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Approximate Vacuums in $(:\phi^{2m}:g(x))_2$ Field Theories and Perturbation Series*

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Abstract. A class of perturbation problems is considered, in which the Rayleigh-Schrödinger perturbation series for the ground state eigenvalue and eigenvector are presumed to diverge. This class includes the $(:\phi^{2m}:g(x))_2$, $(m=2,3)$ quantum field theory models and the quantum mechanical anharmonic oscillator. It is shown that, using matrix elements and vectors which occur in the series coefficients, one may construct convergent approximants to the eigenvalue and eigenvector. Results of a calculation of the ground state energy of the x^4 anharmonic oscillator are given.

I. Introduction

We consider a class of perturbation problems in which the Rayleigh-Schrödinger perturbation series for the ground state eigenvalue and eigenvector of a perturbed operator $H_0+\lambda V$ are presumed to diverge. This class, which is defined in Section II, includes the $(:\phi^{2m}:g(x))_2$, $(m=2,3)$ relativistic quantum field theory models in two-dimensional space-time and the quantum mechanical anharmonic oscillator. In these examples the $R-S$ series are known to diverge [1, 2].

We show that, using matrix elements and vectors which occur in the series coefficients, one may construct convergent approximants to the eigenvalue and eigenvector. Previously, Loeffel *et al.* [3, 2] have shown that the diagonal Padé approximants to the $R-S$ series for the eigenvalues of the x^4 anharmonic oscillator converge and give upper and lower bounds for the eigenvalue $E(\lambda)$, and Simon [4] has shown that the series for the ground state energy of $(:\phi^4:g(x))_2$ is Borel summable to $E(\lambda)$. These results depend on the analytic and asymptotic properties of $E(\lambda)$ for complex λ and do not directly give any information about the eigenvector. Also, the method of Borel summability involves an analytic continuation and hence does not enable one to construct rigorously convergent approximants to $E(\lambda)$. The scheme described in Section III involves solving an approximate eigenvalue problem and gives a monotonically decreasing sequence of upper bounds for $E(\lambda)$ when λ is real

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and positive. One may, of course, construct such approximants to the ground state eigenvalue and eigenvector of an operator H by simply choosing an arbitrary sequence of vectors which spans the Hilbert space and minimizing the form $\langle\psi|H\psi\rangle/\|\psi\|^2$ in the subspace spanned by the first n vectors. We show in Section IV that it is sufficient to choose instead a sequence of vectors which occur in the $R-S$ series for the eigenvector and which may not necessarily span the entire Hilbert space.

In Section V we give the results of a calculation of the ground state energy of the x^4 anharmonic oscillator.

II. Definition of the Class of Problems Considered

The class of problems considered here is defined by the properties $(A-F)$, listed below, of the perturbed Hamiltonian $H_0+\lambda V$ $(\lambda>0)$.

In the $(:\phi^{2m}:g(x))_2$ model $H_0=\int a^+(k)\,a(k)\;(k^2+m_0^2)^{1/2}\,dk$ is the Hamiltonian of a relativistic scalar quantum field in two-dimensional space-time and $V=\int:\phi^{2m}(x):g(x)\,dx$, $(m=2,3)$ is a space cut-off polynomial interaction, where $g(x)$ is a smooth function equal to 1 for small $|x|$ and 0 for large $|x|$. A large amount of information about this model has been derived by Glimm and Jaffe [5] and others. All the properties listed below may be found in [6].

