Approximately Bisimilar Symbolic Models for Incrementally Stable Switched Systems*

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Abstract. Switched systems constitute an important modeling paradigm faithfully describing many engineering systems in which software interacts with the physical world. Despite considerable progress on stability and stabilization of switched systems, the constant evolution of technology demands that we make similar progress with respect to different, and perhaps more complex, objectives. This paper describes one particular approach to address these different objectives based on the construction of approximately equivalent (bisimilar) symbolic models for a switched system. The main contribution of this paper consists in showing that under standard assumptions ensuring incremental stability of a switched system (i.e. existence of common or multiple Lyapunov functions), it is possible to construct a symbolic model that is approximately bisimilar to the original switched system with a precision that can be chosen a priori. To support the computational merits of the proposed approach we present a realistic example of a boost dc-dc converter and show how to synthesize a switched controller that regulates the output voltage at a desired level.

1 Introduction

Switched systems constitute an important modeling paradigm faithfully describing many engineering systems in which software interacts with the physical world. Although this fact already amply justifies its study, switched systems are also quite intriguing from a theoretical point of view. It is well known that by judiciously switching between stable subsystems one can render the overall system unstable. This motivated several researchers over the years to understand which classes of switching strategies or switching signals preserve stability (see

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e.g. [1]). Despite considerable progress on stability and stabilization of switched systems, the constant evolution of technology demands that we make similar progress with respect to different, and perhaps more complex, objectives. These comprise the synthesis of control strategies guiding the switched systems through predetermined operating points while avoiding certain regions in the state space, enforcing limit cycles and oscillatory behavior, reconfiguration upon the occurrence of faults, etc.

This paper describes one particular approach to address these different objectives based on the construction of symbolic models in which sets of states in the switched system are represented by abstract states. When the symbolic models are finite, controller synthesis problems can be efficiently solved by resorting to mature techniques developed in the areas of supervisory control of discrete-event systems [2] and algorithmic game theory [3]. The crucial step is therefore the construction of symbolic models that are detailed enough to capture all the behavior of the original system, but not so detailed that their use for synthesis is as difficult as the original model. This is accomplished, at the technical level, by using the notion of approximate bisimulation. Approximate bisimulation has been introduced in [4] as an approximate version of the usual bisimulation relation [5, 6]. It generalizes the notion of bisimulation by requiring the outputs of two systems to be close instead of being strictly equal. This relaxed requirement makes it possible to compute symbolic models for larger classes of systems as shown recently for incrementally stable continuous control systems [7].

The main contribution of this paper consists in showing that under standard assumptions ensuring incremental stability of a switched system (i.e. existence of common or multiple Lyapunov functions), it is possible to construct a symbolic model that is approximately bisimilar to the original switched system with a precision that can be chosen a priori. The proof is constructive and it is straightforward to derive a procedure for the computation of these symbolic models. Since in problems of practical interest the state space can be assumed to be bounded, the resulting symbolic model is guaranteed to have finitely many states and can thus be used for algorithmic controller synthesis. The technical contribution extends previous work by the authors that considered only purely continuous systems [7]. To support the computational merits of the proposed approach, we present a realistic example of a boost DC-DC converter and show how to synthesize a switched controller that regulates the output voltage at a desired level.

In the following, the symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}_0^+$ denote the set of natural, integer, real, positive and nonnegative real numbers respectively. Given a vector $x \in \mathbb{R}^n$, we denote by x_i its i-th coordinate and by $\|x\|$ its Euclidean norm.

2 Switched systems and incremental stability

2.1 Switched systems

We shall consider the class of switched systems formalized in the following definition.

Definition 1. *A switched system is a quadruple $\Sigma = (\mathbb{R}^n, P, \mathcal{P}, F)$, where:*

- *$\mathbb{R}^n$ is the state space;*
- *$P = \{1, \dots, m\}$ is the finite set of modes;*
- *$\mathcal{P}$ is a subset of $\mathcal{S}(\mathbb{R}_0^+, P)$ which denotes the set of piecewise constant functions from $\mathbb{R}_0^+$ to P, continuous from the right and with a finite number of discontinuities on every bounded interval of $\mathbb{R}_0^+$;*
- *$F = \{f_1, \dots, f_m\}$ is a collection of vector fields indexed by P. For all $p \in P$, $f_p : \mathbb{R}^n \to \mathbb{R}^n$ is a locally Lipschitz continuous map.*

For all $p \in P$, we denote by Σ_p the continuous *subsystem* of Σ defined by the differential equation:

$$\dot{\mathbf{x}}(t) = f_p(\mathbf{x}(t)). \tag{1}$$

We make the assumption that the vector field f_p is such that the solutions of the differential equation (1) are defined on an interval of the form $]a, +\infty[$ with $a < 0$. Sufficient conditions includes linear growth or compact support of the vector field f_p.

A *switching signal* of Σ is a function $\mathbf{p} \in \mathcal{P}$, the discontinuities of $\mathbf{p}$ are called *switching times*. A piecewise $\mathcal{C}^1$ function $\mathbf{x} : \mathbb{R}_0^+ \to \mathbb{R}^n$ is said to be a *trajectory* of Σ if it is continuous and there exists a switching signal $\mathbf{p} \in \mathcal{P}$ such that, at each $t \in \mathbb{R}_0^+$ where the function $\mathbf{p}$ is continuous, $\mathbf{x}$ is continuously differentiable and satisfies:

$$\dot{\mathbf{x}}(t) = f_{\mathbf{p}(t)}(\mathbf{x}(t)).$$

We will use $\mathbf{x}(t, x, \mathbf{p})$ to denote the point reached at time $t \in \mathbb{R}_0^+$ from the initial condition x under the switching signal $\mathbf{p}$. The assumptions on the vector fields $f_1, \dots, f_m$ and the fact that the switching signals have only a finite number of discontinuities on every bounded interval, thus ruling out Zeno behaviors, ensure for all initial conditions and switching signals, existence and uniqueness of the trajectory of Σ. Let us remark that a trajectory of Σ_p is a trajectory of Σ associated with the constant switching signal $\mathbf{p}(t) = p$, for all $t \in \mathbb{R}_0^+$. Then, we will use $\mathbf{x}(t, x, p)$ to denote the point reached by Σ_p at time $t \in \mathbb{R}_0^+$ from the initial condition x.

