REGULAR CONVERGENCE OF OPERATORS AND APPROXIMATE SOLUTION OF EQUATIONS

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The survey is devoted to different concepts of the convergence of linear and nonlinear operators (discrete, regular, compact, stable convergence, etc.). Attention is paid to the applications of these concepts to numerical analysis methods.

INTRODUCTION

The present survey is devoted to certain fundamental concepts of the convergence of operators, arising in the theory of approximate methods, but being, it seems to us, of independent interest. For linear operators A_n, $A \in \mathscr{L}(E, F)$ (E and F are Banach spaces) these concepts are defined as follows: $A_n \to A$ is regular if $\|A_n u - Au\| \to 0$ $\forall u \in E$ and from the boundedness of the sequence $(u_n) \subset E$ and the relative compactness of $(A_n u_n)$ in F follows the relative compactness of (u_n) in E; $A_n \to A$ is stable if $\|A_n u - Au\| \to 0$ $\forall u \in E$ and the inverses $A_n^{-1} \in \mathscr{L}(F, E)$ exist, where $\|A_n^{-1}\| \le \text{const}$ $(n \ge n_0)$; $A_n \to A$ is compact if $\|A_n u - Au\| \to 0$ $\forall u \in E$ and for any bounded sequence $(u_n) \subset E$ the sequence $(A_n u_n)$ is relatively compact in F.

The concepts listed in Sec. 3 are studied in the situation of the so-called discrete convergence (Secs. 1 and 2) when A_n and A act, in general, in different spaces connected with each other in a definite manner. On the basis of these concepts, in Secs. 4 and 5 we study the convergence of solutions of linear and nonlinear equations of form $A_n u_n = v_n$ to the solution of equation $Au = v$, while in Sec. 6 we study the analogous question for eigenvalue problems of form $A_n u_n = \lambda B_n u_n$, $Au = \lambda Bu$ and of form $A_n(\lambda) u_n = 0$, $A(\lambda)u = 0$. In Secs. 7-10 we illustrate the abstract results by application to the method of mechanical quadratures for solving integral equations, to the method of collocation for solving integral and differential equations, and to the difference method for solving boundary-value problems. The main question here is how to establish the regular, stable, or compact convergence of the corresponding operators. Let us mention one typical feature of the convergence theorems in Secs. 4-6. When studying the convergence of solutions of the inhomogeneous equations $A_n u_n = v_n$, $Au = v$ regular convergence $A_n \to A$ and stable convergence $A_n \to A$ lead to equipotent results (Theorems 4.1, 4.2, 5.1). When studying the eigenvalue problems $A_n u_n = \lambda B_n u_n$, $Au = \lambda Bu$ with the aid of the stable convergence $A_n - \lambda B_n \to A - \lambda B$ we are able to prove only the convergence of the eigenvalues, while the eigenelements can diverge; a positive result here holds only when $A_n - \lambda B_n \to A - \lambda B$ regularly for all λ in a neighborhood of the eigenvalue $\lambda_0 \in \sigma(A, B)$ being studied, including the point λ_0 itself. In connection with this, in applications to concrete approximate methods we should try to establish the regular convergence of the corresponding operators without assuming the invertibility of the limit operator. In a number of cases this can be done directly (Secs. 8, 9); in some other cases (Secs. 7, 10) we obtain the result needed on the basis of the following simple assumption: if $B_n \to B$ is stable and $C_n \to C$ is compact and if $\mathscr{R}(B) = F$ then $A_n \equiv B_n + C_n \to B + C \equiv A$ is regular. With the aim of coherence of the basic text (Secs. 1-10) we have provided only the most necessary literature citations. Section 11 contains a survey of the literature and certain supplements.

In the text we have not explained the following standard notation: $\mathscr{L}(E, F)$ is the space of bounded linear operators (defined on all E);

$$\mathscr{N}(A) = \{u \in E : Au = 0\}, \quad \mathscr{R}(A) = \{v \in F : v = Au,\ u \in E\}$$

are the null space and the range of operator $A \in \mathscr{L}(E, F)$. By

$$N = \{1, 2, \ldots, n, \ldots\}$$

we denote the set of positive integers and by N', N", etc., its infinite subsets. Writing $a_n \to a$ $(n \in N')$ denotes the convergence of sequence $(a_n)_{n \in N'}$ to a as $n \to \infty$; writing $a_n \le \text{const}$ $(n \in N')$ denotes that the number sequence $(a_n)_{n \in N'}$ is bounded from above by a constant not depending on n. We shall say that an assertion is valid for almost all n if it is valid for $n \ge n_0$, where n_0 is some number.

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1. Discrete Convergence of Elements and Functionals

1.1. Let E and E_n ($n \in N$) be real ($\mathbf{K} = \mathbf{R}$) or complex ($\mathbf{K} = \mathbf{C}$) Banach spaces. The system $\mathcal{P}=(p_n)_{n\in N}$ of operators $p_n : E \to E_n$ will be called a connecting system for E and E_n ($n \in N$) if

$$\| p_n u \|_{E_n} \to \| u \|_E \ (n\in N) \ \forall u \in E \tag{1.1}$$

and

$$\| p_n (au+a'u') - (ap_n u + a' p_n u')\|_{E_n} \to 0 \\ (n\in N) \quad \forall u,\ u' \in E,\ a,\ a' \in \mathbf{K} \tag{1.2}$$

(asymptotic linearity of p_n). We remark that if the connecting operators p_n are linear and bounded, then by the principle of uniform boundedness (see [53]), from (1.1) we get

$$\| p_n \| \leqslant \text{const}\,(n\in N). \tag{1.3}$$

1.2. Let the Banach spaces E and E_n ($n \in N$) and the connecting system $\mathcal{P}=(p_n)$ for them be given.

<u>Definition 1.1.</u> The sequence $(u_n)_{n\in N'\subseteq N}$ of elements $u_n \in E_n$ $\mathcal{P}$-converges to $u \in E$ if $\|u_n - p_n u\| \to 0$ ($u \in N'$); we denote $u_n \xrightarrow{\mathcal{P}} u$ ($n\in N'$) or simply $u_n \to u$ ($n \in N'$).

<u>Definition 1.2.</u> The sequence $(u_n)_{n\in N}$ of elements $u_n \in E_n$ is $\mathcal{P}$-compact if any subsequence $(u_n)_{n\in N'\subseteq N}$ of it contains a $\mathcal{P}$-convergent subsequence $(u_n)_{n\in N''\subseteq N'}$.

<u>Definition 1.3.</u> The sequence $(u_n^*)_{n\in N'\subseteq N}$ of functionals $u_n^* \in E_n^*$ weakly $\mathcal{P}$-converges to $u^* \in E^*$ if for any $\mathcal{P}$-convergent sequence $(u_n)_{n\in N'}$ we have

$$u_n \xrightarrow{\mathcal{P}} u\,(n\in N') \Rightarrow \langle u_n,\ u_n^* \rangle \to \langle u,\ u^* \rangle \quad (n\in N');$$

we denote $u_n^* \xrightarrow{\mathcal{P}} u^*\,(n\in N')$ or simply $u_n^* \longrightarrow u^*\,(n\in N')$.

<u>Definition 1.4.</u> The sequence $(u_n^*)_{n\in N}$, $u_n^* \in E_n^*$, is weakly $\mathcal{P}$-compact if any subsequence of it contains a weakly $\mathcal{P}$-convergent subsequence.

In case $E_n = E$, $p_n = I$ ($n \in N$) the concepts listed denote, respectively, convergence in the norm of sequence $(u_n)\subset E$, its (relative) compactness, the weak* convergence of sequence $(u_n^*)\subset E^*$, and its weak* compactness.

1.3. From Definition 1.1 and properties (1.1) and (1.2) of connecting operators it follows that $\mathcal{P}$-convergence possesses many "customary" properties of convergence. We list the main properties:

$$u_n \xrightarrow{\mathcal{P}} u,\quad u_n \xrightarrow{\mathcal{P}} u' \Rightarrow u = u';$$

$$u_n \xrightarrow{\mathcal{P}} u,\ u_n' \xrightarrow{\mathcal{P}} u',\ a,\ a' \in \mathbf{K} \Rightarrow a u_n + a' u_n' \xrightarrow{\mathcal{P}} au + a'u';$$

$$u_n \xrightarrow{\mathcal{P}} u\ \ (n\in N),\ N' \subseteq N \Rightarrow u_n \xrightarrow{\mathcal{P}} u\,(n\in N');$$

$$u_n \xrightarrow{\mathcal{P}} u \Rightarrow \| u_n \| \to \| u \|;$$

$$u_n \xrightarrow{\mathcal{P}} 0 \Leftrightarrow \| u_n \| \to 0,$$

in addition, we can $\mathcal{P}$-approximate each $u \in E$, i.e., for it we can find a sequence (u_n), $u_n \in E_n$ (e.g., $u_n = p_n u$), such that $u_n \xrightarrow{\mathcal{P}} u\,(n\in N)$.

If the connecting operators $p_n : E \to E_n$ are linear and bounded, then from (1.3) follows the additional property

$$u^{(n)},\ u \in E,\quad \| u^{(n)} - u \| \to 0 \Rightarrow p_n u^{(n)} \xrightarrow{\mathcal{P}} u. \tag{1.4}$$

1.4. In the case of linear bounded connecting operators $p_n : E \to E_n$, from (1.4) immediately follows the implication

$$(u^{(n)}) \subset E \text{ is relatively compact } \Rightarrow (p_n u^{(n)}) \text{ is } \mathcal{P}\text{-compact}, \tag{1.5}$$

important for applications. The following proposition is a more profound assertion.

Proposition 1.1. Let $p_n : E \to E_n$ ($n \in N$) be linear and bounded. Then the following conditions are equivalent:

(a) the sequence (u_n) is $\mathcal{P}$-compact and the set of its $\mathcal{P}$-limit points is compact in E;

(b) a relatively compact sequence $(u^{(n)}) \subset E$ exists such that $\|u_n - p_n u^{(n)}\| \to 0$.

We note that the set of $\mathcal{P}$-limit points of a $\mathcal{P}$-compact sequence is not necessarily compact in E, but will be compact if E is separable.

1.5. For any bounded sequence $(u_n)_{n \in N}$, $u_n \in E_n$, we define the measure of noncompactness

$$\mu((u_n)) = \inf\{\varepsilon \mid \forall N' \subseteq N \exists (N'' \subseteq N', u'' \in E) : \|u_n - p_n u''\| \leqslant \varepsilon \ (n \in N'')\}.$$

If (u_n) is unbounded, i.e., $\limsup_{n \in N} \|u_n\| = \infty$ then we set $\mu((u_n)) = \infty$. The noncompactness measure possesses the properties

$$0 \leqslant \mu((u_n)) \leqslant \limsup_{n \in N} \|u_n\|;$$

$$\mu((u_n)) = 0 \Leftrightarrow (u_n) \ \mathcal{P}\text{-compact};$$

$$\mu((a u_n)) = |a| \, \mu((u_n)), \quad a \in \mathbf{K};$$

$$\mu((u_n + u_n')) \leqslant \mu((u_n)) + \mu((u_n')).$$

1.6. Proposition 1.2. The following conditions (a) and (b) are equivalent: (a) $u_n^* \xrightarrow{\mathcal{P}} u^*$ $(n \in N)$; (b) $\|u_n^*\| \leq$ const $(n \in N)$; $\langle p_n u', u_n^* \rangle \to \langle u', u^* \rangle$ $(n \in N)$ for each u' from some subset $E' \subseteq E$ dense in E.

By $E^*_{\text{appr}} \subseteq E^*$ we denote the subspace of weakly $\mathcal{P}$-approximatable functionals:

$$u^* \in E^*_{\text{appr}} \Leftrightarrow \exists (u_n^*), \quad u_n^* \in E_n^*, \quad u_n^* \xrightarrow{\mathcal{P}} u^* \ (n \in N).$$

It turns out that not always $E^*_{\text{appr}} = E^*$.

Proposition 1.3. If E is separable then $E^*_{\text{appr}} = E^*$ and, furthermore, for any $u^* \in E^*$ there exists a sequence (u_n^*), $u_n^* \in E_n^*$, such that $u_n^* \xrightarrow{\mathcal{P}} u^*$, $\|u_n^*\| \to \|u^*\|$ $(n \in N)$.

Proposition 1.4. For any $u \in E$, $\|u\| = 1$, let there exist $u' \in E$, $\|u'\| = 1$, such that $\|u - u'\| < 1$ and let the norm in E be Gâteaux-differentiable at point u'. Then E^*_{appr} is total over E, i.e.,

$$u \in E, \quad \langle u, u^* \rangle = 0 \quad \forall u^* \in E^*_{\text{appr}} \Rightarrow u = 0.$$

1.7. A weakly $\mathcal{P}$-compact sequence (u_n^*) is bounded. In many cases the converse also is true.

Proposition 1.5. If E is separable, then each bounded sequence (u_n^*), $u_n^* \in E_n^*$, is weakly $\mathcal{P}$-compact.

Proposition 1.6. Let E possess the property that each bounded sequence of elements of E^* contains a weakly* convergent subsequence. Let $p_n : E \to E_n$ ($n \in N$) be linear and bounded. Then each bounded sequence (u_n^*), $u_n^* \in E_n^*$, is weakly $\mathcal{P}$-compact.

The property mentioned in Proposition 1.6 is possessed by separable spaces, reflexive spaces, and WCG-spaces [144].

1.8. Convergence $u_n \xrightarrow{\mathcal{P}} u$ has been defined for $u_n \in E_n$, $u \in E$; convergence $u_n^* \xrightarrow{\mathcal{P}} u^*$ has been defined for $u_n^* \in E_n^*$, $u^* \in E^*$. In the case of Hilbert spaces both kinds of convergence have meaning for $u_n \in E_n$, $u \in E$.

Proposition 1.7. In the case of Hilbert spaces $E = E^*$, $E_n = E_n^*$ ($n \in N$) there holds the implication

$$u_n \xrightarrow{\mathcal{P}} u \Leftrightarrow u_n \xrightarrow{\mathcal{P}} u, \quad \|u_n\| \to \|u\|.$$

1.9. Remark 1.1. Sometimes in applications the connecting operators have, as their natural domain, only a certain subspace $E' \subseteq E$ dense in E. Let $p_n' : E' \to E_n$ ($n \in N$) satisfy on E' analogs of conditions (1.1) and (1.2). It turns out that then p_n' can be extended up to $p_n : E \to E_n$, such that conditions (1.1) and (1.2) are fulfilled for $\mathcal{P} = (p_n)$ on the whole of E. The extension is not unique; however, the different extensions lead to equivalent concepts of discrete convergence: if $\tilde{\mathcal{P}} = (\tilde{p}_n)$, $\tilde{p}_n : E \to E_n$, are some other extensions with properties (1.1) and (1.2), then

$$u_n \xrightarrow{\mathcal{P}} u \Leftrightarrow u_n \xrightarrow{\tilde{\mathcal{P}}} u.$$

$\mathcal{P}$-convergence can be characterized also without drawing on the extensions

$$u_n \xrightarrow{\mathcal{P}} u \Leftrightarrow \forall \varepsilon > 0 \ \exists u' \in E' : \| u - u' \| \leqslant \varepsilon, \ \limsup \| u_n - p_n' u' \| \leqslant \varepsilon.$$

2. Discrete Convergence of Operators

2.1. Let there be given the Banach spaces E, F, E_n, F_n $(n \in N)$, the connecting system $\mathcal{P} = (p_n)$ for E, E_n $(n \in N)$, and the connecting system $\mathcal{Q} = (q_n)$ for F, F_n $(n \in N)$:

$$\begin{array}{ccc} E & \xrightarrow{A} & F \\ \downarrow p_n & & \downarrow q_n \\ E_n & \xrightarrow{A_n} & F_n. \end{array}$$

Definition 2.1. The sequence of operators $A_n : E_n \to F_n (n \in N)$ $\mathcal{P}\mathcal{Q}$-converges to operator $A : E \to F$ if for any $\mathcal{P}$-convergent sequence (u_n) we have

$$u_n \xrightarrow{\mathcal{P}} u \, (n \in N) \Rightarrow A_n u_n \xrightarrow{\mathcal{Q}} Au \, (n \in N);$$

we denote $A_n \xrightarrow{\mathcal{P}\mathcal{Q}} A$ or simply $A_n \to A$.

Definition 2.2. The sequence of operators $B_n : E_n^* \to F_n^*$ $(n \in N)$ weakly $\mathcal{P}\mathcal{Q}$-converges to $B : E^* \to F^*$ if for any weakly $\mathcal{P}$-convergent sequence (u_n^*) we have

$$u_n^* \xrightarrow{\mathcal{P}} u^* \, (n \in N) \Rightarrow B_n u_n^* \xrightarrow{\mathcal{Q}} Bu^* \, (n \in N);$$

we denote $B_n \xrightarrow{\mathcal{P}\mathcal{Q}} B$ or simply $B_n \to B$.

2.2. The following assertions relate to a number of obvious (or almost obvious) corollaries of the definitions.

Proposition 2.1. The following conditions are equivalent for $A_n \in \mathcal{L}(E_n, F_n)$, $A \in \mathcal{L}(E, F)$

(a)
$$A_n \xrightarrow{\mathcal{P}\mathcal{Q}} A \, (n \in N);$$

(b)
$$\| A_n \| \leqslant \text{const} \, (n \in N), \ \| A_n p_n u' - q_n A u' \| \to 0 \, (n \in N')$$

for each u' in some subset $E' \subseteq E$ dense in E.

We note that Proposition 1.2 is a special case of Proposition 2.1 (the case of $F_n = F = \mathbf{K}$, $q_n = 1$).

Proposition 2.2. The implication

$$A_n \xrightarrow{\mathcal{P}\mathcal{Q}} A \Rightarrow A_n^* \xrightarrow{\mathcal{Q}\mathcal{P}} A^*$$

is valid for $A_n \in \mathcal{L}(E_n, F_n)$, $A \in \mathcal{L}(E, F)$. If each bounded sequence (v_n^*), $v_n^* \in F_n^*$, is weakly $\mathcal{Q}$-compact (see Paragraph 1.7), then

$$A_n \xrightarrow{\mathcal{P}\mathcal{Q}} A \Leftrightarrow A_n^* \xrightarrow{\mathcal{Q}\mathcal{P}} A^*.$$

3. Regular, Stable, and Compact Convergence of Operators

3.1. As in the preceding section, let $\mathcal{P} = (p_n)$ and $\mathcal{Q} = (q_n)$ be connecting systems for the Banach spaces E, E_n $(n \in N)$ and F, F_n $(n \in N)$, respectively.

Definition 3.1. The sequence of operators $A_n \in \mathcal{L}(E_n, F_n)(n \in N)$ converges regularly (or properly) to the operator $A \in \mathcal{L}(E, F)$ if $A_n \xrightarrow{\mathcal{P}\mathcal{Q}} A$ and the following regularity condition is fulfilled: $u_n \in E_n$, $\| u_n \| \leqslant \text{const}$, $(A_n u_n)$ $\mathcal{Q}$-

$$u_n \in E_n, \ \| u_n \| \leqslant \text{const}, \ (A_n u_n) \text{ is } \mathcal{Q}\text{- compact} \Rightarrow (u_n) \text{ is } \mathcal{P}\text{-compact.}$$

Definition 3.2. The sequence of operators $B_n \in \mathcal{L}(E_n, F_n)(n \in N)$ converges stably to the operator $B \in \mathcal{L}(E, F)$ if $B_n \xrightarrow{\mathcal{P}\mathcal{Q}} B$ and the following stability condition is fulfilled: $B_n^{-1} \in \mathcal{L}(F_n, E_n)$ exists for almost all n, where

$$\| B_n^{-1} \| \leqslant \text{const} \quad (n \geqslant n_0).$$

Definition 3.3. The sequence of operators $C_n \in \mathcal{L}(E_n, F_n)(n \in N)$ converges compactly to the operator $C \in \mathcal{L}(E, F)$ if $C_n \xrightarrow{\mathcal{P}\mathcal{Q}} C$ and the following compactness condition is fulfilled:

$$u_n \in E_n, \ \| u_n \| \leqslant \text{const} \, (n \in N) \Rightarrow (C_n u_n) \text{ is } \mathcal{Q}\text{- compact.}$$

3.2. We indicate certain properties of the operators stemming from Definitions 3.1-3.3.

Proposition 3.1. Let $A_n \to A$ be regular. Let the set of $\mathcal{P}$-limit points of any $\mathcal{P}$-compact sequence be compact in E (see Paragraph 1.4). Then we can find n_0 such that

$$\dim \mathcal{N}(A_n) \leqslant \dim \mathcal{N}(A) \ (n \geqslant n_0), \ \dim \mathcal{N}(A) < \infty;$$
$$\mathcal{R}(A_n) \subseteq F_n \ (n \geqslant n_0), \ \mathcal{R}(A) \subseteq F \text{ are closed;};$$
$$\operatorname{ind} A_n \leqslant \operatorname{ind} A \ (n \geqslant n_0), \text{ if } \operatorname{codim} \mathcal{R}(A) < \infty;$$
$$\operatorname{ind} A_n \to -\infty, \text{ if } \operatorname{codim} \mathcal{R}(A) = \infty.$$

Thus, $A \in \mathcal{L}(E, F)$ and $A_n \in \mathcal{L}(E_n, F_n)$ are, at the least, semi-Fredholm operators.

