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Approximated Solutions of Equations with L^1 Data. Application to the H-Convergence of Quasi-Linear Parabolic Equations (*).

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1. – Introduction.

We would like to present in this paper some results about elliptic and parabolic PDE's with L^1 data. In particular we are concerned about recovering a notion of *uniqueness* of the solution. We will also show some applications of these results to the theory of H-convergence of parabolic quasi-linear equations with sub-quadratic and quadratic growth with respect to the gradient (and indeed this is the problem which motivated our research).

Let $\Omega \subset \mathbf{R}^N$ be an open bounded set, $N \geq 2$. We are interested in elliptic problems of the form

$$\text{(1.1)} \qquad \begin{cases} A(u) = -\operatorname{div} a(x, \nabla u) = f + h & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

and in the corresponding parabolic problems

$$\text{(1.2)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div} a(x, t, \nabla u) = f + h & \text{in } Q = \Omega \times (0, T), \\ u(\cdot, 0) = w(\cdot) & \text{in } \Omega, \\ u(x, t) = 0 & \text{on } \Gamma = \partial\Omega \times (0, T). \end{cases}$$

Here the function $a(x, \xi)$: $\Omega \times \mathbf{R}^N \to \mathbf{R}^N$ (the function $a(x, t, \xi)$: $\Omega \times (0, T) \times \mathbf{R}^N \to \mathbf{R}^N$ in the parabolic case) is a Carathéodory function which is strongly monotone and has a growth of order $p - 1$ with respect to ξ, p being a number such that $p > 2 - 1/N$ ($p > 2 - 1/(N+1)$ in the parabolic case). The model of a we have in mind is $a(x, \xi) = a(\xi) = |\xi|^{p-2}\xi$, which corresponds to the p-laplacian: $A(u) = -\Delta_p(u)$.

About the data, we will assume (in the elliptic case) that the right hand side of the

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equation in (1.1) is the sum of a «variational» part $f \in W^{-1,p'}(\Omega) = (W_0^{1,p}(\Omega))'$ and a «non-variational» part $h \in L^1(\Omega)$. We remark that, for $p \leq N$, $L^1(\Omega)$ is not embedded in $W^{-1,p'}(\Omega)$. For the parabolic problems, we will assume $f \in L^{p'}(0, T; W^{-1,p'}(\Omega))$, $h \in L^1(Q)$, while the initial datum w will belong to $L^1(\Omega)$.

The study of linear elliptic equations with L^1 data has been started by STAMPACCHIA (in [St]) by means of duality methods, while in the nonlinear case existence results have been found by BOCCARDO and GALLOUËT in [BG]. More precisely they proved, for $f = 0$, the existence of at least a solution (in the sense of distributions) of problem (1.1) which belongs to the Sobolev space $W_0^{1,q}(\Omega)$ for every $q < \bar{q} = N(p-1)/(N-1)$. The proof of this result is achieved in two steps: first of all an *a priori* L^q estimate on the gradients of the solutions of (1.1) is proved; then the function h is approximated by regular functions, and the previous step is used to show that the solutions of the approximated problems converge to a solution of (1.1). However there is no uniqueness of the solution in the space $W_0^{1,q}(\Omega)$ or even in $\bigcap_{\bar{q}} W^{1,q}(\Omega)$ (see [Se] and [P] for a counterexample), but we observe that it is possible to select a solution which is «better than the others», since it is the only solution which is found by means of approximations: we will call it *Solution Obtained as Limit of Approximations*, or simply *SOLA*.

A similar situation holds in the parabolic case. BOCCARDO and GALLOUËT (in [BG]) proved, if $f = 0$, the existence of a solution of (1.2) in the space $L^q(0, T; W_0^{1,q}(\Omega))$, for every $q < \bar{q}$, where $\bar{q} = p - N/(N+1)$ (this bound must be slightly modified for large p, see Section 4). We will show that the solution of problem (1.2) that one finds by the approximation method does not depend on the approximation chosen for the non regular data. Moreover we will prove that if a solution of (1.2) is regular enough (that is, if its gradient belongs to $L^p(Q)$), then this solution is the SOLA of (1.2). This result, which is particularly useful in the applications, is trivial in the elliptic case, since, from the equation, we immediately obtain $h \in W^{-1,p'}(\Omega)$. On the contrary it requires some effort in the parabolic case, since we have little information on the regularity of the time derivative.

We will be interested in applying these results to the H-convergence of quasi-linear PDE's. Thus it will be useful to recall the definition and the basic properties of linear H-convergence. For $\alpha > 0$, $\beta > 0$, let $\mathcal{M}(\alpha, \beta; \Omega)$ be the class of matrices $a(x) = [a_{ij}(x)]_{i,j=1,\dots,N}$, whose elements are $L^\infty(\Omega)$ functions, satisfying

$$(a(x)\xi, \xi) \geq \alpha|\xi|^2, \qquad (a^{-1}(x)\xi, \xi) \geq \beta^{-1}|\xi|^2$$

for every $\xi \in \mathbf{R}^N$, for a.e. $x \in \Omega$.

Following the notation traditionally used in homogenization theory, we will consider a sequence $\{a_\varepsilon\}_{\varepsilon \in E}$ of matrices in $\mathcal{M}(\alpha, \beta; \Omega)$, ε being the element of an infinitesimal positive sequence E (the typical case in homogenization theory is $a_\varepsilon(x) = a(x/\varepsilon)$, where $a(y) \in \mathcal{M}(\alpha, \beta; \mathbf{R}^N)$ is a periodic matrix with period $[0, 1]^N$). To every matrix a_ε we can associate the elliptic differential operator $A_\varepsilon v = -\operatorname{div}(a_\varepsilon(x)\nabla v)$.

DEFINITION 1.1. – We will say that the sequence $\{a_\varepsilon\}_{\varepsilon \in E}$ H-converge to a matrix $a_0 \in \mathcal{M}(\alpha, \beta; \Omega)$, and we will write

$$a_\varepsilon \overset{H}{\rightharpoonup} a_0 ,$$

if for every $f \in H^{-1}(\Omega)$ the weak solutions $u_\varepsilon \in H_0^1(\Omega)$ of the elliptic equations

$$A_\varepsilon u_\varepsilon = f \quad \text{in } \Omega , \tag{1.3}$$

satisfy

$$u_\varepsilon \rightharpoonup u_0 \quad \text{weakly in } H_0^1(\Omega), \qquad a_\varepsilon \nabla u_\varepsilon \rightharpoonup a_0 \nabla u_0 \quad \text{weakly in } (L^2(\Omega))^N , \tag{1.4}$$

where u_0 is the solution of $A_0 u_0 = f$ in Ω.

The concept of H-convergence was introduced for symmetric matrices by SPAGNOLO in [Sp1], [Sp2] under the name of G-convergence. Later on, MURAT and TARTAR extended the theory to nonsymmetric matrices (see [MT]). This type of convergence is the abstract generalization of the homogenization theory (see [BLP] and [SP]). One of the main results in this theory is the compactness of the class $\mathcal{M}(\alpha, \beta; \Omega)$ with respect to the H-convergence.

A difficulty one encounters when dealing with H-convergence is the fact that the convergence of u_ε to u_0 is in general only weak in $H_0^1(\Omega)$. This is a source of problems especially if one tries to study equations with additional nonlinear terms depending on the gradient, since the weak convergence does not allow to pass to the limit in these terms. To overcome this problem, a sequence of *corrector matrices* $\{p_\varepsilon\}_{\varepsilon \in E}$ is introduced. This sequence, which depends on Ω and $\{a_\varepsilon\}_{\varepsilon \in E}$, satisfies

$$p_\varepsilon \rightharpoonup I \quad \text{weakly in } (L^2(\Omega))^{N^2} , \qquad a_\varepsilon p_\varepsilon \rightharpoonup a_0 \quad \text{weakly in } (L^2(\Omega))^{N^2} ,$$

$$\operatorname{div}(a_\varepsilon p_\varepsilon \xi) \to \operatorname{div}(a_0 \xi) \quad \text{strongly in } H^{-1}(\Omega), \text{ for every } \xi \in \mathbf{R}^N ,$$

where I is the identity matrix. The columns of the matrices p_ε are the gradients of solutions of appropriate elliptic problems. By means of these matrices it is possible to «recover» in some sense the strong convergence of the gradients. More precisely MURAT and TARTAR proved that

$$\nabla u_\varepsilon - p_\varepsilon \nabla u_0 \to 0 \quad \text{strongly in } (L^1(\Omega))^N . \tag{1.5}$$

In Section 3 we will study the behaviour of SOLA's of linear equations with $L^1(\Omega)$ data when the operators H-converge: the main result we will obtain is given by Theorem 3.1, which states that, if $a_\varepsilon \overset{H}{\rightharpoonup} a_0$, and $h_\varepsilon \rightharpoonup h_0$ weakly in $L^1(\Omega)$ then the SOLA's u_ε of the equations $A_\varepsilon u_\varepsilon = f + h_\varepsilon$ converge weakly in $W_0^{1,q}(\Omega)$ (for every $q < \bar{q}$) to u_0, SOLA of $A_0 u_0 = f + h_0$. Moreover we will show that a corrector result of the form (1.5) holds.

A similar situation holds for the H-convergence of parabolic operators of the form

$$\mathcal{P}_\varepsilon v = \frac{\partial v}{\partial t} + A_\varepsilon v = \frac{\partial v}{\partial t} - \operatorname{div}(a_\varepsilon(x)\nabla v).$$

Let us consider the parabolic Cauchy problems:

$$(1.6)\qquad \begin{cases} \mathcal{P}_\varepsilon u_\varepsilon = f & \text{in } Q, \\ u_\varepsilon(x,0) = w(x) & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \Gamma, \end{cases} \qquad \begin{cases} \mathcal{P}_0 u_0 = f & \text{in } Q, \\ u_0(x,0) = w(x) & \text{in } \Omega, \\ u_0 = 0 & \text{on } \Gamma. \end{cases}$$

For every $f \in L^2(0, T; H^{-1}(\Omega))$ and $w \in L^2(\Omega)$, each of the problems (1.6) has a unique solution in the space

$$\mathcal{W}(0, T; \Omega) = \left\{ v \in L^2(0, T; H_0^1(\Omega)) \text{ such that } \frac{\partial u}{\partial t} \in L^2(0, T; H^{-1}(\Omega)) \right\}.$$

The following result extends to parabolic equations the properties stated for elliptic equations:

PROPOSITION 1.1 (see [CS], [BFM]). – *Assume that* $a_\varepsilon \overset{H}{\rightharpoonup} a_0$. *Then, for every* $f \in L^2(0, T; H^{-1}(\Omega))$, *and for every* $w \in L^2(\Omega)$, *the solutions* u_ε, u_0 *of* (1.6) *satisfy*

$$(1.7)\qquad u_\varepsilon \rightharpoonup u_0 \quad \textit{weakly in } L^2(0, T; H_0^1(\Omega)),$$

$$(1.8)\qquad \nabla u_\varepsilon - p_\varepsilon \nabla u_0 \to 0 \quad \textit{strongly in } (L^1(Q))^N.$$

The statement (1.7) basically says that the elliptic H-convergence and the parabolic H-convergence, which can be defined starting directly from problems (1.6), coincide (*). The corrector result (1.8) was proved in [BFM].

We will consider the asymptotic behavior of the SOLA's of problems of the form

$$(1.9)\qquad \begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} + A_\varepsilon u_\varepsilon = f + h_\varepsilon & \text{in } Q, \\ u_\varepsilon(\cdot, 0) = w(\cdot) & \text{in } \Omega, \\ u_\varepsilon(x, t) = 0 & \text{on } \Gamma, \end{cases}$$

under the assumptions that the linear operators A_ε H-converge to A_0, and that $h_\varepsilon \to h_0$

(*) It is important, however, to emphasize that things can be different if the matrices a_ε which define the operators $\mathcal{P}_\varepsilon$ depend also on t: in this case the equivalence between the parabolic H-convergence on one hand, and the elliptic H-convergence for every $t \in (0, T)$ on the other hand, does not hold: see, e.g., [CS], [Sp3], [ZKO].

weakly in $L^1(Q)$. We will show that the SOLA's of (1.9) converge weakly in $L^q(0, T; W_0^{1,q}(\Omega))$ (for every $q < \bar{q}$) to the SOLA of the corresponding limit problem. Moreover we will prove a corrector theorem for these equations.

A vast class of problems related to H-convergence deals with parabolic quasi-linear problems of the form

$$
\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} - \operatorname{div}(a_\varepsilon(x)\nabla u_\varepsilon) + H_\varepsilon(x, t, u_\varepsilon, \nabla u_\varepsilon) = f & \text{in } Q, \\ u_\varepsilon(\cdot, 0) = w(\cdot) \quad \text{in } \Omega, \\ u_\varepsilon(x, t) = 0 \quad \text{on } \Gamma, \end{cases} \tag{1.10}
$$

where $a_\varepsilon \xrightarrow{H} a_0$ and $\{H_\varepsilon\}$ is a sequence of Carathéodory functions. The typical question in this setting is: does the solutions u_ε of (1.10) converge to a solution u_0 of an equation of the same kind? Since the functions H_ε depend on ∇u_ε, the weak convergence of the gradients does not allow to pass to the limit in the nonlinear term. However, if a corrector result like (1.8) holds also for solutions of our quasi-linear equations, then we can show that, if u_ε converges to u_0 weakly in $L^2(0, T; H_0^1(\Omega))$, then u_0 is a solution of

$$
\begin{cases} \dfrac{\partial u_0}{\partial t} - \operatorname{div}(a_0(x)\nabla u_0) + H_0(x, t, u_0, \nabla u_0) = f & \text{in } Q, \\ u_0(\cdot, 0) = w(\cdot) \quad \text{in } \Omega, \\ u_0(x, t) = 0 \quad \text{on } \Gamma, \end{cases}
$$

where $H_0(x, t, s, \xi)$ is the weak L^1-limit of $H_\varepsilon(x, t, s, p_\varepsilon \xi)$. In [BBM] this has been done for quasi-linear elliptic equations with quadratic growth for H_ε with respect to the gradient.