In the anharmonic oscillator $H_0=1/2\left(-\dfrac{d^2}{dx^2}+x^2-1\right)$ and $V=(x^4-3/4)$ or $(x^6-15/8)$. The properties listed below are derived in Simon's article [2]:

A) H_0 is a self-adjoint operator in a Hilbert space $\mathscr{I}$ with ground state eigenvalue (non-degenerate isolated least eigenvalue) 0 and eigenvector Ω. V is a symmetric operator and $\langle\Omega|V\Omega\rangle=0$. $H_0+\lambda V$ is self-adjoint with domain $\mathscr{D}(H_0)\cap\mathscr{D}(V)$ and has a ground state eigenvalue $E(\lambda)$ with eigenvector Ψ_λ:

$$\begin{aligned} H_0\Omega&=0\\ (H_0+\lambda V-E(\lambda))\,\Psi_\lambda&=0\,. \end{aligned} \tag{1}$$

$E(\lambda)$ is a continuous function of λ. The eigenvectors are assumed to be normalized so that $\|\Omega\|=1$, $\Psi_\lambda=\Omega+\Psi_\lambda^\perp$ and $\langle\Omega|\Psi_\lambda^\perp\rangle=0$.

Let $P^\perp$ be the orthogonal projection onto $\mathscr{I}^\perp=\mathscr{I}\ominus\Omega$. If A is an operator in $\mathscr{I}$, let $A^\perp$ be the operator in $\mathscr{I}^\perp$ defined by $A^\perp=P^\perp AP^\perp$. Since 0 is an isolated eigenvalue of H_0, $H_0^\perp$ is a strictly positive self-adjoint operator and therefore has a square root (a self-adjoint operator with a domain containing $\mathscr{D}(H_0^\perp)$) and a bounded inverse and inverse square root (both with domain $\mathscr{I}^\perp$). We will denote these operators by $h^{1/2}$, h^{-1} and $h^{-1/2}$ respectively.

B) $(H_0+\lambda V)^\perp$ is a self-adjoint operator in $\mathscr{I}^\perp$ (this follows because $H_0+\lambda V-P^\perp(H_0+\lambda V)\,P^\perp$ is a bounded operator in $\mathscr{I}$ [7]).

C) A self-adjoint operator N, the "number operator" may be defined in $\mathscr{I}$. N has eigenvalues $0, 1, \ldots$ (not necessarily non-degenerate) and its eigenvectors ψ_n are called "n-particle vectors". A finite sum $\sum_0^M \psi_n$ is called a "finite-particle vector". In the case of the anharmonic oscillator, $N = H_0$ and the n-particle vectors are the harmonic oscillator eigenstates. $\mathscr{D}(V)$ contains all finite-particle vectors.

D) For any finite-particle vector $\chi \in \mathscr{I}^\perp$ there is a constant d such that:

$$\|(h^{-1/2} \lambda V h^{-1/2})^n \chi\| < (dn)^{2n}, \quad n = 1, 2, \ldots .$$

This may be derived from the bilinear form estimates $V^2 < \text{const}((N+1)^3)^2$, $N^2 < \text{const}\, H_0^2$ (which imply $(h^{-1/2} V h^{-1/2})^2 < \text{const}(N^2)^2$) and the fact that V connects only vectors whose particle numbers differ by no more than six.

E) $H_0 + \lambda V$ is essentially self-adjoint on the domain of finite-particle vectors in $\mathscr{D}(H_0)$. As in (B) it follows that $(H_0 + \lambda V)^\perp$ is essentially self-adjoint on the domain $\mathscr{D}$ of finite-particle vectors in $\mathscr{D}(H_0) \cap \mathscr{I}^\perp$. The set of vectors $h^{1/2} \mathscr{D}$ is dense in $\mathscr{I}^\perp$.

F) Let $\Omega_\lambda = \Psi_\lambda / \|\Psi_\lambda\|$. Then $\langle \Omega | \Omega_\lambda \rangle > 0$. For the anharmonic oscillator this is a consequence of the fact that the ground state wave-functions Ω and Ω_λ have no nodes and hence are strictly positive functions. In the "Q-space" representation of the $(:\phi^{2m}: g(x))_2$ models $\Omega = 1$ and Ω_λ is a strictly positive function (almost everywhere).