Proposition 3.2. If $A_n \to A$ is regular and $A \in \mathcal{L}(E, F)$ is a Fredholm operator, then we can find $\gamma > 0$ such that $\mu_{\mathcal{Q}}((A_n u_n)) \geqslant \gamma \mu_{\mathcal{P}}((u_n))$ for any bounded sequence $(u_n))$, $u_n \in E_n$.

Here $\mu_{\mathcal{P}}$ and $\mu_{\mathcal{Q}}$ are noncompactness measures corresponding to E, E_n, $\mathcal{P}$ and F, F_n, $\mathcal{Q}$ (see Paragraph 1.5). We note that the reverse inequality $\mu_{\mathcal{Q}}((A_n u_n)) \leqslant c \mu_{\mathcal{P}}((u_n))$ with $c = \limsup_{n \in N} \| A_n \|$ follows from the $\mathcal{PQ}$-convergence $A_n \to A$ without bringing in the regularity condition.

Proposition 3.2 has the following corollary: if $A_n \to A$ is regular and A is a Fredholm operator, then we can find $\delta > 0$ such that $A_n + A_n' \to A + A'$ is regular for any $A_n' \in \mathcal{L}(E_n, F_n)$ such that $A_n' \xrightarrow{\mathcal{PQ}} A$ and $\| A_n' \| \leqslant \delta$ $(n \in N)$.

Proposition 3.3. If $B_n \to B$ is stable, then $\|Bu\| \geq \gamma \|u\|$ for any $u \in E$, where $\gamma = 1/\sup_{n \geq n_0} \|B_n^{-1}\| > 0$. [But it is not obligatory that $\mathcal{R}(B) = F$.]

Indeed, $\|Bu\| = \lim \|B_n p_n u\| \geq \gamma \lim \|p_n u\| = \gamma \|u\|$.

Almost obvious is the following proposition.

Proposition 3.4. If $C_n \to C$ is compact and if the set of $\mathcal{Q}$-limit points of any $\mathcal{Q}$-compact sequence is compact in F, then operator C is completely continuous.

Remark 3.1. In case $E_n = E$, $p_n = I_E$, $F_n = F$, $q_n = I_F$, A_n, $A \in \mathcal{L}(E, F)$, $\|A_n - A\| \to 0$ $(n \in N)$ we have: $A_n \to A$ is regular if $\dim \mathcal{N}(A) < \infty$ and $\mathcal{R}(A) \subseteq F$ is closed; $A_n \to A$ is stable and regular if $\exists A^{-1} \in \mathcal{L}(F, E)$; $A_n \to A$ is compact if A is completely continuous.

3.3. Let us clarify the interrelation between the three kinds of convergence.

Proposition 3.5. If for B_n, $C_n \in \mathcal{L}(E_n, F_n)$, B, $C \in \mathcal{L}(E, F)$ we have $\mathcal{R}(B) = F$ and

$$B_n \to B \text{ is stable, }, \ C_n \to C \text{ is compact,},$$

then

$$A_n \equiv B_n + C_n \to B + C \equiv A \text{ is regular.}$$

The proof is elementary and follows from the definitions. It is curious that this proposition admits of inversion:

Proposition 3.6. Let $A_n \to A$ be regular, where $A_n \in \mathcal{L}(E_n, F_n)$ and $A \in \mathcal{L}(E, F)$ are Fredholm operators with zero index. Let E^*_{appr} be total over E (see Paragraph 1.6). Then A_n and A are Fredholm operators in the form $A_n = B_n + C_n$, $A = B + C$ such that $B_n \to B$ is stable, $C_n \to C$ is compact, and $B^{-1} \in \mathcal{L}(F, E)$ exists.

4. Convergence of Inverse Operators. Convergence Theorem for Linear Operator Equations

THEOREM 4.1. The following conditions (a), (b), and (c) for operators $A_n \in \mathcal{L}(E_n, F_n)$ $(n \in N)$ and $A \in \mathcal{L}(E, F)$ are equivalent: (a) $A_n \to A$ is regular, A_n $(n \geq n_0)$ are Fredholm operators with zero index, $\mathcal{N}(A) = \{0\}$; (b) $A_n \to A$ is stable, $\mathcal{R}(A) = F$; (c) $A_n \to A$ is stable and regular. If any one of the conditions (a), (b), (c) is fulfilled, then $A^{-1} \in \mathcal{L}(F, E)$ and $A_n^{-1} \in \mathcal{L}(F_n, E_n) (n \geqslant n_0)$ exist and $A_n^{-1} \to A^{-1}$ is stable and regular.

Proof. (a)⇒(б). Because A_n is a Fredholm operator with zero index, the stability condition for A_n reduces to the inequality

$$\| A_n u_n \| \geqslant \gamma \| u_n \| \quad \forall u_n \in E_n, \ n \geqslant n_0,$$

with some $\gamma > 0$. Arguing to the contrary, let us assume that for certain $u_n \in E_n$ we have

$$\|u_n\|=1, \quad \|A_n u_n\| \to 0 \ (n \in N' \subseteq N).$$

On the basis of the regularity condition the sequence (u_n) is $\mathcal{P}$-compact, $u_n \to u$ $(n \in N'' \subseteq N')$, $\|u\|=1$. Then $A_n u_n \to Au$ $(n \in N'')$ and $Au = 0$, despite condition (a). Thus, the stability condition is fulfilled.

To prove that $\mathcal{R}(A)=F$ we take any $v \in F$ and we denote $u_n = A_n^{-1} q_n v$. Then $\|u_n\| \leq$ const, $(A_n u_n)$ is $\mathcal{Q}$-compact, and on the basis of the regularity condition (u_n) is $\mathcal{P}$-compact, $u_n \to u$ $(n \in N' \subseteq N)$. Hence, $A_n u_n \to Au$ $(n \in N')$. But $A_n u_n = q_n v \to v$, therefore $v = Au$ and $v \in \mathcal{R}(A)$.

(b) $\Rightarrow$ (c) To establish the regularity condition it is enough to note that

$$A_n u_n \to v, \quad v = Au \Rightarrow u_n \to u. \tag{4.1}$$

Indeed, $u_n - p_n u = A_n^{-1}[(A_n u_n - q_n v) + (q_n Au - A_n p_n u)]$,

$$\|u_n - p_n u\| \leqslant \text{const}\,[\|A_n u_n - q_n v\| + \|q_n Au - A_n p_n u\|] \to 0.$$

(c) $\Rightarrow$ (a). We only need show that $\mathcal{N}(A)=\{0\}$. This follows from Proposition 3.3.

If one of the conditions (a), (b), (c) is fulfilled, then the ones remaining are fulfilled and $A_n^{-1} \in \mathcal{L}(F_n, E_n)$ $(n \geq n_0)$ exist by virtue of the stability condition, while $A^{-1} \in \mathcal{L}(F, E)$ exists in view of the conditions $\mathcal{N}(A)=\{0\}$, $\mathcal{R}(A)=F$. We rewrite the implication (4.1) established above in the form

$$v_n \to v \Rightarrow A_n^{-1} v_n \to A^{-1} v.$$

Thus, $A_n^{-1} \overset{\mathcal{Q}\mathcal{P}}{\to} A^{-1}$. This convergence is stable since $\|(A_n^{-1})^{-1}\| = \|A_n\| \leq$ const (see Proposition 2.1); from the equivalence of (a) and (b) it follows that the convergence indicated is regular as well. Theorem 4.1 has been proved.

Theorem 4.1 can be treated as a convergence theorem for the linear equations $Au = v$ and $A_n u_n = v_n$ with $A \in \mathcal{L}(E, F)$, $A_n \in \mathcal{L}(E_n, F_n)$ $(n \in N)$.

THEOREM 4.2. Let one of the equivalent conditions (a), (b), (c) in Theorem 4.1 be fulfilled and let $v_n \overset{\mathcal{Q}}{\to} v$ $(n \in N)$, $v_n \in F_n$, $v \in F$. Then the equation $Au = v$ has a unique solution $\tilde{u} \in E$, for almost all n the equation $A_n u_n = v_n$ has the unique solution $\tilde{u}_n \in E_n$, and $\tilde{u}_n \overset{\mathcal{P}}{\to} \tilde{u}$ with the estimate

$$c_1 \|A_n p_n \tilde{u} - v_n\|_{F_n} \leqslant \|\tilde{u}_n - p_n \tilde{u}\|_{E_n} \leqslant c_2 \|A_n p_n \tilde{u} - v_n\|_{F_n}, \tag{4.2}$$

where $c_1 = 1/\sup_{n \geq n_0} \|A_n\| > 0$, $c_2 = \sup_{n \geq n_0} \|A_n^{-1}\| < \infty$.

The proof follows immediately from Theorem 4.1; estimate (4.2) follows from the equality $A_n(\tilde{u}_n - p_n \tilde{u}) = v_n - A_n p_n \tilde{u}$.

5. Convergence Theorems for Nonlinear Equations

5.1. As before let there be given the Banach spaces E, E_n $(n \in N)$ with connecting system $\mathcal{P}=(p_n)$ and the Banach spaces F, F_n $(n \in N)$ with connecting system $\mathcal{Q}=(q_n)$. We consider the equations $\mathcal{A}u=v$ and $\mathcal{A}_n u_n = v_n$, where the (in general, nonlinear) operators $\mathcal{A}$ and $\mathcal{A}_n$ act from E into F and from E_n into F_n, respectively.

At first we formulate one elementary lemma on the solvability of nonlinear equations.

LEMMA 5.1. Let operator $\mathcal{A}$ be Fréchet-differentiable in the sphere $S = \{u \in E : \|u - u^0\| \leq \delta_0\}$. Let $[\mathcal{A}'(u^0)]^{-1} \in \mathcal{L}(F, E)$ and $\|\mathcal{A}'(u^0)\| \leqslant a$, $\|[\mathcal{A}'(u^0)]^{-1}\| \leqslant b$ exist. Finally, let q $(0 \leq q < 1)$ be found such that

$$\|\mathcal{A}'(u) - \mathcal{A}'(u^0)\| \leqslant \frac{q}{b} \quad \text{for } u \in S,$$

$$\|\mathcal{A}u^0 - v\| \leqslant \frac{\delta_0(1-q)}{b}.$$

Then in S the equation $\mathcal{A}u=v$ has a unique solution $\tilde{u}$ and

$$\frac{\alpha}{1+q} \leqslant \|\tilde{u} - u^0\| \leqslant \frac{\alpha}{1-q}, \quad \alpha = \|[\mathcal{A}'(u^0)]^{-1}(\mathcal{A}u^0 - v)\|,$$

$$\frac{\|\mathcal{A}u^0 - v\|}{a} \leqslant \alpha \leqslant b\,\|\mathcal{A}u^0 - v\|.$$

Sketch of the Proof. Having rewritten the equation $\mathcal{A}u=v$ as $u = Tu$ with $Tu=u-[\mathcal{A}'(u^0)]^{-1}(\mathcal{A}u-v)$, on the basis of the lemma's hypotheses we establish that $TS\subseteq S$ and T satisfies on S a Lipschitz condition with constant q, i.e., we apply Banach's fixed point principle.

THEOREM 5.1. Let the following conditions be fulfilled:

(I) The equation $\mathcal{A}u=v$ has the solution $\tilde{u}\in E$ and operator $\mathcal{A}$ is Fréchet-differentiable at point $\tilde{u}$;

(II) we can find $\delta > 0$ such that the operators $\mathcal{A}_n$ $(n\in N)$ are Fréchet-differentiable in the appropriate spheres $\{u_n \in E_n : \|u_n - p_n\tilde{u}\| \leq \delta\}$, and for any $\varepsilon > 0$ we can find $\delta_\varepsilon > 0$ $(\delta_\varepsilon \leq \delta)$ such that $\|\mathcal{A}_n'(u_n)-\mathcal{A}_n'(p_n\tilde{u})\|\leqslant\varepsilon$ $(n\in N)$ for $\|u_n-p_n\tilde{u}\|\leqslant\delta_\varepsilon$ $(n\in N)$;

(III) $\|\mathcal{A}_np_n\tilde{u}-v_n\|\to 0$ $(n\in N)$;

(IV) one of the (equivalent) conditions (a), (b), (c) of Theorem 4.1 is fulfilled for $A=\mathcal{A}'(\tilde{u})\in\mathcal{L}(E, F)$, $A_n=\mathcal{A}_n'(p_n\tilde{u})\in\mathcal{L}(E_n, F_n)$.

Then we can find δ_0 $(0 < \delta_0 \leq \delta)$ such that for almost all n the equation $\mathcal{A}_nu_n=v_n$ has a unique solution $\tilde{u}_n$ in the sphere $\|u_n - p_n\tilde{u}\| \leq \delta_0$. Moreover, $\tilde{u}_n\overset{\mathcal{P}}{\to}\tilde{u}$ with the estimate ($c_1, c_2 =$ const > 0)

$$c_1\|\mathcal{A}_np_n\tilde{u}-v_n\|_{F_n}\leqslant\|\tilde{u}_n-p_n\tilde{u}\|_{E_n}\leqslant c_2\|\mathcal{A}_np_n\tilde{u}-v_n\|_{F_n}. \tag{5.1}$$

Proof. By virtue of the stability condition and of Proposition (2.1),

$$\|\mathcal{A}_n'(p_n\tilde{u})\|\leqslant a=\text{const},\quad \|[\mathcal{A}_n'(p_n\tilde{u})]^{-1}\|\leqslant b=\text{const}\ (n\geqslant n_0). \tag{5.2}$$

We fix $q \in (0, 1)$. By condition (II) we can find $\delta_0 > 0$ such that

$$\sup_{\|u_n-p_n\tilde{u}\|\leqslant\delta_0}\|\mathcal{A}_n'(u_n)-\mathcal{A}_n'(p_n\tilde{u})\|\leqslant\frac{q}{b}\ (n\in N). \tag{5.3}$$

Finally, condition (III) enables us to take it that

$$\|\mathcal{A}_np_n\tilde{u}-v_n\|\leqslant\frac{\delta_0(1-q)}{b}\ (n\geqslant n_0). \tag{5.4}$$

Inequalities (5.2)-(5.4) signify that the hypotheses of Lemma 5.1 have been fulfilled for the equation $\mathcal{A}_nu_n=v_n$ when $n \geq n_0$ (instead of $\mathcal{A}$, E, F, u^0, etc., we should consider $\mathcal{A}_n$, E_n, F_n, $u_n^0=p_n\tilde{u}$, etc.). The theorem's assertion follows immediately from the lemma.

5.2. We shall now reckon the spaces E, F, E_n, F_n $(n \in N)$ to be real. Using the concept of the rotation [76] of completely continuous vector fields (or the equivalent concept of the degree of a mapping), we can establish the convergence of the approximate solutions for a certain class of equations with nondifferentiable operators. We consider the equations $Au+\mathcal{B}u=0$ and $A_nu_n+\mathcal{B}_nu_n=0$ with linear operators $A\in\mathcal{L}(E, F)$, $A_n\in\mathcal{L}(E_n, F_n)$ and nonlinear completely continuous operators $\mathcal{B}:\bar{\Omega}\subset E\to F$, $\mathcal{B}_n:\bar{\Omega}_n\subset E_n\to F_n$. Here $\Omega\subset E$, $\Omega_n\subset E_n$ are nonempty bounded open sets with closures $\bar{\Omega}$, $\bar{\Omega}_n$ and boundaries $\partial\Omega$, $\partial\Omega_n$ consistent among themselves by the following conditions:

$$\begin{gathered}\forall u\in\bar{\Omega}\ \exists(u_n): u_n\in\bar{\Omega}_n,\ u_n\overset{\mathcal{P}}{\to}u\ (n\in N);\\ u_n\in\bar{\Omega}_n,\ u_n\overset{\mathcal{P}}{\to}u\ (n\in N'\subseteq N)\Rightarrow u\in\bar{\Omega};\\ u_n\in\partial\Omega_n,\ u_n\overset{\mathcal{P}}{\to}u\ (n\in N'\subseteq N)\Rightarrow u\in\partial\Omega.\end{gathered} \tag{5.5}$$

THEOREM 5.2. Let the following conditions be fulfilled:

(I) One of the (equivalent) conditions (a), (b), (c) from Theorem 4.1 is fulfilled for $A\in\mathcal{L}(E, F)$, $A_n\in\mathcal{L}(E_n, F_n)$;

(II) $\mathcal{B}_n\to\mathcal{B}$ is compact, i.e.,

$$\begin{gathered}u_n\in\bar{\Omega}_n,\ u_n\overset{\mathcal{P}}{\to}u\Rightarrow\mathcal{B}_nu_n\overset{\mathcal{Q}}{\to}\mathcal{B}u;\\ u_n\in\bar{\Omega}_n\ (n\in N)\Rightarrow(\mathcal{B}_nu_n)\ \text{is}\ \mathcal{Q}\text{-compact};\end{gathered}$$

(III) the equation $Au+\mathcal{B}u=0$ does not have solutions on $\partial\Omega$, and the rotation of the completely continuous vector field $u+A^{-1}\mathcal{B}u$ on boundary $\partial\Omega$ is nonzero:

$$\gamma(I+A^{-1}\mathcal{B};\ \partial\Omega)\neq 0. \tag{5.6}$$

Then the set $\mathcal{U}\subset\bar{\Omega}$ of solutions of equation $Au+\mathcal{B}u=0$ is nonempty, the set $\mathcal{U}_n\subset\bar{\Omega}_n$ of solutions of equation $A_nu_n+\mathcal{B}_nu_n=0$ is nonempty for almost all n. Any sequence $(\tilde{u}_n)$ with $\tilde{u}_n\in\mathcal{U}_n$ is $\mathcal{P}$-compact and its $\mathcal{P}$-limit points belong to $\mathcal{U}$. In particular, if $\mathcal{U}$ consists of one point $\tilde{u}$ [of topological index nonzero by virtue of (5.6)], then $\tilde{u}_n\xrightarrow{\mathcal{P}}\tilde{u}$.

Remark 5.1. Let the open bounded sets $\Omega'\subset F$ and $\Omega_n'\subset F_n$ $(n\in N)$ be consistent analogously to (5.5):

$$\forall v\in\bar{\Omega}'\ \exists(v_n): v_n\in\bar{\Omega}_n',\ v_n\xrightarrow{\mathcal{Q}}v\ (n\in N);$$

$$v_n\in\bar{\Omega}_n',\ v_n\xrightarrow{\mathcal{Q}}v\ (n\in N'\subseteq N)\Rightarrow v\in\bar{\Omega}';$$

$$v_n\in\partial\Omega_n',\ v_n\xrightarrow{\mathcal{Q}}v\ (n\in N'\subseteq N)\Rightarrow v\in\partial\Omega'.$$

Taking as fulfilled the condition (I) in Theorem 5.2, we set $\Omega = A^{-1}\Omega'$, $\Omega_n = A_n^{-1}\Omega_n'$. Then the consistency conditions (5.5) are fulfilled for Ω and Ω_n; condition (5.6) is equivalent to the condition

$$\gamma(I+\mathcal{B}A^{-1};\ \partial\Omega')\neq 0.$$

Remark 5.2. Condition (II) in Theorem 5.2 can be replaced by the less restrictive conditions: $\sup_{u_n\in\Omega_n}\|\mathcal{B}_nu_n\|\leqslant$ const $(n\in N)$ and $A_n+\lambda\mathcal{B}_n\to A+\lambda\mathcal{B}$ is regular $\forall\lambda\in[0,1]$, i.e.,

$$u_n\in\bar{\Omega}_n,\ u_n\xrightarrow{\mathcal{P}}u\Rightarrow\mathcal{B}_nu_n\xrightarrow{\mathcal{Q}}\mathcal{B}u;$$

$$u_n\in\bar{\Omega}_n,\ (A_nu_n+\lambda\mathcal{B}_nu_n)\text{ is }\mathcal{Q}\text{-compact}\Rightarrow(u_n)\text{ is }\mathcal{P}\text{-compact.}$$

In such a formulation Theorem 5.2 remains in force also for a certain class of equations with noncompact operators $\mathcal{B}:\bar{\Omega}\to F$. See [199] regarding this.