2.2 Incremental stability

The results presented in this paper rely on some stability notions. A continuous function $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$. Function γ is said to belong to class $\mathcal{K}_\infty$ if it is a $\mathcal{K}$ function and $\gamma(r) \to \infty$ when $r \to \infty$. A continuous function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to belong to class $\mathcal{KL}$ if for all fixed s, the map $r \mapsto \beta(r, s)$ belongs to class $\mathcal{K}_\infty$ and for all fixed r, the map $s \mapsto \beta(r, s)$ is strictly decreasing and $\beta(r, s) \to 0$ when $s \to \infty$.

Definition 2. *[8] The subsystem Σ_p is incrementally globally asymptotically stable (δ-GAS) if there exists a $\mathcal{KL}$ function β_p such that for all $t \in \mathbb{R}_0^+$, for all $x, y \in \mathbb{R}^n$, the following condition is satisfied:*

$$\|\mathbf{x}(t, x, p) - \mathbf{x}(t, y, p)\| \le \beta_p(\|x - y\|, t).$$

Intuitively, incremental stability means that all the trajectories of the subsystem Σ_p converge to the same reference trajectory independently of their initial condition. This is an incremental version of the notion of global asymptotic stability (GAS) [9]. Let us remark that when f_p satisfies $f_p(0) = 0$ then δ-GAS implies GAS, as all the trajectories of Σ_p converge to the trajectory $\mathbf{x}(t, 0, p) = 0$. Further, if f_p is linear then δ-GAS and GAS are equivalent. Similarly to GAS, δ-GAS can be characterized by dissipation inequalities.

Definition 3. *A smooth function $V_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$ is a δ-GAS Lyapunov function*[4] *for Σ_p if there exist $\mathcal{K}_\infty$ functions $\underline{\alpha}_p$, $\overline{\alpha}_p$ and $\kappa_p \in \mathbb{R}^+$ such that:*

$$\forall x, y \in \mathbb{R}^n, \qquad \underline{\alpha}_p(\|x - y\|) \leq V_p(x, y) \leq \overline{\alpha}_p(\|x - y\|); \tag{2}$$

$$\forall x, y \in \mathbb{R}^n, \;\; \frac{\partial V_p}{\partial x}(x, y) f_p(x) + \frac{\partial V_p}{\partial y}(x, y) f_p(y) \leq -\kappa_p V_p(x, y). \tag{3}$$

The following result completely characterizes δ-GAS in terms of existence of a δ-GAS Lyapunov function.

Theorem 1. *[8] Σ_p is δ-GAS iff it admits a δ-GAS Lyapunov function.*

For the purpose of this paper, we extend the notion of incremental stability to switched systems as follows:

Definition 4. *A switched system $\Sigma = (\mathbb{R}^n, P, \mathcal{P}, F)$ is incrementally globally uniformly asymptotically stable (δ-GUAS) if there exists a $\mathcal{KL}$ function β such that for all $t \in \mathbb{R}_0^+$, for all $x, y \in \mathbb{R}^n$, for all switching signals $\mathbf{p} \in \mathcal{P}$, the following condition is satisfied:*

$$\|\mathbf{x}(t, x, \mathbf{p}) - \mathbf{x}(t, y, \mathbf{p})\| \leq \beta(\|x - y\|, t).$$

Let us remark that the speed of convergence specified by the function β is independent of the switching signal $\mathbf{p}$. Thus, the stability property is uniform over the set of switching signals; hence the notion of incremental global uniform asymptotic stability. Incremental stability of a switched system means that all the trajectories associated with the same switching signal converge to the same reference trajectory independently of their initial condition. This is an incremental version of global uniform asymptotic stability (GUAS) for switched systems [1]. If for all $p \in P$, $f_p(0) = 0$, then δ-GUAS implies GUAS as all the trajectories of Σ converge to the constant trajectory $\mathbf{x}(t, 0, \mathbf{p}) = 0$.

It is well known that a switched system whose subsystems are all GAS may exhibit some unstable behaviors under fast switching signals. The same kind of phenomenon can be observed for switched systems with δ-GAS subsystems. Similarly, the results on common or multiple Lyapunov functions for proving GUAS of switched systems (see e.g. [1]) can be extended to prove δ-GUAS.

[4] In [8], (3) is replaced by $\frac{\partial V_p}{\partial x}(x, y) f_p(x) + \frac{\partial V_p}{\partial y}(x, y) f_p(y) \leq -\rho_p(\|x - y\|)$, where ρ_p is a positive definite function. It is known (see e.g. [1]) that there is no loss of generality in considering $\rho_p(\|x - y\|) = \kappa_p V_p(x, y)$, modifying the δ-GAS Lyapunov function V_p if necessary.

Because of the lack of space, we omit the proofs of the following theorems. Let the $\mathcal{K}_\infty$ functions $\underline{\alpha}$, $\overline{\alpha}$ and the real number κ be given by $\underline{\alpha} = \min(\underline{\alpha}_1, \dots, \underline{\alpha}_m)$, $\overline{\alpha} = \max(\overline{\alpha}_1, \dots, \overline{\alpha}_m)$ and $\kappa = \min(\kappa_1, \dots, \kappa_m)$.

Theorem 2. *Consider a switched system $\Sigma = (\mathbb{R}^n, P, \mathcal{P}, F)$. Let us assume that there exists $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$ which is a common δ-GAS Lyapunov function for subsystems $\Sigma_1, \dots, \Sigma_m$. Then, Σ is δ-GUAS.*

When a common δ-GAS Lyapunov function fails to exist, δ-GUAS of the switched system can be ensured by using multiple δ-GAS Lyapunov functions and a restrained set of switching signals. Let $\mathcal{S}_{\tau_d}(\mathbb{R}_0^+, P)$ denote the set of switching signals with *dwell time* $\tau_d \in \mathbb{R}_0^+$ so that $\mathbf{p} \in \mathcal{S}(\mathbb{R}_0^+, P)$ has dwell time τ_d if the switching times $t_1,\ t_2, \dots$ satisfy $t_1 \geq \tau_d$ and $t_i - t_{i-1} \geq \tau_d$, for all $i \geq 2$.

