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Approximated Solutions to Operator Equations based on the Frame Bounds

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Abstract. We want to find the solution of the problem Ł$u = f$, based on knowledge of frames. Where Ł : $H \to H$ is a boundedly invertible and symmetric operator on a separable Hilbert space H. Inverting the operator can be complicated if the dimension of H is large. Another option is to use an algorithm to obtain approximations of the solution. We will organize an algorithm in order to find approximated solution of the problem depends on the knowledge of some frame bounds and the guaranteed speed of convergence also depends on them.

1. Introduction

The analysis of numerical schemes for operator equations is a field of enormous current interest. In this work we present an algorithm in order to approximate the solution of the operator equation

$$Łu = f, \tag{1.1}$$

where Ł : $H \to H$ is a boundedly invertible and symmetric operator on a separable Hilbert space. As typical example we think of linear differential or integral equations in variational form. In [1, 5, 6], some iterative adaptive methods for solving this system has been developed.

First natural steps were to use multiresolution spaces spanned by wavelets (or correspondingly scaling functions) as test and trial spaces for Galerkin methods. Usually, the operator under consideration is defined on a bounded domain $\Omega \subset \mathbb{R}^d$ or on a closed manifold, and therefore the construction of a wavelet basis with specific properties on this domain or on the manifold is needed. Although there exist by now several construction methods such as e.g., [7, 8], none of them seems to be fully satisfying in the sense that some serious drawbacks such as stability problems cannot be avoided. One way out could be to use a fictitious domain method [9], however, then the compressibility of the problem might be

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reduced. Motivated by these difficulties, we therefore suggest to use a slightly weaker concept, namely frames. During the last 20 years the theory of frames has been growing rapidly, since several new applications have been developed. For example, besides traditional applications as signal processing, image processing, data compression, and sampling theory.

Let H be a separable Hilbert space with dual H^* and Λ be a countable set of indices. A family $\Psi = (\psi_\lambda)_{\lambda\in\Lambda} \subset H$ is a *frame* for H, if there exist constants $0 < A \le B < \infty$ such that for all $f \in H$,

$$A\|f\|_H^2 \le \sum_\lambda |\langle f, \psi_\lambda\rangle|^2 \le B\|f\|_H^2, \tag{1.2}$$

or equivalently (by the Riesz mapping),

$$A\|f\|_{H^*}^2 \le \|f(\Psi)\|_{\ell_2}^2 \le B\|f\|_{H^*}^2, \quad \text{for all } f \in H^*, \tag{1.3}$$

where $f(\Psi) = (f(\psi_\lambda))_\lambda = (\langle f, \psi_\lambda\rangle)_\lambda$. The constants A and B are called *lower* and *upper frame bound* for the frame. Those sequences which satisfy only the upper inequality in (1.2) are called *Bessel sequences*. A frame is *tight*, if $A = B$. If $A = B = 1$, it is called a *Parseval frame*. For an index set $\widetilde{\Lambda} \subset \Lambda$, $(\psi_\lambda)_{\lambda\in\widetilde{\Lambda}}$ is called a *frame sequence* if it is a frame for its closed span.
For the frame Ψ, let $T : \ell_2(\Lambda) \to H$ be the *synthesis operator*

$$T((c_\lambda)_\lambda) = \sum_{\lambda\in\Lambda} c_\lambda \psi_\lambda,$$

and let $T^* : H \to \ell_2(\Lambda)$ (or $T^* : H^* \to \ell_2(\Lambda)$) be the *analysis operator*

$$T^*(f) = (\langle f, \psi_\lambda\rangle)_\lambda.$$

Also, let $S := TT^* : H \to H$ be the *frame operator*

$$S(f) = \sum_\lambda \langle f, \psi_\lambda\rangle \psi_\lambda.$$

Note that T is surjective, T^* is injective and T^* is the adjoint of T. Because of (1.2) or (1.3) T is bounded, in fact we have

$$\|T\| = \|T^*\| \le \sqrt{B}. \tag{1.4}$$

It was shown in [4], S is a positive invertible operator satisfying $AI_H \le S \le BI_H$ and $B^{-1}I_H \le S^{-1} \le A^{-1}I_H$. Also, the sequence

$$\widetilde{\Psi} = (\widetilde{\psi}_\lambda)_{\lambda\in\Lambda} = (S^{-1}\psi_\lambda)_{\lambda\in\Lambda}$$

is a frame (called the *canonical dual frame*) for H with bounds B^{-1}, A^{-1}. Every $f \in H$ has the expansion

$$f = \sum_\lambda \langle f, \widetilde{\psi}_\lambda\rangle \psi_\lambda = \sum_\lambda \langle f, \psi_\lambda\rangle \widetilde{\psi}_\lambda. \tag{1.5}$$

For more information we refer to [2, 3, 4].