5.3. Scheme of the Proof of Theorem 5.2. From conditions (I) and (II) it follows that $A_n^{-1}\mathcal{B}_n\to A^{-1}\mathcal{B}$ is compact. The proof of the invariance of the rotation has a basic difficulty: if the compact convergence $T_n\to T$ holds for the completely continuous operators $T_n:\bar{\Omega}_n\to E_n$ and $T:\bar{\Omega}\to E$ and if T does not have fixed points on $\partial\Omega$, then for almost all n the operator T_n does not have fixed points on $\partial\Omega_n$, and

$$\gamma(I-T_n;\ \partial\Omega_n)=\gamma(I-T;\ \partial\Omega)\quad(n\geqslant n_0).$$

Having established this, we shall have

$$\gamma(I+A_n^{-1}\mathcal{B}_n;\ \partial\Omega_n)=\gamma(I+A^{-1}\mathcal{B};\ \partial\Omega)\neq 0\quad(n\geqslant n_0),$$

and when $n\geq 0$ the equation $A_nu_n+\mathcal{B}_nu_n=0$ has the solution $\tilde{u}_n\in\Omega_n$. From the equality $\tilde{u}_n+A_n^{-1}\mathcal{B}_n\tilde{u}_n=0$ and the $\mathcal{P}$-compactness of $(A_n^{-1}\mathcal{B}_n\tilde{u}_n)$ we obtain the $\mathcal{P}$-compactness of $(\tilde{u}_n)$. If $\tilde{u}_n\to\tilde{u}$ $(n\in N'\subseteq N)$, then $A_n^{-1}\mathcal{B}_n\tilde{u}_n\to A^{-1}\mathcal{B}\tilde{u}$, and $\tilde{u}+A^{-1}\mathcal{B}\tilde{u}=0$, $A\tilde{u}+\mathcal{B}\tilde{u}=0$.

6. The Eigenvalue Problem

6.1. Let E, F, E_n, F_n $(n\in N)$ be complex Banach spaces; $\mathcal{P}=(p_n)$ be the connecting system for E, E_n $(n\in N)$, $\mathcal{Q}=(q_n)$ be the connecting system for F, F_n $(n\in N)$; $A(\lambda)\in\mathcal{L}(E,F)$, $A_n(\lambda)\in\mathcal{L}(E_n,F_n)$ $(n\in N)$ depend on the parameter λ. We consider the problems

$$A(\lambda)u=0\tag{6.1}$$

and

$$A_n(\lambda)u_n=0.\tag{6.2}$$

Let Λ be some domain (an open connected set) in the complex plane. We introduce the conditions:

1°) $A_n(\lambda)\to A(\lambda)$ is regular for each $\lambda\in\Lambda$;

2°) for each $\lambda\in\Lambda$ the operators $A(\lambda)\in\mathcal{L}(E,F)$, $A_n(\lambda)\in\mathcal{L}(E_n,F_n)$ $(n\in N)$ are Fredholm with zero index;

3°) there exists $\lambda'\in\Lambda$ such that $\mathcal{N}(A(\lambda'))=\{0\}$;

4°) the operator-valued functions $A(\cdot):\Lambda\to\mathcal{L}(E,F)$, $A_n(\cdot):\Lambda\to\mathcal{L}(E_n,F_n)$ $(n\in N)$ are holomorphic*;

5°) the operator-valued functions $A_n(\cdot)$ $(n\in N)$ are uniformly bounded on each closed bounded subset $\Lambda_0\subset\Lambda$, i.e., $\|A_n(\lambda)\|\leq c(\Lambda_0)=\text{const}(\lambda\in\Lambda_0;\ n\in N)$.

*We recall [73] that $A(\lambda)$ is holomorphic in Λ if for any $u\in E$, $v^*\in F^*$ the function $a_{u,v^*}:\Lambda\to\mathbf{C}$, $a_{u,v^*}(\lambda)=\langle A(\lambda)u, v^*\rangle$, is holomorphic.

By Σ we denote [the spectrum of problem (6.1) in Λ] the set of those $\lambda \in \Lambda$ for which a bounded inverse to $A(\lambda)$, defined on the whole of F, does not exist; the spectrum Σ_n of problem (6.2) in Λ is defined analogously. From conditions 2°)-4°) it follows (see [123, 80, 81, for instance) that Σ consists of isolated eigenvalues of finite root multiplicity (a condensation of the eigenvalues to the boundary of domain Λ is possible).

THEOREM 6.1. Let conditions 1°)-5°) be fulfilled. Then the following statements are valid.

1. For each $\lambda_0 \in \Sigma$ the sequence (λ_n), $\lambda_n \in \Sigma_n$, $\lambda_n \to \lambda_0$ $(n \geq n_0)$, exists. Conversely, if $\lambda_n \to \lambda_0 \in \Lambda$, $\lambda_n \in \Sigma_n$ $(n \in N' \subseteq N)$, then $\lambda_0 \in \Sigma$.

2. If $u_n^0 \in E_n$, $\|u_n^0\| = 1$, $A_n(\lambda_n) u_n^0 = 0$ and $\lambda_n \to \lambda_0 \in \Lambda$ $(n \in N' \subseteq N)$, then the sequence (u_n^0) is $\mathscr{P}$-compact and for the limit points $u^0 \in E$ we have $\|u^0\| = 1$, $A(\lambda_0)u^0 = 0$.

3. If $v_n^* \in F_n^*$, $\|v_n^*\| = 1$, $[A_n(\lambda_n)]^* v_n^* = 0$ and $\lambda_n \to \lambda_0 \in \Lambda$ $(n \in N' \subseteq N)$, then the sequence (v_n^*) is weakly $\mathscr{Q}$-compact and for the limit points $v^* \in F^*$ we have $v^* \neq 0$, $[A(\lambda_0)]^* v^* = 0$.

Remark 6.1. Statement 1 in Theorem 6.1 remains in force if one of the following (equivalent) conditions is fulfilled instead of 1°):

(a) $A_n(\lambda) \to A(\lambda)$ is regular for each $\lambda \in \Lambda \setminus \Sigma$;

(b) $A_n(\lambda) \to A(\lambda)$ is stable† for each $\lambda \in \Lambda \setminus \Sigma$. The remaining statements in Theorem 6.1 lose force.

6.2. The root subspace $W(\lambda_0)$ of problem (6.1), corresponding to $\lambda_0 \in \Sigma$, is the linear hull of all possible eigenelements and adjointed elements of it, corresponding to $\lambda_0 \in \Sigma$; the Jordan chain $\{u^0, u^1, \ldots, u^k\}$ of length $k + 1$ of elements adjoined to $u^0 \in \mathscr{N}(A(\lambda_0))$ is defined by the conditions

$$
\begin{gathered}
A(\lambda_0) u^0 = 0 \quad (u^0 \neq 0), \\
A(\lambda_0) u^1 + A'(\lambda_0) u^0 = 0, \\
A(\lambda_0) u^2 + A'(\lambda_0) u^1 + \frac{1}{2!} A''(\lambda_0) u^0 = 0, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
A(\lambda_0) u^k + A'(\lambda_0) u^{k-1} + \frac{1}{2!} A''(\lambda_0) u^{k-2} + \ldots + \frac{1}{k!} A^{(k)}(\lambda_0) u^0 = 0.
\end{gathered} \tag{6.3}
$$

The multiplicity $\nu(u^0)$ of eigenelement u^0 is the greatest length of the Jordan chains starting from u^0. The operator-valued function $[A(\lambda)]^{-1}$ has a pole at point λ_0 and its order $\varkappa$ equals the largest multiplicity $\nu(u^0)$ of the eigenelements $u^0 \in \mathscr{N}(A(\lambda_0))$, i.e., $\varkappa = \max\limits_{u^0 \in \mathscr{N}(A(\lambda_0)),\, u^0 \neq 0} \nu(u^0) < \infty$.

The proper subspace $\mathscr{N}(A(\lambda_0))$ is decomposed into the direct sum

$$\mathscr{N}(A(\lambda_0)) = \mathscr{N}_1 + \mathscr{N}_2 + \ldots + \mathscr{N}_l$$

of subspaces $\mathscr{N}_k$ consisting of eigenelements of like multiplicity $\varkappa_k$, so that $\varkappa = \varkappa_1 > \ldots > \varkappa_l \geq 1$. Let $\{u^{1,0}, u^{2,0}, \ldots, u^{m,0}\}$ be a basis of $\mathscr{N}(A(\lambda_0))$ made up of the bases of $\mathscr{N}_1, \ldots, \mathscr{N}_l$ and let $\{u^{j,0}, u^{j,1}, \ldots, u^{j,\nu_j}\}$ be a Jordan chain of maximum length [i.e., $\nu_j = \nu(u^{j,0}) - 1$] starting from $u^{j,0}$ $(j = 1, 2, \ldots, m)$. Then the linear hull of elements $u^{j,k}$, $k = 0, 1, \ldots, \nu_j$, $j = 1, 2, \ldots, m$, coincides with $W(\lambda_0)$.

Let $p_n^0 : W(\lambda_0) \to E_n$ $(n \in N)$ be linear operators such that $\|p_n^0 u - p_n u\| \to 0$ $(n \in N)$ for each $u \in W(\lambda_0)$. We fix some $v_n^* \in \mathscr{N}([A_n(\lambda_n)]^*) \subseteq F_n^*$, $\|v_n^*\| = 1$, and we denote

$$\varepsilon_n^{(0)} = \max_{j=1,\ldots,m} |\langle A_n(\lambda_0) p_n^0 u^{j,0}, v_n^* \rangle|,$$

$$\varepsilon_n^{(1)} = \max_{\substack{j=1,\ldots,m \\ \nu_j \geq 1}} |\langle A_n(\lambda_0) p_n^0 u^{j,1} + A_n'(\lambda_0) p_n^0 u^{j,0}, v_n^* \rangle|,$$

$$\varepsilon_n^{(2)} = \max_{\substack{j=1,\ldots,m \\ \nu_j \geq 2}} |\langle A_n(\lambda_0) p_n^0 u^{j,2} + A_n'(\lambda_0) p_n^0 u^{j,1} + \frac{1}{2!} A_n''(\lambda_0) p_n^0 u^{j,0}, v_n^* \rangle|$$

etc., viz., the approximation errors of the root relations (6.3) for the chosen basis $\mathscr{N}(A(\lambda_0))$. It is clear that $\varepsilon_n^{(k)} \to 0$ $(n \in N)$, $k = 0, 1, \ldots, \varkappa - 1$.

THEOREM 6.2. Let conditions 1°)-5°) be fulfilled. Then for $\lambda_n \in \Sigma_n$, $\lambda_n \to \lambda_0 \in \Sigma$, and $u_n^0 \in \mathscr{N}(A_n(\lambda_n))$, $\|u_n^0\| = 1$, the estimates

$$|\lambda_n - \lambda_0| \leq c\,[(\varepsilon_n^{(0)})^{1/\varkappa} + (\varepsilon_n^{(1)})^{1/(\varkappa-1)} + \ldots + \varepsilon_n^{(\varkappa-1)}], \tag{6.4}$$

†According to Proposition 3.3 stable convergence at points $\lambda \in \Sigma$ is impossible.

$$d_{E_n}(u_n^0, p_n^0\mathcal{N}(A(\lambda_0))) \leqslant c\,[|\lambda_n-\lambda_0| + \max_{j=1,\ldots,m} \| A_n(\lambda_0)\,p_n^0 u^{j,0}\|], \tag{6.5}$$

where c = const and d_{E_n} is the distance in the metric of space E_n, are valid.

Remark 6.2. If $q_n: F \to F_n$ $(n \in N)$ are linear and bounded, then estimate (6.4) can be given the more visible form

$$|\lambda_n-\lambda_0|^{\varkappa} \leqslant c_\delta \varepsilon_{n,\delta}, \tag{6.6}$$

where $c_\delta \leq c/\delta^{\varkappa-1}$, c = const,

$$\varepsilon_{n,\delta} = \max_{|\lambda-\lambda_0|=\delta}\ \max_{u\in W(\lambda_0),\,\|u\|=1} |\langle A_n(\lambda)\,p_n^0 u - q_n A(\lambda)\,u,\ v_n^*\rangle|, \tag{6.7}$$

δ is any positive number such that the disk $|\lambda - \lambda_0| \leq \delta$ is contained in Λ. If $p_n: E \to E_n$ $(n \in N)$ are also linear, then we can set $p_n^0 = p_n$ $(n \in N)$.

6.3. Let us consider the problems with a linear occurrence of the parameter:

$$Au = \lambda Bu, \quad A, B \in \mathcal{L}(E, F); \tag{6.8}$$

$$A_n u_n = \lambda B_n u_n, \quad A_n, B_n \in \mathcal{L}(E_n, F_n). \tag{6.9}$$

For such problems the results of the preceding paragraphs can be extended. By $\sigma(A, B)$ we denote the spectrum of problem (6.8), i.e., the set of those $\lambda \in \mathbf{C}$ for which $A-\lambda B \in \mathcal{L}(E, F)$ does not have a bounded inverse defined on the whole of F. We assume that conditions 2°) and 3°) are fulfilled for $A(\lambda) := A - \lambda B$ and $A_n(\lambda) := A_n - \lambda B_n$. Let $\lambda_0 \in \sigma(A, B) \cap \Lambda$, while $\delta > 0$ is sufficiently small, so that the disk $|\lambda - \lambda_0| \leq \delta$ is contained in Λ and in this disk there are no other points of $\sigma(A, B)$ besides λ_0. We introduce the notation: $W(\lambda_0) \subseteq E$ is the root subspace of problem (6.8) [of the operator-valued function $A(\lambda) = A - \lambda B$] corresponding to λ_0; $W^*(\lambda_0) \subseteq F^*$ is the root subspace of the problem $A^*v^* = \lambda B^*v^*$ adjoint to (6.8), corresponding to λ_0; $W_n(\lambda_0;\delta) \subseteq E_n$ is the linear hull of the root subspaces of problem (6.9), corresponding to the eigenvalues from disk $|\lambda - \lambda_0| \leq \delta$; $W_n^*(\lambda_0;\delta) \subseteq F_n^*$ is the analogous linear hull for the problem adjoint to (6.9).

We note that in the case given conditions (6.3) defining the Jordan chain $\{u^0, u^1, \ldots, u^k\}$ take the form

$$\begin{aligned} (A-\lambda_0 B)\,u^0 &= 0 \quad (u^0 \neq 0),\\ (A-\lambda_0 B)\,u^1 &= Bu^0,\\ &\ldots\ldots\ldots\ldots\\ (A-\lambda_0 B)\,u^k &= Bu^{k-1}. \end{aligned} \tag{6.10}$$

THEOREM 6.3. Let conditions 1°)–3°) be fulfilled for the linear operator-valued functions $A(\lambda) = A - \lambda B$, $A_n(\lambda) = A_n - \lambda B_n$. Then statements 1–3 in Theorem 6.1 [with $\Sigma = \sigma(A, B) \cap \Lambda$, $\Sigma_n = \sigma(A_n, B_n) \cap \Lambda$] and the following statements 4–6 are valid.

4. For each $u \in W(\lambda_0)$ the sequence (u_n), $u_n \in W_n(\lambda_0;\delta)$, $u_n \xrightarrow{\mathcal{P}} u$ $(n \geqslant n_0)$, exists. Conversely, any sequence (u_n) with $u_n \in W_n(\lambda_0;\delta)$, $\|u_n\| = 1$, is $\mathcal{P}$-compact with limit points $u \in W(\lambda_0)$, $\|u\| = 1$. Moreover, for the opening

$$\Theta_{E_n}(W_n(\lambda_0;\delta),\ p_n^0 W(\lambda_0)) = \max\left\{\max_{\substack{u_n\in W_n(\lambda_0;\delta)\\ \|u_n\|=1}} d_{E_n}(u_n, p_n^0 W(\lambda_0)),\ \max_{\substack{u\in W(\lambda_0)\\ \|u\|=1}} d_{E_n}(p_n^0 u, W_n(\lambda_0;\delta))\right\}$$

there holds the convergence $\Theta_{E_n}(W_n(\lambda_0;\delta), p_n^0 W(\lambda_0)) \to 0$, and, hence, $\dim W_n(\lambda_0;\delta) = \dim W(\lambda_0)$ for almost all n. Here $p_n^0: W(\lambda_0) \to E_n$ $(n \in N)$ are any linear operators such that $\|p_n^0 u - p_n u\| \to 0$ $(n \in N)$ $\forall u \in W(\lambda_0)$.

5. For each $v^* \in W^*(\lambda_0)$ the sequence (v_n^*), $v_n^* \in W_n^*(\lambda_0;\delta)$, $v_n^* \xrightarrow{\mathcal{Q}} v^*$ $(n \geqslant n_0)$, exists. Conversely, any sequence (v_n^*) with $v_n^* \in W_n^*(\lambda_0;\delta)$, $\|v_n^*\| = 1$, is weakly $\mathcal{Q}$-compact with limit points $v^* \in W^*(\lambda_0)$, $v^* \neq 0$.

6. The estimates

$$|\lambda_n-\lambda_0| \leqslant c\varepsilon_n^{1/\varkappa}, \quad |\hat{\lambda}_n-\lambda_0| \leqslant c\varepsilon_n, \tag{6.11}$$

$$d_{E_n}(u_n^0,\ p_n^0\mathcal{N}(A-\lambda_0 B)) \leqslant c\left[|\lambda_n-\lambda_0| + \max_{\substack{u\in\mathcal{N}(A-\lambda_0 B)\\ \|u\|=1}} \|(A_n-\lambda_0 B_n)\,p_n^0 u\|_{F_n}\right] \leqslant c\,[|\lambda_n-\lambda_0| + \varepsilon_n'], \tag{6.12}$$

$$\Theta_{E_n}(W_n(\lambda_0;\delta),\ p_n^0 W(\lambda_0)) \leqslant c\varepsilon_n' \tag{6.13}$$

are valid, where c = const, λ_n is any eigenvalue of problem (6.9) from the disk $|\lambda - \lambda_0| \le \delta$, while u_n^0 is any normed eigenelement ($A_n u_n^0 = \lambda_n B_n u_n^0$, $\|u_n^0\| = 1$); $\hat{\lambda}_n$ is the arithmetic mean of the eigenvalues of problem (6.9) from the disk $|\lambda - \lambda_0| \le \delta$, where each of them is taken as many times as its root multiplicity (the dimension of the corresponding root subspace); $\varkappa$ is the greatest length of the Jordan chains of problem (6.8), corresponding to $\lambda_0 \in \sigma(A, B) \cap \Lambda$;

$$\varepsilon_n = \max_{\substack{v_n^* \in W_n^*(\lambda_0;\delta) \\ \|v_n^*\|=1}} \max_{\substack{u,u' \in W(\lambda_0),\|u\|=1 \\ (A-\lambda_0 B)u=Bu'}} |\langle (A_n - \lambda_0 B_n) p_n^0 u - B_n p_n^0 u', v_n^* \rangle|, \tag{6.14}$$

$$\varepsilon_n' = \max_{\substack{u,u' \in W(\lambda_0),\|u\|=1 \\ (A-\lambda_0 B)u=Bu'}} \|(A_n - \lambda_0 B_n) p_n^0 u - B_n p_n^0 u'\|_{F_n}. \tag{6.15}$$

In clarifying the definitions of ε_n and ε_n' we remark that for each $u \in W(\lambda_0)$ there exists a uniquely defined $u' \in W(\lambda_0)$ such that $(A - \lambda_0 B)u = Bu'$ [for $u \in \mathcal{N}(A - \lambda_0 B)$ this will be $u' = 0$]. The quantities ε_n and ε_n' characterize the accuracy of the approximation of the root relations (6.10) by operators A_n and B_n. We note as well that the first of estimates (6.11) somewhat coarsens and simplifies estimate (6.4), which, it is clear, is valid in the given case.

If $q_n : F \to F_n$ ($n \in N$) are linear and bounded, we have

$$\varepsilon_n \leqslant \varepsilon_n' \leqslant c \max_{u \in W(\lambda_0),\|u\|=1} (\|A_n p_n^0 u - q_n A u\| + \|B_n p_n^0 u - q_n B u\|). \tag{6.16}$$

Remark 6.3. Let conditions 2°) and 3°) and one of the following (equivalent) conditions instead of 1°) be fulfilled:

(a) $A_n - \lambda B_n \to A - \lambda B$ is regular for each $\lambda \in \Lambda \setminus \sigma(A, B)$;

(b) $A_n - \lambda B_n \to A - \lambda B$ is stable for each $\lambda \in \Lambda \setminus \sigma(A, B)$ (cf. Remark 6.1).

Then $\dim W_n(\lambda_0; \delta) \ge \dim W(\lambda_0)$ for almost all n and statements 2-6 of Theorem 6.3 lose force. But if for almost all n it happens that $\dim W_n(\lambda_0; \delta) = \dim W(\lambda_0)$, then the convergence $A_n - \lambda B_n \to A - \lambda B$ will be regular also for $\lambda = \lambda_0$ and the statements in Theorem 6.3 will be valid.

