Approximate Waiting-Time for a Thin Liquid Drop Spreading under Gravity

E MOMONIAT

Centre for Differential Equations, Continuum Mechanics and Applications, School of Computational and Applied Mathematics, University of the Witwatersrand, Private Bag 3, Wits 2050, Johannesburg, South Africa
E-mail: ebrahim@cam.wits.ac.za

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Abstract

The method of multiple scales is used to introduce a small-time scale into the non-linear diffusion equation modelling the spreading of a thin liquid drop under gravity. The Lie group method is used to analyse the resulting system. An approximate group invariant solution and an approximation to the waiting-time is obtained. A mathematical description of a spreading drop with non-infinite contact angle is obtained. This application to determining an approximation to the waiting-time is novel as it combines the method of multiple scales and Lie groups.

1 Introduction

The non-linear diffusion equation modelling the spreading of a thin viscous liquid drop under the influence of gravity has been determined by Momoniat et. al. [14] and is given by

$$\frac{\partial h}{\partial t} = \frac{1}{3r}\frac{\partial}{\partial r}\left(rh^3\frac{\partial h}{\partial r}\right). \tag{1.1}$$

If we let $R(t)$ be the radius of the liquid drop at time t, then

$$R(t) = \left(1 + \frac{16}{9}t\right)^{\frac{1}{8}} \tag{1.2}$$

where the group invariant solution, conserving the total volume of the liquid drop, admitted by (1.1) is given by

$$h^*(t,r) = \frac{1}{R^2(t)}\left(1 - \frac{r^2}{R^2(t)}\right)^{\frac{1}{3}}. \tag{1.3}$$

The waiting-time is defined as the time taken for the free surface of a liquid drop to rearrange itself before the drop begins to move. Waiting-time phenomena have been investigated by Kath and Cohen [12] using perturbation techniques. We note from (1.2) that $R(0) = 1$. In this paper the waiting-time satisfies the condition

$$R^*(T) = 1, \tag{1.4}$$

i.e. the drop starts to move only when the radius is 1, where $R^*(t)$ is a radius to be determined. In order to investigate the waiting-time we introduce a smaller time scale using the method of multiple scales (see e.g. [15]). We introduce new variables by defining

$$t_0 = t, \qquad t_1 = \epsilon t, \qquad \epsilon \ll 1. \tag{1.5}$$

The time-derivative can then be rewritten as

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \epsilon \frac{\partial}{\partial t_1} \tag{1.6}$$

and (1.1) is recast as

$$\frac{\partial h}{\partial t_0} + \epsilon \frac{\partial h}{\partial t_1} = \frac{1}{3r}\frac{\partial}{\partial r}\left(r h^3 \frac{\partial h}{\partial r}\right). \tag{1.7}$$

We use the Lie group method to determine an approximate solution admitted by (1.7) of the form

$$h(t_0, t_1, r) = h_0(t_0, t_1, r) + \epsilon h_1(t_0, t_1, r) + \cdots . \tag{1.8}$$

Substituting (1.8) into (1.7) and separating by coefficients of ϵ we obtain the system

$$\frac{\partial h_0}{\partial t_0} = \frac{1}{3r}\frac{\partial}{\partial r}\left(r h_0^3 \frac{\partial h_0}{\partial r}\right), \tag{1.9}$$

$$\frac{\partial h_1}{\partial t_0} + \frac{\partial h_0}{\partial t_1} = \frac{1}{3r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\left[h_0^3 h_1\right]\right). \tag{1.10}$$

The method of multiple scales is traditionally used to get rid of secular terms in a straightforward perturbation. Baikov and Ibragimov [3] have used the method of multiple scales to extend the notion of approximate symmetries as introduced by Baikov et. al. [1, 2]. Another approach is that of finding approximate conditional symmetries admitted by the model equation as presented by Mahomed and Qu [13]. In this paper we use the approach adopted by Fushchich [10], M. Euler [7] and N. Euler [8, 9]. In this paper the application of this approach is novel. We are using this method to introduce a smaller time scale into the non-linear diffusion equation modelling the spreading of the liquid drop under gravity only. This introduces a new parameter which will allow us to get an approximation to the waiting-time. In Section 2 we use the Lie group method to solve the system given by (1.9)–(1.10). An approximation to the waiting-time, T, is calculated in Section 3. Concluding remarks are made in Section 4.