III. Construction of Approximants to Eigenvalues and Eigenvectors

Applying $P^\perp$ and $\langle \Omega |$ to (1) we get a nonlinear system of two equations in $E(\lambda)$ and $\Psi_\lambda^\perp$:

$$0 = \lambda V \Omega + ((H_0 + \lambda V)^\perp - E(\lambda)) \Psi_\lambda^\perp$$
$$E(\lambda) \|\Omega\|^2 = \langle \lambda V \Omega | \Psi_\lambda^\perp \rangle .$$

(The first equation is a vector equation in $\mathscr{I}^\perp$; $V\Omega \in \mathscr{I}^\perp$ because $\langle \Omega | V \Omega \rangle = 0$.)

Eliminating $\Psi_\lambda^\perp$ from these equations we get:

$$E(\lambda) \|\Omega\|^2 = f(E(\lambda), \lambda) \tag{2}$$

where:

$$f(E, \lambda) = -\langle \lambda V \Omega | ((H_0 + \lambda V)^\perp - E)^{-1} \lambda V \Omega \rangle .$$

By (B) and Theorem I below, $f(E, \lambda)$ exists in an interval $-\infty < E < G(\lambda)$ with $G(\lambda) > E(\lambda)$. Since $f(E, \lambda)$ is clearly a negative, monotonically decreasing function of E, the eigenvalue $E(\lambda)$ must be the unique solution of (2) in this interval.

Theorem I. *The lower bound $G(\lambda)$ of $(H_0+\lambda V)^{\perp}$ is greater than $E(\lambda)$.*

Proof. Suppose that the lower bound of $(H_0+\lambda V)^{\perp}$ were equal to $E(\lambda)$. Let $\{\phi_n\}$ be a sequence of vectors in $\mathscr{D}$ such that:

$$\langle\phi_n|(H_0+\lambda V)\,\phi_n\rangle/\|\phi_n\|^2 \xrightarrow[n\to\infty]{} E(\lambda)\,. \tag{3}$$

We may take these vectors to be normalized so that $\phi_n = c_n\Omega_\lambda + \psi_n$ with $c_n>0$, $c_n^2+\|\psi_n\|^2=1$ and $\langle\Omega_\lambda|\psi_n\rangle=0$.

Then:

$$\begin{aligned}\langle\phi_n|(H_0+\lambda V)\phi_n\rangle/\|\phi_n\|^2 &= E(\lambda)\,c_n^2+\langle\psi_n|(H_0+\lambda V)\psi_n\rangle\\ &\geqq E(\lambda)\,c_n^2+(E(\lambda)+m)\,\|\psi_n\|^2 = E(\lambda)+m\,\|\psi_n\|^2\end{aligned}$$

where m is the "isolation distance" of the eigenvalue $E(\lambda)$. To satisfy (3) we must have $\|\psi_n\| \xrightarrow[n\to\infty]{} 0$ which implies $\phi_n \xrightarrow[n\to\infty]{} \Omega_\lambda$ and

$$0=\langle\Omega|\phi_n\rangle \xrightarrow[n\to\infty]{} \langle\Omega|\Omega_\lambda\rangle\,.$$

But since $\langle\Omega|\Omega_\lambda\rangle>0$ this is a contradiction.

Theorem II. *Let $A(E,\lambda)=h^{-1/2}(\lambda V-E)^{\perp}h^{-1/2}$. Then there is a $C(\lambda)>E(\lambda)$ such that if $E<C(\lambda)$ then:*

(i) *$A+1$ is a strictly positive operator, and*

(ii) *A is essentially self-adjoint on any dense domain of finite-particle vectors in $\mathscr{I}^{\perp}$.*

Proof. Since $G(\lambda)>E(\lambda)$ and $E(\lambda)$ is continuous we may find an $\varepsilon>0$ so small that $G\left(\frac{\lambda}{1-\varepsilon}\right)>\left(\frac{1}{1-\varepsilon}\right)E(\lambda)$ and a $C(\lambda)>E(\lambda)$ such that:

$$G\left(\frac{\lambda}{1-\varepsilon}\right)>\left(\frac{1}{1-\varepsilon}\right)C(\lambda)>\left(\frac{1}{1-\varepsilon}\right)E(\lambda)\,.$$