It is important to remark that, if the nonlinear term H_ε depends on the gradient, then, even if $H_\varepsilon = H$ does not change with ε, the limit function H_0 can be different from H, as it can be showed by easy one-dimensional counterexamples.

In Section 6 the corrector result for parabolic equations with L^1 data will be used to study the H-convergence of quasi-linear parabolic equations of the form (1.10), where the functions $H_\varepsilon(x, t, s, \xi)$ are Carathéodory functions with quadratic or sub-quadratic growth with respect to the variable ξ. More precisely we will consider two different situations:

— the case in which the functions $H_\varepsilon(x, t, s, \xi)$ have sub-quadratic growth with respect to ξ, and have the same sign as s, but without any growth restriction with respect to s (case of unbounded solutions);

— the case in which the H_ε's have quadratic growth in ξ, plus some regularity hypotheses on the data which assure that the solutions of (1.10) are bounded.

The key of the proof is the extension to quasi-linear equations of a corrector result. The method that we will use to prove such a result is based on the result about

SOLA's, since a solution of problem (1.10) can be considered, in particular, as a solution of the equation

$$\frac{\partial v}{\partial t} + A_\varepsilon v = f - H_\varepsilon(x, t, u_\varepsilon, \nabla u_\varepsilon),$$

in which the r.h.s. $f - H_\varepsilon(x, t, u_\varepsilon, \nabla u_\varepsilon)$ belongs to $L^{p'}(0, T; W^{-1,p'}(\Omega)) + L^1(Q)$.

A different approach to the problem of uniqueness of solutions of elliptic equations with L^1 data has been studied in [BBGGPV] by defining an *entropy solution*. During the writing of this paper we were informed that LIONS, MURAT and BLANCHARD have studied the problem of uniqueness for elliptic and parabolic equations with L^1 data, through the notion of *renormalized solution* (see [LM], [Mu] and [BlM]).

This paper originated as a chapter of the author's Ph. D. thesis [D]. The plan of the paper is as follows. In Section 2 we will introduce the SOLA's of elliptic equations and prove that the map $h \mapsto u$ which associates to every datum h the corresponding SOLA in continuous from $L^1(\Omega)$ (endowed with the weak topology) to $W_0^{1,q}(\Omega)$ strong for every $q < \bar{q}$. In Section 3 we will deal with the behavior of the SOLA's with respect to the H-convergence of linear operators. In Section 4 and 5 the same scheme will be applied to parabolic equations. Finally, in Section 6, we will give two applications of the theory to the study of the H-convergence of quasi-linear parabolic equations.

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2. – SOLA's of nonlinear elliptic equations.

In this section we will prove the existence and the uniqueness of a SOLA of an elliptic equation with an L^1 datum, and will study its behavior with respect to the weak convergence of the datum. Let Ω be a bounded, open subset of $\boldsymbol{R}^N$, $N \geq 2$. We are interested in nonlinear elliptic equations of the type

$$A(u) = -\operatorname{div} a(x, \nabla u) = f + h \quad \text{in } \Omega, \tag{2.1}$$

with homogeneous Dirichlet boundary conditions. Let p be a real number such that

$$2 - \frac{1}{N} < p \leq N, \tag{2.2}$$

and let $a(x, \xi)$: $\Omega \times \boldsymbol{R}^N \to \boldsymbol{R}^N$ be a Carathéodory function (i.e., continuous with respect to ξ for a.e. $x \in \Omega$, and measurable with respect to x for every $\xi \in \boldsymbol{R}^N$) such that, for a.e. $x \in \Omega$, for every $\xi, \eta \in \mathbf{R}^N$

$$a(x, \xi) \cdot \xi \geq \alpha |\xi|^p , \tag{2.3}$$

$$(a(x, \xi) - a(x, \eta)) \cdot (\xi - \eta) \geq \alpha_1 |\xi - \eta|^p \quad \text{if } 2 \leq p \leq N , \tag{2.4a}$$

$$(a(x, \xi) - a(x, \eta)) \cdot (\xi - \eta) \geq \alpha_1 \frac{|\xi - \eta|^2}{(1 + |\xi| + |\eta|)^{2-p}} \quad \text{if } 2 - \frac{1}{N} < p < 2 , \tag{2.4b}$$

$$|a(x, \xi)| \leq k(x) + \beta |\xi|^{p-1} , \tag{2.5}$$

where α, α_1 and β are positive constants, and $k(x)$ is a function belonging to $L^{p'}(\Omega)$ ($p' = p/(p-1)$). We point out that these assumptions are satisfied, for instance, by the p-Laplacian operator $-\Delta_p$ (i.e. in the case $a(x, \xi) = a(\xi) = |\xi|^{p-2}\xi$). On the right hand side, we assume

$$f \in W^{-1, p'}(\Omega) , \qquad h \in L^1(\Omega) . \tag{2.6}$$

REMARK 2.1. – Note that if $p > N$ we have $L^1(\Omega) \subset W^{-1, p'}(\Omega)$, and then the existence of solutions for (2.1) follows from the classical results on operators acting between Sobolev spaces in duality (see [L]); this explains the upper bound on p given by (2.2). As far as the lower bound on p is concerned, see Remark 2.2 below.

In a general setting, the existence of solutions is guaranteed by the following theorem (see [BG]).

THEOREM 2.1. – *Let $f = 0$, $h \in \mathcal{M}(\Omega)$. Then there exists a solution u of* (2.1), *with $u \in W_0^{1, q}(\Omega)$ for every $q < \bar{q} = N(p-1)/(N-1)$.*

REMARK 2.2. – The assumption $p > 2 - 1/N$ implies $N(p-1)/(N-1) > 1$. Moreover, assumption (2.5) yields $a(x, \nabla u) \in L^1(\Omega)$. If $p \leq 2 - 1/N$, we have $\bar{q} \leq 1$. In this case the solution is not in general in $W_{\text{loc}}^{1,1}(\Omega)$. A definition of a solution if $p \leq 2 - 1/N$, and the proof of its existence and uniqueness, have been given in [BBGG-PV].

In order to define a «good» unique solution (SOLA) of (2.1), we shall use the following result.

PROPOSITION 2.1. – *Suppose that $p \geq 2$. Let $f, \tilde{f} \in W^{-1, p'}(\Omega)$, such that $f - \tilde{f} \in L^1(\Omega)$, and let h, $\tilde{h}$ be two regular function (to fix the ideas, suppose that they belong to $L^\infty(\Omega)$). Let us consider the solutions u, $\tilde{u}$ of*

$$A(u) = f + h , \qquad A(\tilde{u}) = \tilde{f} + \tilde{h} . \tag{2.7}$$

Then, for every $q < \bar{q} = N(p-1)/(N-1)$:

$$\|u - \tilde{u}\|_{W_0^{1,q}(\Omega)} \leq \Psi(\|f - \tilde{f} + h - \tilde{h}\|_{L^1(\Omega)}), \tag{2.8}$$

where Ψ *is a positive function such that*

$$\lim_{s \to 0^+} \Psi(s) = 0. \tag{2.9}$$

PROOF. – Let us define, for $k > 0$, the function $\varphi_k(s) = \min\{(|s| - k)_+, 1\} \operatorname{sign} s$ (here t_+ denotes the positive part of t) and the set $B_k = \{x \in \Omega: k \leq |u(x) - \tilde{u}(x)| < k+1\}$. We use $\varphi_k(u - \tilde{u})$ as test function in (2.7), and we use assumption (2.4*a*) to obtain

$$\alpha_1 \int_{B_k} |\nabla(u - \tilde{u})|^p \, dx \leq \int_{\Omega} |f - \tilde{f} + h - \tilde{h}| \, |\varphi_k(u - \tilde{u})| \, dx \leq \|f - \tilde{f} + h - \tilde{h}\|_{L^1(\Omega)}.$$

Then we continue as in [BG], Lemma 1, keeping track explicitly of $\|(f - \tilde{f}) + h - \tilde{h}\|_{L^1(\Omega)}$ to show that (2.8) holds. It is easy to see that there exist two positive constant c_1 and c_2, depending only on α_1, N, p, q, and Ω, such that $\Psi(s) \leq c_1 s^{1/p}$ if $0 < s < c_2$. ∎

If $2 - 1/N < p < 2$, the proof of a result of the same type as Proposition 1.1 is a little more complicated.

PROPOSITION 2.2. – *Assume that* $2 - 1/N < p < 2$. *Let* $f, \tilde{f}, h, \tilde{h}, u, \tilde{u}$ *as in Proposition* 2.1. *Then, for every* $q < \bar{q} = N(p-1)/(N-1)$, *we have*:

$$\|u - \tilde{u}\|_{W_0^{1,q}(\Omega)} \leq \Lambda(\|f\|_{W^{-1,p'}(\Omega)}, \|\tilde{f}\|_{W^{-1,p'}(\Omega)}, \|h\|_{L^1(\Omega)}, \|\tilde{h}\|_{L^1(\Omega)}, \|(f - \tilde{f}) + h - \tilde{h}\|_{L^1(\Omega)}),$$

where Λ *is a function (that depends also on* α, α_1, q, p, N, Ω*) that tends to zero when* $\|(f - \tilde{f}) + h - \tilde{h}\|_{L^1(\Omega)}$ *tends to* 0 *and all the other norms remain bounded.*

PROOF. – First of all, using hypothesis (2.3) and the technique of [BG], we obtain:

$$\|\nabla u\|_{(L^q(\Omega))^N} \leq c_1, \qquad \|\nabla \tilde{u}\|_{(L^q(\Omega))^N} \leq c_1, \tag{2.10}$$

for every $q < \bar{q}$, where c_1 is a constant depending on α, N, p, $\|f\|_{W^{-1,p'}(\Omega)}$, $\|\tilde{f}\|_{W^{-1,p'}(\Omega)}$, $\|h\|_{L^1}$, $\|\tilde{h}\|_{L^1}$. On the other hand, if we use the same test functions as in the preceding proof, by assumption (2.4*b*) we obtain

$$\alpha_1 \int_{B_k} \frac{|\nabla(u - \tilde{u})|^2}{(1 + |\nabla u| + |\nabla \tilde{u}|)^{2-p}} \, dx \leq \|(f - \tilde{f}) + h - \tilde{h}\|_{L^1(\Omega)},$$

and so, by the Hölder inequality,

$$\int_{B_k} |\nabla(u-\tilde{u})|^q dx = \int_{B_k} \frac{|\nabla(u-\tilde{u})|^q}{(1+|\nabla u|+|\nabla\tilde{u}|)^{q(2-p)/2}}(1+|\nabla u|+|\nabla\tilde{u}|)^{q(2-p)/2} dx \leq$$

$$\leq c_2 \|(f-\tilde{f})+h-\tilde{h}\|_{L^1(\Omega)}^{q/2} \left[\int_{B_k} (1+|\nabla u|+|\nabla\tilde{u}|)^q dx\right]^{(2-p)/2} [\operatorname{meas} B_k]^{(p-q)/2}.$$

Therefore, using the Hölder inequality for series and (2.10)

$$\text{(2.11)} \qquad \int_\Omega |\nabla(u-\tilde{u})|^q dx = \sum_{k=0}^{+\infty} \int_{B_k} |\nabla(u-\tilde{u})|^q dx \leq$$

$$\leq c_2 \|(f-\tilde{f})+h-\tilde{h}\|_{L^1(\Omega)}^{q/2} \left[\int_\Omega (1+|\nabla u|+|\nabla\tilde{u}|)^q dx\right]^{(2-p)/2} \left[\sum_{k=0}^{+\infty} [\operatorname{meas}(B_k)]^{(p-q)/p}\right]^{p/2} \leq$$

$$\leq \Theta \|(f-\tilde{f})+h-\tilde{h}\|_{L^1(\Omega)}^{q/2} \left[\sum_{k=0}^{+\infty} [\operatorname{meas}(B_k)]^{(p-q)/p}\right]^{p/2},$$

where Θ is a quantity which is bounded if $\|f\|_{W^{-1,p'}(\Omega)}$, $\|\tilde{f}\|_{W^{-1,p'}(\Omega)}$, $\|h\|_{L^1(\Omega)}$, and $\|\tilde{h}\|_{L^{-1}(\Omega)}$, are bounded. Let us consider the series on the right hand side of (2.11). If $q^* = Nq/(N-q)$, we have

$$\sum_{k=0}^{+\infty} [\operatorname{meas}(B_k)]^{(p-q)/p} \leq [\operatorname{meas} B_0]^{(p-q)/p} + \sum_{k=1}^{+\infty} \frac{1}{k^{(q^*(p-q))/p}} \left[\int_{B_k} |u-\tilde{u}|^{q^*} dx\right]^{(p-q)/p} \leq$$

$$\leq [\operatorname{meas} \Omega]^{(p-q)/p} + \left[\sum_{k=1}^{+\infty} \frac{1}{k^{(q^*(p-q))/q}}\right]^{q/p} \left[\int_\Omega |u-\tilde{u}|^{q^*} dx\right]^{(p-q)/p}.$$

The series on the right hand side is convergent because the assumption $q < \bar{q} = N(p-1)/(N-1)$ is equivalent to $q^*(p-q)/q > 1$. Hence (2.11) and the Sobolev inequality imply

$$\text{(2.12)} \qquad \int_\Omega |\nabla(u-\tilde{u})|^q dx \leq$$

$$\leq c_3 \Theta \|(f-\tilde{f})+h-\tilde{h}\|_{L^1(\Omega)}^{q/2} \left\{1 + \left[\int_\Omega |\nabla(u-\tilde{u})|^q dx\right]^{q^*(p-q)/2q}\right\}.$$

Now it is easy to check that $q^*(p-q)/2q < 1$, and so $\|\nabla(u-\tilde{u})\|_{(L^q(\Omega))^N}$ is bounded.