2 Lie group analysis

In this section we firstly discuss the application of the Lie group method to systems of equations. A Lie point symmetry generator for the system (1.9)–(1.10) is given by

$$X = \xi^1 \partial_{t_0} + \xi^2 \partial_{t_1} + \xi^3 \partial_r + \eta^1 \partial_{h_0} + \eta^2 \partial_{h_1} \tag{2.1}$$

where $\xi^i = \xi^i(t_0, t_1, r, h_0, h_1)$, $i = 1, 2, 3$ and $\eta^j = \eta^j(t_0, t_1, r, h_0, h_1)$, $j = 1, 2$. The functions ξ^i and η^j are calculated by solving the determining equations

$$X^{[2]}\left(\frac{\partial h_0}{\partial t_0} - \frac{1}{3r}\frac{\partial}{\partial r}\left(rh_0^3\frac{\partial h_0}{\partial r}\right)\right)\bigg|_{(1.9),(1.10)} = 0, \tag{2.2}$$

$$X^{[2]}\left(\frac{\partial h_1}{\partial t_0} + \frac{\partial h_0}{\partial t_1} - \frac{1}{3r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left[h_0^3 h_1\right]\right)\right)\bigg|_{(1.9),(1.10)} = 0. \tag{2.3}$$

The second prolongation, $X^{[2]}$ of X is defined by

$$X^{[2]} = X + \zeta_1^1\partial_{h_{0_{t_0}}} + \zeta_2^1\partial_{h_{0_{t_1}}} + \zeta_3^1\partial_{h_{0_r}} + \zeta_1^2\partial_{h_{1_{t_0}}} + \zeta_3^2\partial_{h_{1_r}} + \zeta_{33}^1\partial_{h_{0_{rr}}} + \zeta_{33}^2\partial_{h_{1_{rr}}}, \tag{2.4}$$

where

$$\zeta_i^1 = D_i\eta^1 - (D_i\xi^j)\frac{\partial h_0}{\partial x_j}, \tag{2.5}$$

$$\zeta_i^2 = D_i\eta^2 - (D_i\xi^j)\frac{\partial h_1}{\partial x_j}, \tag{2.6}$$

$$\zeta_{33}^1 = D_3\zeta_3^1 - (D_3\xi^j)\frac{\partial^2 h_0}{\partial r\partial x_j}, \tag{2.7}$$

$$\zeta_{33}^2 = D_3\zeta_3^2 - (D_3\xi^j)\frac{\partial^2 h_1}{\partial r\partial x_j}. \tag{2.8}$$

The repeated index, j, in (2.5)–(2.8) implies summation where $x_1 = t_0$, $x_2 = t_1$ and $x_3 = r$. The total derivatives are given by

$$D_1 = D_{t_0} = \partial_{t_0} + h_{0_{t_0}}\partial_{h_0} + h_{1_{t_0}}\partial_{h_1} + h_{t_0t_0}\partial_{h_{0_{t_0}}} + \cdots, \tag{2.9}$$

$$D_2 = D_{t_1} = \partial_{t_1} + h_{0_{t_1}}\partial_{h_0} + h_{1_{t_1}}\partial_{h_1} + h_{t_0t_1}\partial_{h_{0_{t_0}}} + \cdots, \tag{2.10}$$

$$D_3 = D_r = \partial_r + h_{0_r}\partial_{h_0} + h_{1_r}\partial_{h_1} + h_{t_0r}\partial_{h_{0_{t_0}}} + \cdots. \tag{2.11}$$

The interested reader is referred to [4, 16]. The determining equations given by (2.2) and (2.3) can be separated out by derivatives of the dependent variables h_0 and h_1. We find that the Lie point symmetry generators admitted by the system (2.2)–(2.3) are infinite dimensional and given by

$$\begin{aligned} X &= (t_0B_2 + A_2(t_1))\,\partial_{t_0} + A_1(t_1)\partial_{t_1} + B_1r\partial_r - \frac{1}{3}(B_2 - 2B_1)h_0\partial_{h_0} \\ &\quad + \left(\frac{1}{3}h_1(2B_1 + 2B_2 - 3A_{1_{t_1}}) - \frac{1}{3}h_0A_{2_{t_1}}\right)\partial_{h_1}, \end{aligned} \tag{2.12}$$