2. **An approximated solution based on the frame bounds**

We construct an algorithm depends on the knowledge of some frame bounds and the guaranteed speed of convergence also depends on them.

Lemma 2.1. *Let $\Psi = (\psi_\lambda)_{\lambda\in\Lambda} \subset H$ be a frame for H with bounds A and B, and Ł be as in (1.1). Then the sequence $\Phi = (Ł\psi_\lambda)_{\lambda\in\Lambda}$ is a frame for H with bounds $\frac{A}{\|Ł^{-1}\|^2}$ and $B\|Ł\|^2$.*

Proof. See [4]. □

For simplicity we denote the lower and upper bound of the frame $\Phi = (Ł\psi_\lambda)_{\lambda\in\Lambda}$ by A and B.

Theorem 2.2. *Let $\Psi = (\psi_\lambda)_{\lambda\in\Lambda} \subset H$ be a frame for H with frame operator S, Ł be as in (1.1) and $\Phi = (Ł\psi_\lambda)_{\lambda\in\Lambda}$ be a frame with bounds A, B and frame operator S'. Let $u_0 = 0$ and for $k \geq 1$, $u_k = u_{k-1} + \frac{2}{A+B}ŁS(f - Łu_{k-1})$. Then $\|u - u_k\| \leq (\frac{B-A}{B+A})^k\|u\|$. In particular the vectors u_k converges to u as $k \to \infty$.*

Proof. By definition of u_k

$$
\begin{aligned}
u - u_k &= u - u_{k-1} - \frac{2}{A+B}ŁSŁ(u - u_{k-1}) \\
&= \left(I - \frac{2}{A+B}ŁSŁ\right)(u - u_{k-1}) \\
&= \left(I - \frac{2}{A+B}ŁSŁ\right)^2(u - u_{k-2}) = \dots \\
&= \left(I - \frac{2}{A+B}ŁSŁ\right)^k(u - u_0),
\end{aligned}
$$

thus

$$
\|u - u_k\| \leq \|I - \frac{2}{A+B}ŁSŁ\|^k\|u\|. \tag{2.1}
$$

But for every $v \in H$ we have

$$
\begin{aligned}
\left\langle \left(I - \frac{2}{A+B}ŁSŁ\right)v, v \right\rangle &= \|v\|^2 - \frac{2}{A+B}\langle ŁSŁv, v\rangle \\
&= \|v\|^2 - \frac{2}{A+B}\langle S'v, v\rangle \\
&= \|v\|^2 - \frac{2}{A+B}\sum_\lambda |\langle v, \phi_\lambda\rangle|^2 \\
&\leq \|v\|^2 - \frac{2A}{A+B}\|v\|^2 \\
&= \left(\frac{B-A}{B+A}\right)\|v\|^2.
\end{aligned}
$$

The last inequality obtains by the frame property of the frame $\Phi = (Ł\psi_\lambda)_{\lambda\in\Lambda}$. Similarly we have

$$-\left(\frac{B-A}{B+A}\right)\|v\|^2 \le \left\langle \left(I - \frac{2}{A+B}ŁSŁ\right)v, v \right\rangle.$$

So we conclude that

$$\left\| I - \frac{2}{A+B}ŁSŁ \right\| \le \frac{B-A}{B+A}. \tag{2.2}$$

Combining this inequality with (2.1) give

$$\|u - u_k\| \le \left(\frac{B-A}{B+A}\right)^k \|u\|. \qquad \square$$

Remark 2.3. If the upper bound of the frame is much larger than the lower bond the convergence might be slow. When the frame is near to tight frame we can obtain faster convergence. If we have a tight frame then $A = B$ and by definition of u_k in previous theorem we have $u_1 = u_0 + \frac{2}{2A}ŁS(f - Łu_0)$ that is $u_1 = \frac{1}{A}ŁSf$. Now,

