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WALKS

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# APPROXIMATING CRITICAL PARAMETERS OF BRANCHING RANDOM WALKS

DANIELA BERTACCHI AND FABIO ZUCCA

**ABSTRACT.** Given a branching random walk on a graph, we consider two kinds of truncations: by inhibiting the reproduction outside a subset of vertices and by allowing at most  $m$  particles per site. We investigate the convergence of weak and strong critical parameters of these truncated branching random walks to the analogous parameters of the original branching random walk. As a corollary, we apply our results to the study of the strong critical parameter of a branching random walk restricted to the cluster of a Bernoulli bond percolation.

**Keywords:** branching random walks, phase transition, critical parameters, graphs.

**AMS subject classification:** 60K35.

## 1. INTRODUCTION

The BRW is a process which serves as a (rough) model for a population living in a spatially structured environment (the vertices of a – possibly oriented – graph  $(X, \mathcal{E}(X))$ ), where each individual lives in a vertex, breeds and dies at random times and each offspring is placed (randomly) in one of the neighbouring vertices. There is no bound on the number of individuals allowed per site. The vertices may be thought as small ecosystems or squares of soil (with their proximity connections – the edges) and individuals as animals or plants. Depending on the parameters involved and on the nature of  $(X, \mathcal{E}(X))$ , the population may face almost sure extinction, global survival (i.e. with positive probability at any time there will be at least one individual alive) or local survival (i.e. with positive probability at arbitrarily large times there will be at least one individual alive in a fixed vertex). These matters have been investigated by several authors ([10], [11], [12], [15], [18], [22] only to mention a few, see [14] for more references).

Let us be more precise as to the definition of the process and of the environment. The graph  $(X, \mathcal{E}(X))$  is endowed with a weight function  $\mu : X \times X \rightarrow [0, +\infty)$  such that  $\mu(x, y) > 0$  if and only if  $(x, y) \in \mathcal{E}(X)$  (in which case we write  $x \rightarrow y$ ). We call the couple  $(X, \mu)$  a weighted graph. We require that there exists  $k > 0$  such that  $k(x) := \sum_{y \in X} \mu(x, y) \leq k$  for all  $x \in X$  (other conditions will be stated in Section 2).

Given  $\lambda > 0$ , the branching random walk (BRW( $X$ ) or briefly BRW) is the continuous-time Markov process  $\{\eta_t\}_{t \geq 0}$ , with configuration space  $\mathbb{N}^X$ , where each existing particle at  $x$  has an exponential lifespan of parameter 1 and, during its life, breeds at the arrival times of a Poisson process of parameter  $\lambda k(x)$  and then chooses to send its offspring to  $y$  with probability  $\mu(x, y)/k(x)$  (note that  $(\mu(x, y)/k(x))_{x, y \in X}$  is the transition matrix of a random walk on  $X$ ). In the literature

one usually finds the particular case  $k(x) = 1$  for all  $x \in X$  (i.e. the breeding rate is constant among locations – no place is more fertile than others).

Two critical parameters are associated to the BRW: the weak (or global) survival critical parameter  $\lambda_w$  and the strong (or local) survival one  $\lambda_s$ . They are defined as

$$\begin{aligned}\lambda_w &:= \inf\{\lambda > 0 : \mathbb{P}^{\delta_{x_0}}(\exists t : \eta_t = \underline{0}) < 1\} \\ \lambda_s &:= \inf\{\lambda > 0 : \mathbb{P}^{\delta_{x_0}}(\exists \bar{t} : \eta_t(x_0) = 0, \forall t \geq \bar{t}) < 1\},\end{aligned}\tag{1.1}$$

where  $x_0$  is a fixed vertex,  $\underline{0}$  is the configuration with no particles at all sites and  $\mathbb{P}^{\delta_{x_0}}$  is the law of the process which starts with one individual in  $x_0$ . Note that these values do not depend on the initial configuration, provided that this configuration is finite (that is, it has only a finite number of individuals), nor on the choice of  $x_0$ . See Section 2 for a discussion on the values of  $\lambda_w$  and  $\lambda_s$ .