Suppose now that $\|f\|_{W^{-1,p'}(\Omega)}$, $\|\tilde{f}\|_{W^{-1,p'}(\Omega)}$, $\|h\|_{L^1(\Omega)}$, and $\|\tilde{h}\|_{L^{-1}(\Omega)}$ are bounded by a constant, so that we can drop Θ. From (2.12), and the inequality $s^\eta \leq s+1$ $(s > 0,$

$\eta = q^*(p-q)/2q$, we obtain

$$\int_\Omega |\nabla(u-\tilde{u})|^q dx \leq c_4 \|(f-\tilde{f}) + h - \tilde{h}\|_{L^1(\Omega)}^{q/2} \left\{ 2 + \int_\Omega |\nabla(u-\tilde{u})|^q dx \right\}.$$

Then, if $\|(f-\tilde{f}) + h - \tilde{h}\|_{L^1(\Omega)}$ is small enough, we have

$$\|\nabla(u-\tilde{u})\|_{L^q(\Omega)} \leq c_5 \|(f-\tilde{f}) + h - \tilde{h}\|_{L^1(\Omega)}^{1/2},$$

and this concludes the proof. ∎

THEOREM 2.2. – *If* $f \in W^{-1,p'}(\Omega)$, $h \in L^1(\Omega)$, *there exists a unique SOLA* u *of* (2.1), *with* u *in* $W_0^{1,q}(\Omega)$ *for every* $q < \bar{q} = N(p-1)/(N-1)$.

PROOF. – Let $\{h_n\}_{n \in N}$ be a sequence of regular functions such that h_n converges to h strongly in $L^1(\Omega)$, and let $u_n \in W_0^{1,p}(\Omega)$ be the solutions of the equations $A(u_n) = f + h_n$. By Proposition 2.1 (or Proposition 2.2 if $p < 2$), applied with $f = \tilde{f}$ it is easily seen that $\{u_n\}_{n \in N}$ is a Cauchy sequence in $W_0^{1,q}(\Omega)$, for every $q < \bar{q}$, and that its limit in the same space is a solution of (2.1).

Moreover, this solution does not depend on the approximating sequence $\{h_n\}$. Actually, if $\{h_n\}_{n \in N}$ and $\{\tilde{h}_n\}_{n \in N}$ are two sequences of regular functions such that $h_n \to h$, $\tilde{h}_n \to h$ strongly in $L^1(\Omega)$, and if u_n and $\tilde{u}_n$ are the solutions in $W_0^{1,p}(\Omega)$ of $A(u_n) = f + h_n$, $A(\tilde{u}_n) = f + \tilde{h}_n$ respectively, then $\{u_n\}$, $\{\tilde{u}_n\}$ converge respectively to u and $\tilde{u}$, which are solutions of (2.1). From Proposition 2.1 (or Proposition 2.2), applied with $\tilde{f} = f$ we easily obtain that $u_n - \tilde{u}_n \to 0$ strongly in $W_0^{1,q}(\Omega)$. Hence, $u = \tilde{u}$.

Let us suppose that the right hand side of (2.1) is decomposed in a different manner, i.e., that:

$$f + h = \tilde{f} + \tilde{h} \quad \text{in } \mathcal{D}'(\Omega), \tag{2.13}$$

with $f, \tilde{f} \in W^{-1,p'}(\Omega)$, $h, \tilde{h} \in L^1(\Omega)$. We remark that from (2.13) it follows that $f - \tilde{f} = \tilde{h} - h \in W^{-1,p'}(\Omega) \cap L^1(\Omega)$. Let $\{h_n\}_{n \in N}$ and $\{\tilde{h}_n\}_{n \in N}$ be two sequences of regular functions such that $h_n \to h$, $\tilde{h}_n \to \tilde{h}$ strongly in $L^1(\Omega)$. As usual, let us consider the solutions $u_n, \tilde{u}_n \in W_0^{1,p}(\Omega)$ of the equations $A(u_n) = f + h_n$, $A(\tilde{u}_n) = \tilde{f} + \tilde{h}_n$ respectively. As it has been shown, u_n and $\tilde{u}_n$ converge in $W_0^{1,q}(\Omega)$ to u and $\tilde{u}$ respectively, solutions of (2.1). As before, we apply Proposition 2.1 (or Proposition 2.2) to prove that $u = \tilde{u}$. ∎

The following result tells us that a solution which is «regular enough» is the SOLA.

THEOREM 2.3. – *Let* $f \in W^{-1,p'}(\Omega)$, $h \in L^1(\Omega)$, *and let* u *be a solution of* (2.1); *suppose that* $u \in W_0^{1,p}(\Omega)$. *Then* u *is the SOLA of* (2.1).

PROOF. – It is enough to observe that $u \in W_0^{1,p}(\Omega)$ implies $h = A(u) - f \in W^{-1,p'}(\Omega)$, and therefore we are in the variational setting. ■

Assume now that $f \in W^{-1,p'}(\Omega)$, and that $\{\tilde{h}_\varepsilon\}_{\varepsilon \in E}$ is a sequence of $L^1(\Omega)$ functions. Let u_ε be the SOLA of

$$(2.14) \qquad A(u_\varepsilon) = f + h_\varepsilon\,, \qquad u_\varepsilon \in W_0^{1,q}(\Omega) \text{ for every } q < \bar{q} = \frac{N(p-1)}{N-1}\,.$$

In the following theorem we will prove a continuous dependence result with respect to the weak-L^1 convergence of the right hand side of (2.14).

THEOREM 2.4. – *If $h_\varepsilon \rightharpoonup h_0$ weakly in $L^1(\Omega)$, then*

$$u_\varepsilon \rightharpoonup u_0 \quad \textit{strongly in } W_0^{1,q}(\Omega), \textit{ for every } q < \bar{q}\,,$$

where u_0 is the SOLA of $A(u_0) = f + h_0$.

PROOF. – For the sake of simplicity, let us suppose that $p \geq 2$. If $2 - 1/N < p < 2$ the proof is the same, the only difference being the use of Proposition 2.2 instead of Proposition 2.1. For $k \in \boldsymbol{N}$, define the function

$$(2.15) \qquad T_k(s) = \begin{cases} k & \text{if } s > k\,, \\ s & \text{if } |s| \leq k\,, \\ -k & \text{if } s < -k\,, \end{cases}$$

and consider the solution $u_\varepsilon^k \in W_0^{1,p}(\Omega)$ of $A(u_\varepsilon^k) = f + T_k(h_\varepsilon)$. Since, for every k, $T_k(h_\varepsilon) \in L^\infty(\Omega) \subset W^{-1,p'}(\Omega)$, for every ε and k there exists a unique solution u_ε^k. Now take $l > k$. Applying Proposition 2.1 to u_ε^l and u_ε^k, we see that $\|u_\varepsilon^l - u_\varepsilon^k\|_{W_0^{1,q}(\Omega)} \leq \Psi(\|T_l(h_\varepsilon) - T_k(h_\varepsilon)\|_{L^1(\Omega)})$ for every $q \in (1, \bar{q})$. On the other hand, we can write

$$\|T_l(h_\varepsilon) - T_k(h_\varepsilon)\|_{L^1(\Omega)} \leq \|h_\varepsilon - T_k(h_\varepsilon)\|_{L^1(\Omega)} \leq \int_{\{|h_\varepsilon| > k\}} |h_\varepsilon|\, dx\,.$$

Since the functions h_ε are equi-integrable, the right hand side can be made arbitrarily small, uniformly with respect to ε, if we choose k large enough. It follows that $\|u_\varepsilon^l - u_\varepsilon^k\|_{W_0^{1,q}(\Omega)} \leq \Psi(\eta_k)$, where $\{\eta_k\}$ is a sequence of positive numbers that converges to zero as k tends to infinity. If we let l tend to $+\infty$, we obtain $\|u_\varepsilon - u_\varepsilon^k\|_{W_0^{1,q}(\Omega)} \leq \Psi(\eta_k)$. By (2.9), this implies that the convergence of u_ε^k to u_ε in $W_0^{1,q}(\Omega)$ as $k \to +\infty$ is uniform with respect to ε, that is

$$(2.16) \qquad \lim_{k \to +\infty} \sup_\varepsilon \|u_\varepsilon - u_\varepsilon^k\|_{W_0^{1,q}(\Omega)} = 0\,.$$

Since $|T_k(h_\varepsilon)| \leq k$, for every fixed k it is possible to find a subsequence of $\{T_k(h_\varepsilon)\}_{\varepsilon \in E}$, still denoted by the same symbol, and a function γ_k such that

$$(2.17) \qquad T_k(h_\varepsilon) \xrightarrow{\varepsilon \to 0} \gamma_k \quad *\text{-weakly in } L^\infty(\Omega)\,.$$

Moreover, by means of a diagonal argument, we can assume that (2.17) holds for every $k \in N$. An immediate consequence of (2.17) is that

$$u_\varepsilon^k \xrightarrow{\varepsilon \to 0} u_0^k \quad \text{strongly in } W_0^{1,p}(\Omega), \text{ for every } k \in N, \tag{2.18}$$

where u_0^k is the solution of $A(u_0^k) = f + \gamma_k$. We remark that, in general, we do not have $\gamma_k = T_k(h_0)$. However, $\gamma_k \to h_0$ strongly in $L^1(\Omega)$ as $k \to \infty$. Indeed, by the weak lower semicontinuity of the norm, we have:

$$\|h_0 - \gamma_k\|_{L^1(\Omega)} \leq \liminf_{\varepsilon \to 0} \|h_\varepsilon - T_k(h_\varepsilon)\|_{L^1(\Omega)} \leq \sup_{\varepsilon \in E} \int_{\{|h_\varepsilon| > k\}} |h_\varepsilon| \leq \eta_k .$$

By (2.18), since u_0 is a SOLA, this implies that

$$u_0^k \xrightarrow{k \to +\infty} u_0 \quad \text{strongly in } W_0^{1,q}(\Omega), \text{ for every } q < \bar{q}. \tag{2.19}$$

Now we can write

$$\|u_\varepsilon - u_0\|_{W_0^{1,q}(\Omega)} \leq \|u_\varepsilon - u_\varepsilon^k\|_{W_0^{1,q}(\Omega)} + \|u_\varepsilon^k - u_0^k\|_{W_0^{1,q}(\Omega)} + \|u_0^k - u_0\|_{W_0^{1,q}(\Omega)} .$$

By (2.17) and (2.19), the first and third term can be made arbitrarily small if we choose k large enough. Once we have fixed k, we will use (2.18) and the fact that $q < p$ to choose ε so that the second term is as small as we want. The proof of the theorem is then finished. ■

3. - H-convergence of SOLA's of linear elliptic equations.

In this section we will study the behavior of the SOLA's of linear elliptic equations with $L^1(\Omega)$ data, with respect to the H-convergence of the operators. In this section we will restrict ourselves to the linear case, and will therefore assume $p = 2$. We will assume that $\{a_\varepsilon\}$ is a sequence of matrices in $\mathcal{M}(\alpha, \beta; \Omega)$ such that $a_\varepsilon \xrightharpoonup{H} a_0$ (see Section 1). Let p_ε be the correction matrices associated with a_ε. We assume that

$$\|p_\varepsilon\|_{(L^\infty(\Omega))^{N^2}} \leq c_1 . \tag{3.1}$$

This regularity hypothesis, which has been considered also in [BoM1], [BD], and [BBM] and is satisfied, for instance, in the case of periodic homogenization, can be weakened by using a different corrector technique, as in [BBDM]. We have the following result.

THEOREM 3.1. - *Let $f \in H^{-1}(\Omega)$, and suppose that $a_\varepsilon \xrightharpoonup{H} a_0$ and that $\{h_\varepsilon\}_{\varepsilon \in E}$ is a sequence of $L^1(\Omega)$ functions such that $h_\varepsilon \rightharpoonup h_0$ weakly in $L^1(\Omega)$. Let u_ε and u_0 be re-*

spectively the SOLA's of the equations $A_\varepsilon u_\varepsilon = f + h_\varepsilon$, $A_0 u_0 = f + h_0$. *Then*

$$u_\varepsilon \xrightarrow{\varepsilon \to 0} u_0 \quad \text{weakly in } W_0^{1,q}(\Omega), \tag{3.2}$$

$$\nabla u_\varepsilon - p_\varepsilon \nabla u_0 \xrightarrow{\varepsilon \to 0} 0 \quad \text{strongly in } (L^q(\Omega))^N, \tag{3.3}$$

for every $q < \bar{q} = N/(N-1)$.

