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Approximating a Convex Body by a Polytope Using the Epsilon-Net Theorem

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Abstract We prove that roughly $\frac{d}{(1-\vartheta)^d} \ln \frac{1}{(1-\vartheta)^d}$ points chosen uniformly and independently from a centered convex body K in $\mathbb{R}^d$ yield a polytope P for which $\vartheta K \subseteq P \subseteq K$ holds with large probability. This gives a joint generalization of results of Brazitikos, Chasapis and Hioni and of Giannopoulos and Milman.

Keywords Approximation by polytopes · Convex body · Epsilon-net theorem · Grünbaum's theorem · VC-dimension

Mathematics Subject Classification 52A27 · 52A20

1 Introduction

A convex body (i.e., a compact convex set with non-empty interior) in $\mathbb{R}^d$ is called *centered*, if its center of mass is the origin.

We study the following problem. Given a centered convex body K in $\mathbb{R}^d$, a positive integer $t \geq d+1$, and $\delta, \vartheta \in (0, 1)$. We want to show that under some assumptions on the parameters d, t, δ, ϑ (and without assumptions on K), the convex hull P of t randomly, uniformly and independently chosen points of K contains ϑK with probability at least $1 - \delta$.

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[4, Thm. 1.1] concerns the case of very rough approximation, that is, where the number t of chosen points is linear in the dimension d. It states that the convex hull of $t = \alpha d$ random points in a centered convex body K is a convex polytope P which satisfies $\frac{c_1}{d} K \subseteq P$, with probability $1 - \delta = 1 - e^{-c_2 d}$, where $c_1, c_2 > 0$ and $\alpha > 1$ are absolute constants. In our first result, we obtain explicit constants.

Theorem 1.1 *Let K be a centered convex body in $\mathbb{R}^d$. Choose $t = 60(d+1)$ points $X_1, \ldots, X_t$ of K randomly, independently and uniformly. Then*

$$\frac{1}{d} K \subseteq \operatorname{conv}\{X_1, \ldots, X_t\} \subseteq K.$$

with probability at least $1 - 4e^{-d-1}$.

Another instance of our general problem is [7, Thm. 5.2], which concerns fine approximation, that is, where the number t of chosen points is exponential in the dimension d. It states that for any $\delta, \gamma \in (0,1)$, if we choose $t = e^{\gamma d}$ random points in any centered convex body K in $\mathbb{R}^d$, then the convex polytope P thus obtained satisfies $c(\delta)\gamma K \subseteq P$, with probability $1 - \delta$. We note that it is not included explicitly in the statement of of [7, Thm. 5.2] that it only holds for sufficiently large d, that is, when $d > d_0$, where d_0 depends on δ and γ. This condition is clearly necessary, as for any γ and any K, with some positive probability, the origin is not in the convex hull of $t = e^{\gamma d}$ random points in K.

[7, Prop. 5.3] follows from the same argument as Theorem 5.2 therein. It states that for any $\delta, \vartheta \in (0,1)$, if we choose $t = c(\delta)\left(\frac{c}{1-\vartheta}\right)^d$ random points in any centered convex body K in $\mathbb{R}^d$, then the convex polytope P thus obtained satisfies $\vartheta K \subseteq P$, with probability $1 - \delta$.

Our main result is the following.

Theorem 1.2 *Let $\vartheta \in (0,1)$, $C \geq 2$. Set*

$$t := \left\lceil C \frac{(d+1)e}{(1-\vartheta)^d} \ln \frac{e}{(1-\vartheta)^d} \right\rceil.$$

Then for any centered convex body K in $\mathbb{R}^d$, if t points $X_1, \ldots, X_t$ of K are chosen randomly, independently and uniformly, then

$$\vartheta K \subseteq \operatorname{conv}\{X_1, \ldots, X_t\} \subseteq K$$

with probability at least $1 - \delta$, *where*

$$\delta := 4\left[11C^2 \left(\frac{(1-\vartheta)^d}{e}\right)^{C-2}\right]^{d+1}.$$

By substituting $\vartheta = \frac{1}{d}$, $C = 6$, we obtain Theorem 1.1.

In order to recover [7, Thm. 5.2], substitute $C = 3$ and $\vartheta = c(\delta)\gamma$ in our Theorem 1.2. Then $t \leq e^{3c(\delta)\gamma d}$, when d is large, and δ is roughly $e^{-c(\delta)\gamma d^2}$. Fixing $c(\delta) = 1/3$ independently of δ yields the result.

We recover [7, Prop. 5.3] in a form which is slightly weaker if ϑ is close to 1, as follows. In our Theorem 1.2, $t \leq \frac{10Cd^2}{(1-\vartheta)^{d+1}}$ (note the exponent $d+1$ instead of d) and $\delta \leq 11C^2/e^{C-2}$. By setting C sufficiently large (depending on the desired δ only), we can make the latter as small as required.