When  $(X, \mu)$  is infinite (and connected), the BRW is, so to speak, unbounded in two respects: the environment, since individuals may live at arbitrarily large distance from their ancestors (actually  $n$ -th generation individuals may live at distance  $n$  from the ancestor), and the colonies' size, since an arbitrarily large number of individuals may pile up on any vertex. Hence it is natural to consider “truncated” BRWs where either space or colonies are bounded, and investigate relationships between these processes and the BRW. Indeed in the literature one often finds problems tackled first in finite or compact spaces and then reached through a “thermodynamical limit” procedure. One can see easily that it is possible to construct the BRW either from the process on finite sets (spatial truncation) or from the process on infinite space and a bound on the number of particles per site (particles truncation). In both cases the truncated process, for any fixed time  $t$ , converges almost surely to the BRW.

First we consider “spatially truncated” BRWs. We choose a family of weighted subgraphs  $\{(X_n, \mu_n)\}_{n \in \mathbb{N}}$ , such that  $X_n \uparrow X$ ,  $\mu_n(x, y) \leq \mu(x, y)$ , and  $\mu_n(x, y) \xrightarrow{n \rightarrow \infty} \mu(x, y)$  for all  $x, y$ . The process  $\text{BRW}(X_n)$  can be seen as the  $\text{BRW}(X)$  with the constraint that reproductions outside  $X_n$  are deleted and the ones from  $x$  to  $y$  ( $x, y$  in  $X_n$ ) are removed with probability  $1 - \mu_n(x, y)/\mu(x, y)$ . It is not difficult to see that for any fixed  $t$ , as  $n$  goes to infinity, the  $\text{BRW}(X_n)$  converges to the BRW almost surely. Our first result is that  $\lambda_s(X_n) \xrightarrow{n \rightarrow \infty} \lambda_s(X)$  (the latter being the strong survival critical parameter of the BRW on  $(X, \mu)$ ). Indeed we prove a slightly more general result (Theorem 3.2) which allows us to prove that if  $X = \mathbb{Z}^d$  and  $X_n$  is the infinite cluster of the Bernoulli bond-percolation of parameter  $p_n$ , where  $p_n \xrightarrow{n \rightarrow \infty} 1$  sufficiently fast, then  $\lambda_s(X_n) \xrightarrow{n \rightarrow \infty} \lambda_s(X)$  almost surely with respect to the percolation probability space (Section 6). We note that results on the spatial approximation, in the special case when  $X = \mathbb{Z}^d$  and  $\mu$  is the transition matrix of the simple random walk, were obtained in [16] using a different approach.

Second, we consider BRWs where at most  $m$  individuals per site are allowed (thus taking values in  $\{0, 1, \dots, m\}^X$ ). We call this process  $\text{BRW}_m$  and denote it by  $\{\eta_t^m\}_{t \geq 0}$ . Note that if  $m = 1$  we get the contact process (indeed the  $\text{BRW}_m$  is sometimes referred to as a “multitype contact

process" – see for instance [17]). It is easily seen that for all fixed  $t$  we have  $\eta_t^m \xrightarrow{m \rightarrow \infty} \eta_t$  almost surely (see for instance [18] where the authors suggest this limit as a way to construct the BRW). Clearly, for all  $m \geq 1$ , one may consider the critical parameters  $\lambda_w^m$  and  $\lambda_s^m$  defined as in (1.1) with  $\eta_t^m$  in place of  $\eta_t$ . One of the questions we investigate in this paper is whether  $\lambda_w^m \xrightarrow{m \rightarrow \infty} \lambda_w$  and  $\lambda_s^m \xrightarrow{m \rightarrow \infty} \lambda_s$ .