$$\begin{aligned}
Ł(\frac{1}{A}ŁSf) &= Ł\left(\frac{1}{A}ŁSŁu\right) \\
&= \frac{1}{A}S'u \\
&= \frac{1}{A}Ł\sum_\lambda \langle u, \phi_\lambda\rangle \phi_\lambda \\
&= Ł\sum_\lambda \langle u, \phi_\lambda\rangle \frac{1}{A}\phi_\lambda \\
&= Ł\sum_\lambda \langle u, \phi_\lambda\rangle S'\phi_\lambda = Łu = f,
\end{aligned}$$

that means

$$u = \frac{1}{A}ŁSf.$$

Note that according the hypothesis of previous theorem for each $v \in H$

$$\begin{aligned}
S'v &= \sum_\lambda \langle v, \phi_\lambda\rangle \phi_\lambda \\
&= \sum_\lambda \langle v, Ł\psi_\lambda\rangle Ł\psi_\lambda \\
&= Ł\sum_\lambda \langle Łv, \psi_\lambda\rangle \psi_\lambda = ŁSŁv,
\end{aligned}$$

therefore $S' = ŁSŁ$.

3. An approximated solution by canonical dual frames

Let $\Psi = (\psi_\lambda)_{\lambda\in\Lambda} \subset H$ be a frame for H and Ł be as in (1.1). By lemma 2.1 the sequence $\Phi = (\text{Ł}\psi_\lambda)_{\lambda\in\Lambda}$ is a frame for H. If we denote by S' its frame operator then by (1.5) each $u \in H$ has the expansion

$$u = \sum_{\lambda\in\Lambda} \langle u, \phi_\lambda \rangle (S')^{-1} \phi_\lambda \tag{3.1}$$

Now because of completeness of the frame Ψ, the equation (1.1) is equivalent to

$$\langle \text{Ł}u, \psi_\lambda \rangle = \langle f, \psi_\lambda \rangle, \quad \text{for all } \lambda \in \Lambda,$$

since Ł is symmetric then

$$\langle u, \phi_\lambda \rangle = \langle f, \psi_\lambda \rangle, \quad \text{for all } \lambda \in \Lambda,$$

hence

$$\sum_{\lambda\in\Lambda} \langle u, \phi_\lambda \rangle (S')^{-1} \phi_\lambda = \sum_{\lambda\in\Lambda} \langle f, \psi_\lambda \rangle (S')^{-1} \phi_\lambda,$$

and by (3.1)

$$u = \sum_{\lambda\in\Lambda} \langle f, \psi_\lambda \rangle (S')^{-1} \phi_\lambda, \tag{3.2}$$

Since H is a space of infinite dimension, it is difficult that we obtain the values of $(S')^{-1}\phi_\lambda$. Our goal is to find a sequence of u_i, related to some frame sequence, such that converges to u in (3.2).

Proposition 3.1. *Let $\Psi = (\psi_\lambda)_{\lambda\in\Lambda} \subset H$ be a frame for H and $\widetilde{\Lambda} \subset \Lambda$ be a finite subset of Λ such that not all elements of $\Psi_{\widetilde{\Lambda}} = (\psi_\lambda)_{\lambda\in\widetilde{\Lambda}}$ are zero. Then $\Psi_{\widetilde{\Lambda}}$ is a frame sequence.*

Proof. See [4]. □

Let $\Lambda_1 \subset \Lambda_2 \subset \Lambda_3 \subset \ldots \nearrow \Lambda$ be finite subsets of Λ where not all elements of Λ_1 are zero. By proposition 3.1 each Ψ_{Λ_i} and consequently Φ_{Λ_i} is a frame sequence. We assume that Ψ_{Λ_i} are frame sequences with common bounds A and B. Also it follows that Φ_{Λ_i} are frame sequences with common bounds $\frac{A}{\|\text{Ł}^{-1}\|^2}$ and $B\|\text{Ł}\|^2$.