PROOF. – Let us define $T_k(s)$ as in (2.15), and consider the solutions $u_\varepsilon^k \in H_0^1(\Omega)$ of the equations $A_\varepsilon u_\varepsilon^k = f + T_k(h_\varepsilon)$. Using Proposition 2.1 as in the proof of Theorem 2.4, we obtain

$$u_\varepsilon^k \xrightarrow{k \to +\infty} u_\varepsilon \quad \text{strongly in } H_0^1(\Omega), \text{ uniformly with respect to } \varepsilon. \tag{3.4}$$

Then we extract a subsequence such that $T_k(h_\varepsilon)$ converges *-weakly in $L^\infty(\Omega)$ to a function γ_k for every $k \in N$. Therefore by the standard properties of the H-convergence, for every fixed k we have

$$u_\varepsilon^k \xrightarrow{\varepsilon \to 0} u_0^k \quad \text{weakly in } W_0^{1,q}(\Omega), \tag{3.5}$$

$$\nabla u_\varepsilon^k - p_\varepsilon \nabla u_0^k \xrightarrow{\varepsilon \to 0} 0 \quad \text{strongly in } (L^1(\Omega))^N, \tag{3.6}$$

where $u_0^k \in H_0^1(\Omega)$ is the solution of $A_0 u_0^k = f + \gamma_k$. Moreover, as in the proof of Theorem 2.4, one can check that $\gamma_k \to h_0$ strongly in $L^1(\Omega)$ as $k \to +\infty$. This yields

$$u_0^k \xrightarrow{k \to +\infty} u_0 \quad \text{strongly in } W_0^{1,q}(\Omega). \tag{3.7}$$

Let $\phi \in W^{-1,q'}(\Omega)$. We can write

$$|\langle \phi, u_\varepsilon - u_0 \rangle_{W^{-1,q'}(\Omega), W_0^{1,q}(\Omega)}| \le \|\phi\|_{W^{-1,q'}(\Omega)} \|u_\varepsilon - u_\varepsilon^k\|_{W_0^{1,q}(\Omega)} +$$
$$+ |\langle \phi, u_\varepsilon^k - u_0^k \rangle_{W^{-1,q'}(\Omega), W_0^{1,q}(\Omega)}| + \|\phi\|_{W^{-1,q'}(\Omega)} \|u_0^k - u_0\|_{W_0^{1,q}(\Omega)}.$$

By (3.4) and (3.7) we can choose k so large that the first and third term of the right hand side are arbitrarily small. Once we have fixed k, we can use (3.5) to conclude that the second term tends to zero as ε tends to zero. Hence, (3.2) is proved. The proof of (3.3) is similar. Indeed we can write

$$\|\nabla u_\varepsilon - p_\varepsilon \nabla u_0\|_{(L^1(\Omega))^N} \le \|\nabla u_\varepsilon - \nabla u_\varepsilon^k\|_{(L^1(\Omega))^N} +$$
$$+ \|\nabla u_\varepsilon^k - p_\varepsilon \nabla u_0^k\|_{(L^1(\Omega))^N} + \|p_\varepsilon\|_{(L^\infty(\Omega))^{N^2}} \|\nabla u_0^k - \nabla u_0\|_{(L^1(\Omega))^N}.$$

Hence, using (3.1), (3.4), (3.6) and (3.7) one shows that $\nabla u_\varepsilon - p_\varepsilon \nabla u_0 \to 0$ strongly in $(L^1(\Omega))^N$. Since $\|\nabla u_\varepsilon - p_\varepsilon \nabla u_0\|_{(L^q(\Omega))^N}$ is bounded for every $q < \bar{q}$, (3.3) holds true, and Theorem 3.1 is proved. ■

4. – SOLA's of nonlinear parabolic equations with L^1 data.

The aim of this section is to extend to parabolic equations results of the same kind of those proved for elliptic equations in the preceding sections. Let $\Omega \subset \boldsymbol{R}^N$ be a bounded, open set, $N \geq 2$. Let T be a positive, real number. We will denote by Q the cylinder $\Omega \times (0, T)$, and by $\Gamma = \partial\Omega \times (0, T)$ the lateral surface of Q. If $t \in (0, T)$ we will define Q_t as the cylinder $\Omega \times (0, t)$. We will suppose that p is a real number such that:

$$2 - \frac{1}{N+1} < p \leq \frac{N(N+2)}{N+1} \tag{4.1}$$

(see Remarks 4.1 and 4.2 for an expanation of these bounds).

Let $a(x, t, \xi)$: $\Omega \times (0, T) \times \boldsymbol{R}^N \to \boldsymbol{R}^N$ be a Carathéodory function such that, for almost every $(x, t) \in Q$, for every $\xi, \eta \in \boldsymbol{R}^N$,

$$a(x, t, \xi)\cdot\xi \geq \alpha|\xi|^p\,, \tag{4.2}$$

$$(a(x, t, \xi) - a(x, t, \eta))\cdot(\xi - \eta) \geq \alpha_1|\xi - \eta|^p \quad \text{if } 2 \leq p < +\infty\,, \tag{4.3a}$$

$$(a(x, t, \xi) - a(x, t, \eta))\cdot(\xi - \eta) \geq \alpha_1 \frac{|\xi - \eta|^2}{(1 + |\xi| + |\eta|)^{2-p}} \quad \text{if } 2 - \frac{1}{N+1} < p < 2\,, \tag{4.3b}$$

$$|a(x, t, \xi)| \leq K(x, t) + \beta|\xi|^{p-1}, \tag{4.4}$$

where α, α_1 and β are positive constants, and $k(x, t)$ is a function belonging to $L^{p'}(Q)$.

As in the elliptic case, we define the differential operator $A(u) = -\operatorname{div} a(x, t, \nabla u)$. We are interested in the study of parabolic problems of the following type:

$$\begin{cases} \dfrac{\partial u}{\partial t} + A(u) = f + h & \text{in } Q\,, \\ u(\cdot, 0) = w(\cdot), & \\ u(x, t) = 0 & \text{on } \Gamma. \end{cases} \tag{4.5}$$

We will assume the following hypotheses on the data f, h, and w:

$$f \in L^{p'}(0, T; W^{-1,p'}(\Omega))\,, \qquad h \in L^1(Q)\,, \qquad w \in L^1(\Omega)\,.$$

It is well known that if the right hand side of equation (3.5) belongs to $L^{p'}(0, T; W^{-1,p'}(\Omega))$ and the initial datum w is in $L^2(\Omega)$ then there exists a unique solution u belonging to $L^p(0, T; W_0^{1,p}(\Omega))$ (see [L]). On the other hand, if the data are $L^1(\Omega)$ functions, or even bounded Radon measures, existence results for these nonlinear problems have been proved by BOCCARDO and GALLOUËT in [BG]. In that paper it is proved (in the case $f = 0$) that, if h and w are bounded Radon measures resp. on Q

and Ω, then there exists a solution u of (4.5) such that

$$u \in L^q(0, T; W^{1,q}(\Omega)) \quad \text{for every } q < \bar{q} = p - \frac{N}{N+1}. \tag{4.6}$$

REMARK 4.1. – We note explicitly that the condition $p > 2 - 1/(N+1)$ implies that $p - N/(N-1) > 1$, so that we always have $\bar{q} > 1$.

REMARK 4.2. – In [BG], as in the proof of Proposition 4.1 below, the Sobolev embedding $W_0^{1,q}(\Omega) \subset L^{q^*}(\Omega)$ (where $q^* = Nq/(N-q)$), which holds for $q < N$, is used. If p is larger than $N(N+2)/(N+1)$, then $p - N/(N+1) > N$, and so the condition $q < N$ it is not automatically satisfied. However, in this case, using the embedding $W_0^{1,q}(\Omega) \subset L^\infty(\Omega)$ if $q > N$, one could repeat every proof changing the bound $q < p - N/(N+1)$ with the (slightly stronger) bound $q < (p - 2 + \sqrt{p^2+4})/2$. Hence, formula (4.6), and all the statements below, can be modified removing the upper bound on p in (4.1) and choosing

$$\bar{q} = \min\left\{p - \frac{N}{N+1}, \frac{p-2+\sqrt{p^2+4}}{2}\right\}.$$

Anyway, all we will need to remember about $\bar{q}$ is that $\bar{q} > 1$ and that $p > \bar{q} > p - 1$, so that, by means of (4.4), $a(x, t, \nabla u) \in (L^1(Q))^N$.

As in the elliptic case, we are interested to the definition of solutions as limit of approximations, or SOLA's. The first step is to give the parabolic counterpart of Propositions 2.1 and 2.2.

PROPOSITION 4.1. – *Let $f, \tilde{f} \in L^{p'}(0, T; W^{-1,p'}(\Omega))$ be such that $f - \tilde{f} \in L^1(Q)$. Let h, $\tilde{h}$, w, $\tilde{w}$ be regular data, and define u, $\tilde{u}$ as the solutions of problems* (4.5) *and*

$$\begin{cases} \dfrac{\partial \tilde{u}}{\partial t} + A(\tilde{u}) = \tilde{f} + \tilde{h} & \text{in } Q, \\ \tilde{u}(\cdot, 0) = \tilde{w}(\cdot), \\ \tilde{u}(x,t) = 0 & \text{on } \Gamma, \end{cases} \tag{4.7}$$

respectively. Then, for every $q < \bar{q}$

$$\|\nabla(u - \tilde{u})\|_{(L^q(Q))^N} \leq \Lambda,$$

where Λ depends on $\|(f-\tilde{f}) + h - \tilde{h}\|_{L^1(Q)}$, $\|w - \tilde{w}\|_{L^1(\Omega)}$, $\|f\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))}$, $\|\tilde{f}\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))}$, $\|h\|_{L^1(Q)}$, $\|\tilde{h}\|_{L^1(Q)}$, $\|w\|_{L^1(\Omega)}$, $\|\tilde{w}\|_{L^1(\Omega)}$, α, α_1, N, p, q, and Q. The dependence of Λ on the various norms is such that, if $\|(f-\tilde{f}) + h - \tilde{h}\|_{L^1(Q)} + \|w - \tilde{w}\|_{L^1(\Omega)}$ tends to zero, and all the other norms are bounded, then Λ converges to zero.

PROOF. – We follow the method of [BG]. Let $k > 0$, and consider the truncation functions $T_k(s)$ defined as in (2.15), and the function $\phi_1(s) = \int_0^s T_1(\sigma)\, d\sigma$. For $\tau \in (0, T)$, we use the function $T_1(u - \tilde{u})\chi_{(0,\tau)}(t)$ as test function in (4.5), (4.7); we easily obtain

$$\int_\Omega \phi_1((u - \tilde{u})(x, \tau))\, dx - \int_\Omega \phi_1((w - \tilde{w})(x))\, dx +$$

$$+ \int_{B_0 \cap Q_\tau} [a(x, t, \nabla u) - a(x, t, \nabla \tilde{u})] \nabla(u - \tilde{u})\, dx\, dt =$$

$$= \int_{Q_\tau} (f - \tilde{f} + h - \tilde{h})\, T_1(u - \tilde{u})\, dx\, dt \leq \|f - \tilde{f} + h - \tilde{h}\|_{L^1(Q)},$$

where, for $k \in N$, we have defined $B_k = \{(x, t) \in Q \colon k \leq |(u - \tilde{u})(x, t)| < k + 1\}$. By means of hypotheses (4.3*a*), or (4.3*b*), and since $|s| - 1/2 \leq \phi_1(s) \leq |s|$, we obtain:

$$\|(u - \tilde{u})(\cdot, \tau)\|_{L^1(\Omega)} \leq \|f - \tilde{f} + h - \tilde{h}\|_{L^1(Q)} + \|w - \tilde{w}\|_{L^1(\Omega)} + \frac{1}{2} \operatorname{meas} \Omega .$$

Since this estimate holds for every $\tau \in (0, T)$, if we define $\varrho = \|f - \tilde{f} + h - \tilde{h}\|_{L^1(Q)} + \|w - \tilde{w}\|_{L^1(\Omega)}$, we can write:

$$\|u - \tilde{u}\|_{L^\infty(0, T; L^1(\Omega))} \leq \varrho + \frac{1}{2} \operatorname{meas} \Omega . \tag{4.8}$$

If we choose $\varphi_k(u - \tilde{u})$ (defined as in the proof of Proposition 2.1) as test function, we obtain:

$$\int_\Omega \psi_k((u - \tilde{u})(x, T))\, dx - \int_\Omega \psi_k((w - \tilde{w})(x))\, dx +$$

$$+ \int_{B_k} [a(x, t, \nabla u) - a(x, t, \nabla \tilde{u})] \cdot \nabla(u - \tilde{u})\, dx\, dt \leq \|f - \tilde{f} + h - \tilde{h}\|_{L^1(Q)},$$

where we have defined $\psi_k(s) = \int_0^s \varphi_k(\sigma)\, d\sigma$. Since $0 \leq \psi_k(s) \leq |s|$, using again (4.3*a*), or (4.3*b*), we get

$$\alpha_1 \int_{B_k} |\nabla(u - \tilde{u})|^p\, dx\, dt \leq \varrho \quad \text{if } p \geq 2, \tag{4.9}$$

$$\alpha_1 \int_{B_k} \frac{|\nabla(u - \tilde{u})|^2}{(1 + |\nabla u| + |\nabla \tilde{u}|)^{2-p}}\, dx\, dt \leq \varrho \quad \text{if } 2 - \frac{1}{N+1} < p < 2 . \tag{4.10}$$