Here is a brief outline of the paper. In Section 2 we state the basic terminology and assumptions needed in the sequel. Section 3 is devoted to the spatial approximation of the strong critical parameter  $\lambda_s$  by finite or infinite sets (see Theorems 3.1 and 3.2 respectively). In Section 4 we establish a series of steps which will be followed to prove the convergence of the sequence  $\lambda_s^m$  to  $\lambda_s$  under some assumptions of self-similarity of the graph (Theorem 4.5). The same approach is used in Section 5 to prove the convergence of the sequence  $\lambda_w^m$  to  $\lambda_w$  when  $X = \mathbb{Z}^d$  (see Theorem 5.1 and Corollary 5.2, and Remark 5.3 for a slightly more general class of graphs) or when  $X$  is a homogeneous tree (Theorem 5.4). The results of Section 3 are applied in Section 6 in order to study the strong critical parameter of a BRW restricted to a random subgraph generated by a Bernoulli edge percolation process. Section 7 is devoted to final remarks and open questions.

## 2. TERMINOLOGY AND ASSUMPTIONS

In this section we state our assumptions on the graph  $(X, \mu)$ ; we also recall the description of the BRW through its generator and the associated semigroup, and discuss the values of  $\lambda_w$  and  $\lambda_s$ .

Given the (weighted) graph  $(X, \mu)$ , the degree of a vertex  $x$ ,  $\deg(x)$  is the cardinality of the set  $\{y \in X : x \rightarrow y\}$ ; we require that  $(X, \mu)$  is with bounded geometry, that is  $\sup_{x \in X} \deg(x) < +\infty$ . Moreover we consider  $(X, \mu)$  connected, which by our definition of  $\mu$  (recall that  $\mu(x, y) > 0$  if and only if  $(x, y) \in \mathcal{E}(X)$ ) is equivalent to  $\mu^{(n)}(x, y) > 0$  for some  $n = n(x, y)$ , where  $\mu^{(n)}$  is the  $n$ -th power of the matrix  $\mu$ . When  $(\mu(x, y))_{x, y}$  is a stochastic matrix (i.e.  $k(x) = 1$  for all  $x \in X$ ), in order to stress this property we use the notation  $P$ ,  $p(x, y)$  and  $p^{(n)}(x, y)$  instead of  $\mu$ ,  $\mu(x, y)$  and  $\mu^{(n)}(x, y)$ .

We need to define the product of two graphs (in our paper these will be space/time products): given two graphs  $(X, \mathcal{E}(X))$ ,  $(Y, \mathcal{E}(Y))$  we denote by  $(X, \mathcal{E}(X)) \times (Y, \mathcal{E}(Y))$  the weighted graph with set of vertices  $X \times Y$  and set of edges  $\mathcal{E} = \{((x, y), (x_1, y_1)) : (x, x_1) \in \mathcal{E}(X), (y, y_1) \in \mathcal{E}(Y)\}$  (in Figure 1 we draw the connected component of  $\mathbb{Z} \times \mathbb{Z}$  containing  $(0, 0)$ ). Besides, by  $(X, \mathcal{E}(X)) \square (Y, \mathcal{E}(Y))$  we mean the graph with the same vertex set as before and vertices  $\mathcal{E} = \{((x, y), (x_1, y_1)) : (x, x_1) \in \mathcal{E}(X), y = y_1 \text{ or } x = x_1, (y, y_1) \in \mathcal{E}(Y)\}$  (see Figure 2).

Let  $\{\eta_t\}_{t \geq 0}$  be the branching random walk on  $X$  with parameter  $\lambda$ , associated to the weight function  $\mu$ : the configuration space is  $\mathbb{N}^X$  and its generator is

$$\mathcal{L}f(\eta) := \sum_{x \in X} \eta(x) \left( \partial_x^- f(\eta) + \lambda \sum_{y \in X} \mu(x, y) \partial_y^+ f(\eta) \right), \quad (2.2)$$

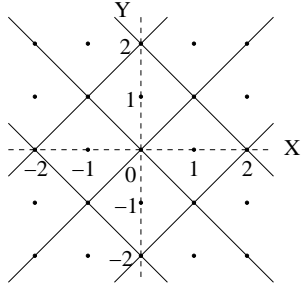


Figure 1:  $X \times Y$  ( $X = Y = \mathbb{Z}$ ).