6.4. Let us consider the case of projection methods. Let E and F be separable complex Banach spaces, $E_n \subseteq E$ and $F_n \subseteq F$ be finite-dimensional subspaces, $\dim E_n = \dim F_n$, and $q_n \in \mathcal{L}(F, F)$ be projectors, $q_n F = F_n$ ($n \in N$). As $A_n, B_n \in \mathcal{L}(E_n, F_n)$ we take the restrictions $A_n = q_n A | E_n$, $B_n = q_n B | E_n$, i.e., Eq. (6.9) has the form

$$q_n (A - \lambda B) u_n = 0 \quad (u_n \in E_n). \tag{6.17}$$

THEOREM 6.4. Let $A, B, C \in \mathcal{L}(E, F)$, where B and C are completely continuous, $\mathcal{R}(A+C) = F$ and $\mathcal{N}(A - \lambda' B) = \{0\}$ for some $\lambda' \in \mathbf{C}$. Let $\|v - q_n v\|_F \to 0$ ($n \in N$) for each $v \in F$ and

$$\|q_n (A + C) u_n\|_F \geqslant \gamma \|u_n\|_E \quad (n \geqslant n_0;\ u_n \in E_n)$$

with some constant $\gamma > 0$ not depending on n and $u_n \in E_n$. Then the statements in Theorems 6.1 and 6.3 with $\Lambda = \mathbf{C}$ are valid for problems (6.8) and (6.17); moreover, estimates (6.11)-(6.13) take the form

$$|\lambda_n - \lambda_0| \leqslant c (\varepsilon_n \varepsilon_n^*)^{1/\varkappa}, \quad |\hat{\lambda}_n - \lambda_0| \leqslant c \varepsilon_n \varepsilon_n^*, \tag{6.18}$$

$$d_E (u_n^0, \mathcal{N}(A - \lambda_0 B)) \leqslant c \varepsilon_n^{1/\varkappa}, \quad \Theta_E (W_n(\lambda_0; \delta), W(\lambda_0)) \leqslant c \varepsilon_n, \tag{6.19}$$

where

$$\varepsilon_n = \max_{u \in W(\lambda_0),\|u\|=1} d_E (u, E_n), \quad \varepsilon_n^* = \max_{v^* \in W^*(\lambda_0),\|v^*\|=1} \|v^* - q_n^* v^*\|_{F^*}, \tag{6.20}$$

$q_n^* \in \mathcal{L}(F^*, F^*)$ is the projector conjugate to $q_n \in \mathcal{L}(F, F)$.

Remark 6.4. Note that in the hypotheses of Theorem 6.4 there exist projectors $p_n \in \mathcal{L}(E, E)$ such that $p_n E = E_n$ and $\|u_n - p_n u\| \to 0$ ($n \in N$) for each $u \in E$. It is clear that

$$\varepsilon_n \leqslant \max_{u \in W(\lambda_0),\|u\|=1} \|u - p_n u\| \leqslant c \varepsilon_n, \quad c = 1 + \limsup \|p_n\| < \infty.$$

6.5. Let E be a separable reflexive complex Banach space continuously and densely imbedded in a Hilbert space H. Then H is continuously and densely imbedded in E^* (which below we take as F):

$$E\subseteq H\subseteq E^*.$$

Let $E_n\subseteq E$ $(n\in N)$ be finite-dimensional subspaces and q_n be orthoprojectors in H, projecting onto E_n; they admit of extension by continuity up to projectors $q_n\in\mathcal{L}(E^*, E^*)$. The projection method (6.17) for solving problem (6.8) is, in the case given, Galerkin's method.

THEOREM 6.5. Let A, B, $C\in\mathcal{L}(E, E^*)$, where B and C are completely continuous, $\mathcal{N}(A-\lambda' B)=\{0\}$ for some $\lambda'\in\mathbf{C}$, and let

$$|(Au+Cu,\ u)|\geqslant\gamma\|u\|_E^2\ \ \forall u\in E \tag{6.21}$$

with some $\gamma > 0$. Let q_n be orthoprojectors in H, $q_nH = E_n$,

$$d_E(u,\ E_n)\to 0\ \ (n\in N)\ \ \forall u\in E. \tag{6.22}$$

Then statements 1-5 in Theorems 6.1 and 6.3 with $\Lambda = \mathbf{C}$ and the estimates

$$|\lambda_n-\lambda_0|\leqslant c(\varepsilon_n\varepsilon_n^*)^{1/\varkappa},\quad |\hat{\lambda}_n-\lambda_0|\leqslant c\varepsilon_n\varepsilon_n^*, \tag{6.23}$$

$$\max_{\substack{u_n\in\mathcal{N}(A_n-\lambda_nB_n)\\ \|u_n\|=1}} d_E(u_n,\ \mathcal{N}(A-\lambda_0B))\leqslant c\varepsilon_n^{1/\varkappa},$$
$$\max_{\substack{u_n\in\mathcal{N}((A_n-\lambda_nB_n)^*)\\ \|u_n\|=1}} d_E(u_n,\ \mathcal{N}((A-\lambda_0B)^*)\leqslant c(\varepsilon_n^*)^{1/\varkappa}, \tag{6.24}$$

$$\Theta_E(W_n(\lambda_0;\ \delta),\ W(\lambda_0))\leqslant c\varepsilon_n,\quad \Theta_E(W_n^*(\lambda_0;\ \delta),\ W^*(\lambda_0))\leqslant c\varepsilon_n^*, \tag{6.25}$$

where

$$\varepsilon_n=\max_{u\in W(\lambda_0),\|u\|=1} d_E(u,\ E_n),\quad \varepsilon_n^*=\max_{u\in W^*(\lambda_0),\|u\|=1} d_E(u,\ E_n),$$

are valid for problems (6.8) and (6.17).

This theorem is as well a corollary of Theorems 6.2 and 6.3 with $A_n=q_nA\,|\,E_n$, $B_n=q_nB\,|\,E_n\in\mathcal{L}(E_n,\ E_n^*)$, space E_n^* coincides, by structure of the elements, with E_n and has the norm $\|u_n^*\|_{E_n^*}=\sup_{u_n\in E_n,\|u_n\|_E=1}|(u_n,\ u_n^*)_H|$ dual to E_n (this norm is a weak norm, induced from E^*, and $\|q_n\|_{\mathcal{L}(E^*,E_n^*)}\leqslant 1$ although, in general, $\|q_n\|_{\mathcal{L}(E^*,E^*)}\to\infty$). Condition (6.22) permits us to construct in a natural manner the connecting system $\mathcal{P}=(p_n)$ for E and E_n $(n\in N)$ such that $u_n\xrightarrow{\mathcal{P}}u$ will denote $\|u_n - u\|\to 0$, $u_n\in E_n$, $u\in E$. The connecting system for E^* and E_n^* $(n\in N)$ will be $Q = (q_n)$. From (6.21) it follows that $A_n + C_n \to A + C$ is stable, $\mathcal{R}(A+C)= F$, while from the complete continuity of B, $C\in\mathcal{L}(E, E^*)$ it follows that $B_n\to B$, $C_n\to C$ is compact [see (1.5)]; on the basis of Proposition 3.5, $A_n - \lambda B_n \to A - \lambda B$ is regular for each $\lambda\in\mathbf{C}$, and Theorems 6.1 and 6.3 are applicable with $\Lambda = \mathbf{C}$.

Results on the convergence of projection methods, analogous to Theorems 6.4 and 6.5, are valid for problem (6.1) with a nonlinear occurrence of parameter λ; the corresponding formulations are obtained on the basis of Theorems 6.1 and 6.2.

Remark 6.5. Let the hypotheses of Theorem 6.5 be fulfilled with $C = 0$ and let $A^* = A$, $B^* = B$. Then $W(\lambda_0)=\mathcal{N}(A-\lambda_0B)$, i.e., estimates (6.23) and (6.24) are valid with $\varkappa = 1$.

7. Method of Mechanical Quadratures

7.1. We consider the integral equation

$$u(t)=\int_D G(t,\ s)\,u(s)\,d\mu(s)+v(t), \tag{7.1}$$

where D is a compact metric space (e.g., a closed bounded domain in $\mathbf{R}^m$), μ is a regular (see [50]) bounded measure on D (e.g., the Lebesgue measure), $v: D\to\mathbf{K}$ and $G: D\times D\to\mathbf{K}$ are continuous functions. Let there be given a certain convergent quadrature formula

$$\int_D w(s)\,d\mu(s)=\sum_{j=1}^{n}\alpha_{jn}w(s_{jn})+\varphi_n(w)\ \ (n\in N) \tag{7.2}$$

with nodes $s_{jn}\in D$ and coefficients $\alpha_{jn}\in\mathbf{K}$ $(j = 1,\ldots,n;\ n\in N)$; the convergence of formula (7.2) signifies that $\varphi_n(w)\to 0$ $(n\in N)$ for any function w continuous on D.

Having replaced the integral in (7.1) by quadrature formula (7.2), in which we drop the remainder term, we arrive at the equation

$$u_n(t)=\sum_{j=1}^{n}\alpha_{jn}G(t, s_{jn})\,u_n(s_{jn})+v(t), \tag{7.3}$$

which is equipotent with the linear system of equations

$$u_{in}=\sum_{j=1}^{n}\alpha_{jn}G(s_{in}, s_{jn})\,u_{jn}+v(s_{in})\quad (i=1,\ldots,n) \tag{7.4}$$

in $u_{in}=u_n(s_{in})$, $i=1,\ldots,n$. Here the equipotency is to be understood in the following sense: if $u_n(t)$ is a solution of Eq. (7.3), then $u_{in}=u_n(s_{in})$ $(i=1,\ldots,n)$ is a solution of system (7.4); conversely, if u_{in} $(i=1,\ldots,n)$ is a solution of system (7.4), then the solution of Eq. (7.3) can be recovered by the formula

$$u_n(t)=\sum_{j=1}^{n}\alpha_{jn}G(t, s_{jn})\,u_{jn}+v(t).$$

We assume $E_n=F_n=E=F=C(D)$ to be the space of continuous functions $u:D\to\mathbf{K}$ with norm $\|u\|=\max_{t\in D}|u(t)|$; $p_n=q_n=I$ $(n\in N)$. By T we denote the integral operator from Eq. (7.1) and by T_n its analog from Eq. (7.3):

$$(Tu)(t)=\int_D G(t, s)\,u(s)\,d\mu(s),\quad (T_nu)(t)=\sum_{j=1}^{n}\alpha_{jn}G(t, s_{jn})\,u(s_{jn}).$$

It is clear that T, $T_n\in\mathscr{L}(C(D), C(D))$ are completely continuous. It turns out that $T_n\to T$ is compact; in the given case this signifies that

$$\|T_nu-Tu\|\to 0\quad \forall u\in C(D);$$

$\|u_n\|\leq 1$ $(n\in N)\Rightarrow(T_nu_n)$ is relatively compact in $C(D)$. The second of these relations is verified without difficulty by using Arzela's theorem, taking into account the uniform continuity of G on the compactum $D\times D$ and the inequality $\sum_{j=1}^{n}|\alpha_{jn}|\leqslant\text{const}$ $(n\in N)$ stemming from the convergence of the quadrature formula. The first of the two stated relations also is obvious:

$$\|T_nu-Tu\|=\max_{t\in D}|\varphi_n(G(t, s)\,u(s))|\to 0,$$

because the system of functions $\mathscr{M}=\{G(t, s)\,u(s)\}_{t\in D}$ is compact in $C(D)$. Thus, $T_n\to T$ is compact. By Proposition 3.5

$$I-\lambda T_n\to I-\lambda T\quad (n\in N)\quad \text{is regular}\quad \forall\lambda\in\mathbf{K}. \tag{7.5}$$

If the homogeneous integral equation $u=Tu$ has only the zero solution, then condition (a) of Theorem 4.1 is fulfilled for the operators $A=I-T$, $A_n=I-T_n\in\mathscr{L}(C(D), C(D))$. According to Theorem 4.2, Eq. (7.3) has a unique $\tilde{u}_n$ for almost all n and the sequence $(\tilde{u}_n)$ converges uniformly to the solution $\tilde{u}$ of Eq. (7.1) with the estimate

$$c_1\delta_n\leqslant\max_{t\in D}|\tilde{u}_n(t)-\tilde{u}(t)|\leqslant c_2\delta_n,\ \delta_n=\max_{t\in D}|\varphi_n(G(t, s)\,\tilde{u}(s))|.$$

7.2. By virtue of (7.5), the hypotheses of Theorems 6.1 and 6.3 (here we set $A=A_n=I$, $B=T$, $B_n=T_n$) are fulfilled for the problems

$$u(t)=\lambda\int_D G(t, s)\,u(s)\,d\mu(s) \tag{7.6}$$

and

$$u_n(t)=\lambda\sum_{j=1}^{n}\alpha_{jn}G(t, s_{jn})\,u_n(s_{jn}). \tag{7.7}$$

Without repeating the rather lengthy formulations of the statements of these theorems, we merely remark that estimates (6.11)-(6.13) are, in the case given, valid with

$$\varepsilon_n\leqslant\varepsilon_n'\leqslant c\max_{u\in W(\lambda_0),\|u\|=1}\ \max_{t\in D}|\varphi_n(G(t, s)\,u(s))|.$$

Problem (7.7) is equivalent to the eigenvalue problem of linear algebra

$$u_{in}=\lambda\sum_{j=1}^{n}\alpha_{jn}G(s_{in}, s_{jn})u_{jn} \quad (i=1, \ldots, n).$$

7.3. Let us consider the eigenvalue problem

$$A(\lambda)u\equiv u(t)-\int_D G(t, s, \lambda)u(s)\,d\mu(s)=0 \tag{7.8}$$

with a nonlinear occurrence of the parameter λ. An application of the quadrature formula (7.2) leads to the problem

$$A_n(\lambda)u_n\equiv u_n(t)-\sum_{j=1}^{n}\alpha_{jn}G(t, s_{jn}, \lambda)u_n(s_{jn}), \tag{7.9}$$

equivalent to the system of equations

$$u_{in}=\sum_{j=1}^{n}\alpha_{jn}G(s_{in}, s_{jn}, \lambda)u_{jn} \quad (i=1,\ldots, n).$$

We introduce the following assumptions:

$$G(t, s, \lambda) \text{ is holomorphic in } \lambda\in\Lambda\subseteq\mathbf{C};$$

$$G(t, s, \lambda),\ \frac{\partial G(t, s, \lambda)}{\partial\lambda} \text{ is continuous in } (t, s)\in D\times D;$$

$$\max_{t\in D}\int_D\left|\frac{G(t, s, \lambda+\Delta\lambda)-G(t, s, \lambda)}{\Delta\lambda}-\frac{\partial G(t, s, \lambda)}{\partial\lambda}\right|d|\mu|(s)\to 0 \quad \text{as* } \Delta\lambda\to 0;$$

Equation (7.8) has only the zero solution for some $\lambda'\in\Lambda$. Under these assumptions the hypotheses of Theorems 6.1 and 6.2 are fulfilled for problems (7.8) and (7.9). Estimates (6.4) and (6.5) take the form

$$|\lambda_n-\lambda_0|\leqslant c_\delta\varepsilon_n^{1/\varkappa}, \quad d(u_n^0, \mathcal{N}(A(\lambda_0)))\leqslant c\varepsilon_n^{1/\varkappa},$$

$$\varepsilon_n=\max_{u\in W(\lambda_0),\|u\|=1}\ \max_{|\lambda-\lambda_0|=\delta}\ \max_{t\in D}|\varphi_n(G(t, s, \lambda)u(s))|.$$

7.4. We consider the nonlinear integral equation

$$\mathcal{A}u\equiv u(t)-\int_D G(t, s, u(s))\,d\mu(s)=0. \tag{7.10}$$

An application of quadrature formula (7.2) leads to the equation

$$\mathcal{A}_n u_n\equiv u_n(t)-\sum_{j=1}^{n}\alpha_{jn}G(t, s_{jn}, u_n(s_{jn}))=0, \tag{7.11}$$

equivalent to the system of equations

$$u_{in}=\sum_{j=1}^{n}\alpha_{jn}G(s_{in}, s_{jn}, u_{jn}) \quad (i=1,\ldots, n)$$

in $u_{in}=u_n(s_{in})$, $i=1,\ldots, n$. Let the following conditions be fulfilled: Equation (7.10) has the solution $\tilde{u}(t)$; $G(t, s, u)$ is continuous and has a continuous partial derivative with respect to u when $t\in D$, $s\in D$, $|u-\tilde{u}(s)|\leq\delta$; the linear homogeneous equation

$$u(t)=\int_D H(t, s)u(s)\,d\mu(s), \quad H(t, s)=\frac{\partial G(t, s, \tilde{u}(s))}{\partial u},$$

has only the zero solution. Then the hypotheses of Theorem 5.1 are fulfilled for operators $\mathcal{A}$ and $\mathcal{A}_n$, and by this theorem Eq. (7.11) has, for almost all n, a unique solution $\tilde{u}_n$ close to $\tilde{u}$; the estimate

$$c_1\delta_n\leqslant\max_{t\in D}|\tilde{u}_n(t)-\tilde{u}(t)|\leqslant c_2\delta_n, \quad \delta_n=\max_{t\in D}|\varphi_n(G(t, s, \tilde{u}(s))|,$$

is valid.

The hypotheses of Theorem 5.2 are fulfilled if $G(t, s, u)$ is only continuous and $\gamma(I-T; \partial\Omega)\neq 0$, where $T: C(D)\to C(D)$ is the integral operator of Eq. (7.10), $\Omega\subset C(D)$ is some bounded open set.

*Here $|\mu|$ is the total variation of measure μ ($|\mu|=\mu$ if μ is positive).

8. Method of Collocation

8.1. Let ρ be some measurable function on $[a, b]$ such that

$$\int_a^b \rho(t)\,dt < \infty, \quad \int_a^b \frac{dt}{\rho(t)} < \infty.$$

By L^2_ρ we denote the space of functions square-summable on $[a, b]$ with weight ρ,

$$\|v\|_{L^2_\rho} = \left(\int_a^b \rho(t)\,|v(t)|^2 dt\right)^{1/2}.$$

By H^m_ρ we denote the space of functions $u : [a, b] \to \mathbf{K}$ absolutely continuous together with their derivatives up to order $m - 1$ and such that $u^{(m)} \in L^2_\rho$,

$$\|u\|_{H^m_\rho} = \sum_{k=0}^{m-1} \max_{a \leqslant t \leqslant b} |u^{(k)}(t)| + \|u^{(m)}\|_{L^2_\rho}.$$

We do not exclude the case $m = 0$ and take $H^0_\rho = L^2_\rho$. Finally, by $\mathscr{R}$ we denote the space of bounded functions Riemann-integrable on $[a, b]$,

$$\|v\|_{\mathscr{R}} = \sup_{a \leqslant t \leqslant b} |v(t)|.$$

The spaces L^2_ρ, H^m_ρ, $\mathscr{R}$ are Banach spaces. Obviously, the imbedding $\mathscr{R} \subset L^2_\rho$ is continuous.

We consider the problem

$$u^{(m)} + \mathscr{C}u = f, \quad b_j u = \varphi_j \ (j = 1, \ldots, m), \tag{8.1}$$

where $f \in \mathscr{R}$, $\varphi_j \in \mathbf{K}$, $\mathscr{C} \in \mathscr{L}(H^m_\rho, \mathscr{R})$ is some completely continuous operator, b_j are certain linear continuous functionals over H^m_ρ (when $m = 0$ the "boundary" conditions are absent). We seek an approximate solution of problem (8.1) in the form

$$u_n(t) = \sum_{k=0}^{m+n} a_k t^k \quad \left(\text{or } u_n(t) = \sum_{k=0}^{m+n} b_k \pi_k(t)\right), \tag{8.2}$$

where $\pi_k(t)$ are polynomials orthogonal with weight ρ (π_k is a polynomial of degree k):

$$\int_a^b \rho(t)\,\pi_k(t)\,\pi_j(t)\,dt = \delta_{kj} \ (k, j = 0, 1, 2, \ldots).$$

The coefficients a_k (or b_k) are determined from the conditions

$$[u_n^{(m)} + \mathscr{C}u_n - f]_{t=t_i} = 0 \ (i = 0, 1, \ldots, n),$$
$$b_j u_n = \varphi_j \ (j = 1, \ldots, m), \tag{8.3}$$

where $t_i = t_{i,n} \in [a, b]$ ($i = 0, 1, \ldots, n$) are the zeros of polynomial $\pi_{n+1}(t)$.

By P_n we denote the Lagrange projector that associates with any function v its interporation polynomial of degree $\leq n$, such that $(P_n v)(t_i) = v(t_i)$ ($i = 0, 1, \ldots, n$). Conditions (8.2) and (8.3) are equivalent to the problem

$$u_n^{(m)} + P_n \mathscr{C} u_n = P_n f, \quad b_j u_n = \varphi_j \ (j = 1, \ldots, m). \tag{8.4}$$

We set $E_n = E = H^m_\rho$, $F_n = F = L^2_\rho \times \mathbf{K}^m$, $p_n = I_E$, $q_n = I_F$, $A = \left[\frac{d^m}{dt^m} + \mathscr{C}, b_1, \ldots, b_m\right]$, $A_n = \left[\frac{d^m}{dt^m} + P_n\mathscr{C}, b_1, \ldots, b_m\right]$.