Theorem 3. *Let $\tau_d \in \mathbb{R}_0^+$, consider a switched system $\Sigma_{\tau_d} = (\mathbb{R}^n, P, \mathcal{P}, F)$ with $\mathcal{P} \subseteq \mathcal{S}_{\tau_d}(\mathbb{R}_0^+, P)$. Let us assume that for all $p \in P$, there exists a δ-GAS Lyapunov function V_p for subsystem $\Sigma_{\tau_d,p}$ and that in addition there exists $\mu \geq 1$ such that:*

$$\forall x, y \in \mathbb{R}^n,\ \forall p, p' \in P,\ V_p(x, y) \leq \mu V_{p'}(x, y). \tag{4}$$

If $\tau_d > \frac{\log \mu}{\kappa}$, then Σ_{τ_d} is δ-GUAS.

In the following, we show that under the assumptions of Theorems 2 or 3, it is possible to compute approximately equivalent symbolic models of switched systems. We will make the following supplementary assumption on the δ-GAS Lyapunov functions: for all $p \in P$, there exists a $\mathcal{K}_\infty$ function γ_p such that

$$\forall x, y, z \in \mathbb{R}^n,\ |V_p(x, y) - V_p(x, z)| \leq \gamma_p(\|y - z\|). \tag{5}$$

Note that γ_p is not a function of the variable x; let the $\mathcal{K}_\infty$ function γ be given by $\gamma = \max(\gamma_1, \dots, \gamma_m)$. We will discuss this assumption later in the paper and we will show that it is not restrictive provided we are interested in the dynamics of the switched system on a compact subset of the state space $\mathbb{R}^n$.

3 Approximate bisimulation

In this section, we present a notion of approximate equivalence which will relate a switched system to the symbolic models that we construct. We start by introducing the class of transition systems which allows us to model switched and symbolic systems in a common framework.

Definition 5. *A transition system is a sextuple $T = (Q, L, \longrightarrow, O, H, I)$ consisting of:*

- *a set of states Q;*
- *a set of labels L;*
- *a transition relation $\longrightarrow \subseteq Q \times L \times Q$;*
- *an output set O;*

– *an output function* $H : Q \to O$*;*
– *a set of initial states* $I \subseteq Q$*.*

T is said to be metric if the output set O is equipped with a metric d, countable if Q and L are countable sets, finite, if Q and L are finite sets.

The transition $(q, l, q') \in \longrightarrow$ will be denoted $q \xrightarrow{l} q'$. The transition relation captures the dynamics of the transition system: $q \xrightarrow{l} q'$ means that the system can evolve from state q to state q' under the action labelled by l.

Transition systems can serve as abstract models for describing switched systems. Given a switched system $\Sigma = (\mathbb{R}^n, P, \mathcal{P}, F)$ where $\mathcal{P} = \mathcal{S}(\mathbb{R}_0^+, P)$, we define the associated transition system $T(\Sigma) = (Q, L, \longrightarrow, O, H, I)$, where the set of states is $Q = \mathbb{R}^n$; the set of labels is $L = P$; the transition relation is given by $q \xrightarrow{l} q'$ iff there exists a trajectory $\mathbf{x}$ of the subsystem Σ_l such that $\mathbf{x}(\tau, q, l) = q'$ for some $\tau \in \mathbb{R}^+$; the set of outputs is $O = \mathbb{R}^n$; the observation map H is the identity map over $\mathbb{R}^n$; the set of initial states is $I = \mathbb{R}^n$. The transition system $T(\Sigma)$ is metric when the set of outputs $O = \mathbb{R}^n$ is equipped with the metric $d(q, q') = \|q - q'\|$. Note that the state space of $T(\Sigma)$ is infinite.

Usual equivalence relationships between transition systems rely on the equality of the languages. In this paper, we are mostly interested in bisimulation equivalence [5, 6]. Intuitively, a bisimulation relation between two transition systems T_1 and T_2 is a relation between their set of states explaining how a trajectory of T_1 can be transformed into a trajectory of T_2 with the same associated sequence of outputs, and vice versa. The requirement of equality of output sequences, as in the classical formulation of bisimulation [5, 6] is quite strong for metric transition systems. We shall relax this, by requiring output sequences to be close where closeness is measured with respect to the metric on the output space. This relaxation leads to the notion of approximate bisimulation relation introduced in [4].

Definition 6. *Let $T_1 = (Q_1, L, \underset{1}{\longrightarrow}, O, H_1, I_1)$, $T_2 = (Q_2, L, \underset{2}{\longrightarrow}, O, H_2, I_2)$ be metric transition systems with the same sets of labels L and outputs O equipped with the metric d. Let $\varepsilon \in \mathbb{R}_0^+$ be a given precision, a relation $R \subseteq Q_1 \times Q_2$ is said to be an ε-approximate bisimulation relation between T_1 and T_2 if for all $(q_1, q_2) \in R$:*

– $d(H_1(q_1), H_2(q_2)) \leq \varepsilon$*;*
– *for all $q_1 \xrightarrow[1]{l} q_1'$, there exists $q_2 \xrightarrow[2]{l} q_2'$, such that $(q_1', q_2') \in R$;*
– *for all $q_2 \xrightarrow[2]{l} q_2'$, there exists $q_1 \xrightarrow[1]{l} q_1'$, such that $(q_1', q_2') \in R$.*

The transition systems T_1 and T_2 are said to be approximately bisimilar with precision ε (denoted $T_1 \sim_\varepsilon T_2$) if:

– *for all $q_1 \in I_1$, there exists $q_2 \in I_2$, such that $(q_1, q_2) \in R$;*
– *for all $q_2 \in I_2$, there exists $q_1 \in I_1$, such that $(q_1, q_2) \in R$.*