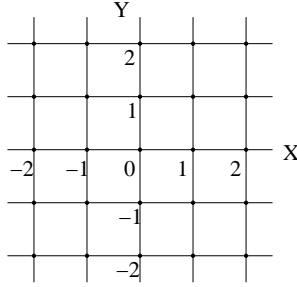


Figure 2:  $X \square Y$  ( $X = Y = \mathbb{Z}$ ).

where  $\partial_x^\pm f(\eta) := f(\eta \pm \delta_x) - f(\eta)$ . Analogously the generator of the BRW $_m$   $\{\eta_t^m\}_{t \geq 0}$  is

$$\mathcal{L}_m f(\eta) := \sum_{x \in X} \eta(x) \left( \partial_x^- f(\eta) + \lambda \sum_{y \in X} \mu(x, y) \mathbb{1}_{[0, m-1]}(\eta(y)) \partial_y^+ f(\eta) \right), \quad (2.3)$$

Note that the configuration space is still  $\mathbb{N}^X$  (though one may consider  $\{0, 1, \dots, m\}^X$  as well). The semigroup  $S_t$  is defined as

$$S_t f(\eta) := \mathbb{E}^\eta(f(\eta_t)),$$

(where  $f$  is any function on  $\mathbb{N}^X$  such that the expected value is defined).

The weak and strong survival critical parameters of the BRW clearly depend on the weighted graph  $(X, \mu)$ ; we denote them by  $\lambda_s(X, \mu)$  and  $\lambda_w(X, \mu)$  (or simply by  $\lambda_s(X)$  and  $\lambda_w(X)$ ). Analogously we denote by  $\lambda_s^m(X, \mu)$  and  $\lambda_w^m(X, \mu)$  (or simply by  $\lambda_s^m(X)$  and  $\lambda_w^m(X)$ ) the critical parameters of the BRW $_m$  on  $(X, \mu)$ . It is known (see for instance [3]) that  $\lambda_s(X) = R_\mu := 1/\limsup_n \sqrt[n]{\mu^{(n)}(x, y)}$  (which is easily independent of  $x, y \in X$  since the graph is connected). On the other hand the explicit value of  $\lambda_w(X)$  is not known in general. Nevertheless in many cases it is possible to prove that  $\lambda_w(X) = 1/\limsup_n \sqrt[n]{\sum_{y \in X} \mu^{(n)}(x, y)}$  (see [3]). In particular if  $k(x) = K$  for all  $x \in X$  then  $\lambda_w(X) = 1/K$ ; thus if  $(\mu(x, y))_{x, y}$  is a stochastic matrix then  $\lambda_w(X) = 1$ .

The two critical parameters coincide (i.e. there is no pure weak phase) in many cases: if  $X$  is finite, or, when  $\mu = P$  is stochastic, if  $R = 1$ . Here are two sufficient conditions for  $R = 1$ :

- (1)  $(X, P)$  is the simple random walk on a non-oriented graph and the ball of radius  $n$  and center  $x$  has subexponential growth ( $\sqrt[n]{|B_n(x)|} \rightarrow 0$  as  $n \rightarrow \infty$ ).

Indeed for any reversible random walk the following universal lower bound holds

$$p^{(2n)}(x, x) \geq v(x)/v(B_n(x))$$

(see [4, Lemma 6.2]) where  $v$  is a reversibility measure. If  $P$  is the simple random walk then  $v$  is the counting measure and the claim follows.

An explicit example is the simple random walk on  $\mathbb{Z}^d$  or on  $d$ -dimensional combs (see [23, Section 2.21] for the definition of comb).

- (2)  $(X, P)$  a symmetric, irreducible random walk on an amenable group (see [23]).

### 3. SPATIAL APPROXIMATION

In this section  $\{X_n\}_{n \in \mathbb{N}}$  will be a sequence of finite subsets of  $X$  such that  $X_n \subseteq X_{n+1}$  and  $\bigcup_{n=1}^{\infty} X_n = X$ ; we denote by  ${}_n\mu$  the truncation matrix defined by  ${}_n\mu := \mu|_{X_n \times X_n}$ . We define  ${}_nR_\mu := 1/\limsup_{k \rightarrow \infty} \sqrt[k]{{}_n\mu^{(k)}(x, y)}$ .