For such a choice of spaces the operators A, $A_n \in \mathscr{L}(E, F)$ are Fredholm and their indices equal zero. Problems (8.1) and (8.4) can be written as the equations $Au = v$ and $A_n u_n = v_n$, where $v = (f, \varphi_1, \ldots, \varphi_n) \in F$, $v_n = (P_n f, \varphi_1, \ldots, \varphi_n) \in F$. Let us show that

$$\|A_n - A\|_{\mathscr{L}(E,F)} \to 0 \ (n \in N).$$

Indeed, by the Erdős – Turan theorem (see [115]) $\|v - P_n v\|_{L^2_\rho} \to 0 \ \forall v \in \mathscr{R}$, hence $\sup_{v \in \mathscr{M}} \|v - P_n v\|_{L^2_\rho} \to 0$ for any relatively compact set $\mathscr{M} \subset \mathscr{R}$. The set $\mathscr{M} = \{v \mid v = \mathscr{C}u, u \in H^m_\rho, \|u\|_{H^m_\rho} \leqslant 1\}$ is such a set and, therefore,

$$\|A_n - A\| = \sup_{\|u\|_{H^m_\rho} \leqslant 1} \|P_n \mathscr{C} u - \mathscr{C} u\|_{L^2_\rho} \to 0.$$

Let us assume that the homogeneous problem corresponding to (8.1) has only the zero solution in H_ρ^m. Then condition (a) in Theorem 4.1 is fulfilled (see Remark 3.1). By Theorem 4.2, problem (8.4) is uniquely solvable for almost all n, i.e., conditions (8.2) and (8.3) define the unique approximation $\tilde{u}_n$ to the solution $\tilde{u}$ of problem (8.1). The sequence $(\tilde{u}_n)$ converges in the norm in H_ρ^m to $\tilde{u}$; the estimate

$$c_1 \| \tilde{u}^{(m)} - P_n \tilde{u}^{(m)} \|_{L_\rho^2} \leqslant \| \tilde{u}_n - \tilde{u} \|_{H_\rho^m} \leqslant c_2 \| \tilde{u}^{(m)} - P_n \tilde{u}^{(m)} \|_{L_\rho^2} \tag{8.5}$$

is valid [note that $\tilde{u}^{(m)} \in \mathscr{R}$ by virtue of the conditions imposed on problem (8.1)].

8.2. Just as simply we obtain a convergence theorem for the eigenvalue problem of form

$$u^{(m)} + \mathscr{C}u = \lambda \mathscr{B}u, \quad b_j u - \lambda c_j u = 0 \quad (j = 1, \ldots, m),$$

where $\mathscr{C}$, $\mathscr{B} \in \mathscr{L}(H_\rho^m, \mathscr{R})$ are completely continuous, b_j, $c_j \in (H_\rho^m)^*$. Here the collocation method leads to the problem

$$u_n^{(m)} + P_n \mathscr{C} u_n = \lambda P_n \mathscr{B} u_n, \quad b_j u_n - \lambda c_j u_n = 0 \quad (j = 1, \ldots, m).$$

In the given case estimates (6.11)-(6.13) are valid with

$$\varepsilon_n \leqslant \varepsilon_n' \leqslant c \max_{u \in W(\lambda_0), \|u\|_{H_\rho^m} = 1} \| u^{(m)} - P_n u^{(m)} \|_{L_\rho^2}.$$

8.3. We now assume that the operators $\mathscr{C}: H_\rho^m \to \mathscr{R}$, $b_j: H_\rho^m \to \mathbf{K}$ (j = 1, ..., m) are nonlinear, and, as before, $\mathscr{C}$ is completely continuous and $f \in \mathscr{R}$. Let problem (8.1) have the solution $\tilde{u} \in H_\rho^m$, let the operators mentioned be continuously Fréchet-differentiable at point $\tilde{u}$, and let the homogeneous problem

$$u^{(m)} + \mathscr{C}'(\tilde{u}) u = 0, \quad b_j'(\tilde{u}) u = 0 \quad (j = 1, \ldots, m)$$

have only the zero solution in H_ρ^m. Under the assumptions made the hypotheses of Theorem 5.1 are fulfilled, according to which conditions (8.2) and (8.3) define, for almost all n, the unique approximation $\tilde{u}_n$ in a neighborhood of $\tilde{u}$; the convergence $\| \tilde{u}_n - \tilde{u} \|_{H_\rho^m} \to 0$ holds with estimate (8.5).

When the operators b_j are linear we can apply Theorem 5.2 as well, which permits us to waive the differentiability condition on operator $\mathscr{C}$.

9. A Difference Method for Ordinary Differential Equations

9.1. We consider the numerical differentiation formula

$$u^{(k)}(t) = h^{-k} \sum_{j=-r_k}^{s_k} b_{j,k} u(t + jh) \equiv (D_h^{(k)} u)(t), \tag{9.1}$$

where $b_{j,k} \in \mathbf{R}$, $r + s \geq k$. It is said to be convergent if $(D_h^{(k)}u)(t) \to u^{(k)}(t)$ for any smooth function (say, for $u \in C^\infty$) at any point of smoothness. The simplest convergent numerical differentiation formula is given by the operator $\partial_h^k = (\partial_h)^k$, where $\partial_h = h^{-1}(S_h - I)$ is the simplest first-order difference operator, $(S_h u)(t) = u(t + h)$.

Proposition 9.1. The numerical differentiation formula (9.1) converges if and only if $D_h^{(k)}$ is representable in the form

$$D_h^{(k)} = \left(\sum_{j=-r_k}^{s_k - k} \beta_{j,k} S_h^j \right) \partial_h^k, \quad \beta_{j,k} \in \mathrm{R}, \quad \sum_{j=-r_k}^{s_k - k} \beta_{j,k} = 1.$$

The polynomial $\sum_{j=0}^{r_k + s_k - k} \beta_{j - r_k, k} \zeta^j$ is called characteristic for the (convergent) numerical differentiation formula (9.1). The location of its roots plays an important role in the question on the convergence of the difference methods constructed on the basis of formula (9.1).

9.2. We consider the equation

$$Au \equiv \sum_{k=0}^{m} a_k(t) u^{(k)} = v(t), \tag{9.2}$$

whose coefficients $a_k(t)$ and free term v(t) are continuous and ω-periodic, and $a_m(t) \neq 0$ for all $t \in \mathbf{R}$. We seek an ω-periodic solution of Eq. (9.2). We shall seek the approximate solution in the form of an ω-periodic net function $u_h : \mathbf{R}_h \to \mathbf{K}$, where $\mathbf{R}_h = \{ih \mid i = 0, \pm 1, \pm 2, \ldots\}$, $h = \omega/n$; the ω-periodicity of u_h signifies that $u_h(ih + nh) = u_h(ih)$ $(i = 0, \pm 1, \ldots)$. We replace the derivatives in Eq. (9.2) with the aid of some convergent numerical differentiation formula of form (9.1), except that for the approximation of the highest derivative ($k = m$) we shall require that the roots of the corresponding characteristic polynomial not lie on the unit circle $|\zeta| = 1$ in the complex plane. Thus, the difference approximation of problem (9.2) has the form

$$A_h u_h \equiv \sum_{k=0}^{m} a_k(t) D_h^{(k)} u_h(t) = v(t), \quad t \in \mathbf{R}_h. \tag{9.3}$$

Equation (9.3) actually represents a system of n (intrinsic) equations in the n (intrinsic) coordinates of the ω-periodic net function u_h.

We assume that $E = C^{(m)}$, $F = C$, $E_h = C_h^{(m)}$, $F_h = C_h$ are spaces of ω-periodic functions and net functions with the usual norms

$$\|u\|_m = \|u\|_{C^{(m)}} = \max_{0 \le k \le m} \|u^{(k)}\|_0, \quad \|v\|_0 = \|v\|_C = \max_{0 \le t \le \omega} |v(t)|,$$

$$\|u_h\|_m = \|u_h\|_{C_h^{(m)}} = \max_{0 \le k \le m} \|\partial_h^k u_h\|_0, \quad \|v_h\|_0 = \|v_h\|_{C_h} = \max_{0 \le j \le n} |v_h(jh)|.$$

Further, we define the connecting operators $p_h \in \mathscr{L}(E, E_h)$ and $q_h \in \mathscr{L}(F, F_h)$ as operators of restriction of a function on a net: $(p_h u)(t) = u(t)$, $(q_h v)(t) = v(t)$ for $t \in \mathbf{R}_h$. Although $\mathscr{P} = (p_h)$ and $\mathscr{Q} = (q_h)$ and consist of like operators, the discrete convergences corresponding to them are different because of the difference of the norms in E_h and F_h.

Proposition 9.2. For the operators $A_h \in \mathscr{L}(C_h^{(m)}, C_h)$ and $A \in \mathscr{L}(C^{(m)}, C)$ defined in (9.2) and (9.3) there holds the regular $\mathscr{P}\mathscr{Q}$-convergence $A_h \to A$ $(n \in N)$.

Proof. The $\mathscr{P}\mathscr{Q}$-convergence $A_h \to A$ follows easily from the convergence of numerical differentiation formula (9.1) and from Proposition 9.1. Let us demonstrate the verification of the regularity condition

$$\|u_h\|_m \le 1, \ (A_h u_h) \text{ is } \mathscr{Q}\text{-compact} \Rightarrow (u_h) \text{ is } \mathscr{P} \text{ compact.} \tag{9.4}$$

First of all we note that

$$\|v_h\|_0 \le c_0, \quad \|\partial_h v_h\|_0 \le c_1 \Rightarrow (v_h) \text{ is } \mathscr{Q}\text{- compact}.$$

Indeed, the polygonal lines $v^{(n)} \in C$ $(n \in N)$ with $v^{(n)}(ih) = v_h(ih)$ $(i = 0, \pm 1, \ldots)$ satisfy the conditions

$$|v^{(n)}(t)| \le c_0, \quad |v^{(n)}(t_1) - v^{(n)}(t_2)| \le c_1 |t_1 - t_2| \ (n \in N),$$

and by Arzela's theorem the sequence $(v^{(n)})$ is relatively compact in C, while the sequence (v_h), in view of (1.5) and of the equality $v_h = q_h v^{(n)}$, is $\mathscr{Q}$-compact.

Thus, from the inequalities $\|u_h\|_m \le 1$ $(n \in N)$ it follows that the sequences $(\partial_h^k u_h)$, $k = 0, 1, \ldots, m-1$, are $\mathscr{Q}$-compact, and, together with them, so are the sequences $(D_h^{(k)} u_h)$, $k = 0, 1, \ldots, m-1$, $\left(\sum_{k=0}^{m-1} a_k D_h^{(k)} u_h\right)$. Along with the $\mathscr{Q}$-compactness of $(A_h u_h)$ this implies the $\mathscr{Q}$-compactness of $(D_h^{(m)} u_h)$. Below it is shown that

$$D_h^{(m)} u_h \xrightarrow{\mathscr{Q}} v \ (n \in N' \subseteq N) \Rightarrow \partial_h^{(m)} u_h \xrightarrow{\mathscr{Q}} v \ (n \in N'). \tag{9.5}$$

Hence, $(\partial_h^{(m)} u_h)$ is $\mathscr{Q}$-compact as well. Thus, the sequences $(\partial_h^k u_h)$, $k = 0, 1, \ldots, m$, are $\mathscr{Q}$-compact, which is equivalent to the $\mathscr{P}$-compactness of (u_h) and proves (9.4).

Let us clarify (9.5). By Proposition 9.1,

$$D_h^{(m)} = V_h S_h^{-r_m} \partial_h^m, \quad V_h = \sum_{j=0}^{r_m + s_m - m} \beta_{j - r_m, m} S_h^j,$$

i.e., V_h is a polynomial of the isometric $S_h \in \mathscr{L}(C_h, C_h)$. Since the roots of the (characteristic) polynomial, by hypothesis, do not lie on the circle $|\zeta| = 1$, we have that $V_h^{-1} \in \mathscr{L}(C_h, C_h)$ exists, $\|V_h^{-1}\| \le \text{const}$. Thus, $V_h \xrightarrow{\mathscr{Q}\mathscr{Q}} I$ is stable. By Theorem 4.1, $V_h^{-1} \xrightarrow{\mathscr{Q}\mathscr{Q}} I$, which proves (9.5). Proposition 9.2 has been proved.

If the homogeneous equation $Au = 0$ has only the zero solution in the class of ω-periodic solutions, then condition (a) of Theorem 4.1 is fulfilled for Eqs. (9.2) and (9.3). By Theorem 4.2, Eq. (9.2) has the unique solution $\tilde{u} \in C^{(m)}$, Eq. (9.3) has for almost all n the unique solution $\tilde{u}_h \in C_h^{(m)}$, and $\|\tilde{u}_h - p_h\tilde{u}\|_m \to 0$ with estimate

$$\| \tilde{u}_h - p_h \tilde{u} \|_m \leqslant c \max_{0 \leqslant k \leqslant m} \| D_h^{(k)} p_k \tilde{u} - q_h D^k \tilde{u} \|_0, \tag{9.6}$$

where $D^k = d^k/dt^k$. The rapidity of convergence depends on the quality of the numerical differentiation formula (9.1) used and on the smoothness of solution $\tilde{u}$.

9.3. We obtain an analogous result, in particular, estimate (9.6), on the basis of Theorem 5.1 for the nonlinear equation

$$f(t, u, u', \ldots, u^{(m)}) = 0 \tag{9.7}$$

and for its discretization by a difference method

$$f(t, D_h^{(0)} u_h(t), D_h^{(1)} u_h(t), \ldots, D_h^{(m)} u_h(t)) = 0, \; t \in \mathbf{R}_h. \tag{9.8}$$

For the application of Theorem 5.1 to f we should impose the following conditions: function $f(t, z_0, z_1, \ldots, z_m)$ is ω-periodic in t and Eq. (9.7) has an ω-periodic solution $\tilde{u}$; $f(t, z_0, z_1, \ldots, z_m)$ is continuous and has continuous partial derivatives with respect to $z_0, \ldots, z_m$ for $-\infty < t < \infty$, $|z_k - \tilde{u}^{(k)}(t)| \leq \delta$ $(k = 0, 1, \ldots, m)$; the linear homogeneous equation

$$\sum_{k=0}^{m} a_k(t) u^{(k)} = 0, \quad a_k(t) = \frac{\partial f(t, \tilde{u}(t), \ldots, \tilde{u}^{(m)}(t))}{\partial z_k},$$

has only the zero ω-periodic solution and $a_m(t) \neq 0$ for all $t \in \mathbf{R}$.

9.4. Theorem 5.2 is applicable for equations of form

$$Au + \mathcal{B}u \equiv \sum_{k=0}^{m} a_k(t) u^{(k)} + g(t, u, u', \ldots, u^{(m-1)}) = 0, \tag{9.9}$$

retaining the conditions in Paragraph 9.2 for $Au = \sum_{k=0}^{m} a_k(t) u^{(k)}$ and its discretization and requiring that g be continuous in all its variables and be ω-periodic in t. Let $\Omega' \subset C$ be an open bounded set such that $A\tilde{u} \notin \partial\Omega'$ for the ω-periodic solutions $\tilde{u}$ of problem (9.9) and $\gamma(I + \mathcal{B}A^{-1}; \partial\Omega') \neq 0$. Then Eq. (9.9) has at least one ω-periodic solution $\tilde{u}$ such that $A\tilde{u} \in \Omega'$, while the equation

$$\sum_{k=0}^{m} a_k(t) D_h^{(k)} u_h(t) + g(t, D_h^{(0)} u_h(t), \ldots, D_h^{(m-1)} u_h(t)) = 0, \; t \in \mathbf{R}_h,$$

has for almost all n at least one ω-periodic solution $\tilde{u}_h$ such that the polygonal line constructed from the $A_h\tilde{u}_h$ also belongs to Ω'. The sequence $(\tilde{u}_h)$ is $\mathcal{P}$-compact and its limit points are solutions of problem (9.9); in particular, $\|\tilde{u}_h - p_h\tilde{u}\|_m \to 0$ if Eq. (9.9) has a unique ω-periodic solution $\tilde{u}$ with $A\tilde{u} \in \Omega'$.

Here, besides Theorem 5.2 we have taken advantage further of Remark 5.1, having constructed from $\Omega' \subset C$ the set $\Omega_h' \subset C_h$ by the following rule: $v_h \in \Omega_h'$ if for the polygonal line $v^{(n)} \in C$ with $v^{(n)}(ih) = v_h(ih)$ $(i = 0, +1, \ldots)$ we have $v^{(n)} \in \Omega'$. As is easy to see, the consistency conditions for Ω' and Ω_h' are fulfilled.

10. A Difference Method for Elliptic Equations

10.1. Let $\Omega \subset \mathbf{R}^d$ be a open bounded domain,

$$Au = \sum_{|\alpha|, |\beta| \leqslant m} D^\alpha (a_{\alpha\beta} D^\beta u), \; Bu = \sum_{\substack{|\alpha|, |\beta| \leqslant m \\ |\alpha + \beta| < 2m}} D^\alpha (b_{\alpha\beta} D^\beta u)$$

be differential operators with bounded piecewise-continuous coefficients $a_{\alpha\beta}, b_{\alpha\beta} : \bar{\Omega} \to \mathbf{K}$, where A is uniformly elliptic, i.e., the coefficients of the principal part of A satisfy the inequality

$$(-1)^m \sum_{|\alpha| = |\beta| = m} a_{\alpha\beta}(x) t_\alpha \bar{t}_\beta \geqslant \gamma \sum_{|\alpha| = m} |t_\alpha|^2, \; \gamma > 0, \tag{10.1}$$

for any $x \in \Omega$ and any $t_\alpha \in \mathbf{K}$ $(|\alpha| = m)$. We consider the boundary-value problems

$$Au = v, \; u \in H_0^m(\Omega) \tag{10.2}$$

$[v \in H^{-m}(\Omega)$ is given] and

$$Au = \lambda Bu, \; u \in H_0^m(\Omega). \tag{10.3}$$

Here we have used the notation: $H^m(\Omega)$ is the Sobolev space of functions having square-summable generalized derivatives of order $\leq m$,

$$\|u\|_m = \|u\|_{H^m(\Omega)} = \Big(\sum_{|\alpha|\leq m} \|D^\alpha u\|^2_{L^2(\Omega)}\Big)^{1/2};$$

$H_0^m(\Omega)$ is the closure in $H^m(\Omega)$ of the set of infinitely differentiable functions with supports in Ω; $H^{-m}(\Omega)$ is the space adjoint to $H_0^m(\Omega)$, adjoint with respect to the scalar product in $L_2(\Omega)$,

$$\|v\|_{-m} = \|v\|_{H^{-m}(\Omega)} = \sup_{u\in H_0^m(\Omega),\, \|u\|_m<1} |(u, v)|.$$

It is clear that $A, B\in\mathscr{L}(H_0^m(\Omega), H^{-m}(\Omega))$, and, by virtue of the imbedding theorems for Sobolev spaces, B is completely continuous.

We approximate the differential operators A and B by finite operators of form

$$A_h u_h = \sum_{|\alpha|,|\beta|\leq m} D_h^\alpha (a_{\alpha\beta}\bar{D}_h^\beta u),\quad B_h u_h = \sum_{\substack{|\alpha|,|\beta|\leq m\\ |\alpha+\beta|<2m}} D_h^\alpha (b_{\alpha\beta}\bar{D}_h^\beta u_h),$$

where

$$D_h^\alpha = \partial^{\alpha-\gamma_\alpha}\bar{\partial}^{\gamma_\alpha},\quad \bar{D}_h^\alpha = \bar{\partial}^{\alpha-\gamma_\alpha}\partial^{\gamma_\alpha},\quad 0\leq\gamma_\alpha\leq\alpha,$$

$\partial^\alpha u$ and $\bar{\partial}^\alpha u$ are, respectively, the forward and the backward differences of order α on a net R_h^d of step $h > 0$ with respect to all variables x_k:

$$R_h^d = \{x\in R^d : x = hj,\ j = (j_1, \ldots, j_d),\ j_k = 0, \pm 1, \pm 2, \ldots;\ k = 1, \ldots, d\}.$$

Further, we select a net domain $\Omega_h\subset R_h^d$, for example, in the following manner. We denote $\bar{\Omega}_h = \bar{\Omega}\cap R_h^d$ and we define $\Omega_h\subset\bar{\Omega}_h$ by the condition: $x\in\Omega_h$ if and only if the expression $(A_h u)(x)$ attracts only nodes from $\bar{\Omega}_h$. From problems (10.2) and (10.3) we set up the difference problems

$$(A_h u_h)(x) = v_h(x) \quad \text{for } x\in\Omega_h,\quad u_h(x) = 0 \quad \text{for } x\in R_h^d\setminus\Omega_h \tag{10.4}$$

and

$$(A_h u_h)(x) = \lambda (B_h u_h)(x) \quad \text{for } x\in\Omega_h,\quad u_h(x) = 0 \text{ for } x\in R_h^d\setminus\Omega_h. \tag{10.5}$$

Here $u_h : R_h^d \to K$ is a net function.