where B_i, $i = 1, 2$ are arbitrary constants and $A_j = A_j(t_1)$, $j = 1, 2$. To be able to proceed further we make the assumption that

$$A_1(t_1) = 0, \qquad A_2(t_1) = A_2 = \text{const.} \tag{2.13}$$

The group invariant solutions for h_0 and h_1 corresponding to the Lie point symmetry generator (2.12) are calculated by solving

$$X\left(h_i - \Phi_i(t_0, t_1, r)\right)\big|_{h_i=\Phi_i(t_0,t_1,r)} = 0, \qquad i = 0, 1. \tag{2.14}$$

The resulting first-order quasi-linear partial differential equations can be solved to give

$$h_0(t_0, t_1, r) = (A_2 + t_0 B_2)^{\frac{2B_1 - B_2}{3B_2}} F(k_1, k_2), \tag{2.15}$$

$$h_1(t_0, t_1, r) = (A_2 + t_0 B_2)^{\frac{2(B_1 + B_2)}{3B_2}} G(k1, k2), \tag{2.16}$$

$$k_1 = t_1, \qquad k_2 = r(A_2 + t_0 B_2)^{-\frac{B_1}{B_2}}. \tag{2.17}$$

By substituting (2.15)–(2.17) into (1.9) and (1.10) we obtain the following system of second-order non-linear partial differential equations:

$$k_2(B_2 - 2B_1)F + 3k_2^2 B_1 \frac{\partial F}{\partial k_2} + 3k_2 F^2 \left(\frac{\partial F}{\partial k_2}\right)^2 + F^3 \left(\frac{\partial F}{\partial k_2} + k_2 \frac{\partial^2 F}{\partial k_2^2}\right) = 0, \tag{2.18}$$

$$\begin{aligned}&\left(3k_2^2 B_1 + F^3 + 6k_2 F^2 \frac{\partial F}{\partial k_2}\right) \frac{\partial G}{\partial k_2} + k_2 \left(F^3 \frac{\partial^2 G}{\partial k_2^2} - 3\frac{\partial F}{\partial k_2}\right) \\ &\quad + G\left(-2k_2(B_1 + B_2) + 6k_2 F \left(\frac{\partial F}{\partial k_2}\right)^2 + 3F^2 \left(\frac{\partial F}{\partial k_2} + k_2 \frac{\partial^2 F}{\partial k_2^2}\right)\right) = 0.\end{aligned} \tag{2.19}$$

The Lie point symmetry generators admitted by the system (2.18)–(2.19) are determined in a similar manner as indicated above. We find that the Lie point symmetry generator admitted by this system is given by

$$Y = \left(M_1 - \frac{5}{3}k_1 M_2\right) \partial_{k_1} - M_2 k_2 \partial_{k_2} - \frac{2}{3} M_2 F \partial_F + M_2 G \partial_G, \tag{2.20}$$

where M_i, $i = 1, 2$ are arbitrary constants. The group invariant solutions F and G admitted by the Lie point symmetry generator (2.20) can be determined by solving

$$Y\left(F - \Gamma_1(k_1, k_2)\right)\big|_{F = \Gamma_1(k_1, k_2)} = 0, \qquad Y\left(G - \Gamma_2(k_1, k_2)\right)\big|_{G = \Gamma_2(k_1, k_2)} = 0. \tag{2.21}$$

The resulting system of first-order quasi-linear partial differential equations can be solved to give

$$F(k_1, k_2) = (5k_1 M_2 - 3M_1)^{\frac{2}{5}} P(z), \tag{2.22}$$

$$G(k_1, k_2) = (5k_1 M_2 - 3M_1)^{-\frac{3}{5}} Q(z), \tag{2.23}$$

$$z = k_2 \left(5k_1 M_2 - 3M_1\right)^{-\frac{3}{5}}. \tag{2.24}$$

Substituting (2.22)–(2.24) into (2.18) and (2.19) we obtain the system of second-order non-linear ordinary differential equations

$$\begin{aligned}&z(B_2 - 2B_1)P(z) + 3z^2 B_1 P'(z) + 3zP(z)^2 P'(z)^2 \\ &\quad + P(z)^3\big(P'(z) + zP''(z)\big) = 0,\end{aligned} \tag{2.25}$$

$$\begin{aligned}&-6zP(z)\big(M_2 - Q(z)P'(z)^2\big) + z(-2(B_1 + B_2)Q(z) \\ &\quad + 3z(3M_2 P'(z) + B_1 Q'(z)) + P(z)^3(Q'(z) + zQ''(z)) \\ &\quad + 3P(z)^2(2zP'(z)Q'(z) + Q(z)(P'(z) + zP''(z)) = 0.\end{aligned} \tag{2.26}$$