We say that  $(X, \mu)$  is quasi-transitive if there exists a finite partition of  $X$  such that for all couples  $(x, y)$  in the same class there exists a bijection  $\gamma$  on  $X$  satisfying  $\gamma(x) = y$  and, for all  $a, b \in X$ ,  $\mu(\gamma(a), \gamma(b)) = \mu(a, b)$  (when the last condition holds we say that  $\mu$  is  $\gamma$ -invariant). For instance if  $\mu(x, y) = p(x, y)$  where  $P$  is the simple random walk on  $X$  then it is  $\gamma$ -invariant for any automorphism  $\gamma$ .

**Lemma 3.1.** *Let  $\{X_n\}_{n \in \mathbb{N}}$  be such that  $(X_n, {}_n\mu)$  is connected for all  $n$ . Then  ${}_nR_\mu \geq {}_{n+1}R_\mu$  for all  $n$  and when  $X_n \subsetneq X_{n+1}$  we have  ${}_nR_\mu > {}_{n+1}R_\mu$ . Moreover  ${}_nR_\mu \downarrow R_\mu$ .*

*Proof.* This is essentially Theorem 6.8 of [19]. □

The next result is a generalization of this lemma and it goes beyond the pure spatial approximation by finite subsets.

**Theorem 3.2.** *Let  $\{(Y_n, \mu_n)\}_{n \in \mathbb{N}}$  be a sequence of connected weighted graphs and let  $\{X_n\}_{n \in \mathbb{N}}$  be such that  $Y_n \supseteq X_n$ . Let us suppose that  $\mu_n(x, y) \leq \mu(x, y)$  for all  $n \in \mathbb{N}$ ,  $x, y \in Y_n$  and  $\mu_n(x, y) \rightarrow \mu(x, y)$  for all  $x, y \in X$ . If  $(X_n, {}_n\mu)$  is connected for every  $n \in \mathbb{N}$  then  $\lambda_s(Y_n, \mu_n) \geq \lambda_s(X, \mu)$  and  $\lambda_s(Y_n, \mu_n) \xrightarrow{n \rightarrow \infty} \lambda_s(X, \mu)$ .*

*Proof.* We note that, for all finite  $A \subset X$ , eventually  $A \subset Y_m$ . Hence  $\mu_n(x, y)$  is well-defined for all sufficiently large  $n$ . By Lemma 3.1 for any  $\varepsilon > 0$  there exists  $n_0$  such that, for all  $n \geq n_0$ ,  $\lambda_s(X_n, {}_n\mu) < \lambda_s(X, \mu) + \varepsilon/2$ . Define  $\rho_n = \mu_n|_{X_{n_0} \times X_{n_0}}$ . Since  $X_{n_0}$  is finite and  $\rho_n \rightarrow {}_{n_0}\mu$  then  $\lambda_s(X_{n_0}, \rho_n) \rightarrow \lambda_s(Y_{n_0}, {}_{n_0}\mu)$ . Indeed  $\lambda_s(X_{n_0}, \rho_n)$  and  $\lambda_s(Y_{n_0}, {}_{n_0}\mu)$  are the Perron-Frobenius eigenvalues of  $\rho_n$  and  ${}_{n_0}\mu$  respectively and, by construction, for any  $\delta > 0$ , eventually  $(1 - \delta)_{n_0}\mu \leq \rho_n \leq {}_{n_0}\mu$ . If we define  $n_1 \geq n_0$  such that  $\lambda_s(X_{n_0}, \rho_n) < \lambda_s(X_{n_0}, {}_{n_0}\mu) + \varepsilon/2$  for all  $n \geq n_1$  then

$$\lambda_s(Y_n, \mu_n) \leq \lambda_s(X_{n_0}, \rho_n) < \lambda_s(X_{n_0}, {}_{n_0}\mu) + \varepsilon/2 < \lambda_s(X, \mu) + \varepsilon,$$

holds for all  $n \geq n_1$ . □

A simple situation where the previous theorem applies, is the non-oriented case ( $\mu(x, y) > 0$  if and only if  $\mu(y, x) > 0$ ) where  $X_n = Y_n$  is the ball of radius  $n$  with center at a fixed vertex  $x_0$  of  $X$ .