We choose the "discrete" spaces $L^2(\Omega_h)$, $H_0^m(\Omega_h)$, $H^{-m}(\Omega_h)$, consisting of net functions $u_h : R_h^d \to K$ with supports in Ω_h (i.e., equal to zero outside Ω_h) and provided with the norms

$$\|u_h\|_0 = \|u_h\|_{L^2(\Omega_h)} = \sqrt{(u_h, u_h)},\quad (u_h, v_h) = h^d \sum_{x\in R_h^d} u(x)\overline{v(x)},$$

$$\|u_h\|_m = \|u_h\|_{H_0^m(\Omega_h)} = \Big(\sum_{|\alpha|\leq m} \|\partial^\alpha u_h\|_0^2\Big)^{1/2},$$

$$\|v_h\|_{-m} = \|v_h\|_{H^{-m}(\Omega_h)} = \sup_{u_h\in H_0^m(\Omega_h),\, \|u_h\|_m=1} |(u_h, v_h)|.$$

For $u\in L^2(\Omega)$ we defined $p_h u\in L^2(\Omega_h)$ by the formula

$$(p_h u)(x) = \begin{cases} 0 & \text{for } x\in R_h^d\setminus\Omega_h,\\ h^{-d}\int_{\omega_h(x)} u(y)\,dy & \text{for } x\in\Omega_h,\end{cases}$$

where

$$\omega_h(x) = \Big\{y\in R^d : |y_k - x_k| \leq \frac{h}{2},\ k = 1, \ldots, d\Big\}$$

and we take it that $u(x) = 0$ for $x\notin\Omega$. It turns out that

$$\|p_h u\|_m \to \|u\|_m \quad \forall u\in H_0^m(\Omega),$$
$$\|p_h u\|_0 \to \|u\|_0 \quad \forall u\in L^2(\Omega),$$
$$\|p_h v\|_{-m} \to \|v\|_{-m} \quad \forall v\in L^2(\Omega).$$

These relations signify that $\mathcal{P}=(p_h)$ is a connecting system for $H_0^m(\Omega)$ and $H_0^m(\Omega_h)$ for $L^2(\Omega)$ and $L^2(\Omega_h)$, and, after a suitable continuation (see Remark 1.1), also for $H^{-m}(\Omega)$ and $H^{-m}(\Omega_h)$. Correspondingly we shall speak of $\mathcal{P}_m$-, $\mathcal{P}_0$- and $\mathcal{P}_{-m}$-convergences: $u_h \xrightarrow{\mathcal{P}_k} u$ if $\|u_k - p_k u\|_k \to 0$ (k = m, 0 or − m); instead of $u_h \xrightarrow{\mathcal{P}_k} u$ we shall write $u_h \xrightarrow{k} u$.

Proposition 10.1. For any $\lambda \in K$ the sequence of operators $A_h - \lambda B_h \in \mathcal{L}(H_0^m(\Omega_h), H^{-m}(\Omega_h))$ $\mathcal{P}_m\mathcal{P}_{-m}$-converges regularly to $A - \lambda B \in \mathcal{L}(H_0^m(\Omega), H^{-m}(\Omega))$.

The proof can be constructed on the basis of Proposition 3.5, having set

$$\hat{A}u = \sum_{|\alpha|=|\beta|=m} D^\alpha (a_{\alpha\beta} D^\beta u) + \gamma \sum_{|\alpha|<m} (-1)^{|\alpha|} D^{2\alpha} u,$$

$$\hat{A}_h u_h = \sum_{|\alpha|=|\beta|=m} D_h^\alpha (a_{\alpha\beta} \bar{D}_h^\beta u_h) + \gamma \sum_{|\alpha|<m} (-1)^{|\alpha|} D_h^\alpha \bar{D}_h^\alpha u_h$$

and shown that $\mathcal{R}(\hat{A}) = H^{-m}(\Omega)$,

$$\hat{A}_h \to \hat{A} \text{ is stable,}$$

$$B_h \to B \text{ is compact,} \quad A_h - \hat{A}_h \to A - \hat{A} \text{ is compact.}$$

The $\mathcal{P}_m\mathcal{P}_{-m}$-convergence of the operators indicated stems from the following remarks: if $u_h \xrightarrow{m} u$ and $|\alpha| \le m$, then $\partial^\alpha u_h \xrightarrow{0} D^\alpha u$; if $v_h \xrightarrow{0} v$ and $a : \bar{\Omega} \to K$ is bounded and piecewise-continuous, then $av_h \xrightarrow{0} av$; if $w_h \xrightarrow{0} w$ and $|\alpha| \le m$, then $\partial^\alpha w_h \xrightarrow{-m} D^\alpha w$. The compactness conditions for B_h and $A_h - \hat{A}_h$ stem from the following remarks that are certain discrete analogs of the imbedding theorems for Sobolev spaces: if $\|u_h\|_m \le$ const. $|\alpha| < m$, then $(D_h^\alpha u_h)$ is $\mathcal{P}_0$-compact; if $\|w_h\|_0 \le$ const, $|\alpha| < m$, then $(D_h^\alpha w_h)$ is $\mathcal{P}_{-m}$-compact. Finally, by an integration, respectively, a summation by parts, on the basis of (10.1), we can establish that

$$(\hat{A}u, u) \geqslant \gamma \| u \|_m^2 \quad \forall u \in H_0^m(\Omega),$$

$$(\hat{A}_h u_h, u_h) \geqslant \gamma \| u_h \|_m^2 \quad \forall u_h \in H_0^m(\Omega_h).$$

Hence

$$\| \hat{A}u \|_{-m} \geqslant \gamma \| u \|_m, \quad \| \hat{A}^* u \|_{-m} \geqslant \gamma \| u \|_m \quad \forall u \in H_0^m(\Omega),$$

$$\| \hat{A}_h u_h \|_{-m} \geqslant \gamma \| u_h \|_m \quad \forall u_h \in H_0^m(\Omega_h),$$

i.e., $\mathcal{R}(\hat{A}) = H^{-m}(\Omega)$ and the stability condition is fulfilled for operators $\hat{A}_h$. Proposition 10.1 has been proved.

Let us assume that equation Au = 0 has only the zero solution in $H_0^m(\Omega)$. Then the hypotheses of Theorem 4.2 are fulfilled for operators A and A_h and for almost all h the difference problem (10.4) has the unique solution $\tilde{u}_h$: if $v_h \xrightarrow{-m} v$, then $\|\tilde{u}_h - p_h\tilde{u}\|_m \to 0$, where $\tilde{u}$ is the solution of problem (10.2); the estimate

$$c_1 \| A_h p_h \tilde{u} - v_h \|_{-m} \leqslant \| \tilde{u}_h - p_h \tilde{u} \|_m \leqslant c_2 \| A_h p_h \tilde{u} - v_h \|_{-m}$$

is valid.

Theorems 6.2 and 6.3 with $\Lambda = \mathbf{C}$ are applicable for problems (10.3) and (10.5). Under certain smoothness conditions on the coefficients and on the boundary of domain Ω the estimates (6.11)-(6.13) are valid with $\varepsilon_n \le ch$, $\varepsilon_n' \le ch^{1/2}$.

11. Supplements and Bibliographic Comments

To Secs. 1-2. The idea of discrete convergence is not new; it was always (sometimes, tacitly) used in the study of the convergence of difference methods (e.g., see [108, 41, 42, 79]). The systematic study of discrete convergence was undertaken by Stummel [178, 183, 192]; also see his other papers. Our presentation, based on the specification of the connecting system $\mathcal{P} = (p_n)$ for operators $p_n : E \to E_n$ with properties (1.1) and (1.2) is equivalent to Stummel's approach in [178]; a number of more general ideas were developed in [192]. We cite the most general one of them. Let X, Y, X_n, Y_n be sets, $A : X \to Y$, $A_n : X_n \to Y_n$, and let there be given certain mappings $\lim^X$ from $\prod_n X_n$ into X and $\lim^Y$ from $\prod_n Y_n$ into Y. The discrete convergence $x_n \to x$ signifies that (x_n) belongs to the domain of mapping $\lim^X$; the discrete convergence $A_n \to A$ is then defined by the usual condition $x_n \to x \Rightarrow A_n x_n \to Ax$.

Another approach is possible: instead of the connecting mappings $p_n : E \to E_n$ we take as given the linear operators $\pi_n : E_n \to E$ with $\mathcal{N}(\pi_n) = \{0\}$ and with the aid of these operators we identify E_n with the subspace

$\pi_n E_n$ of space E. The discrete convergence $u_n \to u$ is now defined by the condition $\|\pi_n u_n - u\| \to 0$. Such an approach goesback to the old, already classic papers [58, 59] of Kantorovich. This approach is particularly natural when considering projection methods, including the method of finite elements; for a difference method such an approach is somewhat artificial.

Some authors take as simultaneously specified both $p_n: E \to E_n$ and $\pi_n: E_n \to E$ ($n \in N$); see [100, 164], for instance.

In some papers (see [43, 79], for example) the spaces E_n are taken to be factor spaces, $E_n = E/E^{(n)}$, while $p_n: E \to E/E^{(n)}$ are taken to be the corresponding canonic mappings. A very similar situation is encountered in papers ([14, 21], for instance) in which the E_n are arbitrary, but on the connecting mappings are imposed the conditions of linearity, boundedness, (1.1), and

$$p_n E = E_n, \inf_{u \in E,\, p_n u = u_n} \|u\|_E \leqslant c \|u_n\|_{E_n} \ \forall u_n \in E_n \ (n \in N).$$

Let us mention the original papers on the material in Secs. 1-2; some of them contain additional material. Regarding: Paragraphs 1.1-1.3 see [178]; Paragraph 1.4 see [27]; Paragraph 1.5 see [30, 196]; Paragraph 1.6: on Proposition 1.2 see [178], Proposition 1.3 see [152], Proposition 1.4 see [196, 198]; Paragraph 1.7 see [178]; Paragraph 1.8 see [30, 196]; Paragraph 1.9 see [27, 196]; Sec. 2 see [178, 179].

To Sec. 3. The concepts of stability, of the stable convergence (stable approximation) of operators are widely known and deeply rooted in computational mathematics. We brought this concept in only for linear operators; other definitions are possible for nonlinear operators (see [168, 183, 188, 192]).

The role of the compact convergence of operators was first noted by Sobolev [118]. He proves a convergence theorem for the linear equations $u = Tu + v$ and $u = T_n u + v$ in the situation when $T, T_n \in \mathscr{L}(E, E)$ are completely continuous, $I - T$ is invertible, $T_n \to T$ is compact,* and also applies this abstract result to the method of mechanical quadratures for solving integral equations. Abroad these results were repeated, deepened, and extended by Anselone and his co-workers [127-131] (collectively compact approximation); in particular, the eigenvalue problem $Tu = \lambda u$, $T_n u = \lambda u$ was studied. Vainikko [14, 21] and Stummel [178] studied compact convergence within the framework of discrete convergence.

The concept of regular convergence of operators, fundamental in the present survey, was first introduced, it seems, by Petryshyn [169] in the special case of projection schemes† and by Karma [65] (y_0-approximation) and Grigorieff [151] (a-regularity) within the framework of discrete convergence; in [32, 33] it is used under the name proper convergence. Grigorieff's approach is formally more general than the one presented in the survey: he examines not necessarily bounded operators in not necessarily complete normed space. However, by bringing in the norm of a graph, the operators turn into bounded ones, while Grigorieff's a-regularity leads to the regularity condition in our sense; we shall return to this question in the comments to Sec. 4.

It is most complicated to prove Propositions 3.1 and 3.2 from the results in Sec. 3. Proposition 3.1 has been proved in [196, 198]; a part of its assertions were established earlier in [151]; also see [203]; Proposition 3.2 in toto was proved in [198], and under certain simplifying assumptions, in [30, 196]. As we saw, Proposition 3.3 is trivial; Propositions 3.4 and 3.5 are almost trivial (see [30, 196] for the proofs). Proposition 3.6 was established in [30, 196, 198].

For $A_n, A \in \mathscr{L}(E, F)$ we can introduce one more concept of convergence by the following conditions: (a) $\|A_n u - Au\| \to 0 \ \forall u \in E$; (b) $(A_n u_n - A u_n)$ is relatively compact in F for any bounded sequence $(u_n) \subset E$ (i.e., $A_n - A \to 0$ is compact). This concept attracted a lot of attention and was studied in [17, 21, 105-107, 148, 40, 60, 84-90, 92-96]. Obviously, from the compact convergence $A_n - A \to 0$ it follows, in the case of a completely continuous A, that $A_n \to A$ is compact, while in the case of a Fredholm A, that $A_n \to A$ is regular. Therefore, the majority of the results in the papers listed follow from the results presented in Secs. 3-6. An exception here is the result on the equality of indices $\operatorname{ind} A_n = \operatorname{ind} A$ ($n \geq n_0$) for a compact convergence $A_n - A \to 0$ and for a Fredholm A [compare with Proposition 3.1, which under the conditions of regular convergence $A_n \to A$ yields only the inequality $\operatorname{ind} A_n \leq \operatorname{ind} A$ ($n \geq n_0$)].

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*Sobolev assumes that $\|T_n u - Tu\| \to 0 \ \forall u \in E$ and that operators T_n are completely continuous in the aggregate, i.e., $\bigcup T_n \Omega$ is relatively compact in E for any bounded subset $\Omega \subset E$. It is clear that these conditions are equivalent to the compact convergence $T_n \to T$.

† It should be noted that in computational mathematics the concept of regular convergence was used implicitly earlier. Thus, e.g., Lebedev [91], in the study of the convergence of difference schemes, essentially proves the regular convergence of difference operators because for the principal parts of the operators he establishes stable convergence while for the subordinate parts, compact convergence (see Proposition 3.5).

To Sec. 4. As we could convince ourselves, the results in Sec. 4 are rather elementary. These questions are discussed in [30, 196, 152, 153]. Below we indicate a simple generalization of Theorem 4.1 to unbounded operators.

Let E, F, E_n, F_n ($n \in N$) be Banach spaces, $\mathscr{P}=(p_n)$ and $\mathscr{Q}=(q_n)$ be the connecting systems for E, E_n and F, F_n, respectively. By $\mathscr{C}(E, F)$ we denote the set of closed operators with domain $\mathscr{D}(A)\subseteq E$ and with a range in F.

Definition. The sequence of operators $A_n \in \mathscr{C}(E_n, F_n)$ $(n\in N)$ is $\mathscr{P}\mathscr{Q}$-consistent with $A\in\mathscr{C}(E, F)$ if for any $u\in\mathscr{D}(A)$ we can find a sequence (u_n), $u_n\in\mathscr{D}(A_n)$, such that $u_n\overset{\mathscr{P}}{\to}u$, $A_nu_n\overset{\mathscr{Q}}{\to}Au$ $(n\in N)$.

We shall take it that A_n and A are $\mathscr{P}\mathscr{Q}$-consistent. On G = D(A) and $G_n=\mathscr{D}(A_n)$ we introduce graph norms:

$$\|u\|_G=\|u\|_E+\|Au\|_F,\quad u\in\mathscr{D}(A),$$
$$\|u_n\|_{G_n}=\|u_n\|_{E_n}+\|A_nu_n\|_{F_n},\quad u_n\in\mathscr{D}(A_n).$$

By the same token, G and G_n are turned into Banach spaces, and $A: G \to F$, $A_n: G_n \to F_n$ into bounded operators. We define the operators $r_n: G \to G_n$ ($n \in N$) in the following way. First, for each $u' \in G$, using the consistency condition, we fix (u_n'), $u_n' \in G_n$, such that $u_n'\overset{\mathscr{P}}{\to}u'$, $A_nu_n'\overset{\mathscr{Q}}{\to}Au'$ $(n\in N)$. Next we set $r_nu' = u_n'$ ($u' \in G$, $n \in N$). It is not difficult to see that $\mathscr{R}=(r_n)$ is the connecting system for G, G_n ($n \in N$), and $u_n\overset{\mathscr{R}}{\to}u$ is equivalent to the requirement $u_n\overset{\mathscr{P}}{\to}u$, $A_nu_n\overset{\mathscr{Q}}{\to}Au$.

From the definition immediately stems the following proposition.

Proposition. If $A_n\in\mathscr{C}(E_n, F_n)$ and $A\in\mathscr{C}(E, F)$ are $\mathscr{P}\mathscr{Q}$-consistent, then $A_n\overset{\mathscr{R}\mathscr{Q}}{\to}A\,(n\in N)$.

It is also easy to observe that the stability condition for the $\mathscr{R}\mathscr{Q}$-convergence $A_n \to A$ is equivalent to the existence for almost all n of the inverses $A_n^{-1}\in\mathscr{L}(F_n, E_n)$, $\|A_n^{-1}\|_{\mathscr{L}(F_n, E_n)}\leq\text{const}\,(n\geq n_0)$, while the regularity condition for the convergence indicated takes the following form: if the sequence (u_n), $u_n\in\mathscr{D}(A_n)$, is such that $\|u_n\|_{E_n} \leq$ const and (A_nu_n) is $\mathscr{Q}$-compact, then this sequence (u_n) itself is $\mathscr{P}$-compact, and from $u_n\overset{\mathscr{P}}{\to}u$, $A_nu_n\overset{\mathscr{Q}}{\to}v(n\in N'\subseteq N)$ it follows that $u\in\mathscr{D}(A)$, $Au=v$. Correspondingly, we shall speak of the stable and of the regular $\mathscr{P}\mathscr{Q}$-consistency of operators A_n and A. From Theorem 4.1 we obtain the following result.

THEOREM. The following conditions are equivalent for operators $A\in\mathscr{C}(E, F)$ and $A_n\in\mathscr{C}(E_n, F_n)$ ($n \in N$):

(a) A_n and A are regularly $\mathscr{P}\mathscr{Q}$-consistent, $\mathscr{N}(A)=\{0\}$, A_n are Fredholm with zero index for almost all n;

(b) A_n and A are stably $\mathscr{P}\mathscr{Q}$-consistent, $\mathscr{R}(A)=F$;

(c) A_n and A are stably and regularly $\mathscr{P}\mathscr{Q}$-consistent;

(d) the inverse $A^{-1}\in\mathscr{L}(F, E)$ exists, the inverses $A_n^{-1}\in\mathscr{L}(F_n, E_n)$ exist for almost all n, and $A_n^{-1}\overset{\mathscr{Q}\mathscr{P}}{\to}A^{-1}$.

An analogous transfer is possible for the remaining results in Secs. 3-6; on this matter see [37]. From the point of view of applications the necessity of these generalizations is questionable because by a suitable choice of spaces we can take care in advance that the operators being studied are bounded.

To Sec. 5. Lemma 5.1 was established in [12] and has been used by the author in a number of papers [12, 17, 19, 21, 30, 195, 196, 32, 77]. In the formulation presented Theorem 5.1 was proved in [32]; an equivalent formulation in the case when E_n and F_n ($n \in N$) are subspaces or factor spaces of E and F, respectively, was already encountered in [17, 21]. Similar considerations exist in [163, 154, 188].

The first results on the convergence of approximate methods, based on the concept of the rotation of vector fields, are due to Krasnosel'skii [75, 76] and concern Galerkin's method. Next, this approach was developed by Vainikko [10, 12, 17, 19, 21, 26, 199] and Bobylev [3]. Theorem 5.2 has been proved in [26, 199].

See [76, 78, 116, 117, 112, 5, 199, 143, 136, 167] for an exposition of the concept of vector field rotation or of the equivalent concept of mapping degree for different classes of operators.

We did not touch upon one more approach for nonlinear equations, based on the theory of monotone operators. For this see [6, 146, 116, 117].

To Sec. 6. The convergence of projection methods in the eigenvalue problem was established by Pol'skii [104]; Troitskaya [122] extended this result to approximate methods, placed within the framework of the general theory of approximate methods of Kantorovich [58, 59]. The rapidity of convergence of the methods mentioned was studied by Vainikko [7, 8, 11, 13, 77]; in particular, results close to Theorems 6.4 and 6.5 were established in [11, 7], except that the estimates of the arithmetic mean $\hat{\lambda}_n$ are from a later period (the idea of such estimates goes back to Bramble and Osborn [134, 135, 166]. Asymptotically exact estimates for self-adjoint problems are given in [8]; cruder estimates occur in [39]. Our presentation of projection methods in Sec. 6 is not a standard one. In this presentation we wished to stress that the theory of projection methods is included in a natural way in the general theory developed on the basis of the regular convergence of operators. The norming of E_n^* used in Paragraph 6.5 and the interesting consequences in discrete convergence were borrowed from Oja [101, 102, 35]. The case of many parameters has been studied in [124].

