exist an index $\lambda \in \Lambda$ and a map f_λ: $E_\lambda \to P$ such that

(1) $$f \simeq f_\lambda q_\lambda .$$

(M2) If P is a polyhedron, $\lambda \in \Lambda$, and f, f': $E_\lambda \to P$ are maps such that

(2) $$fq_\lambda \simeq f'q_\lambda .$$

then there exists a $\lambda'\geqslant\lambda$ such that

$$fq_{\lambda\lambda'}\simeq f'q_{\lambda\lambda'}\,. \tag{3}$$

THEOREM 15. *Let $\underline{q}\colon E\to\underline{E}$ be a resolution of E. Then $[\underline{E}]$ is associated with E via $[\underline{q}]$, i.e.* (M1) *and* (M2) *hold.*

The proof is based on several known facts, which we state as lemmas. Their proofs can be found, e.g., in [2].

LEMMA 4. *Let K be a simplicial complex and let $\mathscr{K}$ be the star-covering of K. If two maps $f,f'\colon E\to|K|$ are $\mathscr{K}$-near, then $f\simeq f'$.*

This is Theorem 2.2 of [2].

LEMMA 5. *Let K be a simplicial complex and let $f\colon E\to|K|$ be a map. Then for any normal covering $\mathscr{U}$ of E, $\mathscr{U}\geqslant f^{-1}(\mathscr{K})$, there exists a simplicial map g of the nerve $N(\mathscr{U})$ into K such that for any canonical map $p\colon E\to|N(\mathscr{U})|$ one has*

$$(gp,f)\leqslant\mathscr{K}\,. \tag{4}$$

See, e.g., the proof of Theorem 2.3 of [2].

LEMMA 6. *Let $P=|K|$ be a polyhedron, E a space, $\mathscr{U}$ a normal covering of E, $p_{\mathscr{U}}\colon E\to|N(\mathscr{U})|$ a canonical map and $f_0,f_1\colon|N(\mathscr{U})|\to E$ two maps such that*

$$f_0p_{\mathscr{U}}\simeq f_1p_{\mathscr{U}}\,. \tag{5}$$

Then there exists a normal covering $\mathscr{U}'$ of E such that $\mathscr{U}'\geqslant\mathscr{U}$ and

$$f_0p_{\mathscr{U}\mathscr{U}'}\simeq f_1p_{\mathscr{U}\mathscr{U}'}\,; \tag{6}$$

here $p_{\mathscr{U}\mathscr{U}'}\colon N(\mathscr{U}')\to N(\mathscr{U})$ is any simplicial map, which sends a vertex $U'\in\mathscr{U}'$ into a vertex $U\in\mathscr{U}$ for which $U'\subseteq U$.

This is Theorem 2.5 of [2] (also see [13], § 3, Lemma 10).

Proof of Theorem 15. Let $\underline{q}\colon E\to\underline{E}$ be a resolution of E. We must show that (M1) and (M2) are fulfilled. We assume that P is a polyhedron and L a simplicial complex such that $P=|L|$. We denote by $\mathscr{L}$ the star-covering of L.

Proof of (M1). Let $f\colon E\to P$ be a map. Applying property (R1) to P and $\mathscr{L}$, we obtain a $\lambda\in\Lambda$ and a map $f_\lambda\colon E_\lambda\to P$ such that $(f,f_\lambda q_\lambda)\leqslant\mathscr{L}$. Then Lemma 4 implies $f\simeq f_\lambda q_\lambda$.

Proof of (M2). Let $\lambda\in\Lambda$ and let $f,f'\colon E_\lambda\to P$ be maps such that (2) holds. Applying (R2) to P and $\mathscr{L}$ we obtain an open covering $\mathscr{V}'$ of P. Let K' (K'') denote the first (second) barycentric subdivision of K and let $\mathscr{K}'$ ($\mathscr{K}''$) be the corresponding star-covering. Notice that $\mathscr{K}\leqslant^*\mathscr{K}'\leqslant^*\mathscr{K}''$. By Lemma 5, there exist a normal covering $\mathscr{U}$ of E and simplicial maps $g,g'\colon N(\mathscr{U})\to K''$ such that for any canonical map $p_{\mathscr{U}}\colon E\to|N(\mathscr{U})|$

$$(fq_\lambda,gp_{\mathscr{U}})\leqslant\mathscr{K}''\,, \tag{6}$$

$$(f'q_\lambda,g'p_{\mathscr{U}})\leqslant\mathscr{K}''\,. \tag{7}$$

Lemma 4 and the assumption (2) imply

$$gp_{\mathscr{U}}\simeq fq_\lambda\simeq f'q_\lambda\simeq g'p_{\mathscr{U}}\,. \tag{8}$$

Therefore, by Lemma 6, there exists a normal covering $\mathscr{U}'$ of E, $\mathscr{U}\leqslant\mathscr{U}'$, such that for any projection $p_{\mathscr{U}\mathscr{U}'}\colon N(\mathscr{U}')\to N(\mathscr{U})$ one has

$$gp_{\mathscr{U}\mathscr{U}'}\simeq g'p_{\mathscr{U}\mathscr{U}'}\,. \tag{9}$$

Let $p_{\mathscr{U}'}\colon E\to|N(\mathscr{U}')|$ be a canonical map. Then the maps $p_{\mathscr{U}\mathscr{U}'}p_{\mathscr{U}'}$ and $p_{\mathscr{U}}$ are contiguous with respect to $N(\mathscr{U})$. Since $g\colon N(\mathscr{U})\to K''$ is a simplicial map, the maps $gp_{\mathscr{U}\mathscr{U}'}p_{\mathscr{U}'}$ and $gp_{\mathscr{U}}$ are contiguous with respect to K''. Consequently,

$$(gp_{\mathscr{U}\mathscr{U}'}p_{\mathscr{U}'},gp_{\mathscr{U}})\leqslant\mathscr{K}''\,, \tag{10}$$

which together with (6) implies

$$(gp_{\mathscr{U}\mathscr{U}'}p_{\mathscr{U}'},fq_\lambda)\leqslant\mathscr{K}'\,. \tag{11}$$