Thus if $E<C(\lambda)$ we have the bilinear form relation on $\mathscr{D}\otimes\mathscr{D}$:

$$\left[H_0+\frac{\lambda}{1-\varepsilon}V\right]^{\perp} > G\left(\frac{\lambda}{1-\varepsilon}\right)>\left(\frac{1}{1-\varepsilon}\right)E$$

or

$$[H_0+\lambda V]^{\perp}-E>\varepsilon H_0^{\perp}$$

which is equivalent to the relation $1+A>\varepsilon$ on $(h^{1/2}\mathscr{D}\otimes h^{1/2}\mathscr{D})$. Using property (D) and Definition I below we see that any finite-particle vector is a Stieltjes vector for A. Thus, by (i) and Theorem III below, A is essentially self-adjoint on any dense domain of finite-particle vectors in $\mathscr{I}^{\perp}$.

Definition I. χ is called a Stieltjes vector for an operator B if $\chi\varepsilon\bigcap_{n=1}^{\infty}\mathscr{D}(B^n)$ and:

$$\sum_{n=0}^{\infty}\|B^n\chi\|^{-1/2n}=\infty\,.$$

Theorem III. *Let B be a symmetric semi-bounded operator in a Hilbert space $\mathscr{H}$. If B has a set of Stieltjes vectors $\mathscr{S}$ which is dense in $\mathscr{H}$, then B is essentially self-adjoint.*

Proof. See [8, 9].

From (C) and (E) it is not hard to see that the operators $h^{1/2}$, $(1+A)h^{1/2}$ and $h^{1/2}(1+A)h^{1/2}$ are defined on all vectors in $\mathscr{D}$, so we may write

$$([H_0+\lambda V]^{\perp}-E)\chi = h^{1/2}(1+A)h^{1/2}\chi\,, \quad \chi\in\mathscr{D}\,.$$

By (E), $\mathscr{S}=h^{1/2}\mathscr{D}$ is a dense set of finite-particle vectors. By (D), Definition I and Theorem III, if $E<C(\lambda)$ then A is essentially self-adjoint on $\mathscr{S}$. Hence, if $E<C(\lambda)$ then $(1+A)\mathscr{S}$ is dense in $\mathscr{I}^{\perp}$ and by (E) $([H_0+\lambda V]^{\perp}-E)\mathscr{D}$ is dense in $\mathscr{I}^{\perp}$. This allows us to write:

$$([H_0+\lambda V]^{\perp}-E)^{-1} = h^{-1/2}(1+A)^{-1}h^{-1/2}$$

and

$$f(E,\lambda) = -\langle h^{-1/2}\lambda V\Omega|(1+A)^{-1}h^{-1/2}\lambda V\Omega\rangle\,.$$

We approximate $f(E,\lambda)$ by the Method of Moments which is briefly described in the next paragraph.

Let B be a self-adjoint operator bounded below by M and let $\phi\in\bigcap_{n=0}^{\infty}\mathscr{D}(B^n)$. Let $\mathscr{H}_n$ be the space spanned by the vectors $\{B^k\phi|k=0,\ldots,n-1\}$, let E_n be the orthogonal projection onto $\mathscr{H}_n$, and let $B_n=E_nBE_n$. Clearly B_n is also bounded below by M. The Method of Moments approximant to $(B-\mu)^{-1}$ is defined as the inverse of $B_n-\mu$ considered as an $n\times n$ matrix operator in $\mathscr{H}_n$. If $P_n(\mu)=\det(\mu I-B_n)$ is the characteristic polynomial of B_n (so that $P_n(B_n)=0$) then:

$$\frac{1}{B_n-\mu} = -\frac{1}{P_n(\mu)}\,\frac{P_n(B_n)-P_n(\mu)}{B_n-\mu}\,.$$

Note that this is a polynomial of degree $n-1$ in B_n and that:

$$\left\langle\phi\left|\frac{1}{B_n-\mu}\phi\right.\right\rangle = -\frac{1}{P_n(\mu)}\left\langle\phi\left|\frac{P_n(B_n)-P_n(\mu)}{B_n-\mu}\phi\right.\right\rangle = -\frac{Q_n(\mu)}{P_n(\mu)}$$

is a rational function with numerator $Q_n(\mu)$ of degree $n-1$.