We know from [14] that (2.25) admits a solution of the form

$$P(z) = a_1 \left(1 - y^2\right)^{\frac{1}{3}}. \tag{2.27}$$

We look for solutions for $Q(z)$ of the form

$$Q(z) = a_2 \left(1 - y^2\right)^{a_3}. \tag{2.28}$$

The system (2.25)–(2.26) admit solutions of the form (2.27) and (2.28) provided

$$a_3 = -\frac{2}{3}, \qquad B_1 = \frac{2}{9}a_1^3, \qquad B_2 = \frac{16}{9}a_1^3, \qquad M_2 = -\frac{8}{9}a_1^2 a_2. \tag{2.29}$$

The group invariant solutions for h_0 and h_1 are then given by

$$h_0(t_0, t_1, r) = a_1 V^{-1}(t_0) W^2(t_1) \left(1 - \frac{r^2}{V(t_0)W^6(t_1)}\right)^{\frac{1}{3}}, \tag{2.30}$$

$$h_1(t_0, t_1, r) = a_2 V^3(t_0) W^{-3}(t_1) \left(1 - \frac{r^2}{V(t_0)W^6(t_1)}\right)^{-\frac{2}{3}}, \tag{2.31}$$

$$V(t_0) = \left(A_2 + \frac{16}{9}a_1^3 t_0\right)^{\frac{1}{4}}, \tag{2.32}$$

$$W(t_1) = \left(-\frac{40}{9}a_1^2 a_2 t_1 - 3M_1\right)^{\frac{1}{5}}. \tag{2.33}$$

We impose the condition $\lim\limits_{t_1 \to 0} h_0(t_0, t_1, r) \to h^*(t_0, r)$, then

$$a_1 = 1, \qquad A_2 = 1, \qquad M_1 = -\frac{1}{3}. \tag{2.34}$$

Therefore

$$h_0(t_0, t_1, r) = \left(\frac{R_2(t_1)}{R_1(t_0)}\right)^2 \left(1 - \frac{r^2}{R_1^2(t_0)R_2^6(t_1)}\right)^{\frac{1}{3}}, \tag{2.35}$$

$$h_1(t_0, t_1, r) = a_2 \left(\frac{R_1^2(t_0)}{R_2(t_1)}\right)^3 \left(1 - \frac{r^2}{R_1^2(t_1)R_2^6(t_1)}\right)^{-\frac{2}{3}}, \tag{2.36}$$

$$R_1(t_0) = R(t_0) = \left(1 + \frac{16}{9}t_0\right)^{\frac{1}{8}}, \tag{2.37}$$

$$R_2(t_1) = \left(1 - \frac{40}{9}a_2 t_1\right)^{\frac{1}{5}}. \tag{2.38}$$

Equations (2.35)–(2.38) represent an approximate group invariant solution of the form (1.8) admitted by (1.7).

3 Approximation to waiting-time

To get an approximation to the waiting-time we substitute (2.35)–(2.38) into (1.8) and then separate terms to first order in ϵ we find that an approximate group invariant solution admitted by (1.7) is given by

$$h(t,r) = \frac{1}{R^2(t)}\left[\left(1-\frac{r^2}{R^2(t)}\right)^{\frac{1}{3}} + \epsilon a_2\left(1-\frac{r^2}{R^2(t)}\right)^{-\frac{2}{3}}\right]. \tag{3.1}$$

We firstly calculate the new radius $R^*(t)$ such that

$$h(t,R^*(t)) = 0. \tag{3.2}$$

We find that

$$R^*(t) = R(t)\sqrt{1+a_2\epsilon}. \tag{3.3}$$

Imposing (1.4) on (3.3) we find that

$$T = \frac{9}{16}\left((1+a_2\epsilon)^{-4} - 1\right). \tag{3.4}$$

Taking (3.4) to first order in ϵ we obtain

$$T = -\frac{9}{4}a_2\epsilon, \qquad a_2 < 0. \tag{3.5}$$

No restriction is placed on the constant a_2. Since $a_2 < 0$, (3.3) places a restriction on the values that a_2 can take, i.e.