4 Approximately bisimilar symbolic models

In the following, we will work with a sub-transition system of $T(\Sigma)$ obtained by selecting the transitions of $T(\Sigma)$ that describe trajectories of duration τ_s for some chosen $\tau_s \in \mathbb{R}^+$. This can be seen as a sampling process. Moreover, we suppose that switching instants can only occur at times of the form $i\tau_s$ with $i \in \mathbb{N}$. This is a natural constraint when the switching in Σ has to be controlled by a microprocessor with clock period τ_s. Given a switched system $\Sigma = (\mathbb{R}^n, P, \mathcal{P}, F)$ where $\mathcal{P} = \mathcal{S}(\mathbb{R}_0^+, P)$, and a time sampling parameter $\tau_s \in \mathbb{R}^+$, we define the associated transition system $T_{\tau_s}(\Sigma) = (Q_1, L_1, \xrightarrow[1]{}, O_1, H_1, I_1)$ where the set of states is $Q_1 = \mathbb{R}^n$; the set of labels is $L_1 = P$; the transition relation is given by $q \xrightarrow[1]{l} q'$ iff $\mathbf{x}(\tau_s, q, l) = q'$; the set of outputs is $O_1 = \mathbb{R}^n$; the observation map H_1 is the identity map over $\mathbb{R}^n$; the set of initial states is $I_1 = \mathbb{R}^n$. The transition system $T_{\tau_s}(\Sigma)$ is metric when the set of outputs $O_1 = \mathbb{R}^n$ is equipped with the metric $d(q, q') = \|q - q'\|$.

4.1 Common Lyapunov function

We first examinate the case when there exists a common δ-GAS Lyapunov function V for subsystems $\Sigma_1, \ldots, \Sigma_m$. We start by approximating the set of states $Q_1 = \mathbb{R}^n$ by the lattice:

$$[\mathbb{R}^n]_\eta = \left\{ q \in \mathbb{R}^n \;\middle|\; q_i = k_i \frac{2\eta}{\sqrt{n}},\ k_i \in \mathbb{Z},\ i = 1, ..., n \right\},$$

where $\eta \in \mathbb{R}^+$ is a state space discretization parameter. By simple geometrical considerations, we can see that for all $x \in \mathbb{R}^n$, there exists $q \in [\mathbb{R}^n]_\eta$ such that $\|x - q\| \leq \eta$.

Let us define the transition system $T_{\tau_s,\eta}(\Sigma) = (Q_2, L_2, \xrightarrow[2]{}, O_2, H_2, I_2)$, where the set of states is $Q_2 = [\mathbb{R}^n]_\eta$; the set of labels remains the same $L_2 = L_1 = P$; the transition relation is given by $q \xrightarrow[2]{l} q'$ iff $\|\mathbf{x}(\tau_s, q, l) - q'\| \leq \eta$; the set of outputs remains the same $O_2 = O_1 = \mathbb{R}^n$; the observation map H_2 is the natural inclusion map from $[\mathbb{R}^n]_\eta$ to $\mathbb{R}^n$, i.e. $H_2(q) = q$; the set of initial states is $I_2 = [\mathbb{R}^n]_\eta$. Note that the transition system $T_{\tau_s,\eta}(\Sigma)$ is countable. Moreover, it is metric when the set of outputs $O_2 = \mathbb{R}^n$ is equipped with the metric $d(q, q') = \|q - q'\|$.

We now give the result that relates the existence of a common δ-GAS Lyapunov function for the subsystems $\Sigma_1, \ldots, \Sigma_m$ to the existence of approximately bisimilar symbolic models for the transition system $T_{\tau_s}(\Sigma)$.

Theorem 4. *Consider a switched system $\Sigma = (\mathbb{R}^n, P, \mathcal{P}, F)$ with $\mathcal{P} = \mathcal{S}(\mathbb{R}_0^+, P)$, time and state space sampling parameters $\tau_s, \eta \in \mathbb{R}^+$ and a desired precision $\varepsilon \in \mathbb{R}^+$. Let us assume that there exists $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$ which is a common δ-GAS Lyapunov function for subsystems $\Sigma_1, \ldots, \Sigma_m$ and such that equation (5) holds for some $\mathcal{K}_\infty$ function γ. If*

$$\eta \leq \min\left\{ \gamma^{-1}\left((1 - e^{-\kappa\tau_s})\underline{\alpha}(\varepsilon) \right), \overline{\alpha}^{-1}\left(\underline{\alpha}(\varepsilon) \right) \right\} \qquad (6)$$

then, the transition systems $T_{\tau_s}(\Sigma)$ and $T_{\tau_s,\eta}(\Sigma)$ are approximately bisimilar with precision ε.

Proof. We start by showing that the relation $R \subseteq Q_1 \times Q_2$ defined by $(q_1, q_2) \in R$ iff $V(q_1, q_2) \leq \underline{\alpha}(\varepsilon)$, is an ε-approximate bisimulation relation. Let $(q_1, q_2) \in R$, then $\|q_1 - q_2\| \leq \underline{\alpha}^{-1}(V(q_1, q_2)) \leq \varepsilon$. Thus, the first condition of Definition 6 holds. Let $q_1 \xrightarrow[1]{l} q_1'$, then $q_1' = \mathbf{x}(\tau_s, q_1, l)$. There exists $q_2' \in [\mathbb{R}^n]_\eta$ such that $\|\mathbf{x}(\tau_s, q_2, l) - q_2'\| \leq \eta$. Then, we have $q_2 \xrightarrow[2]{l} q_2'$. Let us check that $(q_1', q_2') \in R$. From equation (5), $|V(q_1', q_2') - V(q_1', \mathbf{x}(\tau_s, q_2, l))| \leq \gamma(\|q_2' - \mathbf{x}(\tau_s, q_2, l))\|) \leq \gamma(\eta)$. It follows that

$$\begin{aligned} V(q_1', q_2') &\leq V(q_1', \mathbf{x}(\tau_s, q_2, l)) + \gamma(\eta) = V(\mathbf{x}(\tau_s, q_1, l), \mathbf{x}(\tau_s, q_2, l)) + \gamma(\eta) \\ &\leq e^{-\kappa\tau_s} V(q_1, q_2) + \gamma(\eta) \end{aligned} \tag{7}$$

because V is a δ-GAS Lyapunov function for subsystem Σ_l. Then, from equation (6) and since γ is a $\mathcal{K}_\infty$ function, $V(q_1', q_2') \leq e^{-\kappa\tau_s}\underline{\alpha}(\varepsilon) + \gamma(\eta) \leq \underline{\alpha}(\varepsilon)$. Hence, $(q_1', q_2') \in R$. In a similar way, we can prove that, for all $q_2 \xrightarrow[2]{l} q_2'$, there is $q_1 \xrightarrow[1]{l} q_1'$ such that $(q_1', q_2') \in R$. Hence R is an ε-approximate bisimulation relation between $T_\tau(\Sigma)$ and $T_{\tau,\eta}(\Sigma)$.