We compare our Theorem 1.2 with the main result, [4, Thm. 1.2], which states the following. Let $\beta \in (0, 1)$. There exist a constant $\alpha = \alpha(\beta) > 1$ depending only on β and an absolute constant $c > 0$ with the following property. Let K be a centered convex body in $\mathbb{R}^d$, $\alpha d \leq t \leq e^d$, and choose t points uniformly distributed in K. Then the convex polytope thus obtained contains ϑK, where $\vartheta = \frac{c\beta \ln(t/d)}{d}$ with probability $1-\delta$, where $\delta \leq \exp(-t^{1-\beta}d^{\beta})$.

When ϑ is of order $1/d$, the two results are the same up the constants involved, see our Theorem 1.1 and the discussion preceding it. For fine approximation, that is, when ϑ is a constant, by setting $C = \frac{1}{(1-\vartheta)^{d/2}}$, we obtain roughly $t \approx \exp(\vartheta d/2)$ and $\delta \approx \exp[-\vartheta d^2 \exp(\vartheta d/2)]$. In the mean time, [4, Thm. 1.2] gives roughly $t \approx \exp(\vartheta d/(c\beta))$ and $\delta \approx \exp[-\exp((1-\beta)\vartheta d/(c\beta))d^{\beta}]$.

In Sect. 2, we present a generalization of a classical result of Grünbaum [10], according to which any half-space containing the center of mass of a convex body contains at least a $1/e$ fraction of its volume. In Sect. 3, we state a specific form of the ε-net theorem, a result from combinatorics obtained by Haussler and Welzl [11] building on ideas of Vapnik and Chervonenkis [21], and then refined by Komlós et al. [12]. In Sect. 4, we combine these two to obtain Theorem 1.2. Finally, in Sect. 5, using a recent result of Fradelizi et al. [6], we extend our main result to approximating a linear section of a centered convex body.

For surveys on the topic of approximation of convex bodies by polytopes, cf. [2,5,9], and for some further recent results on approximation in the Banach–Mazur distance (or, geometric distance) when the vertices are not necessarily picked randomly and uniformly from the body, see [3,16].

We note that, in a similar vein, Gordon, Litvak, Pajor and Tomczak-Jaegermann [8, Thm. 3.1] showed that if K is an origin-symmetric convex body in $\mathbb{R}^d$ and $t = (4/\varepsilon)^{2d}$ random points $X_1, \dots, X_t$ are chosen from it uniformly and independently, then, with probability larger than $1 - \exp(-(8/\varepsilon)^d/2)$, these t points form a *metric ε-net* of K with respect to K, that is, $K \subseteq \bigcup_{i=1}^{t}(X_i + \varepsilon K)$. We will use the term '$\varepsilon$-net' in a different, combinatorial sense, to be defined in Sect. 3.

2 Convexity: A stability Version of a Theorem of Grünbaum

Grünbaum's theorem [10] states that for any centered convex body K in $\mathbb{R}^d$, and any half-space F_0 that contains the origin we have

$$\mathrm{vol}(K)/e \leq \mathrm{vol}(K \cap F_0), \tag{1}$$

where vol $(\cdot)$ denotes volume.

We say that a half-space F *supports K from outside* if the boundary of the half-space intersects bd K, but F does not intersect the interior of K. Lemma 2.1, is a stability version of Grünbaum's theorem.

Lemma 2.1 *Let K be a convex body in $\mathbb{R}^d$ with centroid at the origin. Let $0 < \vartheta < 1$, and F be a half-space that supports ϑK from outside. Then*

$$\operatorname{vol}(K)\frac{(1-\vartheta)^d}{e} \leq \operatorname{vol}(K \cap F). \tag{2}$$

Proof Let F_0 be a translate of F containing o on its boundary, and let F_1 be a translate of F that supports K from outside. Finally, let $p \in \operatorname{bd} F_1 \cap K$. Then $\vartheta p + (1-\vartheta)(K \cap F_0)$ (that is, the homothetic copy of $K \cap F_0$ with homothety center p and ratio $1-\vartheta$) is in $K \cap F$. Its volume is $(1-\vartheta)^d \operatorname{vol}(K \cap F_0)$, which by (1), is at least $(1-\vartheta)^d \operatorname{vol}(K)/e$, finishing the proof. □

3 Combinatorics: The ε-Net Theorem of Haussler and Welzl

Definition 3.1 Let $\mathcal{F}$ be a family of subsets of some set U. The *Vapnik–Chervonenkis dimension* (*VC-dimension*, in short) of $\mathcal{F}$ is the maximal cardinality of a subset V of U such that V is shattered by $\mathcal{F}$, that is, $\{F \cap V \ : \ F \in \mathcal{F}\} = 2^V$.