$$-\frac{1}{\epsilon} < a_2 < 0. \tag{3.6}$$

The solution (3.1) is plotted in Fig. 1. The case when $\epsilon = 0$ is plotted in Fig. 2. We note that the introduction of a small-time has changed the height and slope of the free surface. When the small-time is included the initial height of the drop is smaller than one. Also, the angle

$$\left.\frac{\partial h^*}{\partial r}\right|_{r=R(t)} \neq \left.\frac{\partial h}{\partial r}\right|_{r=R(t)}. \tag{3.7}$$

In fact

$$\left.\frac{\partial h^*}{\partial r}\right|_{r=R(t)} = -\infty, \qquad \left.\frac{\partial h}{\partial r}\right|_{r=R^*(t)} = -\frac{2\sqrt{1+a_2\epsilon}}{(a_2\epsilon)^{\frac{2}{3}}R^3(t)}. \tag{3.8}$$

4 Concluding remarks

The approach taken in this paper to determine an approximation to the waiting-time is novel. It combined both the method of multiple scales and Lie groups. Three important results are obtained in this paper. Firstly, we have found an approximation to the waiting-time which is of concern in the engineering of coating flows. Delays in the flattening of the drop need to be taken into account during manufacturing processes. Secondly, we have found that the contact angle of the initial profile does not have to be infinite for the drop to move. This is consistent with observations where not all spreading drops have infinite contact angles (see e.g. [5, 6, 11]). Thirdly, the effect of waiting-time on the radius of the liquid drop has been determined. The difference in radii has been plotted in Fig. 2. The delay caused by the waiting-time significantly affects the radius of the liquid drop. This can be seen from Fig. 2 where the difference between the two radii, (1.2) and (3.3), increases over time. We have obtained a parameter a_2 in our approximate solution (3.1) and (3.3). This constant can be determined from experiment.

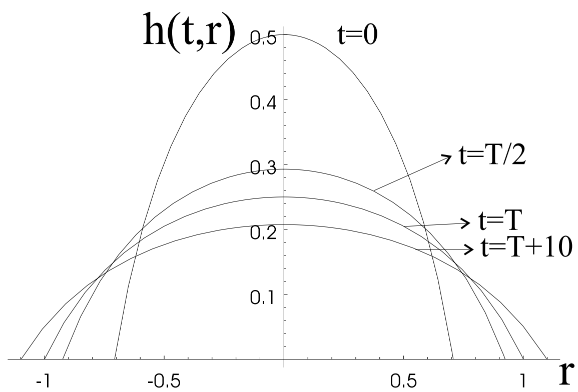

Figure 1. Plot of the (3.1) where $a_2 = -50$ and $\epsilon = 0.01$. T is calculated from (3.5) as $T = 1.125$.

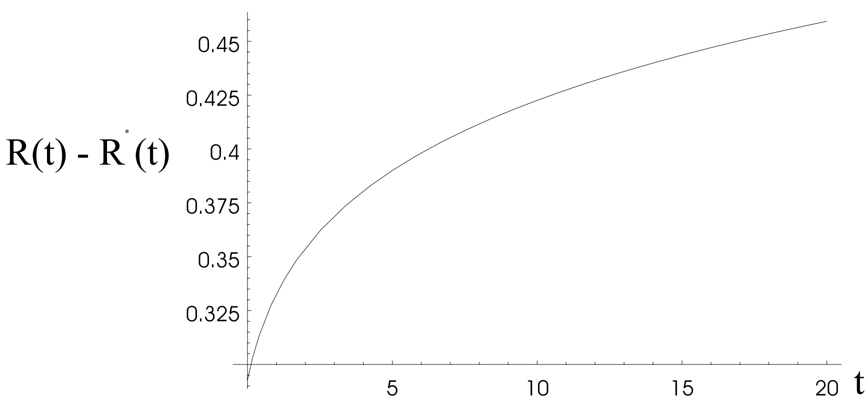

Figure 2. Plot of $R(t) - R^*(t)$ for $t \in [0, 20]$.

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