If $p \geq 2$, from (4.9), by the Hölder inequality, we obtain

$$\int_Q |\nabla(u - \tilde{u})|^q \, dx \, dt = \int_{B_0} |\nabla(u - \tilde{u})|^q \, dx \, dt + \sum_{k=1}^{+\infty} \int_{B_k} |\nabla(u - \tilde{u})|^q \, dx \, dt \leq$$

$$\leq \alpha_1^{-q/p} (\operatorname{meas} Q)^{(p-q)/p} \varrho^{q/p} + \alpha_1^{-q/p} \varrho^{q/p} \sum_{k=1}^{+\infty} (\operatorname{meas} B_k)^{(p-q)/p} \leq$$

$$\leq c_1 \varrho^{q/p} \left\{ 1 + \sum_{k=1}^{+\infty} \frac{1}{k^{r(p-q)/p}} \left[\int_{B_k} |u - \tilde{u}|^r \, dx \, dt \right]^{(p-q)/p} \right\},$$

where $r = (N+1)q/N$. Let us study the right hand side. By means of the Hölder inequality for series:

$$\sum_{k=1}^{+\infty} \frac{1}{k^{r(p-q)/p}} \left[\int_{B_k} |u - \tilde{u}|^r \, dx \, dt \right]^{(p-q)/p} \leq \left[\int_Q |u - \tilde{u}|^r \, dx \, dt \right]^{(p-q)/p} \left[\sum_{k=1}^{+\infty} \frac{1}{k^{r(p-q)/p}} \right]^{q/p}.$$

By the hypotheses on r and q, we easily obtain that $r(p-q)/q > 1$, so that the latter series is finite. Hence

$$\int_Q |\nabla(u - \tilde{u})|^q \, dx \, dt \leq c_2 \varrho^{q/p} \left\{ 1 + \left[\int_Q |u - \tilde{u}|^r \, dx \, dt \right]^{(p-q)/p} \right\}. \tag{4.11}$$

Since $1 < r < q^*$, the interpolation inequality implies that, for almost every $t \in (0, T)$,

$$\|(u - \tilde{u})(\cdot, t)\|_{L^r(\Omega)} \leq \|(u - \tilde{u})(\cdot, t)\|_{L^1(\Omega)}^{\theta} \|(u - \tilde{u})(\cdot, t)\|_{L^{q^*}(\Omega)}^{1-\theta},$$

where $\theta \in (0, 1)$ is such that $1/r = \theta + (1 + \theta)/q^*$. By simple calculations, $r\theta = q/N$, $r(1 - \theta) = q$, so that

$$\int_Q |u - \tilde{u}|^r \, dx \, dt \leq \int_0^T \|(u - \tilde{u})(\cdot, t)\|_{L^1(\Omega)}^{r\theta} \|(u - \tilde{u})(\cdot, t)\|_{L^{q^*}(\Omega)}^{r(1-\theta)} \, dt \leq$$

$$\leq \|(u - \tilde{u})\|_{L^\infty(0, T; L^1(\Omega))}^{q/N} \int_0^T \|(u - \tilde{u})(\cdot, t)\|_{L^{q^*}(\Omega)}^q \, dt.$$

By (4.8) and the Sobolev embedding we deduce:

$$\int_Q |u - \tilde{u}|^r \, dx \, dt \leq c_3 \left(\varrho + \frac{\operatorname{meas} \Omega}{2} \right)^{q/N} \int_Q |\nabla(u - \tilde{u})|^q \, dx \, dt.$$

Inequality (4.11) can be rewritten as

$$\int_Q |\nabla(u-\tilde{u})|^q\,dx\,dt \leq c_4\varrho^{q/p}\left[1+\left(\varrho+\frac{\operatorname{meas}\Omega}{2}\right)^{(q(p-q))/Np}\left[\int_Q |\nabla(u-\tilde{u})|^q\,dx\,dt\right]^{(p-q)/p}\right\}.$$

Since $(p-q)/p<1$, the left hand side is bounded. Moreover, if ϱ is small enough, we have

$$\int_Q |\nabla(u-\tilde{u})|^q\,dx\,dt \leq c_5\varrho^{q/p},$$

where c_5 is a positive constant that depends on Q, α_1, N, p, q. This concludes the proof of Proposition 4.1 in the case $p \geq 2$.

If $2-1/(N+1)<p<2$, the proof is slightly more complicated. As first step, using the techniques of [BG], we obtain for u and $\tilde{u}$,

$$\|\nabla u\|_{(L^q(Q))^N} \leq c_6, \qquad \|\nabla\tilde{u}\|_{(L^q(Q))^N} \leq c_7 \qquad \text{for every } q<\bar{q}, \tag{4.12}$$

where c_6 is a positive constant that depends on $\|f\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))}$, $\|h\|_{L^1(Q)}$, $\|w\|_{L^1(\Omega)}$, Q, α, N, p, and q, while c_7 depends on the norms of $\tilde{f}$, $\tilde{h}$ and $\tilde{w}$ in the same spaces. Using (4.10) and (4.12), we have

$$\int_{B_k} |\nabla(u-\tilde{u})|^q\,dx\,dt = \int_{B_k} \frac{|\nabla(u-\tilde{u})|^2}{(1+|\nabla u|+|\nabla\tilde{u}|)^{q(2-p)/2}}(1+|\nabla u|+|\nabla\tilde{u}|)^{q(2-p)/2}\,dx\,dt \leq$$

$$\leq \left[\int_{B_k} \frac{|\nabla(u-\tilde{u})|^2}{(1+|\nabla u|+|\nabla\tilde{u}|)^{2-p}}\,dx\,dt\right]^{q/2}\left[\int_{B_k}(1+|\nabla u|+|\nabla\tilde{u}|)^{q(2-p)/(2-q)}\,dx\,dt\right]^{(2-q)/2} \leq$$

$$\leq \left(\frac{\varrho}{\alpha_1}\right)^{q/2}\left[\int_{B_k}(1+|\nabla u|+|\nabla\tilde{u}|)^{q(2-p)/(2-q)}\,dx\,dt\right]^{(2-q)/2},$$

and, going on as in the proof of Proposition 2.2,

$$\int_Q |\nabla(u-\tilde{u})|^q\,dx\,dt \leq c_8\varrho^{q/2}\left[\int_Q (1+|\nabla u|+|\nabla\tilde{u}|)^q\,dx\,dt\right]^{(2-p)/2}\left[\sum_{k=0}^{+\infty}[\operatorname{meas}B_k]^{(p-q)/p}\right]^{p/2}.$$

By (4.12), the quantity $\int_Q (1+|\nabla u|+|\nabla\tilde{u}|)^q\,dx\,dt$ is bounded by some positive constant Θ, depending on $\|f\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))}$, $\|\tilde{f}\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))}$, $\|h\|_{L^1(Q)}$, $\|\tilde{h}\|_{L^1(Q)}$, $\|w\|_{L^1(\Omega)}$, and $\|\tilde{w}\|_{L^1(\Omega)}$, so that

$$\int_Q |\nabla(u-\tilde{u})|^q\,dx\,dt \leq c_9\Theta^{q/2}\left[\sum_{k=0}^{+\infty}[\operatorname{meas}B_k]^{(p-q)/p}\right]^{p/2}.$$

The series that appears in the right hand side can be studied exactly as in the case

$p \geq 2$, obtaining

$$\left[\sum_{k=0}^{+\infty} [\operatorname{meas} B_k]^{(p-q)/p}\right]^{p/2} \leq c_{10}(1 + \|u - \tilde{u}\|_{L^\infty(0,T;L^1(\Omega))}^{(q(p-q))/2N} \|\nabla(u - \tilde{u})\|_{(L^q(Q))^N}^{(q(p-q))/2}).$$

By (4.8), this implies:

$$\|\nabla(u - \tilde{u})\|_{(L^q(Q))^N}^q \leq c_{11}\,\Theta\varrho^{q/2}(1 + (1+\varrho)^{(q(p-q))/2N} \|\nabla(u - \tilde{u})\|_{(L^q(Q))^N}^{(q(p-q))/2}).$$

Since $q(p-q)/2 < q$, $\|\nabla(u - \tilde{u})\|_{(L^q(Q))^N}$ is bounded by a constant (depending on $\|f\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))}$, $\|\tilde{f}\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))}$, $\|h\|_{L^1(Q)}$, $\|\tilde{h}\|_{L^1(Q)}$, $\|w\|_{L^1(\Omega)}$, $\|\tilde{w}\|_{L^1(\Omega)}$, α, α_1, N, p, q, and Q). If ϱ is small enough, and all the other norms are bounded, then $\|\nabla(u - \tilde{u})\|_{(L^q(Q))^N} \leq c_{12}\sqrt{\varrho}$, and this concludes the proof of Proposition 4.1. ∎

THEOREM 4.1. – *There exists a unique SOLA u (which belongs to $L^q(0, T; W_0^{1,q}(\Omega))$ for every $q < \bar{q}$) of problem* (4.5).

PROOF. – Consider two sequences $\{h_n\}_{n\in N}$ and $\{w_n\}_{n\in N}$ of regular functions such that $h_n \to h$ strongly in $L^1(Q)$, $w_n \to w$ strongly in $L^1(\Omega)$. For $n \in N$ let $u_n \in L^p(0, T; W_0^{1,p}(\Omega))$ be the solutions of problems

$$\begin{cases} \dfrac{\partial u_n}{\partial t} + A(u_n) = f + h_n & \text{in } Q, \\ u_n(\cdot, 0) = w_n(\cdot), & \\ u_n(x, t) = 0 & \text{on } \Gamma. \end{cases} \tag{4.13}$$

Applying Proposition 4.1 to these problems, we deduce that the sequence $\{u_n\}_{n\in N}$ is a Cauchy sequence in $W_0^{1,q}(Q)$, for every $q < \bar{q}$. Hence u_n converges in this space to a function u, which is a solution of problem (4.5). Indeed, it is easily seen that u satisfies the equation in the sense of distributions in Q (and obviously satisfies the boundary condition on Γ). To pass to the limit on the initial condition, we recall the estimate $\|u_n\|_{L^\infty(0,T;L^1(\Omega))} \leq c_1$, which implies $\|u_n\|_{L^\infty(0,T;W^{-1,r}(\Omega))} \leq c_2$, for every $r < N/(N-1)$. On the other hand, by (4.13) we obtain $\partial u_n/\partial t = f + h_n - A(u_n)$, and so

$$\|u_n(t+\delta) - u_n(t)\|_{W^{-1,r}(\Omega)} = \left\| \int_t^{t+\delta} \frac{\partial u_n}{\partial t}(s)\,ds \right\|_{W^{-1,r}(\Omega)} \leq$$

$$\leq \int_t^{t+\delta} (\|f(s)\|_{W^{-1,r}(\Omega)} + \|h_n(s)\|_{W^{-1,r}(\Omega)} + \|A(u_n)(s)\|_{W^{-1,r}(\Omega)})\,ds.$$

If we choose $r < q/(p-1)$ (note that this implies that $r < p'$, $r < N/(N-1)$), and use hypothesis (4.4), then the last integral can be made small (uniformly with respect to n and t) if δ is small. It follows, by a generalization of the Ascoli-Arzelà theorem (see, e.g., [Si]), that $\{u_n\}_{n\in N}$ is relatively compact in $C([0, T]; W^{-1,r}(\Omega))$; hence u_n converges to u in this space, so that $u_n(\cdot, 0)$ tends to $u(\cdot, 0)$ in $W^{-1,r}(\Omega)$; this implies that u satisfies the initial condition $u(\cdot, 0) = w(\cdot)$.

Now, let $\{\tilde{h}_n\}_{n \in N}$ and $\{\tilde{w}_n\}_{n \in N}$ be a different approximation of h and w respectively. As we have shown before, the solutions $\tilde{u}_n$ of the parabolic problems with data $\tilde{h}_n$, $\tilde{w}_n$, converge in $L^q(0, T; W_0^{1,q}(\Omega))$ to a solution $\tilde{u}$ of (4.5). Applying Proposition 4.1 to $u_n - \tilde{u}_n$, we have that $\|u_n - \tilde{u}_n\|_{L^q(0, T; W_0^{1,q}(\Omega))} \to 0$, and so $u = \tilde{u}$. If we assume to have a different decomposition of the right hand side of equation (4.5), i.e., if $f + h = \tilde{f} + \tilde{h}$, where $f, \tilde{f} \in L^{p'}(0, T; W^{-1,p'}(\Omega))$, and $h, \tilde{h} \in L^1(Q)$, the solution found by any approximation is again the same, as it is shown by reasoning as in the proof of Theorem 2.2. ∎

Our next step, which is of fundamental importance for the applications, consists in proving that a solution of (4.5) which belongs to $L^p(0, T; W_0^{1,p}(\Omega))$ is the SOLA of (4.5). The proof of this fact is more complicated in the parabolic case than in the elliptic one, mainly because the time derivative $\partial u/\partial t$ of a solution does not belong to any «classical» dual distributional space, since (as it can be seen from the equation) it is the sum of a term belonging to the dual space $L^{p'}(0, T; W^{-1,p'}(\Omega))$, and a term in $L^1(Q)$. Due to this fact, it is not clear how to treat the term containing the time derivative $\partial u/\partial t$ after multiplication by a test function. Lemma 4.2 below will show what happens of that term if $T_k(u)$ is used as a test function.

We begin by recalling an useful approximation lemma, that can be found in [BMP] (for $p = 2$) or [G] (in the general case); although in these works the initial datum ω is supposed to be in $L^p(\Omega)$, the proof works also under the weaker hypothesis $w \in L^1(\Omega)$.