A *transversal* of the set family $\mathcal{F}$ is a subset Q of U that intersects each member of $\mathcal{F}$.

Let $\varepsilon \in (0, 1)$ be given. When U is equipped with a probability measure for which each member of $\mathcal{F}$ is measurable, then a transversal of those members of $\mathcal{F}$ that are of measure at least ε is called an *ε-net*.

It follows from Radon's lemma (cf. [13, Thm. 1.3.1], or [19, Thm. 1.1.5]) that if U is any subset of $\mathbb{R}^d$, and $\mathcal{F}$ is a family of half-spaces of $\mathbb{R}^d$, then the VC-dimension of $\mathcal{F}$ is at most $d+1$.

The ε-Net Theorem was first proved by Haussler and Welzl [11], and then improved by Komlós et al. [12]. We state a slightly weaker form of Theorem 3.1 of [12] than the original, in order to have an explicit bound on the probability δ of failure.

Lemma 3.2 (ε-Net Theorem). *Let $0 < \varepsilon < 1/e$, $C \geq 2$, and let D be a positive integer. Let $\mathcal{F}$ be a family of some measurable subsets of a probability space (U, μ), where the probability of each member F of $\mathcal{F}$ is $\mu(F) \geq \varepsilon$. Assume that the VC-dimension of $\mathcal{F}$ is at most D. Set*

$$t := \left\lceil C\frac{D}{\varepsilon}\ln\frac{1}{\varepsilon}\right\rceil.$$

Choose t elements $X_1, \dots, X_t$ of V randomly, independently according to μ. Then $\{X_1, \dots, X_t\}$ is a transversal of $\mathcal{F}$ with probability at least $1-\delta$, where

$$\delta := 4\left[11C^2\varepsilon^{C-2}\right]^D.$$

Proof We provide an outline of the first, conceptual part of the proof closely following [17, Thm. 15.5]. Then, we continue with a detailed computation to obtain the bound on the probability stated in Lemma 3.2.

Let $T > t$ be an integer, to be set later. We select (with repetition) independently t random elements of U with respect to μ, call it the first sample, and denote it by x. Then, we choose another $T - t$ elements, call it the second sample, and denote it by y. For any $F \in \mathcal{F}$, and any finite sequence w of elements of U, let $I(F, w)$ denote the number of elements of w in F with multiplicity. Let m_F denote the median of $I(F, y)$.

Note that $I(F, y)$ is a binomial variable, and hence, its mean and median are close to each other. More precisely,

$$m_F \geq (T - t)\varepsilon - 1. \tag{3}$$

It is not hard to see that

$$\mu(\exists F \in \mathcal{F} : I(F, x) = 0) \leq 2\mu(\exists F \in \mathcal{F} : I(F, x) = 0 \text{ and } I(F, y) \geq m_F). \tag{4}$$

Denote the concatenation of the two sequences x and y by $\overline{xy}$. Fix any length T sequence z of elements of U.

It is simple to obtain a bound on the following conditional probability:

$$\mu\Big(\exists F \in \mathcal{F} : I(F, x) = \; 0 \text{and} I(F, y) \geq m_F \,|\, \overline{xy} = z\Big) \\ \leq \chi[I(F, z) \geq m_F]\Big(1 - \frac{t}{T}\Big)^{m_F}, \tag{5}$$

where χ denotes the indicator function of an event, that is, it is one if the event holds, and zero otherwise.

The key idea follows. Consider z as a set. Then, by the Shatter function lemma (cf. of [17, Thm. 15.4] or [13, Lem. 10.2.5]) proved independently by Shelah [20], Sauer [18] and Vapnik and Chervonenkis [21], z has at most

$$\sum_{i=0}^{D} \binom{T}{i}$$

distinct intersections with members of $\mathcal{F}$. Thus by (3) and (5), we have

$$\mu\Big(\exists F \in \mathcal{F} : I(F, x) = 0 \text{ and } I(F, y) \geq m_F \,|\, \overline{xy} = z\Big) \leq \sum_{i=0}^{D} \binom{T}{i} \Big(1 - \frac{t}{T}\Big)^{(T-t)\varepsilon - 1}. \tag{6}$$

Let E be the 'bad' event, that is, when $\{X_1, \ldots, X_t\}$ is not a transversal of $\mathcal{F}$. So far, by (4) and (6), we obtained that for any integer $T > t$, we have that the probability of the event E is

$$\mu(E) < 2 \sum_{i=0}^{D} \binom{T}{i} \Big(1 - \frac{t}{T}\Big)^{(T-t)\varepsilon - 1}.$$

From this point on, we describe the computations in detail, in order to obtain the bound on the probability stated in Lemma 3.2.