**Remark 3.3.** *If  $Y_n$  is finite for all  $n$ , then  $\lambda_w(Y_n) = \lambda_s(Y_n)$ , hence  $\lambda_w(Y_n) \rightarrow \lambda_w(X)$  if and only if  $\lambda_w(X) = \lambda_s(X)$ .*

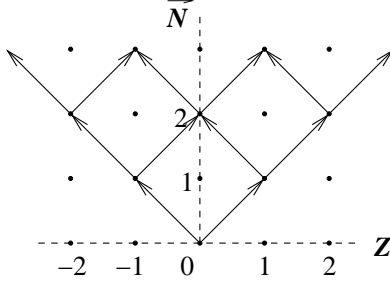


Figure 3:  $\mathbb{Z} \times \vec{\mathbb{N}}$ .

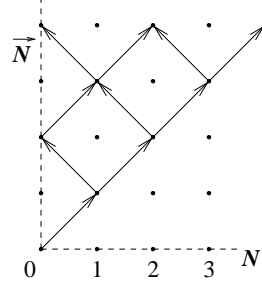


Figure 4:  $\mathbb{N} \times \vec{\mathbb{N}}$ .

#### 4. APPROXIMATION OF $\lambda_s$ BY $m$ -PARTICLE BRWS

From now on, we suppose that  $X$  is countable (otherwise  $\lambda_w^n(X) = \lambda_s^n(X) = +\infty$ ). Our proof of the convergence of  $\lambda_s^m$  and  $\lambda_w^m$  is essentially divided in four steps.

**Fact 1:** We find a graph  $(I, \mathcal{E}(I))$  such that the Bernoulli percolation on  $(I, \mathcal{E}(I)) \times \vec{\mathbb{N}}$  has two phases (where we denote by  $\vec{\mathbb{N}}$  the oriented graph on  $\mathbb{N}$ , that is,  $(i, j)$  is an edge if and only if  $j = i + 1$ ). Note that since the (oriented) Bernoulli bond-percolation on  $\mathbb{Z} \times \vec{\mathbb{N}}$  and  $\mathbb{N} \times \vec{\mathbb{N}}$  has two phases, it is enough to find a copy of the graph  $\mathbb{Z}$  or  $\mathbb{N}$  as a subgraph of  $I$ . This is true for instance for any infinite non-oriented graph (in this paper, we choose either  $I = \mathbb{Z}$ , or  $I = \mathbb{N}$ , or  $I = X$ ). Figures 3 and 4 respectively show the connected components of the products  $\mathbb{Z} \times \vec{\mathbb{N}}$  and  $\mathbb{N} \times \vec{\mathbb{N}}$  containing  $(0, 0)$ .

**Fact 2:** If  $\lambda$  is sufficiently large ( $\lambda > \lambda_s$  or  $\lambda > \lambda_w$ ), then for every  $\varepsilon > 0$  there exists a collection of disjoint sets  $\{A_i\}_{i \in I}$  ( $A_i \subset X$  for all  $i \in I$ ),  $\bar{t} > 0$ , and  $k \in \mathbb{N}$ , such that,  $\forall i \in I$ ,

$$\mathbb{P} \left( \forall j : (i, j) \in \mathcal{E}(I), \sum_{x \in A_j} \eta_{\bar{t}}(x) \geq k \mid \sum_{x \in A_i} \eta_0(x) = k \right) > 1 - \varepsilon. \quad (4.4)$$

**Fact 3:** If Fact 2 holds, then there exists  $m \in \mathbb{N}$ ,  $m \geq k$ , such that

$$\mathbb{P} \left( \forall j : (i, j) \in \mathcal{E}(I), \sum_{x \in A_j} \eta_{\bar{t}}^m(x) \geq k \mid \sum_{x \in A_i} \eta_0^m(x) = k \right) > 1 - 2\varepsilon. \quad (4.5)$$