By definition of $I_2 = [\mathbb{R}^n]_\eta$, for all $q_1 \in I_1 = \mathbb{R}^n$, there exists $q_2 \in I_2$ such that $\|q_1 - q_2\| \leq \eta$. Then, $V(q_1, q_2) \leq \overline{\alpha}(\|q_1 - q_2\|) \leq \overline{\alpha}(\eta) \leq \underline{\alpha}(\varepsilon)$ because of equation (6) and $\overline{\alpha}$ is a $\mathcal{K}_\infty$ function. Hence, $(q_1, q_2) \in R$. Conversely, for all $q_2 \in I_2$, $q_1 = q_2 \in \mathbb{R}^n = I_1$, then $V(q_1, q_2) = 0$ and $(q_1, q_2) \in R$. Therefore, $T_{\tau_s}(\Sigma)$ and $T_{\tau_s,\eta}(\Sigma)$ are approximately bisimilar with precision ε. ■

Let us remark that, for a given time sampling parameter τ_s and a desired precision $\varepsilon \in \mathbb{R}^+$, there always exists $\eta \in \mathbb{R}^+$ sufficiently small such that equation (6) holds. This means that for switched systems admitting a common δ-GAS Lyapunov function there exists approximately bisimilar symbolic models and any precision can be reached for all sampling rates.

4.2 Multiple Lyapunov functions

If a common δ-GAS Lyapunov function does not exist, it remains possible to compute approximately bisimilar symbolic models provided we restrict the set of switching signals using a dwell time τ_d. In this section, we consider a switched system $\Sigma_{\tau_d} = (\mathbb{R}^n, P, \mathcal{P}, F)$ where $\mathcal{P} = \mathcal{S}_{\tau_d}(\mathbb{R}_0^+, P)$. Let τ_s be a time sampling parameter; for simplicity, we will assume that the dwell time τ_d is an integer multiple of τ_s: there exists $N \in \mathbb{N}$ such that $\tau_d = N\tau_s$. Representing Σ_{τ_d} using a transition system is a bit less trivial than previously as we need to record inside the state of the transition system the time elapsed since the latest switching occured. Thus, the transition system associated with Σ_{τ_d} is $T_{\tau_s}(\Sigma_{\tau_d}) = (Q_1, L_1, \xrightarrow[1]{}, O_1, H_1, I_1)$ where:

– the set of states is $Q_1 = \mathbb{R}^n \times P \times \{0, \ldots, N-1\}$, a state $(x, p, i) \in Q_1$ means that the current state of Σ_{τ_d} is x, the current value of the switching signal is p and the time elapsed since the latest switching is exactly $i\tau_s$ if $i < N-1$ or at least $(N-1)\tau_s$ if $i = N-1$.
– the set of labels is $L_1 = P$;
– the transition relation is given by $(x, p, i) \xrightarrow[1]{l} (x', p', i')$ iff $l = p$ and one the following holds:
  - $i < N-1$, $x' = \mathbf{x}(\tau_s, x, p)$, $p' = p$ and $i' = i+1$: switching is not allowed because the time elapsed since the latest switch is strictly smaller than the dwell time;
  - $i = N-1$, $x' = \mathbf{x}(\tau_s, x, p)$, $p' = p$ and $i' = N-1$: switching is allowed but no switch occurs;
  - $i = N-1$, $x' = \mathbf{x}(\tau_s, x, p)$, $p' \neq p$ and $i' = 0$: switching is allowed and a switch occurs.
– the set of outputs is $O_1 = \mathbb{R}^n$;
– the observation map H_1 is given by $H_1((x, p, i)) = x$;
– the set of initial states is $I_1 = \mathbb{R}^n \times P \times \{0\}$.

One can verify that the output trajectories of $T_{\tau_s}(\Sigma_{\tau_d})$ are the output trajectories of $T_{\tau_s}(\Sigma)$ associated with switching signals with dwell time $\tau_d = N\tau_s$. The approximation of the set of states of $T_{\tau_s}(\Sigma_{\tau_d})$ by a symbolic model is done using a lattice, as previously. Let $\eta \in \mathbb{R}^+$ be a state space discretization parameter, we define the transition system $T_{\tau_s,\eta}(\Sigma_{\tau_d}) = (Q_2, L_2, \xrightarrow[2]{}, O_2, H_2, I_2)$ where:
– the set of states is $Q_2 = [\mathbb{R}^n]_\eta \times P \times \{0, \ldots, N-1\}$.
– the set of labels remains the same $L_2 = L_1 = P$;
– the transition relation is given by $(x, p, i) \xrightarrow[1]{l} (x', p', i')$ iff $l = p$ and one of the following holds:
  - $i < N-1$, $\|\mathbf{x}(\tau_s, x, p) - x'\| \leq \eta$, $p' = p$ and $i' = i+1$;
  - $i = N-1$, $\|\mathbf{x}(\tau_s, x, p) - x'\| \leq \eta$, $p' = p$ and $i' = N-1$;
  - $i = N-1$, $\|\mathbf{x}(\tau_s, x, p) - x'\| \leq \eta$, $p' \neq p$ and $i' = 0$;
– the set of outputs remains the same $O_2 = O_1 = \mathbb{R}^n$;
– the observation map H_2 is given by $H_2((x, p, i)) = x$;
– the set of initial states is $I_2 = [\mathbb{R}^n]_\eta \times P \times \{0\}$.

Note that the transition system $T_{\tau_s,\eta}(\Sigma_{\tau_d})$ is countable. Moreover, $T_{\tau_s}(\Sigma_{\tau_d})$ and $T_{\tau_s,\eta}(\Sigma_{\tau_d})$ are metric when the set of outputs $O_1 = O_2 = \mathbb{R}^n$ is equipped with the metric $d(x, x') = \|x - x'\|$. The following theorem establishes the approximate equivalence of $T_{\tau_s}(\Sigma_{\tau_d})$ and $T_{\tau_s,\eta}(\Sigma_{\tau_d})$.