A study of the properties of the polynomials P_n, Q_n [10, Chapter I] shows that if $\mu<M$ then $\langle\phi|(B_n-\mu)^{-1}\phi\rangle$ is positive and increases monotonically as $n\to\infty$. In addition it can be shown [9] that if $\mu<M$ and ϕ is a Stieltjes vector for B then $(B_n-\mu)^{-1}\phi\xrightarrow[n\to\infty]{}(B-\mu)^{-1}\phi$.

Since $A > -1+\varepsilon$ if $E < C(\lambda)$, we may take $B = A$, $\mu = -1$ and $\phi = h^{-1/2}\lambda V\Omega \in \mathscr{J}^{\perp}$. If we define $f_n(E,\lambda) = -\langle\phi|(1+A_n)^{-1}\phi\rangle$ it follows that, for $E < C(\lambda)$, $f_n(E,\lambda)$ is negative and decreases monotonically to $f(E,\lambda)$ as $n\to\infty$. Thus, if $E_n(\lambda)$ is the least solution of the equation $E\|\Omega\|^2 = f_n(E,\lambda)$ then $E_n(\lambda)$ must decrease monotonically to $E(\lambda)$ as $n\to\infty$. Furthermore, since $h^{-1/2}$ is bounded,

$$h^{-1/2}(1+A_n)^{-1}\phi \xrightarrow[n\to\infty]{} h^{-1/2}(1+A)^{-1}\phi = ([H_0+\lambda V]^{\perp} - E)^{-1}\lambda V\Omega$$

and if $\Psi_n = \Omega - h^{-1/2}[1+A_n(E_n(\lambda),\lambda)]^{-1}h^{-1/2}\lambda V\Omega$ then it is not hard to show that $\langle\Psi_n|[H_0+\lambda V - E_n(\lambda)]\,\Psi_n\rangle = 0$ so that $\langle\Psi_n|(H_0+\lambda V)\,\Psi_n\rangle / \|\Psi_n\|^2 = E_n(\lambda) \xrightarrow[n\to\infty]{} E(\lambda)$. By an argument similar to that used in the proof of Theorem I it follows that $\Psi_n \xrightarrow[n\to\infty]{} \Psi_{\lambda}$.

IV. Relationship to the Rayleigh-Schrödinger Series

By definition the polynomial $(1+A_n)^{-1} = \sum_{k=0}^{n-1} C_k A_n^k$ satisfies the equations:

$$\left\langle A_n^j\phi \middle| \left\{1-(1+A_n)\sum_{k=0}^{n-1} C_k A_n^k\right\}\phi\right\rangle = 0\,, \quad j = 0,\ldots,n-1\,.$$

Since $\langle\phi|A_n^{\alpha}\phi\rangle = \langle\phi|A^{\alpha}\phi\rangle$ for $\alpha = 0,\ldots,2n-1$ this is equivalent to the system:

$$\sum_{k=0}^{n-1} C_k\langle A^j\phi|(1+A)\,A^k\phi\rangle = \langle\phi|A^j\phi\rangle\,, \quad j = 0,\ldots,n-1\,.$$