LEMMA 4.1. – *Let $u \in L^p(0, T; W_0^{1,p}(\Omega))$ be a function such that $\partial u/\partial t = \alpha_1 + \alpha_2$ in the sense of distributions, with $\alpha_1 \in L^{p'}(0, T; W^{-1,p'}(\Omega))$, and $\alpha_2 \in L^1(Q)$; suppose that $u(\cdot, 0) = w(\cdot) \in L^1(\Omega)$. Then there exists a sequence $\{u_n\}_{n \in N}$ of functions such that u_n, $\partial u_n/\partial t \in L^p(0, T; W_0^{1,p}(\Omega))$ (note that this implies that $u_n \in C([0, T]; W_0^{1,p}(\Omega))$),*

$$u_n \to u \quad \text{strongly in } L^p(0, T; W_0^{1,p}(\Omega)), \qquad u_n(\cdot, 0) \to w(\cdot) \quad \text{strongly in } L^1(\Omega),$$

$$\frac{\partial u_n}{\partial t} = \alpha_1^n + \alpha_2^n \in L^{p'}(0, T; W^{-1,p'}(\Omega)) + L^1(Q),$$

where

$$\alpha_1^n \to \alpha_1 \quad \text{strongly in } L^{p'}(0, T; W^{-1,p'}(\Omega)), \qquad \alpha_2^n \to \alpha_2 \quad \text{strongly in } L^1(Q).$$

LEMMA 4.2. – *Suppose that $u \in L^p(0, T; W_0^{1,p}(\Omega))$, and that $\partial u/\partial t = \alpha_1 + \alpha_2 \in L^{p'}(0, T; W^{-1,p'}(\Omega)) + L^1(Q)$, $u(\cdot, 0) \in L^1(\Omega)$. Then $\phi_k(u(\cdot, t)) \in L^1(\Omega)$ for* *every* $t \in [0, T]$, *and*

$$\langle\!\langle \alpha_1, T_k(u) \rangle\!\rangle_Q + \int_Q \alpha_2 T_k(u)\, dx\, dt = \int_\Omega \phi_k(u(T))\, dx - \int_\Omega \phi_k(u(0))\, dx\,,$$

where $\phi_k = \int_0^s T_k(\sigma)\, d\sigma$ *and* $\langle\!\langle \cdot, \cdot \rangle\!\rangle_Q$ *denotes the duality pairing between* $L^{p'}(0, T; W^{-1,p'}(\Omega))$ *and* $L^p(0, T; W_0^{1,p}(\Omega))$.

PROOF. – Let $\{u_n\}$ be the approximating sequence given by Lemma 4.1. Since u_n is «regular»,

$$\text{(4.14)}\qquad \langle\!\langle \alpha_1^n, T_k(u_n) \rangle\!\rangle_Q + \int_Q \alpha_2^n T_k(u_n)\, dx\, dt =$$

$$= \int_Q \frac{\partial u_n}{\partial t} T_k(u_n)\, dx\, dt = \int_\Omega \phi_k(u_n(T))\, dx - \int_\Omega \phi_k(u_n(0))\, dx\,.$$

Since $T_k(u_n) \to T_k(u)$ strongly in $L^p(0, T; W_0^{1,p}(\Omega))$ and *-weakly in $L^\infty(Q)$, $\alpha_1^n \to \alpha_1$ strongly in $L^{p'}(0, T; W^{-1,p'}(\Omega))$, and $\alpha_2^n \to \alpha_2$ strongly in $L^1(Q)$, it is possible to pass to the limit in the l.h.s. of (4.14). To finish the proof, we only have to pass to the limit in the r.h.s. To this aim we introduce the functions

$$z_n(t) = \int_\Omega \phi_k(u_n(x, t))\, dx\,, \qquad z(t) = \int_\Omega \phi_k(u(x, t))\, dx\,.$$

We will show that $z_n \to z$ uniformly (and therefore pointwise) on $[0, T]$. First of all we observe that $z_n \to z$ strongly in $L^1(0, T)$. Indeed

$$\int_0^T |z_n(t) - z(t)|\, dt \le \int_0^T \int_\Omega |\phi_k(u_n(x, t)) - \phi_k(u(x, t))|\, dx\, dt\,,$$

and the last term tends to zero as $n \to +\infty$ by the Vitali theorem, since $\phi_k(u_n)$ converges to $\phi_k(u)$ in measure on Q, and $|\phi_k(u_n) - \phi_k(u)| \le k(|u_n| + |u|)$. Let $0 \le t_1 < t_2 \le T$, and define $Q_{t_1, t_2} = \Omega \times (t_1, t_2)$. For every $n \in N$ we have, since u_n is «regular»:

$$|z_n(t_2) - z_n(t_1)| = \left| \int_\Omega [\phi_k(u_n(x, t_2)) - \phi_k(u_n(x, t_1))]\, dx \right| =$$

$$= \left| \int_{t_1}^{t_2} \int_\Omega \frac{\partial u_n}{\partial t}(x, t)\, T_k(u_n(x, t))\, dx\, dt \right| \le \left| \int_{t_1}^{t_2} \langle \alpha_1^n(t), T_k(u_n(t)) \rangle_{W^{-1,p'}(\Omega),\, W_0^{1,p}(\Omega)}\, dt \right| +$$

$$+ \int_{Q_{t_1, t_2}} |\alpha_2^n(x, t)\, T_k(u_n(x, t))|\, dx\, dt \le c \|\alpha_1^n\|_{L^{p'}(t_1, t_2;\, W^{-1,p'}(\Omega))} + k \|\alpha_2^n\|_{L^1(Q_{t_1, t_2})}\,.$$

Since the last term is small (uniformly with respect to n) if t_1 and t_2 are close enough, the functions $z_n(t)$ are equi-continuous (and obviously equi-bounded, since $z_n(0) = \int_\Omega \phi_k(u_n(0))\,dx$ converges to $\int_\Omega \phi_k(w)\,dx$). By the Ascoli-Arzelà theorem, the sequence $\{z_n\}$ is relatively compact in $C([0, T])$. This, together with the convergence in $L^1(0, T)$, implies that z_n tends to z in $C([0, T])$. ■

THEOREM 4.2. – *Let* $f \in L^{p'}(0, T; W^{-1,p'}(\Omega))$, $h \in L^1(Q)$, $w \in L^1(\Omega)$, *and let* u *be a solution of* (4.5) in the sense of distributions *such that* $u \in L^p(0, T; W_0^{1,p}(\Omega))$. *Then* u *is the SOLA of* (4.5).

PROOF. – Once again, suppose that $\{h_n\}_{n\in N}$ and $\{w_n\}_{n\in N}$ are sequences of regular functions that approximate h and w respectively. Let u_n be the solution of (4.13). As we have seen, u_n converges in $L^q(0, T; W_0^{1,q}(\Omega))$, for every $q < \bar{q}$, to a function v, which is the SOLA of (4.5). We are going to prove that $u = v$. Formally, the idea is to use the function $T_1(u - u_n)\chi_{(0,\tau)}(t)$ as test function in (4.5) and (4.13). The function $u - u_n$ satisfies the hypotheses of Lemma 4.2, since $u - u_n \in L^p(0, T; W_0^{1,p}(\Omega))$ by the hypothesis on u, and $(\partial/\partial t)(u - u_n) = -(A(u) - A(u_n)) + (h - h_n) \in L^{p'}(0, T; W^{-1,p'}(\Omega)) + L^1(Q)$. By Lemma 4.2

$$-\langle\!\langle A(u) - A(u_n), T_1(u - u_n)\rangle\!\rangle_{Q_\tau} + \int_{Q_\tau} (h - h_n)\,T_1(u - u_n)\,dx\,dt =$$

$$= \int_\Omega \phi_1((u - u_n)(\tau))\,dx - \int_\Omega \phi_1(w - w_n)\,dx\,.$$

Since the first term is negative, we obtain

$$\int_\Omega \phi_1((u - u_n)(\tau))\,dx \le \int_{Q_\tau} (h - h_n)\,T_1(u - u_n)\,dx\,dt +$$

$$+ \int_\Omega \phi_1(w - w_n)\,dx \le \|h - h_n\|_{L^1(Q)} + \|w - w_n\|_{L^1(\Omega)}\,.$$

Hence $\lim_{n\to+\infty} \int_\Omega \phi_1((u - u_n)(\tau))\,dx = 0$ for every $\tau \in (0, T)$. It is easily seen that this implies that u_n converges to u in measure. Hence, $u = v$, and the proof is complete. ■

In view of the applications of the following section, our next step will be to study the behavior of the SOLA's of parabolic equations if the right hand side converges weakly in $L^1(Q)$.

THEOREM 4.3. – *Let $f \in L^{p'}(0, T; W^{-1,p'}(\Omega))$, $w \in L^1(\Omega)$, and suppose that $\{h_\varepsilon\}_{\varepsilon \in E}$ is a sequence of functions in $L^1(Q)$ such that $h_\varepsilon \rightharpoonup h_0$ weakly in $L^1(Q)$. Let u_ε and u_0 be the SOLA's of the problems*

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} + A(u_\varepsilon) = f + h_\varepsilon & \text{in } Q, \\ u_\varepsilon(\cdot, 0) = w(\cdot), \\ u_\varepsilon(x, t) = 0 \quad \text{on } \Gamma, \end{cases} \qquad \begin{cases} \dfrac{\partial u_0}{\partial t} + A(u_0) = f + h_0 & \text{in } Q, \\ u_0(\cdot, 0) = w(\cdot), \\ u_0(x, t) = 0 \quad \text{on } \Gamma, \end{cases}$$

respectively. Then $u_\varepsilon \to u_0$ strongly in $L^q(0, T; W_0^{1,q}(\Omega))$, for every $q < \bar{q}$.

PROOF. – Let $k \in N$, and consider the solutions u_ε^k of the problems

$$\begin{cases} \dfrac{\partial u_\varepsilon^k}{\partial t} + A(u_\varepsilon^k) = f + T_k(h_\varepsilon) & \text{in } Q, \\ u_\varepsilon^k(\cdot, 0) = T_k(w)(\cdot), \\ u_\varepsilon^k(x, t) = 0 \quad \text{on } \Gamma. \end{cases}$$

Since the right hand side of the equation belongs to $L^{p'}(0, T; W^{-1,p'}(\Omega))$, it is well known that there exists a unique solution $u_\varepsilon^k \in L^p(0, T; W_0^{1,p}(\Omega))$ for every k. To conclude the proof, it is enough to follow the lines of the proof of Theorem 2.4. ∎

5. – H-convergence of SOLA's of parabolic equations.

Our next step will be, as announced, the study of the H-convergence of operators. We will assume that $p = 2$ and that the operators are linear. Moreover, we will require that the matrices a_ε that define such operators depend only the space variable x. We define $A_\varepsilon v = -\operatorname{div}(a_\varepsilon(x)\nabla v)$, and suppose that $a_\varepsilon \xrightharpoonup{H} a_0$. Under these hypotheses the convergence (1.7) and the corrector result (1.8) hold. Aim of Theorem 5.1 below is to prove for the SOLA's an analogous result if the r.h.s. of the equation is weakly convergent in $L^1(Q)$. To prove this result, we will need an easy preliminary remark.

REMARK 5.1. – If $\{f_\varepsilon\}_{\varepsilon \in E}$ and $\{h_\varepsilon\}_{\varepsilon \in E}$ are two sequences such that $f_\varepsilon \to f_0$ strongly in $L^2(0, T; H^{-1}(\Omega))$, $h_\varepsilon \rightharpoonup h_0$ weakly in $L^2(Q)$, and if $w \in L^2(\Omega)$, then the solutions $u_\varepsilon, u_0 \in L^2(0, T; H_0^1(\Omega))$ of

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} + A_\varepsilon u_\varepsilon = f_\varepsilon + h_\varepsilon & \text{in } Q, \\ u_\varepsilon(0) = w, \end{cases} \qquad \begin{cases} \dfrac{\partial u_0}{\partial t} + A_0 u_0 = f_0 + h_0 & \text{in } Q, \\ u_0(0) = w, \end{cases}$$

satisfy $u_\varepsilon \rightharpoonup u_0$ weakly in $L^2(0, T; H_0^1(\Omega))$, $\nabla u_\varepsilon - p_\varepsilon \nabla u_0 \to 0$ strongly in $L^1(Q)$. To prove this, we introduce the solutions z_ε of the auxiliary problems

$$\begin{cases} \dfrac{\partial z_\varepsilon}{\partial t} + A_\varepsilon z_\varepsilon = f_0 + h_0 \quad \text{in } Q, \\ u_\varepsilon(0) = w. \end{cases}$$

By Proposition 1.1, $z_\varepsilon \rightharpoonup u_0$ weakly in $L^2(0, T; H_0^1(\Omega))$, and $\nabla z_\varepsilon - p_\varepsilon \nabla u_0 \to 0$ strongly in $(L^1(Q))^N$. Since $u_\varepsilon - z_\varepsilon$ converges to zero strongly in $L^2(0, T; H_0^1(\Omega))$, our statement is proved.