Theorem 5. *Consider $\tau_d \in \mathbb{R}_0^+$, a switched system $\Sigma_{\tau_d} = (\mathbb{R}^n, P, \mathcal{P}, F)$ with $\mathcal{P} = \mathcal{S}_{\tau_d}(\mathbb{R}_0^+, P)$, time and state space sampling parameters $\tau_s, \eta \in \mathbb{R}^+$ and a desired precision $\varepsilon \in \mathbb{R}^+$. Let us assume that for all $p \in P$, there exists a δ-GAS Lyapunov function V_p for subsystem $\Sigma_{\tau_d,p}$ and that equations (4) and (5) hold for some $\mu \geq 1$ and $\mathcal{K}_\infty$ functions $\gamma_1, \ldots, \gamma_m$. If $\tau_d > \frac{\log \mu}{\kappa}$ and*

$$\eta \leq \min\left\{\gamma^{-1}\left(\frac{\frac{1}{\mu} - e^{-\kappa\tau_d}}{1 - e^{-\kappa\tau_d}}(1 - e^{-\kappa\tau_s})\underline{\alpha}(\varepsilon)\right), \overline{\alpha}^{-1}\left(\underline{\alpha}(\varepsilon)\right)\right\} \tag{8}$$

then, the transition systems $T_{\tau_s}(\Sigma_{\tau_d})$ and $T_{\tau_s,\eta}(\Sigma_{\tau_d})$ are approximately bisimilar with precision ε.

Proof. Let us define the relation $R \subseteq Q_1 \times Q_2$ by

$$R = \{(x_1, p_1, i_1, x_2, p_2, i_2) \in Q_1 \times Q_2 |\ p_1 = p_2 = p,\ i_1 = i_2 = i, V_p(x_1, x_2) \leq \delta_i\}$$

where $\delta_0, \dots, \delta_N$ are given recursively by $\delta_0 = \underline{\alpha}(\varepsilon),\ \delta_{i+1} = e^{-\kappa\tau_s}\delta_i + \gamma(\eta)$. Let us remark that:

$$\delta_i = e^{-i\kappa\tau_s}\underline{\alpha}(\varepsilon) + \gamma(\eta)\frac{1 - e^{-i\kappa\tau_s}}{1 - e^{-\kappa\tau_s}} = \frac{\gamma(\eta)}{1 - e^{-\kappa\tau_s}} + e^{-i\kappa\tau_s}\left(\underline{\alpha}(\varepsilon) - \frac{\gamma(\eta)}{1 - e^{-\kappa\tau_s}}\right) \quad (9)$$

From equation (4), $\mu \geq 1$; then, from equation (8) and since γ is a $\mathcal{K}_\infty$ function, $\gamma(\eta) \leq (1 - e^{-\kappa\tau_s})\underline{\alpha}(\varepsilon)$. It follows from (9) that $\delta_0 \geq \delta_1 \geq \dots \geq \delta_{N-1} \geq \delta_N$. From equation (8), and since γ is a $\mathcal{K}_\infty$ function and $\tau_d = N\tau_s$,

$$\delta_N = e^{-\kappa\tau_d}\underline{\alpha}(\varepsilon) + \gamma(\eta)\frac{1 - e^{-\kappa\tau_d}}{1 - e^{-\kappa\tau_s}} \leq e^{-\kappa\tau_d}\underline{\alpha}(\varepsilon) + \left(\frac{1}{\mu} - e^{-\kappa\tau_d}\right)\underline{\alpha}(\varepsilon) = \frac{\underline{\alpha}(\varepsilon)}{\mu}.$$

We can now prove that R is an ε-approximate bisimulation relation between $T_{\tau_s}(\Sigma_{\tau_d})$ and $T_{\tau_s,\eta}(\Sigma_{\tau_d})$. Let $(x_1, p, i, x_2, p, i) \in R$, then

$$\begin{aligned} \|H_1(x_1, p, i) - H_2(x_2, p, i)\| = \|x_1 - x_2\| &\leq \underline{\alpha}^{-1}\left(V_p(x_1, x_2)\right) \\ &\leq \underline{\alpha}^{-1}(\delta_i) \leq \underline{\alpha}^{-1}(\delta_0) = \varepsilon. \end{aligned}$$

Hence, the first condition of Definition 6 holds. Let us prove that the second condition holds as well. Let $(x_1, p, i) \xrightarrow[1]{p} (x_1', p', i')$, then $x_1' = \mathbf{x}(\tau_s, x_1, p)$. There exists a transition $(x_2, p, i) \xrightarrow[2]{p} (x_2', p', i')$ with $\|x_2' - \mathbf{x}(\tau_s, x_2, p)\| \leq \eta$. From equation (5) and since V_p is a δ-GAS Lyapunov function for subsystem Σ_p we can show, similarly to equation (7), that

$$V_p(x_1', x_2') \leq e^{-\kappa\tau_s}V_p(x_1, x_2) + \gamma(\eta) \leq e^{-\kappa\tau_s}\delta_i + \gamma(\eta) = \delta_{i+1}. \quad (10)$$

We now examinate three separate cases:

- $i < N - 1$, then $p' = p$ and $i' = i + 1$; since $V_p(x_1', x_2') \leq \delta_{i+1}$, it follows that $(x_1', p, i + 1, x_2', p, i + 1) \in R$.
- $i = N - 1$ and $p' = p$ then $i' = N - 1$; from (10), $V_p(x_1', x_2') \leq \delta_N \leq \delta_{N-1}$, it follows that $(x_1', p, N - 1, x_2', p, N - 1) \in R$.
- $i = N - 1$ and $p' \neq p$ then $i' = 0$; from (10), $V_p(x_1', x_2') \leq \delta_N \leq \delta_0/\mu$. From equation (5), it follows that $V_{p'}(x_1', x_2') \leq \mu V_p(x_1', x_2') \leq \delta_0$. Therefore, $(x_1', p', 0, x_2', p', 0) \in R$.

Similarly, we can show that for any transition $(x_2, p, i) \xrightarrow[2]{l} (x_2', p', i')$, there exists a transition $(x_1, p, i) \xrightarrow[1]{l} (x_1', p', i')$ such that $(x_1', p', i', x_2', p', i') \in R$. Hence, R is an ε-approximate bisimulation relation.