If one solves for the coefficients $\{C_k\}$ in terms of determinants by Cramer's rule one may derive the explicit formula:

$$\left\langle\phi\middle|\frac{1}{A_n+1}\phi\right\rangle = -\frac{\begin{vmatrix} 0 & \langle\phi|\phi\rangle & \cdots & \langle\phi|A^{n-1}\phi\rangle \\ \langle\phi|\phi\rangle & \langle\phi|(A+1)\,\phi\rangle & \cdots & \langle\phi|(A+1)\,A^{n-1}\phi\rangle \\ \langle A\phi|\phi\rangle & \langle A\phi|(A+1)\,\phi\rangle & \cdots & \langle A\phi|(A+1)\,A^{n-1}\phi\rangle \\ \vdots & \vdots & & \vdots \\ \langle A^{n-1}\phi|\phi\rangle & \langle A^{n-1}\phi|(A+1)\,\phi\rangle & \cdots & \langle A^{n-1}\phi|(A+1)\,A^{n-1}\phi\rangle \end{vmatrix}}{\begin{vmatrix} \langle\phi|(A+1)\,\phi\rangle & \cdots & \langle\phi|(A+1)\,A^{n-1}\phi\rangle \\ \vdots & & \vdots \\ \langle A^{n-1}\phi|(A+1)\,\phi\rangle & \cdots & \langle A^{n-1}\phi|(A+1)\,A^{n-1}\phi\rangle \end{vmatrix}}$$

From this and Eq. (2) one finds that $E_n(\lambda)$ is the least zero in E of $\Delta_n(E, \lambda)$, a polynomial in E and λ:

$$\Delta_n(E,\lambda)=\begin{vmatrix} -E\|\Omega\|^2 & \langle\phi|\phi\rangle & \cdots & \langle\phi|A^{n-1}\phi\rangle \\ \langle\phi|\phi\rangle & \langle\phi|(A+1)\phi\rangle & \cdots & \langle\phi|(A+1)A^{n-1}\phi\rangle \\ \vdots & \vdots & & \vdots \\ \langle A^{n-1}\phi|\phi\rangle & \langle A^{n-1}\phi|(A+1)\phi\rangle & \cdots & \langle A^{n-1}\phi|(A+1)A^{n-1}\phi\rangle \end{vmatrix}$$

The following formal considerations show that our approximate to $E(\lambda)$ and Ψ_λ require roughly the same information that is needed to calculate the corresponding $R-S$ series coefficients.

We introduce the notation:

$$[\alpha_1\alpha_2\ldots\alpha_n]=\langle\Omega|\lambda V h^{-\alpha_1}\lambda V h^{-\alpha_2}\lambda V\ldots\lambda V h^{-\alpha_n}\lambda V\Omega\rangle$$
$$|\alpha_1\ldots\alpha_n)=h^{-\alpha_1}\lambda V\ldots\lambda V h^{-\alpha_n}\lambda V\Omega$$

so that, for example:

$$h^{-1/2}A^2\phi=E^2|3)-E|12)-E|21)+|111)$$
$$\langle\phi|A^3\phi\rangle=E^3[4]-E^2([22]+2[13])+E([121]+2[112])-[1111]\,.$$

Note that the expression for $\langle\phi|A^n\phi\rangle$ contains all matrix elements $[\alpha_1\ldots\alpha_k]$ such that $\alpha_1+\cdots+\alpha_k=n+1$ and that the expression for $h^{-1/2}A^n\phi$ contains all vectors $|\alpha_1\ldots\alpha_k)$ such that $\alpha_1+\cdots+\alpha_k=n+1$. Thus any matrix element will eventually occur in the coefficients of the polynomial $\Delta_n(E,\lambda)$ for some n. Since the approximate eigenvectors Ψ_n are of the form $\Psi_n=\Omega+h^{-1/2}R_{n-1}(A)\phi$ where $R_{n-1}(A)$ is a polynomial of degree $n-1$ in A we see that any vector $|\alpha_1\ldots\alpha_k)$ will eventually occur in Ψ_n for some n.