THEOREM 5.1. – *Let $f \in L^2(0, T; H^{-1}(\Omega))$, $w \in L^1(\Omega)$, and let $\{h_\varepsilon\}_{\varepsilon \in E}$ be a sequence of functions in $L^1(Q)$ such that $h_\varepsilon \rightharpoonup h_0$ weakly in $L^1(Q)$. Assume that $\{a_\varepsilon\}_{\varepsilon \in E}$ is a sequence of matrices in $\mathcal{M}(\alpha, \beta; \Omega)$ which H-converges to a matrix a_0. Let p_ε be the correction matrices for a_ε, and suppose that $\|p_\varepsilon\|_{(L^\infty(Q))^{N^2}} \leq c$. Let u_ε and u_0 be the SOLA's of the problems*

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} + A_\varepsilon u_\varepsilon = f + h_\varepsilon \quad \text{in } Q, \\ u_\varepsilon(\cdot, 0) = w(\cdot), \\ u_\varepsilon(x, t) = 0 \quad \text{on } \Gamma, \end{cases} \qquad \begin{cases} \dfrac{\partial u_0}{\partial t} + A_0 u_0 = f + h_0 \quad \text{in } Q, \\ u_0(\cdot, 0) = w(\cdot), \\ u_0(x, t) = 0 \quad \text{on } \Gamma, \end{cases}$$

respectively. Then

$$u_\varepsilon \rightharpoonup u_0 \quad \textit{weakly in } L^q(0, T; W_0^{1,q}(\Omega)), \tag{5.1}$$

$$\nabla u_\varepsilon - p_\varepsilon \nabla u_0 \to 0 \quad \textit{strongly in } (L^q(Q))^N, \tag{5.2}$$

for every $q < \bar{q} = (N+2)/(N+1)$.

PROOF. – Let us consider the solutions u_ε^k of the problems

$$\begin{cases} \dfrac{\partial u_\varepsilon^k}{\partial t} + A_\varepsilon u_\varepsilon^k = f + T_k(h_\varepsilon) \quad \text{in } Q, \\ u_\varepsilon^k(\cdot, 0) = T_k(w)(\cdot), \\ u_\varepsilon^k(x, t) = 0 \quad \text{on } \Gamma. \end{cases}$$

Then, as it is easily checked by Proposition 4.1,

$$u_\varepsilon^k \xrightarrow{k \to +\infty} u_\varepsilon \quad \text{strongly in } L^q(0, T; W_0^{1,q}(\Omega)), \text{ uniformly with respect to } \varepsilon.$$

By diagonal selection, it is possible to extract a subsequence such that, for every fixed $k \in N$, $T_k(h_\varepsilon) \rightharpoonup \eta_k$ *-weakly in $L^\infty(Q)$. By Remark 5.1, we have, for every k:

$$u_\varepsilon^k \xrightarrow{\varepsilon \to 0} u_0^k \quad \text{weakly in } L^2(0, T; H_0^1(\Omega)),$$

$$\nabla u_\varepsilon^k - p_\varepsilon \nabla u_0^k \xrightarrow{\varepsilon \to 0} 0 \quad \text{strongly in } L^1(Q),$$

where u_0^k is the solution of

$$\begin{cases} \dfrac{\partial u_0^k}{\partial t} + A_0 u_0^k = f + \gamma_k & \text{in } Q, \\ u_0^k(\cdot, 0) = T_k(w)(\cdot), \\ u_0^k(x, t) = 0 \quad \text{on } \Gamma. \end{cases}$$

On the other hand, as $k \to +\infty$, $\gamma_k \to h_0$, $T_k(w) \to w$ strongly in $L^1(Q)$, $L^1(\Omega)$ respectively and this implies

$$u_0^k \xrightarrow{k \to +\infty} u_0 \quad \text{strongly in } L^q(0, T; W_0^{1,q}(\Omega)).$$

At this point it is possible to proceed as in Theorem 3.1 to prove (5.1) and (5.2). ∎

6. - Some applications: H convergence of parabolic equations with nonlinear first order terms.

This section will be devoted to some applications of the theory developed in the preceding sections to the H-convergence of parabolic equations with nonlinear first order terms. The study of the elliptic problems of this type as been carried out in [BoM1], [BD], [BBM]. All these papers are based on approximation results with correctors (see [MT]), that are used in order to identify the weak limit of the lower order terms. We will follow the same scheme, that is, we will prove that a corrector result also holds for the solutions of our nonlinear parabolic problems. The main idea of the proof is to consider the nonlinear lower order term as a datum, and to think the solutions of the original problems as solutions of linear problems with data in $L^2(0, T; H^{-1}(\Omega)) + L^1(Q)$. The basic remark is that, due to their regularity, the solutions we consider are the SOLA's of these problems with non regular data.

Throughout this section, we will assume that $\{a_\varepsilon(x)\}_{\varepsilon \in E}$ is a sequence of matrices in $\mathcal{M}(\alpha, \beta; \Omega)$ such that $a_\varepsilon \xrightarrow{H} a_0$. We will denote by A_ε, A_0 the linear elliptic operators associated to a_ε, a_0 respectively. Let p_ε be the correction matrices associated to a_ε. We assume that

$$\|p_\varepsilon\|_{(L^\infty(Q))^{N^2}} \leq c_1 . \tag{6.1}$$

The following proposition, that has been proved in [BoM1] (see also [BBM]), is the main tool that will allow us to pass to the limit in the nonlinear term, as soon as we have obtained a corrector result for the solutions.

PROPOSITION 6.1. - *Let $\{H_\varepsilon\}_{\varepsilon \in E}$ be a sequence of Carathéodory functions $H_\varepsilon(x, t, s, \xi)$: $\Omega \times (0, T) \times \boldsymbol{R} \times \boldsymbol{R}^N \to \boldsymbol{R}$. Assume that H_ε is such that, for every*

$s, \tilde{s} \in \boldsymbol{R}$, for every $\xi, \tilde{\xi} \in \boldsymbol{R}^N$, for almost every $(x, t) \in Q$,

$$|H_\varepsilon(x, t, s, \xi) - H_\varepsilon(x, t, s, \tilde{\xi})| \leq b_1(|s|)(1 + |\xi|^{\gamma-1} + |\tilde{\xi}|^{\gamma-1})|\xi - \tilde{\xi}|, \tag{6.2}$$

$$|H_\varepsilon(x, t, s, \xi) - H_\varepsilon(x, t, \tilde{s}, \xi)| \leq b_2(|s - \tilde{s}|)(1 + |\xi|^\gamma), \tag{6.3}$$

$$|H_\varepsilon(x, t, 0, 0)| \leq c_0, \tag{6.4}$$

where $c_0 > 0$, $1 < \gamma \leq 2$, and b_1, b_2 are continuous, positive, increasing functions, with $b_2(0) = 0$. Then there exists a subsequence (still denoted by the same symbol), and a function $H_0(x, t, s, \xi)$ which satisfies ((6.2)-(6.4) for $\varepsilon = 0$ (up to a multiplicative constant) such that, for every $s \in \boldsymbol{R}$, for every $\xi \in \boldsymbol{R}^N$

$$H_\varepsilon(x, t, s, p_\varepsilon \xi) \rightharpoonup H_0(x, t, s, \xi) \quad \textit{weakly in } L^1(Q). \tag{6.5}$$

Moreover, if $\{z_\varepsilon\}_{\varepsilon \in E}$ is a sequence of functions such that $z_\varepsilon \to z$ almost everywhere in Q, $\|z_\varepsilon\|_{L^\infty(Q)} \leq c_2$, and if $\phi \in (L^2(Q))^N$, then

$$H_\varepsilon(x, t, z_\varepsilon, p_\varepsilon \phi) \rightharpoonup H_0(x, t, z, \phi) \quad \textit{weakly in } L^1(Q). \tag{6.6}$$

We remark explicitly that, even if the functions H_ε do not change with ε, i.e., if $H_\varepsilon = H$ for every ε, the limit H_0 given by Proposition 6.1 is, in general, different from H.

6.1. – *First application: H-convergence of quasi-linear parabolic equations with first order terms with sub-quadratic growth (unbounded solutions).*

We are going to study the following sequence of parabolic problems:

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} + A_\varepsilon u_\varepsilon + g_\varepsilon(x, t, u_\varepsilon, \nabla u_\varepsilon) = f & \text{in } Q, \\ u_\varepsilon(\cdot, 0) = w(\cdot), \\ u_\varepsilon(x, t) = 0 & \text{on } \Gamma. \end{cases} \tag{6.7}$$

We will assume that $f \in L^2(0, T; H^{-1}(\Omega))$, $w \in L^2(\Omega)$, and that $g_\varepsilon(x, t, s, \xi)$: $\Omega \times (0, T) \times \boldsymbol{R} \times \boldsymbol{R}^N \to \boldsymbol{R}$ are Carathéodory functions such that, for almost every $(x, t) \in Q$, for every $s, \tilde{s} \in \boldsymbol{R}$ for every $\xi, \tilde{\xi} \in \boldsymbol{R}^N$:

$$g_\varepsilon(x, t, s, \xi)s \geq 0 \tag{6.8}$$

$$g_\varepsilon(x, t, \cdot, \xi) \quad \text{is an increasing function}, \tag{6.9}$$

$$|g_\varepsilon(x, t, s, \xi) - g_\varepsilon(x, t, s, \tilde{\xi})| \leq b_1(|s|)(1 + |\xi|^{\gamma-1} + |\tilde{\xi}|^{\gamma-1})|\xi - \tilde{\xi}|, \tag{6.10}$$

$$|g_\varepsilon(x, t, s, \xi) - g_\varepsilon(x, t, \tilde{s}, \xi)| \leq b_2(|s - \tilde{s}|)(1 + |\xi|^\gamma). \tag{6.11}$$

In (6.10) and (6.11), γ is a real number such that $1 < \gamma < 2$, b_1 and b_2 are as in the statement of Proposition 6.1. We remark that no growth restriction is assumed with respect to s.

REMARK 6.1. – Condition (6.8) implies that $g_\varepsilon(x, t, 0, \xi) = 0$ almost everywhere on Q; hence, by (6.11),

$$|g_\varepsilon(x, t, s, \xi)| \leqslant b_2(|s|)(1 + |\xi|^\gamma), \tag{6.12}$$

for almost every $(x, t) \in Q$, for every $s \in \boldsymbol{R}$, for every $\xi \in \boldsymbol{R}^N$.

Under these hypotheses, we can apply Proposition 6.1 to the sequence $\{g_\varepsilon\}$, and extract a subsequence such that, for every $\phi \in (L^2(Q))^N$, for every bounded sequence z_ε in $L^\infty(Q)$ which converges almost everywhere in Q to a function z, we have $g_\varepsilon(x, t, z_\varepsilon, p_\varepsilon\phi) \rightharpoonup g_0(x, t, z, \phi)$ weakly in $L^1(Q)$. It is easy to prove that the limit function g_0 satisfies sign and monotonicity conditions similar to (6.8) and (6.9).

Before giving the results of this section, we have to specify in which sense a function u is solution of (6.7).

DEFINITION 6.1. – We will say that a function u_ε is a weak solution of problem (6.7) if $u_\varepsilon \in L^2(0, T; H_0^1(\Omega))$, $g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) \in L^1(Q)$, $u_\varepsilon g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) \in L^1(Q)$ and

$$\frac{\partial u_\varepsilon}{\partial t} + A_\varepsilon u_\varepsilon + g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) = f \quad \text{in } \mathcal{D}'(Q), \tag{6.13}$$

$$u_\varepsilon(\cdot, 0) = w(\cdot). \tag{6.14}$$

Note that from (6.13) we obtain $\partial u_\varepsilon/\partial t = (f - A_\varepsilon u_\varepsilon) - g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)$ in the sense of distributions, and the right hand side belongs to $L^2(0, T; H^{-1}(\Omega)) + L^1(Q)$. This implies that $u_\varepsilon \in C([0, T]; W^{-1,r}(\Omega))$ for r small enough, so that the initial condition (6.14) makes sense. The existence of a weak solution of problems of type (6.7) has been proved in [BoM2].

Assume that $a_\varepsilon \xrightarrow{H} a_0$, and that hypotheses (6.8)-(6.11), and (6.1), hold. Let us consider a sequence $\{u_\varepsilon\}_{\varepsilon \in E}$ of weak solutions of (6.7). Using Lemma 4.2, we obtain:

$$\int_\Omega \phi_k(u_\varepsilon(x, T))\,dx = \int_\Omega \phi_k(w(x))\,dx - \langle\!\langle A_\varepsilon u_\varepsilon - f, T_k(u_\varepsilon)\rangle\!\rangle_Q - \int_Q g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)\, T_k(u_\varepsilon)\,dx\,dt\,.$$

If we let k tend to $+\infty$, the right hand side converges to

$$\frac{1}{2}\int_\Omega |w|^2 dx - \langle\!\langle A_\varepsilon u_\varepsilon - f, u_\varepsilon\rangle\!\rangle_Q - \int_Q g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)\, u_\varepsilon\,dx\,dt$$

(we have used (6.8), the fact that $u_\varepsilon g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) \in L^1(Q)$ and the dominated convergence theorem to study the last term). Hence, by the monotone convergence theorem, since $\phi_k(u_\varepsilon)$ increases to $u_\varepsilon^2/2$ as k tends to infinity,

$$\frac{1}{2}\int_\Omega |u_\varepsilon(x, T)|^2 dx - \frac{1}{2}\int_\Omega |w(x)|^2 dx + \langle\langle A_\varepsilon u_\varepsilon - f, u_\varepsilon \rangle\rangle + \int_Q g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) u_\varepsilon \, dx\, dt = 0 .$$

In particular, using the ellipticity of the matrices:

$$\alpha \|u_\varepsilon\|^2_{L^2(0, T; H_0^1(\Omega))} + \int_Q g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) u_\varepsilon \, dx\, dt \leqslant$$

$$\leqslant \|f\|_{L^2(0, T; H^{-1}(\Omega))} \|u_\varepsilon\|_{L^2(0, T; H_0^1(\Omega))} + \frac{1}{2}\|w\|_{L^2(\Omega)} .$$

This implies, by (6.8), that

$$\|u_\varepsilon\|_{L^2(0, T; H_0^1(\Omega))} \leqslant c_1 \tag{6.15}$$

and

$$\int_Q g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) u_\varepsilon \, dx\, dt \leqslant \|f\|_{L^2(0, T; H^{-1}(\Omega))} \|u_\varepsilon\|_{L^2(0, T; H_0^1(\Omega))} + \frac{1}{2}\|w\|_{L^2(\Omega)} \leqslant c_2 . \tag{6.16}$$

Moreover, we can prove that the sequence $\{g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)\}_{\varepsilon \in E}$ is relatively compact in $L^1(Q)$ weak. To see this, using (6.8), (6.12), (6.15) and (6.16), we obtain, for every measurable set $E \subset Q$, and for every $m > 0$,

$$\int_E |g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)| \, dx\, dt = \int_{E \cap \{|u_\varepsilon| \leqslant m\}} |g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)| \, dx\, dt + \int_{E \cap \{|u_\varepsilon| > m\}} |g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)| \, dx\, dt \leqslant$$

$$\leqslant b_2(m)(\operatorname{meas} E + (\operatorname{meas} E)^{1-\gamma/2} \|\nabla u_\varepsilon\|^\gamma_{(L^2(Q))^N}) + \frac{1}{m}\int_Q g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) u_\varepsilon \, dx\, dt \leqslant$$

$$\leqslant b_2(m)(\operatorname{meas} E + (\operatorname{meas} E)^{1-\gamma/2} c_1^\gamma) + \frac{c_2}{m} .$$

Since m can be chosen arbitrarily large, this implies that $\{g_\varepsilon\}_{\varepsilon \in E}$ is equi-integrable, hence weakly compact in $L^1(Q)$. Therefore there exist $u_0 \in L^2(0, T; H_0^1(\Omega))$, $h \in L^1(Q)$ such that (up to subsequences) $u_\varepsilon \rightharpoonup u_0$ weakly in $L^2(0, T; H_0^1(\Omega))$, $g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) \rightharpoonup h$ weakly in $L^1(Q)$.