We set $T = \lfloor \frac{\varepsilon t^2}{D} \rfloor$ and use $\sum_{i=0}^{D} \binom{T}{i} \le \left(\frac{eT}{D}\right)^D$, to obtain that

$$\mu(E) < 2\Big(\frac{eT}{D}\Big)^D\Big(1-\frac{t}{T}\Big)^{(T-t)\varepsilon-1} < 2\Big(\frac{e\varepsilon t^2}{D^2}\Big)^D\Big(1-\frac{D}{\varepsilon t}\Big)^{\left(\varepsilon t^2/D-t-1\right)\varepsilon-1}$$
$$< 2\Big(\frac{e\varepsilon t^2}{D^2}\Big)^D e^{-\varepsilon t+D+D/t+D/(\varepsilon t)},$$

which, after substituting the expression for t in some places and using $\varepsilon < 1/e$, is at most

$$\big(2e^{1/C+1/(eC)}\big)\left(\frac{e^2\varepsilon t^2}{D^2}\right)^D \varepsilon^{CD},$$

which, using $C \ge 2$ is at most

$$\big(2e^{1/C+1/(eC)}\big)\left(\frac{e^2(1+1/(2e))^2C^2\ln^2(1/\varepsilon)}{\varepsilon}\right)^D \varepsilon^{CD} < 4\big(11C^2\varepsilon^{C-2}\big)^D,$$

completing the proof of Lemma 3.2. □

For more on the theory of ε-nets, see [1,13,15,17].

4 Proof of Theorem 1.2

Proof of Theorem 1.2 We consider the following set system on the base set K:

$$\mathcal{F} := \{K \cap F : F \text{ is a half-space that supports } \vartheta K \text{ from outside}\}.$$

Clearly, the VC-dimension of $\mathcal{F}$ is at most $D := d+1$. Let μ be the Lebesgue measure restricted to K, and assume that $\mathrm{vol}(K) = 1$, that is, that μ is a probability measure. By (2), we have that each set in $\mathcal{F}$ is of measure at least $\varepsilon := \frac{(1-\vartheta)^d}{e}$. Lemma 3.2 yields that if we choose t points of K independently with respect to μ (that is, uniformly), then with probability at least $1-\delta$, we obtain a set $Q \subseteq K$ that intersects every member of $\mathcal{F}$. The latter is equivalent to $\vartheta K \subseteq \mathrm{conv}\, Q$, completing the proof. □

5 Approximating a Section of a Convex Body

Let K be a centered convex body in $\mathbb{R}^d$, and V a linear subspace of $\mathbb{R}^d$. Now, $K \cap V$ may not be centered however, we may still want to approximate $K \cap V$ with a polytope $P \subset K \cap V$ such that $\vartheta(K \cap V) \subset P$ for some not too small ϑ.

The main result of [6] (for further results, see also [14]) states that there is an absolute constant $c > 0$ such that for every centered convex body K in $\mathbb{R}^d$, every

$(d-k)$-dimensional linear subspace V of $\mathbb{R}^d$, $0 \le k \le d-1$, and any $u \in V$ unit vector, we have

$$\mathrm{vol}_{d-k}(K \cap V \cap u^+) \ge \frac{c}{(k+1)^2}\left(1+\frac{k+1}{d-k}\right)^{-(d-k-2)} \mathrm{vol}_{d-k}(K \cap V), \tag{7}$$

where $u^+ = \{x \in \mathbb{R}^d : \langle u, x\rangle \ge 0\}$ is the half-space with inner normal vector u.

Using this result, our proof of Theorem 1.2 immediately yields the following.

Theorem 5.1 *Let $\vartheta \in (0,1)$, $C \ge 2$. Let K be a centered convex body in $\mathbb{R}^d$ and V be $(d-k)$-dimensional linear subspace of $\mathbb{R}^d$ with $0 \le k \le d-1$. Set*

$$t := \left\lceil C\frac{(d-k+1)(k+1)^2}{c\big(1+\frac{k+1}{d-k}\big)^{d-k-2}(1-\vartheta)^{d-k}} \ln \frac{(k+1)^2}{c\big(1+\frac{k+1}{d-k}\big)^{d-k-2}(1-\vartheta)^{d-k}}\right\rceil,$$

where c is the universal constant from (7). Choose t points $X_1, \ldots, X_t$ of $K \cap V$ randomly, independently and uniformly with respect to the $(d-k)$-dimensional Lebesgue measure on V. Then

$$\vartheta(K \cap V) \subseteq \mathrm{conv}\{X_1, \ldots, X_t\} \subseteq K \cap V,$$

with probability at least $1-\delta$, where

$$\delta := 4\left[11C^2\left(\frac{c\big(1+\frac{k+1}{d-k}\big)^{d-k-2}(1-\vartheta)^{d-k}}{(k+1)^2}\right)^{C-2}\right]^{d-k+1}.$$

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