For all initial states $(x_1,p,0) \in I_1$, there exists $(x_2,p,0) \in I_2$ such that $\|x_1 - x_2\| \leq \eta$. Then, $V_p(x_1,x_2) \leq \overline{\alpha}(\eta) \leq \underline{\alpha}(\varepsilon)$ because of equation (8) and $\overline{\alpha}$ is $\mathcal{K}_\infty$ function. Hence, $V_p(x_1,x_2) \leq \delta_0$ and $(x_1,p,0,x_2,p,0) \in R$. Conversely, for all $(x_2,p,0) \in I_2$, $(x_1,p,0) = (x_2,p,0) \in I_1$. Then, $V_p(x_1,x_2) = 0 \leq \delta_0$ and $(x_1,p,0,x_2,p,0) \in R$. Thus, $T_{\tau_s}(\Sigma_{\tau_d})$ and $T_{\tau_s,\eta}(\Sigma_{\tau_d})$ are approximately bisimilar with precision ε. ■

Provided that $\tau_d > \frac{\log \mu}{\kappa}$, for a given time sampling parameter and a desired precision, there always exists $\eta \in \mathbb{R}^+$ sufficiently small such that equation (8) holds. Thus, if the dwell time is large enough, we can compute symbolic models of arbitrary precision of the switched system. Let us remark that the lower bound we obtain on the dwell time is the same than the one in Theorem 3 ensuring incremental stability of the switched system. Theorem 4 can be seen as a corollary of Theorem 5. Indeed, existence of a common δ-GAS Lyapunov function is equivalent to equation (4) with $\mu = 1$. Then, no constraint is necessary on the dwell time and equation (8) becomes equivalent to (6).

The previous Theorems also give indications on the practical computation of these symbolic models. The sets of states of $T_{\tau_s,\eta}(\Sigma)$ or $T_{\tau_s,\eta}(\Sigma_{\tau_d})$ are countable but infinite. However, if we are interested in the dynamics of the switched system only on a compact subset $C \subseteq \mathbb{R}^n$, then we can restrict the set of states of $T_{\tau_s,\eta}(\Sigma)$ or $T_{\tau_s,\eta}(\Sigma_{\tau_d})$ to the sets $[\mathbb{R}^n]_\mu \cap C$ or $([\mathbb{R}^n]_\mu \cap C) \times P \times \{0,\dots,N-1\}$ which are finite. The computation of the transition relations is then relatively simple since it mainly involves the numerical computation of the points $\mathbf{x}(\tau_s,x,p)$ with $x \in [\mathbb{R}^n]_\mu \cap C$ and $p \in P$. This can be done by simulation of the subsystems $\Sigma_1,\dots,\Sigma_m$. Numerical errors in the computation of these points can be taken into account: it is sufficient to replace η by $\eta + e$, where e is an evaluation of the error, in Theorems 4 and 5.

Finally, we would like to discuss the assumption made in equation (5). This assumption is quite strong because the inequality has to hold for any triple in $\mathbb{R}^n$, and the function γ_p must be independent of x. However, if we are interested in the dynamics of the switched system on the compact subset $C \subseteq \mathbb{R}^n$, we only need this assumption to hold for all $x,y,z \in C$. Then, it is sufficient to assume that V_p is $\mathcal{C}^1$ on C. Indeed, for all $x,y,z \in C$,

$$|V_p(x,y) - V_p(x,z)| \leq \left(\max_{x,y \in C} \left\| \frac{\partial V_p}{\partial y}(x,y) \right\| \right) \|y - z\| = \gamma_p(\|y - z\|).$$

In this case, equation (5) holds. This means that the existence of approximately bisimilar symbolic models on an arbitrary compact subset of $\mathbb{R}^n$ does not need more assumptions than existence of common or multiple Lyapunov functions ensuring incremental stability of the switched system.

5 Symbolic models for the boost DC-DC converter

In this section, we use our methodology to compute symbolic models of a concrete switched system: the boost DC-DC converter (see Figure 1). This is an example of electrical power convertor that has been studied from the point of view of hybrid control in [10–13].

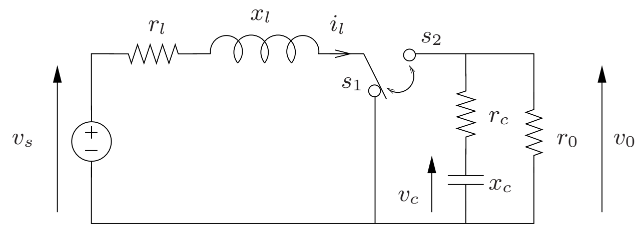

Fig. 1. boost DC-DC converter.

The boost converter has two operation modes depending on the position of the switch. The state of the system is $x(t) = [i_l(t)\ v_c(t)]^T$ where $i_l(t)$ is the inductor current and $v_c(t)$ the capacitor voltage. The dynamics associated with both modes are affine of the form $\dot{x}(t) = A_p x(t) + b$ $(p = 1, 2)$ with

$$A_1 = \begin{bmatrix} -\frac{r_l}{x_l} & 0 \\ 0 & -\frac{1}{x_c}\frac{1}{r_0+r_c} \end{bmatrix},\ A_2 = \begin{bmatrix} -\frac{1}{x_l}(r_l+\frac{r_0 r_c}{r_0+r_c}) & -\frac{1}{x_l}\frac{r_0}{r_0+r_c} \\ \frac{1}{x_c}\frac{r_0}{r_0+r_c} & -\frac{1}{x_c}\frac{1}{r_0+r_c} \end{bmatrix},\ b = \begin{bmatrix} \frac{v_s}{x_l} \\ 0 \end{bmatrix}.$$

It is clear that the boost DC-DC converter is an example of a switched system. In the following, we use the numerical values from [11], that is, in the per unit system, $x_c = 70$ p.u., $x_l = 3$ p.u., $r_c = 0.005$ p.u., $r_l = 0.05$ p.u., $r_0 = 1$ p.u. and $v_s = 1$ p.u.. The goal of the boost DC-DC converter is to regulate the output voltage across the load r_0. This control problem is usually reformulated as a current reference scheme. Then, the goal is to keep the inductor current $i_l(t)$ around a reference value i_l^{ref}. This can be done, for instance, by synthesizing a controller that keeps the state of the switched system in an invariant set $\mathcal{I}$ centered around the reference value.