THEOREM 6.1. – *The function u_0 is solution of the problem*

$$\begin{cases} \dfrac{\partial u_0}{\partial t} + A_0 u_0 + g_0(x, t, u_0, \nabla u_0) = f & \text{in } Q, \\ u_0(x, 0) = w(x), & \\ u_0(x, t) = 0 & \text{on } \Gamma, \end{cases} \tag{6.17}$$

where g_0 is limit of g_ε in the sense of Proposition 6.1.

PROOF. – We apply the theory, developed in the proceding sections, of SOLA's of equations with L^1 data. Indeed, the solutions u_ε of (6.7) can be viewed as solutions of the problems

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} + A_\varepsilon u_\varepsilon = f + h_\varepsilon & \text{in } Q, \\ u_\varepsilon(\cdot, 0) = w(\cdot), \\ u_\varepsilon(x, t) = 0 \quad \text{on } \Gamma, \end{cases}$$

where we have defined $h_\varepsilon = -g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)$. The right hand side of these equations is, as we have seen, the sum of a term belonging to the «natural» space $L^2(0, T: H^{-1}(\Omega))$, and of a term $g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)$, which is weakly convergent in $L^1(Q)$. Moreover, and this is the main point, u_ε belongs to $L^2(0, T; H_0^1(\Omega))$, so that, by Theorem 4.2, it is the SOLA of this equation. Thus, we can apply Theorem 5.1 to obtain

$$u_\varepsilon \rightharpoonup u \quad \text{weakly in } L^q(0, T; W_0^{1,q}(\Omega)), \qquad \nabla u_\varepsilon - p_\varepsilon \nabla u \to 0 \quad \text{strongly in } (L^q(Q)),$$

for every $q < \bar{q}$, where u is the SOLA of

$$\begin{cases} \dfrac{\partial u}{\partial t} + A_0 u = f - h & \text{in } Q, \\ u(x, 0) = w(x), \\ u(x, t) = 0 \quad \text{on } \Gamma. \end{cases}$$

This implies that $u_0 = u$. Moreover, since $\|\nabla u_\varepsilon - p_\varepsilon \nabla u_0\|_{(L^2(Q))^N} \leq c_3$, then

$$\nabla u_\varepsilon - p_\varepsilon \nabla u_0 \to 0 \quad \text{strongly in } (L^q(Q))^N, \text{ for every } q < 2. \tag{6.18}$$

Now we only have to identify h. We will use the corrector result (6.18) to show that $h = g_0(u_0, \nabla u_0)$. To prove this, remark that, as a direct consequence of Proposition 6.1, we have

$$g_\varepsilon(T_k(u_\varepsilon), p_\varepsilon \nabla u_0) \rightharpoonup g_0(T_k(u_0), \nabla u_0) \quad \text{weakly in } L^1(Q), \text{ for every } k \in N. \tag{6.19}$$

Using (6.1), (6.10) and the Hölder inequality, we obtain:

$$\|g_\varepsilon(T_k(u_\varepsilon), \nabla u_\varepsilon) - g_\varepsilon(T_k(u_\varepsilon), p_\varepsilon \nabla u_0)\|_{L^1(Q)} \leq$$

$$\leq b_1(k) \int_Q (1 + |\nabla u_\varepsilon|^{\gamma-1} + |p_\varepsilon \nabla u_0|^{\gamma-1}) |\nabla u_\varepsilon - p_\varepsilon \nabla u_0| \, dx\, dt \leq$$

$$\leq c_5 b_1(k) \left[\int_Q (1 + |\nabla u_\varepsilon|^2 + |p_\varepsilon \nabla u_0|^2) \, dx\, dt \right]^{(\gamma-1)/2} \|\nabla u_\varepsilon - p_\varepsilon \nabla u_0\|_{(L^{2/(3-\gamma)}(Q))^N} \leq$$

$$\leq c_6 b_1(k) \|\nabla u_\varepsilon - p_\varepsilon \nabla u_0\|_{(L^{2/(3-\gamma)}(Q))^N}.$$

Since $2/(3-\gamma)<2$, by (6.18) the last term vanishes as $\varepsilon \to 0$. This, together with (6.19), yields

$$(6.20) \qquad g_\varepsilon(T_k(u_\varepsilon), \nabla u_\varepsilon) \rightharpoonup g_0(T_k(u_0), \nabla u_0) \quad \text{weakly in } L^1(Q), \text{ for every } k \in N.$$

Therefore, for every $k \in N$, we have, by (6.9):

$$\|g_0(T_k(u_0), \nabla u_0)\|_{L^1(Q)} \leqslant \liminf_{\varepsilon \to 0} \|g_\varepsilon(T_k(u_\varepsilon), \nabla u_\varepsilon)\|_{L^1(Q)} \leqslant$$
$$\leqslant \liminf_{\varepsilon \to 0} \|g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)\|_{L^1(Q)} < +\infty .$$

If we let k tend to infinity, the monotone convergence theorem implies that $g_0(u_0, \nabla u_0) \in L^1(Q)$. Moreover, if $\phi \in L^\infty(Q)$, we can write:

$$(6.21) \qquad \left| \int_Q [g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) - g_0(u_0, \nabla u_0)]\, \phi \, dx\, dt \right| \leqslant \int_Q |g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) - g_\varepsilon(T_k(u_\varepsilon), \nabla u_\varepsilon)|\, |\phi|\, dx\, dt$$
$$+ \left| \int_Q [g_\varepsilon(T_k(u_\varepsilon), \nabla u_\varepsilon) - g_0(T_k(u_0), \nabla u_0)]\, \phi\, dx\, dt \right| +$$
$$+ \int_Q |g_0(T_k(u_0), \nabla u_0) - g_0(u_0, \nabla u_0)|\, |\phi|\, dx\, dt .$$

Let us study the first term of the right hand side of (6.21). Using the hypotheses on g_ε and (6.17) we get:

$$\int_Q |g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) - g_\varepsilon(T_k(u_\varepsilon), \nabla u_\varepsilon)|\, |\phi|\, dx\, dt \leqslant \int_{\{|u_\varepsilon| > k\}} |g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)|\, |\phi|\, dx\, dt \leqslant$$
$$\leqslant \frac{\|\phi\|_{L^\infty(Q)}}{k} \int_Q g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)\, u_\varepsilon\, dx\, dt \leqslant \frac{c_7 \|\phi\|_{L^\infty(Q)}}{k} .$$

Since g_0 enjoys the same properties of g_ε (i.e., the sign condition and the monotonicity), we can write for the third term:

$$\int_Q |g_0(T_k(u_0), \nabla u_0) - g_0(u_0, \nabla u_0)|\, |\phi|\, dx\, dt \leqslant \|\phi\|_{L^\infty(Q)} \int_{\{|u| > k\}} |g_0(u_0, \nabla u_0)|\, dx\, dt .$$

Both these terms can be made arbitrarily small if k is large enough. Hence (6.20) and (6.21) imply that the weak limit of $g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)$ in $L^1(Q)$ is $g_0(u_0, \nabla u_0)$. Finally we note that it is possible to prove, as in [BD], that $g_0(u_0, \nabla u_0)\, u_0 \in L^1(Q)$, and so u_0 is a weak solution of the problem according to Definition 6.1. This concludes the proof of Theorem 6.1. ■

6.2. – *Second application: Quasi-linear parabolic equations with quadratic growth (bounded solutions).*

Let $H_\varepsilon(x, t, s, \xi)$: $\Omega \times (0, T) \times \boldsymbol{R} \times \boldsymbol{R}^N$ be a sequence of Carathéodory functions satisfying (6.2)-(6.4), with $\gamma = 2$, and

$$|H_\varepsilon(x, t, s, \xi)| \leq c_0(1 + |\xi|^2) \tag{6.22}$$

for some $c_0 > 0$.

Let $f \in L^\infty(0, T; W^{-1,\infty}(\Omega))$. Let us consider the problems

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} + A_\varepsilon u_\varepsilon + H_\varepsilon(x, t, u_\varepsilon, \nabla u_\varepsilon) = f & \text{in } Q, \\ u_\varepsilon(x, 0) = 0, \\ u_\varepsilon(x, t) = 0 \quad \text{on } \Gamma. \end{cases} \tag{6.23}$$

In this case, the nonlinear term has a quadratic growth, so that we assume more regularity on the datum f. However, we no longer need the sign condition on H_ε. These equations have been studied by various authors (see, e.g., [BMP], [Mo], [OP], [G]). The following result has been proved in [OP].

THEOREM 6.2. – *For every fixed* ε *there exists a solution* $u_\varepsilon \in L^\infty(Q) \cap \cap L^2(0, T; H_0^1(\Omega))$ *of problem* (6.23).

Moreover, as a consequence of the proof of Theorem 6.2, we have the following estimates:

$$\|u_\varepsilon\|_{L^\infty(Q)} \leq c_1, \qquad \|u_\varepsilon\|_{L^2(0, T; H_0^1(\Omega))} \leq c_2,$$

for some positive constants c_1 and c_2 independent of ε. Hence, up to subsequences we can assume that

$$u_\varepsilon \rightharpoonup u_0 \quad \text{weakly in } L^2(0, T; H_0^1(\Omega)), \quad \text{and } *\text{-weakly in } L^\infty(Q).$$

Using the theory of SOLA's, we can show the following result:

THEOREM 6.3. – *The function* u_0 *is a solution of*

$$\begin{cases} \dfrac{\partial u_0}{\partial t} + A_0 u_0 + H_0(x, t, u_0, \nabla u_0) = f & \text{in } Q, \\ u_0(x, 0) = 0, \\ u_0(x, t) = 0 \quad \text{on } \Gamma, \end{cases}$$

where H_0 *is limit of* H_ε *in the sense of Proposition* 6.1.

PROOF. – As in the preceding application, we can consider the functions u_ε as solutions of the equations

$$\frac{\partial u_\varepsilon}{\partial t} + A_\varepsilon u_\varepsilon = f - h_\varepsilon \in L^2(0, T; H^{-1}(\Omega)) + L^1(Q), \tag{6.24}$$

where $h_\varepsilon = H_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)$. The function u_ε belongs to $L^2(0, T; H_0^1(\Omega))$, and so, by Proposition 4.2, it is the SOLA of (6.24). To apply the theory developed in the preceding section, we have to show that the sequence $\{H_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)\}$ is weakly relatively compact in $L^1(Q)$. To see this, we use the following Meyers type regularity result for these quasi-linear equations.

THEOREM 6.4 (see [GS], [B]). – *There exist a real number $p > 2$, and a positive constant c_3 such that*

$$\|\nabla u_\varepsilon\|_{(L^p(Q))^N} \leq c_3 .$$

By means of Theorem 6.4, it is easily seen that the functions $H_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)$ are bounded in $L^{1+\delta}(Q)$ $(\delta > 0)$, hence relatively compact in $L^1(Q)$ weak. Thus, it is possible to extract a subsequence such that $H_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) \rightharpoonup h$ weakly in $L^1(Q)$. By Theorem 5.1, u_0 is the SOLA of the equation

$$\frac{\partial u_0}{\partial t} + A_0 u_0 = f - h .$$

Our last step consists in the identification of h. Again by Theorem 5.1, and by Theorem 6.4, it is easy to prove that

$$\nabla u_\varepsilon - p_\varepsilon \nabla u_0 \to 0 \quad \text{strongly in } (L^2(Q))^N .$$

Starting from this corrector result, we deduce (exactly as in [BBM]) that

$$H_\varepsilon(x, t, u_\varepsilon, \nabla u_\varepsilon) \rightharpoonup H_0(x, t, u_0, \nabla u_0) \quad \text{weakly in } L^1(Q),$$

and this concludes the proof. ■

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