Within the framework of compact and regular convergence of operators the convergence of approximate methods in the eigenvalue problem was studied by Atkinson, Anselone, Vainikko, Karma, Stummel, Grigorieff, Jeggle, etc. (see [131, 128, 14, 17, 21, 30, 195-197, 33, 61-65, 179, 155-158, 160, 161, 124). Theorems 6.1 and 6.2 with a reduced degree of generality were proved in [33]†; however, the substance of the results were already available in the earlier papers of Karma; convergence without an indication of rapidity of convergence was studied by Grigorieff and Jeggle [158]. Theorem 6.3 in full was established in [30, 197]; however, its analogs on the base of a stable convergence $A_n \to A$ and a compact convergence $B_n \to B$ already existed in [21]; also see [171]. Grigorieff's paper [156] also was devoted to estimates of convergence. In contrast to our exposition, he did not prove the equality $\dim W_n(\lambda_0; \delta) = \dim W(\lambda_0)$ but assumed it; Remark 6.3, also established in [30, 197], clarifies the situation. Within the framework of stable convergence (under the conditions of Remark 6.3) the convergence of approximate methods has been studied in a group of papers [137-142] by Chatelin. Her papers, as well as those of Grigorieff and Jeggle, touch on unbounded closed operators; see [37] and the comments to Sec. 4.

In [57] there is an interesting approach, drawing on the measure of noncompactness, for the eigenvalue problem for operators A_n, $A \in \mathscr{L}(E, E)$.

To Sec. 7. The method of mechanical quadratures served as a point of application of various abstract theories. Linear integral equations were touched on in [58, 59, 118, 129, 127, 128, 1, 99, 14, 24, 21, 30, 196, 36, 41, 159, 180, 162]; of course, the results in these papers overlap to a considerable extent and it is somewhat difficult to name the author of the general formulation given at the end of Paragraph 7.1. The convergence of the method for integral equations with discontinuous kernels has been studied in [159] and in [24, 21] from different viewpoints; the case of unbounded kernels was considered in [36] and the case of an unbounded measure, in [162].

The eigenvalue problem was studied in [131, 132, 31, 14, 21, 30, 64]; estimates of convergence were established in [31, 14, 21, 64] and were repeated in [132]. The convergence of the method for nonlinear integral equations was studied in [2, 12, 77, 24, 21, 30, 196, 4, 189].

To Sec. 8. The first results on the convergence of the collocation method for boundary-value problems were obtained by Karpilovskaya [67] (her results are also derived in [59]); next, this method, for various problems (integrodifferential equations, equations with lag, more general boundary-value problems, etc.), was studied in a number of papers, in particular, see [103, 68-71, 74, 9, 10, 20, 22, 23, 72, 77]. The exposition in Sec. 8 follows the approach suggested in [9], but is carried out in a somewhat more general situation. In this exposition we wanted to emphasize that in the convergence question a secondary role is played by the structure of the equation (integral equation when $m = 0$, differential or integrodifferential equation, etc., depending on the actual form of operator $\mathscr{C}$). Analogous results are valid for the method of moments (see [51, 15, 77]; an interesting approach using expansions in Chebyshev polynomials for a system of first-order differential equations was worked out in [194]; also see [145, 200]).

In the case of periodic differential equations it is natural to seek the approximate solution not as an algebraic but a trigonometric polynomial. Here in the case of equally spaced interpolation nodes results completely analogous to those in Sec. 8 are valid. Certain results in this direction exist in [114]; see [193] on the Galerkin method for the same problem. An autonomous case, when the period of the desired solution also is subject to determination, was studied in [34].

† The estimate for $|\lambda_n - \lambda_0|$ in [33] was given in form (6.6); however, estimate (6.4) can be substantiated by insignificant modifications of the arguments.

Collocation methods based on spline approximations have become widely prevalent in recent times. The investigation of the convergence of spline-collocation methods can be found in [165, 172-174, 176, 175, 170, 133, 201, 177, 147].

To Sec. 9. See Samarskii's monograph [113] for an extensive bibliography on difference methods. In Sec. 9 we reproduced the results in [28, 195]; a detailed exposition exists also in [30]. In [18, 21, 64] similar results were established for boundary-value problems with arbitrary bounded conditions (also see [16, 25]), but the presentation is unsuccessful in view of the complications arising during the reduction to operator equations of form $x = Tx + f$ and $x_n = T_n x_n + f_n$ (a compact convergence $T_n \to T$ can be proved for them). A simplified presentation on the base of regular convergence is available in [66, 30]. Similar results for general boundary-value problems are in [82, 83, 125, 126, 45-49]; in them the stability condition is established directly. Also see [149, 150]. The convergence of a difference method for equations with discontinuous coefficients on the basis of compact and regular convergence was studied in [54-56]; in [109-111] the same ideology is used to investigate a difference method for equations with a deviating argument.

To Sec. 10. See [113, 52] for an extensive literature on the difference method for elliptic equations. In Sec. 10 we reproduced certain results of Tamme [121] on the eigenvalue problem. Results concerning inhomogeneous equations are very well known in the main (see, e.g., [133, 52]); the Dirichlet problem too has been studied for nonlinear equations (e.g., see [44, 52, 97, 119]), and here, in particular, the theory of monotone operators showed itself to be a successful tool. In [38] an analysis analogous to Sec. 10 but in stronger forms was carried out for elliptic equations with periodic coefficients.

The theory of difference schemes for elliptic equations still strongly lags behind the general theory of differential equations. An ideal situation would be when to each existence theorem (to the theorem on the isomorphism between the given problems and the ones desired) there would correspond a convergence theorem for the difference schemes in appropriate discrete norms, or a description of the class of such schemes. We are very far from such an ideal. In particular, very little is known on the convergence in stronger norms of the difference scheme described in Sec. 10 (see [52, 120]).

Recently projection-difference methods (the method of finite elements) have gained wide prevalence. These are projection methods in which the coordinate functions used have "small" supports. The question on the convergence of such methods is usually resolved within the framework of projection methods, and the central point of the investigation turns on the theory of approximation of various classes of functions by special functions with small supports. There is an extensive bibliography on this subject, but we restrict reference to the papers of Mikhlin [98], Varga [39], and Aubin [100].

LITERATURE CITED

1. P. M. Anselone, "Theory of approximation of collectively compact operators and its application to transfer equations," in: Variation-Difference Methods in Mathematical Physics [in Russian], Novosibirsk (1976), pp. 173-181.
2. A. N. Baluev, "On the approximate solution of nonlinear integral equations," Uch. Zap. Leningr. Gos. Univ., No. 271, 28-31 (1958).
3. N. A. Bobylev, "On the theory of factor-methods for the approximate solution of nonlinear problems," Dokl. Akad. Nauk SSSR, 199, No. 1, 9-12 (1971).
4. N. A. Bobylev, "Method of mechanical quadratures in the problem of periodic solutions," Usp. Mat. Nauk, 27, No. 4, 203-204 (1972).
5. Yu. G. Borisovich, V. G. Zvyagin, and Yu. I. Sapronov, "Nonlinear Fredholm mappings and the Leray – Schauder theory," Usp. Mat. Nauk, 32, No. 4, 3-54 (1977).
6. M. M. Vainberg, The Variation Method and the Method of Monotone Operators in the Theory of Nonlinear Equations [in Russian], Nauka, Moscow (1972).
7. G. M. Vainikko, "Asymptotic estimates of errors in projection methods in the eigenvalue problem," Zh. Vychisl. Mat. Mat. Fiz., 4, No. 3, 405-425 (1964).
8. G. M. Vainikko, "Error estimates for the Bubnov – Galerkin method in the eigenvalue problem," Zh. Vychisl. Mat. Mat. Fiz., 5, No. 4, 587-607 (1965).
9. G. M. Vainikko, "On the convergence and stability of the collocation method," Differents. Uravn., 1, No. 2, 244-254 (1965).
10. G. M. Vainikko, "On the convergence of the collocation method for nonlinear differential equations," Zh. Vychisl. Mat. Mat. Fiz., 6, No. 1, 35-42 (1966).

11. G. M. Vainikko, "On the rapidity of convergence of certain Galerkin-type approximate methods in the eigenvalue problem," Izv. Vyssh. Uchebn. Zaved., Mat., No. 2, 37-45 (1966).
12. G. M. Vainikko, "The perturbed Galerkin method and the general theory of approximate methods for nonlinear equations," Zh. Vychisl. Mat. Mat. Fiz., 7, No. 4, 723-751 (1967).
13. G. M. Vainikko, "On the rapidity of convergence of approximate methods in the eigenvalue problem," Zh. Vychisl. Mat. Mat. Fiz., 7, No. 5, 977-987 (1967).
14. G. M. Vainikko, "Compact approximation of linear completely continuous operators by operators in factor spaces," Tartu Ülikooli Toimetised, No. 220, 190-204 (1968).
15. G. M. Vainikko, "On the rapidity of convergence of the method of moments for ordinary differential equations," Sib. Mat. Zh., 9, No. 1, 21-28 (1968).
16. G. M. Vainikko, "On the connection between the methods of mechanical quadratures and of finite differences," Zh. Vychisl. Mat. Mat. Fiz., 9, No. 2, 259-270 (1969).
17. G. M. Vainikko, "Compact approximation principle in the theory of approximate methods," Zh. Vychisl. Mat. Mat. Fiz., 9, No. 4, 739-761 (1969).
18. G. M. Vainikko, "On a difference method for ordinary differential equations," Zh. Vychisl. Mat. Mat. Fiz., 9, No. 5, 1057-1074 (1969).
19. G. M. Vainikko, "Compact approximation of operators and approximate solution of operator equations," Dokl. Akad. Nauk SSSR, 189, No. 2, 237-240 (1969).
20. G. M. Vainikko, "On the approximation of linear and nonlinear operator equations," Doctoral Dissertation, Voronezh (1969).
21. G. M. Vainikko, Compact Approximation of Operators and Approximate Solution of Equations [in Russian], Tartusk. Univ., Tartu (1970).
22. G. M. Vainikko, "On the convergence of the collocation method for multidimensional integral equations," Tartu Ülikooli Toimetised, No. 253, 244-257 (1970).
23. G. M. Vainikko, "On the stability of the collocation method," Tartu Riikliku Ülikooli Toimetised, No. 281, 190-196 (1971).
24. G. M. Vainikko, "On the convergence of the method of mechanical quadratures for integral equations with discontinuous kernels," Sib. Mat. Zh., 12, No. 1, 40-53 (1971).
25. G. M. Vainikko, "On the convergence of quadrature-difference methods for linear integrodifferential equations," Zh. Vychisl. Mat. Mat. Fiz., 11, No. 3, 770-776 (1971).
26. G. M. Vainikko, "On the approximation of the fixed points of completely continuous operators," Tartu Ülikooli Toimetised, No. 342, 225-236 (1974).
27. G. M. Vainikko, "Discrete-compact sequences," Zh. Vychisl. Mat. Mat. Fiz., 14, No. 3, 572-583 (1974).
28. G. M. Vainikko, "On the convergence of a difference method in the problem on periodic solutions of ordinary differential equations," Zh. Vychisl. Mat. Mat. Fiz., 15, No. 1, 87-100 (1975).
29. G. M. Vainikko, "Certain problems connected with discrete convergence of operators," Tr. Nauchn.-Issled. Inst. Mat. Voronezh. Univ., No. 19, 21-26 (1975).
30. G. M. Vainikko, Analysis of Discretization Methods. Special Course [in Russian], Tartusk. Univ., Tartu (1976).
31. G. M. Vainikko and A. M. Dement'eva, "On the rapidity of convergence of the method of mechanical quadratures in the eigenvalue problem," Zh. Vychisl. Mat. Mat. Fiz., 8, No. 5, 1105-1110 (1968).
32. G. M. Vainikko and O. O. Karma, "On the convergence of approximate methods of solving linear and nonlinear operator equations," Zh. Vychisl. Mat. Mat. Fiz., 14, No. 4, 828-837 (1974).
33. G. M. Vainikko and O. O. Karma, "On the rapidity of convergence of approximate methods in the eigenvalue problem with a nonlinear occurrence of the parameter," Zh. Vychisl. Mat. Mat. Fiz., 14, No. 6, 1393-1408 (1974).
34. G. M. Vainikko and P. Miidla, "On the convergence of approximate methods of seeking self-oscillations," Abh. Akad. Wess. DDR, Abt. Math. Naturwiss. Techn., No. 4, 347-353 (1977).
35. G. M. Vainikko and P. P. Oja, "On the convergence and the rapidity of convergence of the Galerkin method for abstract evolution equations," Differents. Uravn., 11, No. 7, 1269-1277 (1975).
36. G. M. Vainikko and A. Pedas, "On the solution of integral equations with a logarithmic singularity by the method of mechanical quadratures," Tartu Riikliku Ülikooli Toimetised, No. 281, 201-210 (1971).
37. G. M. Vainikko and S. I. Piskarev, "On regularly consistent operators," Izv. Vyssh. Uchebn. Zaved., Mat., No. 10, 25-36 (1977).
38. G. M. Vainikko and É. É. Tamme, "Convergence of a difference method in the problem on periodic solutions of elliptic equations," Zh. Vychisl. Mat. Mat. Fiz., 16, No. 3, 652-664 (1976).

39. R. S. Varga, Functional Analysis and Approximation Theory in Numerical Analysis, S.I.A.M., Philadelphia (1971).
40. Yu. N. Vladimirskii, "Remarks on compact approximation in Banach spaces," Sib. Mat. Zh., 15, No. 1, 200-204 (1974).
41. N. K. Gavurin, Lectures of Computation Methods [in Russian], Nauka, Moscow (1972).
42. S. K. Godunov and V. S. Ryaben'kii, Difference Schemes. Introduction to the Theory [in Russian], Nauka, Moscow (1977).
43. N. N. Gudovich, "On an abstract scheme of a difference method," Zh. Vychisl. Mat. Mat. Fiz., 6, No. 5, 916-921 (1966).
44. N. N. Gudovich, "On the application of a difference method to solving nonlinear elliptic equations," Dokl. Akad. Nauk SSSR, 179, No. 6, 1257-1260 (1968).
45. N. N. Gudovich, "On the construction of stable difference schemes of any preassigned order of approximation for linear ordinary differential equations," Dokl. Akad. Nauk SSSR, 217, No. 2, 264-267 (1974).
46. N. N. Gudovich, "On a new method of constructing stable difference schemes of any preassigned order of approximation for linear ordinary differential equations," Zh. Vychisl. Mat. Mat. Fiz., 15, No. 4, 931-945 (1975).
47. N. N. Gudovich, "On the stability of difference schemes for linear ordinary differential equations," Dokl. Akad. Nauk SSSR, 224, No. 4, 748-751 (1975).
48. N. N. Gudovich, "Stability of difference schemes for ordinary differential equations. I," in: Differential Equations and Their Applications [in Russian], No. 11, Vilnius (1975), pp. 9-33.
49. N. N. Gudovich, "Stability of difference schemes for ordinary differential equations. II," in: Differential Equations and Their Applications [in Russian], No. 12, Vilnius (1975), pp. 9-35.
50. N. Dunford and J. T. Schwartz, Linear Operators, Pt. I: General Theory, Wiley, New York (1958).
51. I. K. Daugavet, "On the method of moments for ordinary differential equations," Sib. Mat. Zh., 6, No. 1, 70-85 (1965).
52. E. G. D'yakonov, Difference Methods for Solving Boundary-Value Problems. No. 1 (Stationary Problems) [in Russian], Moscow Univ., Moscow (1971).
53. K. Yoshida, Functional Analysis (5th ed.), Springer-Verlag, Berlin – Heidelberg – New York (1978).
54. H. Jokk, "On the convergence of difference methods for second-order ordinary differential equations," ENSV Tead. Akad. Toimetised, Füüs., Mat., 22, No. 1, 31-36 (1973).
55. H. Jokk, "On the convergence of difference methods for second-order ordinary differential equations on a nonuniform net," ENSV Tead. Akad. Toimetised, Füüs., Mat., 22, No. 3, 227-232 (1973).
56. H. Jokk, "Results of an investigation of the convergence of difference methods for second-order nonlinear differential equations on a nonuniform net," ENSV Tead. Akad. Toimetised, Füüs., Mat., 23, No. 1, 86-88 (1974).
57. M. Kamenskii, "Measures of noncompactness and perturbation theory of linear operators," Tartu Ülikooli Toimetised, No. 430, 112-122 (1977).
58. L. V. Kantorovich, "Functional analysis and applied mathematics," Usp. Mat. Nauk, 3, No. 6, 89-187 (1948).
59. L. V. Kantorovich and G. P. Akilov, Functional Analysis in Normed Spaces [in Russian], Fizmatgiz, Moscow (1959).
60. I. V. Karklin'sh and V. S. Levchenkov, "Invariance of the index of linear homomorphisms in Banach spaces under a sequentially compact approximation," Latv. Mat. Ezhegodnik, 17, 3-23 (1976).
61. O. O. Karma, "On a compact approximation of operator-valued functions," Tartu Ülikooli Toimetised, No. 277, 194-204 (1971).
62. O. O. Karma, "Asymptotic error estimates of approximate eigenvalues of holomorphic Fredholm operator-valued functions," Zh. Vychisl. Mat. Mat. Fiz., 11, No. 3, 559-568 (1971).
63. O. O. Karma, "On the approximation of operator-valued functions and the convergence of approximate eigenvalues," Tr. Vychisl. Tsentra, Tartus. Univ., No. 24, 3-143 (1971).
64. O. O. Karma, "On the convergence of discretized methods of seeking eigenvalues of integral and differential operators holomorphically dependent on a parameter," Tr. Vychisl. Tsentra, Tartus. Univ., No. 24, 144-159 (1971).
65. O. O. Karma, "On the approximation of operator-valued functions and the convergence of approximate eigenvalues," Master's Thesis, Tartu (1971).
66. O. O. Karma, "On the convergence of a difference method in nonlinear eigenvalue problems for linear differential equations," Tartu Ülikooli Toimetised, No. 374, 211-228 (1975).

67. É. B. Karpilovskaya, "On the convergence of an interpolation method for ordinary differential equations," Usp. Mat. Nauk, 8, No. 3, 111-118 (1953).
68. É. B. Karpilovskaya, "On the convergence of the collocation method," Dokl. Akad. Nauk SSSR, 151, No. 4, 766-769 (1963).
69. É. B. Karpilovskaya, "On the convergence of the collocation method for certain boundary-value problems of mathematical physics," Sib. Mat. Zh., 4, No. 3, 632-640 (1963).
70. É. B. Karpilovskaya, "On the collocation method for integrodifferential equations with a biharmonic principle part," Zh. Vychisl. Mat. Mat. Fiz., 10, No. 6, 1537-1541 (1970).
71. É. B. Karpilovskaya, "On the convergence of the collocation method in the eigenvalue problem," Diferents. Uravn., 11, No. 12, 2249-2260 (1975).
72. M. F. Kaspshitskaya and N. I. Tukalevskaya, "On the convergence of the collocation method," Ukr. Mat. Zh., 19, No. 4, 48-56 (1967).
73. T. Kato, Perturbation Theory of Linear Operators, Springer-Verlag, Berlin – Heidelberg – New York (1966).
74. O. Kis, "On the convergence of the coincidence method," Acta Math. Acad. Sci. Hung., 17, No. 3-4, 433-442 (1966).
75. M. A. Krasnosel'skii, "Some problems in nonlinear analysis," Usp. Mat. Nauk, 9, No. 3, 57-114 (1954).
76. M. A. Krasnosel'skii, Topological Methods in the Theory of Nonlinear Integral Equations [in Russian], Gostekhizdat, Moscow (1956).
77. M. A. Krasnosel'skii, G. M. Vainikko, P. P. Zabreiko, Ya. B. Rutitskii, and V. Ya. Stetsenko, Approximate Solution of Operator Equations [in Russian], Nauka, Moscow (1969).
78. M. A. Krasnosel'skii and P. P. Zabreiko, Geometric Methods of Nonlinear Analysis [in Russian], Nauka, Moscow (1975).
79. S. G. Krein, Linear Differential Equations in Banach Space [in Russian], Nauka, Moscow (1967).
80. S. G. Krein and V. P. Trofimov, "On Fredholm operators holomorphically dependent on a parameter," in: Collection of Articles on Functional Spaces and Operator Equations [in Russian], Voronezh (1970), pp. 63-85.
81. S. G. Krein and V. P. Trofimov, "On the multiplicity of a characteristic point of a holomorphic operator-valued function," in: Mathematical Investigations [in Russian], Vol. 5, No. 4, Akad. Nauk Mold. SSR, Kishinev (1970), pp. 105-114.
82. S. G. Krein and L. N. Shablitskaya, "On the stability of difference schemes for a Cauchy problem," Zh. Vychisl. Mat. Mat. Fiz., 6, No. 4, 648-664 (1966).
83. S. G. Krein and L. N. Shablitskaya, "Necessary stability conditions for difference schemes and the eigenvalues of difference operators," Zh. Vychisl. Mat. Mat. Fiz., 13, No. 3, 647-657 (1973).
84. V. I. Labeev, "On certain properties of the compact approximation of closed linear mappings," Uch. Zap. Latv. Univ., 222, 92-108 (1975).
85. V. I. Labeev, "On certain properties of the c-convergence of closed linear mappings," Uch. Zap. Latv. Univ., 222, 109-127 (1975).
86. V. I. Labeev, "On certain properties of the sequentially compact approximation of linear mappings in normed spaces," Uch. Zap. Latv. Univ., 236, No. 1, 39-58 (1975).
87. V. I. Labeev, "Stability of properties of the spectrum of linear mappings under a sequentially compact approximation in Banach spaces," Uch. Zap. Latv. Univ., 236, No. 1, 59-75 (1975).
88. V. I. Labeev, "On the stability of sequential compactness of mappings in topological vector spaces under sequentially precompact approximation," Uch. Zap. Latv. Univ., 236, No. 1, 76-90 (1975).
89. V. I. Labeev, "On the comparison of certain convergences of linear mappings," Uch. Zap. Latv. Univ., 257, No. 2, 26-31 (1976).
90. V. I. Labeev, "On an estimate of the norms of linear perturbations under a sequentially compact approximation," Uch. Zap. Latv. Univ., 257, No. 2, 32-36 (1976).
91. V. I. Lebedev, "Difference analogs of orthogonal expansions of fundamental differential operators and of certain boundary-value problems of mathematical physics," Zh. Vychisl. Mat. Mat. Fiz., 4, No. 4, 649-659 (1964).
92. V. S. Levchenkov, "Preservation of the index of closed linear mappings with a closed range in Banach space under a sequentially compact approximation," Uch. Zap. Latv. Univ., 236, No. 1, 91-96 (1975).
93. V. S. Levchenkov, "On the connection of sequentially compact approximation with convergence in norm for sequences of linear continuous mappings in normed spaces," Uch. Zap. Latv. Univ., 236, No. 1, 103-107 (1975).