Let $V_i = \text{span}\Phi_{\Lambda_i} = \text{span}(\text{Ł}\psi_\lambda)_{\lambda\in\Lambda_i}$ and $S_i' : V_i \to V_i$ be its frame operator, then for every $f \in V_i$ we have

$$\begin{aligned} S_i' f &= \sum_{\lambda\in\Lambda_i} \langle f, \text{Ł}\psi_\lambda \rangle \text{Ł}\psi_\lambda \\ &= \text{Ł}\Big(\sum_{\lambda\in\Lambda_i} \langle f, \text{Ł}\psi_\lambda \rangle \psi_\lambda \Big) \\ &= \text{Ł}\Big(\sum_{\lambda\in\Lambda_i} \langle \text{Ł}f, \psi_\lambda \rangle \psi_\lambda \Big) \\ &= \text{Ł}S_i\text{Ł}f \end{aligned}$$

therefore $S_i' = Ł_i S_i Ł_i$, where $Ł_i = Ł\,|_{V_i}$ and S_i is the frame operator of the frame sequence Ψ_{Λ_i}. Specially we obtain $(S_i')^{-1} = Ł_i^{-1} S_i^{-1} Ł_i^{-1}$, thus

$$\sup_{i\in\mathbb{N}} \|(S_i')^{-1}\| \leq \frac{\|Ł^{-1}\|^2}{A}, \quad \text{for all } i \in \mathbb{N}. \tag{3.3}$$

Now we are ready to present the main result that is the following theorem.

Theorem 3.2. *Let Ψ_Λ, Φ_Λ and Λ_i be as above. Assume that*

$$u_i = \sum_{\lambda\in\Lambda_i} \langle f, \psi_\lambda\rangle (S_i')^{-1}\phi_\lambda,$$

then $u_i \to u$ as $i \to \infty$.

Proof. Let $(c_\lambda)_{\lambda\in\Lambda} \in \ell_2(\Lambda)$ and $\sum_{\lambda\in\Lambda_i} c_\lambda \phi_\lambda = 0$. Using (3.3) we have

$$\begin{aligned}\left\| (S_i')^{-1} \sum_{\lambda\in\Lambda_i} c_\lambda\phi_\lambda \right\|_{V_i} &\leq \|(S_i')^{-1}\| \left\| \sum_{\lambda\in\Lambda_i} c_\lambda\phi_\lambda \right\|_{V_i^*} \\ &\leq \frac{\|Ł^{-1}\|^2}{A} \left\| \sum_{\lambda\in\Lambda_i} c_\lambda\phi_\lambda \right\|_{V_i^*},\end{aligned}$$

hence

$$(S_i')^{-1} \sum_{\lambda\in\Lambda_i} c_\lambda\phi_\lambda \to 0, \quad as \quad i \to \infty. \tag{3.4}$$

Since $(\langle f, \psi_\lambda\rangle)_{\lambda\in\Lambda} \in \ell_2(\Lambda)$ and $\ell_2(\Lambda) = Ran(T^*) \oplus Ker(T)$ then

$$(\langle f, \psi_\lambda\rangle)_{\lambda\in\Lambda} = (\langle f', \phi_\lambda\rangle)_{\lambda\in\Lambda} + (c_\lambda)_{\lambda\in\Lambda}$$

for some $f' \in H$ and $(c_\lambda)_{\lambda\in\Lambda} \in N(T)$. Thus

$$\sum_{\lambda\in\Lambda_i} \langle f, \psi_\lambda\rangle (S_i')^{-1}\phi_\lambda = \sum_{\lambda\in\Lambda_i} \langle f', \phi_\lambda\rangle (S_i')^{-1}\phi_\lambda + \sum_{\lambda\in\Lambda_i} c_\lambda (S_i')^{-1}\phi_\lambda,$$

therefore

$$\sum_{\lambda\in\Lambda_i} \langle f, \psi_\lambda\rangle (S_i')^{-1}\phi_\lambda = \sum_{\lambda\in\Lambda_i} \langle f', \phi_\lambda\rangle (S_i')^{-1}\phi_\lambda + (S_i')^{-1} \sum_{\lambda\in\Lambda_i} c_\lambda\phi_\lambda. \tag{3.5}$$

Also because of $(c_\lambda)_{\lambda\in\Lambda} \in N(T)$, it follows

$$\begin{aligned}u &= \sum_{\lambda\in\Lambda} \langle f, \psi_\lambda\rangle (S')^{-1}\phi_\lambda \\ &= \sum_{\lambda\in\Lambda} \langle f', \phi_\lambda\rangle (S')^{-1}\phi_\lambda + (S')^{-1}\sum_{\lambda\in\Lambda} c_\lambda\phi_\lambda \\ &= f',\end{aligned}$$

combining this with (3.4) and (3.5) induce the result as $i \to \infty$. □

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