The coefficients of the $R-S$ series for the eigenvalue and eigenvector:

$$\begin{aligned} E(\lambda)&=0+0+\varepsilon_2\lambda^2+\varepsilon_3\lambda^3+\varepsilon_4\lambda^4+\cdots \\ &=-[1]+[11]+([2][1]-[111])+\cdots \\ \Psi_\lambda&=\Omega+\lambda|1\rangle+\lambda^2|2\rangle+\lambda^3|3\rangle+\cdots \\ &=\Omega|1)+|11)+\{[1]|2)-|111)\}+\cdots \end{aligned}$$

may be generated by the recursive scheme [11]:

$$|1\rangle=-h^{-1}V\Omega$$
$$|n\rangle=h^{-1}(-V|n-1\rangle+\varepsilon_2|n-2\rangle+\varepsilon_3|n-3\rangle+\cdots+\varepsilon_{n-1}|1\rangle)$$
$$\varepsilon_n=\langle n-1|V\Omega\rangle$$

where we have used the relations $H_0\Omega = 0$, $\langle\Omega|V\Omega\rangle = \varepsilon_1 = 0$ and $\langle n|\Omega\rangle = 0$. Since $|n\rangle$ is built up by successive applications of the operators h^{-1} and $h^{-1}V$ to $|1\rangle$ it is not hard to see that any vector $|\alpha_1 \ldots \alpha_k)$ will eventually occur in one of the vector coefficients $|n\rangle$. Since $\varepsilon_n = \langle n-1|V\Omega\rangle$ it follows that any matrix element $[\alpha_1 \ldots \alpha_k]$ will eventually occur in the expression for some ε_n.

This also shows, incidentally, that the eigenvector Ψ_λ may be approximated arbitrarily well in norm by finite linear combinations of the vectors $|\alpha_1 \ldots \alpha_k)$ which occur in the $R-S$ vector coefficients.

V. Calculation of the Ground State Energy of the Anharmonic Oscillator

We have calculated $E_n(\lambda)$ for the anharmonic oscillator $H_0 + \lambda V = 1/2\left(-\frac{d^2}{dx^2} + x^2 - 1\right) + \lambda(x^4 - 3/4)$ for $\lambda = 0.05$ and $\lambda = 0.5$.

In order to compare our results with the $[n, n]$ Padé approximant calculations [2, 3] we have computed the quantity $E_n^\beta = 2E_n(\lambda) + 1 + 3\lambda/2$ which converges to the ground state E^β of the Hamiltonian

$$\left(-\frac{d^2}{dx^2} + x^2 + \beta x^4\right)$$

when $\beta = 2\lambda$. Since E_n^β and $[n, n]$ both involve the interaction V to order $2n+1$, we have compared these quantities in Table 1. Note that the $[n, n]$ Padé gives a lower bound for E^β while E_n^β gives an upper bound. The apparent rate of convergence is about the same in both methods.

Table 1. *Comparison of E_n^β with the $[n, n]$ Padé Approximants to the $R-S$ series*

	$\beta = 0.1$		$\beta = 1$	
N	E_n^β	$[n, n]$	E_n^β	$[n, n]$
2	1.065 305 203 600	1.065 217 852 490	1.428 681 532 418	1.348 289 096 707
3	1.065 287 172 105	1.065 280 680 051	1.408 347 896 118	1.373 799 864 956
4	1.065 285 688 647	1.065 285 049 128	1.400 101 384 802	1.383 756 497 228
5	1.065 285 532 721	1.065 285 455 329	1.396 365 475 043	1.388 075 603 389
6	1.065 285 513 009	1.065 285 502 030	1.394 537 935 828	1.390 103 754 651
7	1.065 285 510 126	1.065 285 508 357	1.393 590 998 055	1.391 116 612 108
8	1.065 285 509 651	1.065 285 509 335	1.393 077 586 429	1.391 648 018 148
9	1.065 285 509 565	1.065 285 509 503	1.392 788 729 636	1.391 938 365 335
10	1.065 285 509 548	1.065 285 509 535	1.392 621 095 433	1.392 102 495 074
11	1.065 285 509 544	1.065 285 509 541	1.392 521 200 261	1.392 198 009 942

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