94. V. S. Levchenkov, "On one generalization of the concept of sequentially compact approximation of linear mappings," Uch. Zap. Latv. Univ., 257, 37-45 (1976).
95. A. Kh. Luepin'sh, "On sequentially compact approximation," Uch. Zap. Latv. Univ., 236, No. 1, 28-38 (1975).
96. A. Kh. Luepin'sh, "New topologies in which the limit of a sequence of compact mappings is a compact mapping," Uch. Zap. Latv. Univ., 257, 46-51 (1976).
97. A. D. Lyashko, "Difference schemes for quasilinear elliptic equations of any order," Izv. Vyssh. Uchebn. Zaved., Mat., No. 9, 46-53 (1973).
98. S. G. Mikhlin, "Variation-net approximation," Zap. Nauch. Sem. Leningr. Otd. Mat. Inst. Akad. Nauk SSSR, 48, 32-188 (1974).
99. I. P. Mysovskikh, "On the convergence of the method of mechanical cubatures for solving integral equations," in: Computation Methods [in Russian], No. 4, Leningrad Univ. (1967), pp. 63-72.
100. J.-P. Aubin, Approximation of Elliptic Boundary-Value Problems, Wiley, New York (1972).
101. P. P. Oja, "On the convergence and stability of Galerkin's method for parabolic equations with differentiable operators," Tartu Ülikooli Toimetised, No. 374, 194-210 (1975).
102. P. P. Oja, "On the stability of Galerkin's method for evolution equations," ENSV Tead. Akad. Toimetised. Füüs., Mat., 25, No. 3, 219-226 (1976).
103. I. Petersen, "On the convergence of approximate interpolation methods for ordinary differential equations," ENSV Tead. Akad. Toimetised, Füüs.-Mat. ja Tehn. Teaduste Seer., 10, No. 1, 3-12 (1961).
104. N. I. Pol'skii, "On the convergence of certain approximate methods of analysis," Ukr. Mat. Zh., 7, No. 1, 56-70 (1955).
105. L. S. Rakovshchik, "On compact approximation of normally solvable operators," Zh. Vychisl. Mat. Mat. Fiz., 11, No. 5, 1312-1318 (1971).
106. L. S. Rakovshchik, "Stability of index and semistability of defect numbers under compact approximation," Sib. Mat. Zh., 13, No. 3, 630-637 (1972).
107. L. S. Rakovshchik, "Stability of certain properties of normally solvable linear operators under compact approximation" (Editorial Board "Sib. Math. J. Acad. Sci. USSR"), Novosibirsk (1973) (Manuscript deposited in VINITI on 29 August 1973 as No. 6710-73).
108. R. D. Richtmyer and K. W. Morton, Difference Methods for Initial-Value Problems (2nd ed.), Wiley, New York (1967).
109. I. R. Saarniit, "On an estimate of the convergence of the finite-difference method in the case of differential equations with a deviating argument," Tartu Ülikooli Toimetised, No. 277, 205-216 (1971).
110. I. R. Saarniit, "Construction and convergence of h^2-schemes for a linear boundary-value problem for a second-order differential equation with deviating argument," Zh. Vychisl. Mat. Mat. Fiz., 12, No. 1, 105-111 (1972).
111. I. R. Saarnitt, "On the finite-different method solution of a boundary-value problem for a differential equation with deviating argument," Zh. Vychisl. Mat. Mat. Fiz., 16, No. 2, 372-384 (1976).
112. B. N. Sadovskii, "Limit compact and contracting operators," Usp. Mat. Nauk, 27, No. 1, 81-146 (1972).
113. A. A. Samarskii, Introduction to the Theory of Difference Schemes [in Russian], Nauka, Moscow (1971).
114. A. M. Samoilenko and N. I. Ronto, Numerical-Analytical Methods for Investigating Periodic Solutions [in Russian], Vishcha Shkola, Kiev (1976).
115. G. Szegö, Orthogonal Polynomials (4th ed.), Amer. Math. Soc., Providence, Rhode Island (1975).
116. I. V. Skrypnik, High-Order Nonlinear Elliptic Equations [in Russian], Naukova Dumka, Kiev (1973).
117. I. V. Skrypnik, "Solvability and properties of solutions of nonlinear elliptic equations," in: Contemporary Problems in Mathematics [in Russian] (Itogi Nauki i Tekhniki, VINITI Akad. Nauk SSSR), Moscow (1976), pp. 131-254.
118. S. L. Sobolev, "Some remarks on the numerical solution of integral equations," Izv. Akad. Nauk SSSR, Ser. Mat., 20, No. 4, 413-436 (1956).
119. P. E. Sobolevskii and M. F. Tiunchik, "On a difference method for the approximate solution of quasilinear elliptic and parabolic equations," Tr. Mat. Fak. Voronezh. Univ., No. 2, 82-106 (1970).
120. P. E. Sobolevskii and M. F. Tiunchik, "On a difference method for the approximation solution of elliptic equations," Tr. Mat. Fak. Voronezh. Univ., No. 4, 117-127 (1970).
121. É. É. Tamme, "On the regular convergence of difference approximations of the Dirichlet problem," ENSV Tead. Akad. Toimetised, Füüs., Mat., 26, No. 1, 3-8 (1977).
122. E. A. Troitskaya, "On eigenvalues and eigenvectors of completely continuous operators," Izv. Vyssh. Uchebn. Zaved., Mat., No. 3, 148-156 (1961).

123. V. P. Trofimov, "On root subspaces of operators analytically dependent on a parameter," Mat. Issled., 3, No. 3, 117-125 (1968).
124. V. P. Trofimov, "On the characteristic points of Fredholm operator-valued functions," Tr. Mat. Fak. Voronezh. Univ., No. 6, 94-97 (1972).
125. L. N. Shablitskaya, "On the application of a finite-difference method to the solution of linear boundary-problems," Proc. Seminar Functional Analysis [in Russian], No. 9, Voronezh. Univ., Voronezh (1967), pp. 178-183.
126. L. N. Shablitskaya, "Stability of difference schemes for nonlinear ordinary differential equations," Tr. Mat. Fak. Voronezh. Univ., No. 3, 61-67 (1972).
127. P. M. Anselone, "Collectively compact approximations of integral operators with discontinuous kernels," J. Math. Anal. Appl., 22, No. 3, 582-590 (1968).
128. P. M. Anselone, Collectively Compact Approximation Theory, Prentice-Hall, Englewood Cliffs, New Jersey (1971).
129. P. M. Anselone and R. H. Moore, "Approximate solutions of integral and operator equations," J. Math. Anal. Appl., 9, No. 2, 268-277 (1964).
130. P. M. Anselone and T. W. Palmer, "Spectral analysis of collectively compact, strongly convergent operator sequences," Pac. J. Math., 25, No. 3, 423-431 (1968).
131. K. E. Atkinson, "The numerical solution of the eigenvalue problem for compact integral operators," Trans. Am. Math. Soc., 129, No. 3, 458-465 (1967).
132. K. E. Atkinson, "Convergence rates for approximate eigenvalues of compact integral operators," SIAM J. Numer. Anal., 12, No. 2, 213-222 (1975).
133. C. de Boor and B. Swartz, "Collocation at Gaussian points," SIAM J. Numer. Anal., 10, No. 4, 582-606 (1973).
134. J. H. Bramble, "On the approximation of eigenvalues of non-self-adjoint operators," Proc. EQUADIFF III, Third Czech. Conf. Diff. Eqts. and Appl., Brno, 1972, J. E. Purkyně Univ., Brno (1973), pp. 15-19.
135. J. H. Bramble and J. E. Osborn, "Rate of convergence estimates for non-self-adjoint eigenvalue approximations," Math. Comput., 27, No. 123, 525-549 (1973).
136. F. E. Browder and W. V. Petryshyn, "The topological degree and Galerkin approximations for noncompact operators in Banach spaces," Bull. Am. Math. Soc., 74, No. 4, 641-646 (1968).
137. F. Chatelin, "Convergence of approximation methods to compute eigenelements of linear operations," SIAM J. Numer. Anal., 10, No. 5, 939-948 (1973).
138. F. Chatelin, "Approximation de spectre d'un opérateur linéaire; application aux opérateurs différentiels elliptiques non autoadjoints," Numer. Math., 20, No. 3, 193-204 (1973).
139. F. Chatelin, "La méthode de Galerkin. Ordre de convergence des éléments propres," C. R. Acad. Paris, A278, No. 18, 1213-1215 (1974).
140. F. Chatelin-Laborde, "Méthodes d'approximation des valeurs propres d'opérateurs linéaires dans un espace de Banach. Critère de stabilité," C. R. Acad. Sci. Paris, 271, No. 19, A949-A952 (1970).
141. F. Chatelin-Laborde, "Étude de la stabilité de méthodes d'approximation des éléments propres d'opérateurs linéaires," C. R. Acad. Sci. Paris, 272, No. 10, A673-A675 (1971).
142. F. Chatelin-Laborde, "Étude de la continuité du spectre d'un opérateur linéaire," C. R. Acad. Sci. Paris, 274, No. 4, A328-A331 (1972).
143. K. Deimling, Nichtlineare Gleichungen und Abbildungsgrade, Springer-Verlag, Berlin – Heidelberg – New York (1974).
144. J. Diestel, Geometry of Spaces – Selected Topics, Lecture Notes Math., Vol. 485, Springer-Verlag, Berlin – Heidelberg – New York (1975).
145. M. Fujii, "Numerical solution of boundary-value problems with nonlinear boundary conditions in Chebyshev series," Bull. Fukuoka Univ. Educ. Nat. Sci., 25, 27-45 (1975).
146. H. Gajewski, K. Gröger, and K. Zacharias, Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen, Akad. Verlag, Berlin (1974).
147. J. Gladwell and D. J. Mullings, "On the effect of boundary conditions in collocation by polynomial splines for the solution of boundary-value problems in ordinary differential equations," J. Inst. Math. Appl., 16, No. 1, 93-107 (1975).
148. S. Goldberg, "Perturbations of semi-Fredholm operators by operators converging to zero compactly," Proc. Am. Math. Soc., 45, No. 1, 93-98 (1974).
149. R. D. Grigorieff, "Die Konvergenz des Rand- und Eigenwert-Problems linearer gewöhnlicher Differenzengleichungen," Numer. Math., 15, No. 1, 15-48 (1970).

150. R. D. Grigorieff, "Über die Koerzitivität gewöhnlicher Differenzenoperatoren und die Konvergenz von Mehrschrittverfahren," Numer. Math., 15, No. 3, 196-218 (1970).
151. R. D. Grigorieff, "Über die Fredholm-Alternative bei linearen approximationsregulären Operatoren," Appl. Anal., 2, No. 3, 217-227 (1972).
152. R. D. Grigorieff, "Zur Theorie linearer approximationsregulärer Operatoren. I," Math. Nachr., 55, No. 1-6, 233-249 (1973).
153. R. D. Grigorieff, "Zur Theorie linearer approximationsregulärer Operatoren. II," Math. Nachr., 55, No. 1-6, 251-263 (1973).
154. R. D. Grigorieff, "Über diskrete Approximationen nichtlinearer Gleichungen 1. Art," Math. Nachr., 69, 253-272 (1975).
155. R. D. Grigorieff, "Diskrete Approximation von Eigenwertproblemen. I. Qualitative Konvergenz," Numer. Math., 24, No. 4, 355-374 (1975).
156. R. D. Grigorieff, "Diskrete Approximation von Eigenwertproblemen. II. Konvergenzordnung," Numer. Math., 24, No. 5, 415-433 (1975).
157. R. D. Grigorieff, "Diskrete Approximation von Eigenwertproblemen. III. Asymptotische Entwicklungen," Numer. Math., 25, No. 1, 79-97 (1975).
158. R. D. Grigorieff and H. Jeggle, "Approximation von Eigenwertproblemen bei nichtlinearer Parameterabhängigkeit," Manuscr. Math., 10, No. 3, 245-271 (1973).
159. R. L. James, "Uniform convergence of positive operators," Math. Z., 120, No. 2, 124-142 (1971).
160. H. Jeggle, "Über die näherungsweise Lösung der Eigenwertaufgabe für kompakte Operatoren," Z. Angew. Math. Mech., 50, Sonderh. 1-4, 54-55 (1970).
161. H. Jeggle, "Über die Approximation von linearen Gleichungen zweiter Art und Eigenwertproblemen in Banach-Raumen," Math. Z., , 124, No. 4, 319-342 (1972).
162. H. Jeggle, "Uniformly convergent approximations for integral equations on noncompact manifolds," Appl. Anal., 5, No. 3, 227-248 (1976).
163. H. B. Keller, "Approximation methods for nonlinear problems with application to two-point boundary-value problems," Math. Comput., 29, No. 130, 464-474 (1975).
164. P. Linz, "A general theory for the approximate solution of operator equations of the second kind," SIAM J. Numer. Anal., 14, No. 3, 543-554 (1977).
165. T. R. Lucas and W. J. Reddien, "Some collocation methods for nonlinear boundary-value problems," SIAM J. Numer. Anal., 9, No. 2, 341-356 (1972).
166. J. E. Osborn, "Spectral approximation for compact operators," Math. Comput., 29, No. 131, 712-725 (1975).
167. W. Petry, "Existence theorems for a class of nonlinear operator equations," J. Math. Anal. Appl., 43, No. 1, 250-260 (1973).
168. W. V. Petryshyn, "Projection methods in nonlinear numerical functional analysis," J. Math. Mech., 17, No. 4, 353-372 (1967).
169. W. V. Petryshyn, "On projectional-solvability and the Fredholm alternative for equations involving linear A-proper operators," Arch. Rat. Mech. Anal., 30, No. 4, 270-284 (1968).
170. J. L. Phillips, "The use of collocation as a projection method for solving linear operator equations," SIAM J. Numer. Anal., 9, No. 1, 14-28 (1972).
171. R. Rannacher, "Zur asymptotischen Störungstheorie für Eigenwertaufgaben mit diskreten Teilspektren," Math. Z., 141, No. 3, 219-233 (1975).
172. G. W. Reddien, "Approximation methods for two-point boundary-value problems with nonlinear boundary conditions," SIAM J. Numer. Anal., 13, No. 3, 405-411 (1976).
173. G. W. Reddien, "Some projection methods for the eigenvalue problem," Appl. Anal., 6, No. 1, 61-73 (1976).
174. G. W. Reddien and L. L. Schumaker, "On a collocation method for singular two-point boundary-value problems," Numer. Math., 25, No. 4, 427-432 (1976).
175. R. D. Russell, "Collocation for systems of boundary-value problems," Numer. Math., 23, No. 2, 119-133 (1974).
176. R. D. Russell and L. F. Shampine, "A collocation method for boundary-value problems," Numer. Math., 19, No. 1, 1-28 (1972).
177. M. Sakai, "Cubic spline function and difference method," Mem. Fac. Sci. Kyushu Univ., A28, No. 1, 43-58 (1974).
178. F. Stummel, "Diskrete Konvergenz linearer Operatoren. I," Math. Ann., 190, No. 1, 45-92 (1970).
179. F. Stummel, "Diskrete Konvergenz linearer Operatoren. II," Math. Z., 120, No. 3, 231-264 (1971).

180. F. Stummel, "Diskrete Konvergenz linearer Operatoren. III," Proc. Oberwolfach Conf. Linear Operators and Approximations, Birkhauser, Basel (1972).
181. F. Stummel, "Singular perturbations of elliptic sesquilinear forms," in: Lecture Notes Math., Vol. 280, Springer-Verlag, Berlin – Heidelberg – New York (1972), pp. 155-180.
182. F. Stummel, Approximation Methods in Analysis, Lecture Notes Ser. Math. Inst. Aarhus Univ., Vol. 35, Aarhus (1973).
183. F. Stummel, "Discrete convergence of mappings," in: Topics in Numerical Analysis, Academic Press, London – New York (1973), pp. 285-310.
184. F. Stummel, "Difference methods for linear initial-value problems," in: Lecture Notes Math., Vol. 395, Springer-Verlag, Berlin – Heidelberg – New York (1974), pp. 123-135.
185. F. Stummel, "Discretely uniform approximation of continuous functions," J. Approx. Theory, 13, No. 2, 178-191 (1975).
186. F. Stummel, "Perturbation theory for Sobolev spaces," Proc. R. Soc. Edinburgh, A73, 5-49 (1975).
187. F. Stummel, "Biconvergence, bistability and consistency of one-step methods for the numerical solution of initial-value problems in ordinary differential equations," in: Topics in Numerical Analysis 2, Academic Press, London – New York (1975), pp. 197-211.
188. F. Stummel, "Stability and discrete convergence of differentiable mappings," Rev. Roum. Math. Pures Appl., 21, No. 1, 63-96 (1976).
189. F. Stummel, "Perturbations of nonlinear integral operators," Proc. R. Soc. Edinburgh, A74, 55-70 (1976).
190. F. Stummel, "Perturbation of domains in elliptic boundary-value problems," in: Lecture Notes Math., Vol. 503, Springer-Verlag, Berlin – Heidelberg – New York (1976), pp. 110-136.
191. F. Stummel, "Weak stability and weak discrete convergence of continuous mappings," Numer. Math., 26, No. 3, 301-315 (1976).
192. F. Stummel and J. Reinhardt, "Discrete convergence of continuous mappings in metric spaces," in: Lecture Notes Math., Vol. 333, Springer-Verlag, Berlin – Heidelberg – New York (1973), pp. 218-242.
193. M. Urabe, "Galerkin's procedure for nonlinear periodic systems," Arch. Rat. Mech. Anal., 20, No. 2, 120-152 (1965).
194. M. Urabe, "Numerical solution of multi-point boundary-value problems in Chebyshev series. Theory of the method," Publs. Res. Inst. Math. Sci., B, No. 9, 341-366 (1967).
195. G. Vainikko, "Konvergenzuntersuchungen der Näherungsmethoden für lineare und nichtlineare Operatorgleichungen und Eigenwertprobleme mit Anwendungen zum Differenzenverfahren," Wiss. Schriftenr. Techn. Hochsh. Karl-Marx-Stadt, No. 3, 501-531 (1975).
196. G. Vainikko, Funktionalanalysis der Diskretisierungsmethoden, Teubner, Verlagsges., Leipzig (1976).
197. G. Vainikko, "Über die Konvergenz und Divergenz von Näherungsmethoden bei Eigenwertproblemen," Math. Nachr., 78, 145-164 (1977).
198. G. Vainikko, "Über Konvergenzbegriffe für lineare Operatoren in der Numerischen Mathematik," Math. Nachr., 78, 165-183 (1977).
199. G. Vainikko, "Über die Invarianz der Rotation bei Approximation der Vektorfelder," Abh. Akad. Wiss. DDR, Abt. Math. Naturwiss. Techn., No. 1, 265-271 (1977).
200. H. Voss, "Projektionsverfahren fur Randwertaufgaben mit nichtlinearen Randbedingungen," Numer. Math., 24, No. 4, 317-329 (1975).
201. R. Winther, "A collocation method for eigenvalue problems," BIT (Sver.), 14, No. 1, 96-105 (1974).
202. K. A. Wittenbrink, "High-order projection methods of moment and collocation type for nonlinear boundary-value problems," Computing, 11, No. 3, 255-274 (1973).
203. R. Wolf, "Über lineare approximationsregulare Operatoren," Math. Nachr., 59, No. 1-6, 325-341 (1974).