It can be shown by solving a set of 2 linear matrix inequalities that the subsystems associated with the two operation modes are both incrementally stable and that they share a common δ-GAS Lyapunov function of the form $V(x,y) = \sqrt{(x-y)^T M(x-y)}$, where M is positive definite symmetric. For a better numerical conditioning, we rescale the second variable of the system (i.e. the state of the system becomes $x(t) = [i_l(t)\ 5v_c(t)]^T$; the matrices A_1, A_2 and vector b are modified accordingly). The δ-GAS Lyapunov function that we obtain has the following characteristics: $\underline{\alpha}(s) = s$, $\overline{\alpha}(s) = \gamma(s) = 1.0127s$, $\kappa = 0.014$, and we set the sampling period to $\tau_s = 0.5$. Then, a symbolic model can be computed for the boost DC-DC converter using the procedure described in Section 4. According to Theorem 4, a desired precision ε can be achieved by choosing a state space discretization parameter η satisfying $\eta \leq \varepsilon/145$. In this example, the ratio between the precision of the symbolic approximation and the state space discretization parameter is quite large. This is explained by the fact that the subsystems are quite weakly stable since the value of κ is small.

We consider two different values of the precision parameter ε. We first choose a precision $\varepsilon = 2.6$ which can be achieved by choosing $\eta = \frac{1}{40\sqrt{2}}$. This precision is

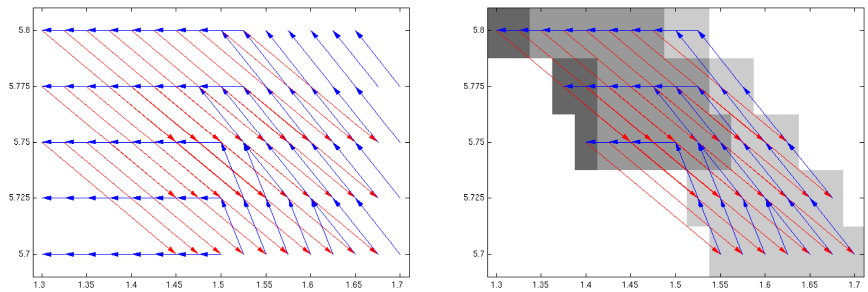

Fig. 2. Symbolic model of the DC-DC converter for $\eta = \frac{1}{40\sqrt{2}}$ (left); Controller for the symbolic model (right) (dark gray: mode 1, light gray: mode 2, medium gray: both modes are acceptable, white: the invariance property cannot be ensured from these states).

quite poor and makes the computed symbolic model of no practical use. However, it helps to understand the second experiment related further. On Figure 2, the symbolic model of the boost DC-DC converter is shown on the left, red and blue arrows represent the transitions associated with mode 1 and 2, respectively. We only represented the transitions that keep the state of the symbolic model in the set $\mathcal{I}' = [1.3, 1.7] \times [5.7, 5.8]$. Using supervisory control [2], we synthesized a controller that keeps the state of the symbolic model inside $\mathcal{I}'$. It is shown on the right figure: dark and light gray means that for these states of the symbolic model the controller has to use mode 1 and 2, respectively; medium gray means that for these states the controller can use either mode 1 or mode 2; white means that from these states there does not exist any switching sequence that keeps the state of the symbolic model in $\mathcal{I}'$. From this controller, using the approach presented in [14], one could derive a controller for the boost DC-DC converter that keeps the state of the switched system in $\mathcal{I} = [1.3-\varepsilon, 1.7+\varepsilon] \times [5.7-\varepsilon, 5.8+\varepsilon]$ which is not useful in practice.

The second value we consider for the precision parameter is $\varepsilon = 0.026$. This precision can be achieved by choosing $\eta = \frac{1}{4000\sqrt{2}}$. We do not show the symbolic model as it has too many states (642001) to be represented graphically. We repeat the same experiment with this model, the supervisory controller that keeps the state of the symbolic model in $\mathcal{I}'$ is shown in Figure 3, on the left. The computation of the symbolic model and the synthesis of the supervisory controller, implemented in MATLAB, takes overall around 80 seconds. From the controller of the symbolic model, we derive a controller for the boost DC-DC converter that keeps the state of the switched system in $\mathcal{I} = [1.3 - \varepsilon, 1.7 + \varepsilon] \times [5.7 - \varepsilon, 5.8 + \varepsilon]$. We apply a lazy control strategy, when the controller can choose both modes 1 and 2, it just keeps the current operation mode unchanged. A state trajectory of the controlled boost DC-DC converter is shown in Figure 3, on the right. We can see that the trajectory remains in the invariant set.

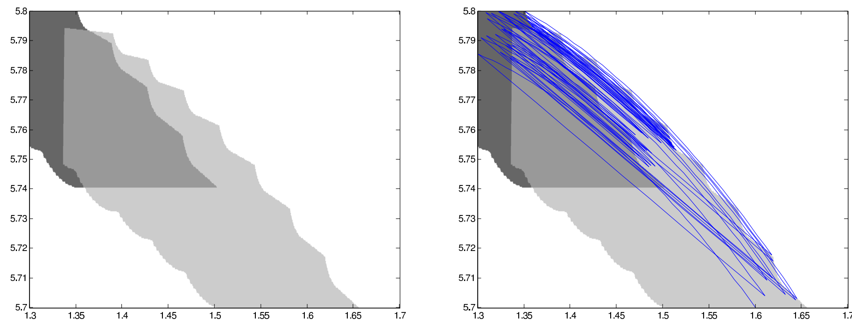

Fig. 3. Controller for a symbolic model of the DC-DC converter for $\eta = \frac{1}{4000\sqrt{2}}$ (left); Trajectory of the boost DC-DC converter using the previous controller (right).

6 Conclusion

In this paper, we showed, under standard assumptions ensuring incremental stability, the existence of approximately bisimilar symbolic abstractions for switched systems. The abstractions are effectively computable and any precision can be achieved. An example of application has been showed on the DC-DC converter.

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