Similarly, we obtain

$$(g'p_{\mathscr{U}\mathscr{U}'}p_{\mathscr{U}'},f'q_\lambda)\leqslant\mathscr{K}'\,. \tag{12}$$

Now consider an open covering $\mathscr{V}$ of $|N(\mathscr{U}')|$, which refines both coverings $(gp_{\mathscr{U}\mathscr{U}'})^{-1}(\mathscr{K}')$ and $(g'p_{\mathscr{U}\mathscr{U}'})^{-1}(\mathscr{K}')$. By property (R1), there exists a $\lambda'\geqslant\lambda$ and there is a map $h\colon E_{\lambda'}\to|N(\mathscr{U}')|$ such that

$$(hq_{\lambda'},p_{\mathscr{U}'})\leqslant\mathscr{V}\,. \tag{13}$$

We thus obtain

$$(gp_{\mathscr{U}\mathscr{U}'}hq_{\lambda'},gp_{\mathscr{U}\mathscr{U}'}p_{\mathscr{U}'})\leqslant\mathscr{K}'\,, \tag{14}$$

$$(g'p_{\mathscr{U}\mathscr{U}'}hq_{\lambda'},g'p_{\mathscr{U}\mathscr{U}'}p_{\mathscr{U}'})\leqslant\mathscr{K}'\,. \tag{15}$$

Now (11) and (14) imply

$$(fq_\lambda,gp_{\mathscr{U}\mathscr{U}'}hq_{\lambda'})\leqslant\mathscr{K}\,. \tag{16}$$

Similarly we obtain

$$(f'q_\lambda,g'p_{\mathscr{U}\mathscr{U}'}hq_{\lambda'})\leqslant\mathscr{K}\,. \tag{17}$$

Since $\mathscr{K}\geqslant\mathscr{V}'$, by the choice of $\mathscr{V}'$, we conclude that there is a $\lambda''\geqslant\lambda'$ such that

$$(fq_{\lambda\lambda''},gp_{\mathscr{U}\mathscr{U}'}hq_{\lambda'\lambda''})\leqslant\mathscr{L} \tag{18}$$

and

$$(f'q_{\lambda\lambda''},g'p_{\mathscr{U}\mathscr{U}'}hq_{\lambda'\lambda''})\leqslant\mathscr{L}\,. \tag{19}$$

Hence, by Lemma 4,

$$fq_{\lambda\lambda''}\simeq gp_{\mathscr{U}\mathscr{U}'}hq_{\lambda'\lambda''}\,, \tag{21}$$

$$f'q_{\lambda\lambda''}\simeq g'p_{\mathscr{U}\mathscr{U}'}hq_{\lambda'\lambda''}\,. \tag{22}$$

Finally, (2) implies the desired conclusion $fq_{\lambda\lambda''}\simeq f'q_{\lambda\lambda''}$.

Remark 10. There are several known methods of assigning to a space E a polyhedral (ANR) associated system, e.g. assigning to E its Čech system [19] (also see [13]). Theorem 15 shows that the proofs of Theorems 11 and 13 offer alternative methods, which generalize the original Mardešić–Segal ANR-system approach to shape [18].

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On the k-pseudo-symmetrical approximate differentiability *

by

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Abstract. The purpose of this paper is to establish a connection between two ways of generalizing the notion of derivative.

1. It is well-known that a number of significant properties of differentiable functions can be expressed in terms of some symmetrical or, generally, bilateral differential quotients (see, for instance, [4] and [3]). On the other hand, a powerful way of generalizing the notion of derivative is that of picking up only these values of the differential quotient that correspond to a suitable set having positive density at a given point: so one obtains, e.g., the approximate (or asymptotical) derivative (see, for instance, [1] and [3]).

Within the present paper, our purpose is to establish a transparent connection between the first and the second way to get a notion of derivative; more precisely, we shall give a theorem who clarifies the relation between the usual approximate derivative and a new one, here called *k-pseudo-symmetrical approximate* (or *asymptotical*) *derivative*.

Such a theorem shows that this new definition, based on a method introduced elsewhere [4] by one of us (S. V.), gives place to an approximate derivative that exists, at least almost everywhere, in any measurable set where the usual one does.

As for a complete understanding of the demonstration it will be useful the knowledge of a deep and elegant theorem by A. Kintchine [2], we report here its statement: *let $f(x)$ be a measurable function, assigned on a measurable set E. Then almost all points of E do belong to one of the following sets*

$E_1 \equiv \{x \in E$: *the approximate derivative of $f(x)$ exists* (¹)$\}$;

$E_2 \equiv \{x \in E$: *its upper (lower) approximate derivates are both* $+\infty\,(-\infty)\}$.

2. Let $f(x)$ be a real function of a real variable, i.e. let $A \subset R$ and $f(x)\colon A \to R$. It is well-known that one can give the notion of *approximate* (or *asymptotical*) *derivative* of $f(x)$ at the point $x \in R$ in the following way [1]:

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