Indeed let  $N_t$  be the total number of particles ever born in the BRW before time  $t$ ; it is clear that  $N_t$  is a process bounded above by a branching process with birth rate  $\lambda$  and death rate 0. If  $N_0 < +\infty$  almost surely then for all  $t > 0$  we have  $N_t < +\infty$  almost surely; hence for all  $t > 0$  and  $\varepsilon > 0$  there exists  $n(t, \varepsilon)$  such that  $\mathbb{P}(N_t \leq n(t, \varepsilon)) > 1 - \varepsilon$ . Define  $\bar{n} = n(\bar{t}, \varepsilon)$ . We note that if  $A$  is such that  $\mathbb{P}(A) > 1 - \varepsilon$  then  $\mathbb{P}(A | N_t \leq \bar{n}) \geq \mathbb{P}(A, N_t \leq \bar{n}) \geq 1 - 2\varepsilon$ . If we choose  $m \geq \max(\bar{n}, k)$ , given that (4.4) holds we get (4.5).

**Fact 4:** Consider an edge  $((i, n), (j, n + 1))$  in  $(I, \mathcal{E}(I)) \times \vec{\mathbb{N}}$ : let it be open if starting at time  $n\bar{t}$  with  $k$  individuals in  $A_i$ , the process  $\{\eta_t^m\}_{t \geq 0}$  has at least  $k$  individuals in  $A_j$  at time  $(n + 1)\bar{t}$ . Thus (given Fact 2) the probability of weak survival is bounded from



below by the probability that there exists an infinite cluster containing  $(i_0, 0)$  in a Bernoulli one-dependent and oriented bond-percolation in  $I \times \vec{\mathbb{N}}$ , with parameter  $1 - \varepsilon$  (this kind of comparison has been widely used in the literature, see for instance [5], [1], [21], [20] and [6]). Thus if Fact 2 is proven for  $\lambda > \lambda_w$  then we deduce that  $\lambda_w^m \xrightarrow{m \rightarrow \infty} \lambda_w$ . On the other hand, if Fact 2 is proven for  $\lambda > \lambda_s$ , to prove  $\lambda_s^m \xrightarrow{m \rightarrow \infty} \lambda_s$  we need to pick  $A_i$  finite for all  $i$  and  $I$  containing a copy of  $\mathbb{N}$  as a subgraph. Indeed the infinite open cluster in a supercritical Bernoulli bond percolation in  $\mathbb{Z} \times \vec{\mathbb{N}}$  or  $\mathbb{N} \times \vec{\mathbb{N}}$  with probability 1 has an infinite intersection with the set  $\{(0, n) : n \in \mathbb{N}\}$ . As a consequence, in the supercritical case we have, with positive probability, an infinite open cluster in  $\mathbb{Z} \times \vec{\mathbb{N}}$  (resp.  $\mathbb{N} \times \vec{\mathbb{N}}$ ) which contains the origin  $(0, 0)$  and infinite vertices of the set  $\{(0, n) : n \geq 0\}$ .

**Remark 4.1.** *The previous set of steps represents the skeleton of every proof in this paper but one: in Theorem 5.1 we need a generalization of this approach. We sketch here the main differences. We choose an oriented graph  $(W, \mathcal{E}(W))$  and a family of subsets of  $X$ ,  $\{A_{(i,n)}\}_{(i,n) \in W}$  such that*

- *$W$  is a subset of the set  $\mathbb{Z} \times \mathbb{N}$  (note that this is an inclusion between sets not between graphs);*
- *for all  $n \in \mathbb{N}$  we have that  $\{A_{(i,n)}\}_i$  is a collection of disjoint subsets of  $X$  (where the index  $i$  is running in the set  $\{j : (j, n) \in W\}$ );*
- *$(i, n) \rightarrow (j, m)$  implies  $m = n + 1$ .*

*The analog of Fact 2 is the following: if  $\lambda$  is sufficiently large ( $\lambda > \lambda_s$  or  $\lambda > \lambda_w$ ), then for every  $\varepsilon > 0$  there exists  $\bar{t} > 0$  and  $k \in \mathbb{N}$ , such that, for all  $n \in \mathbb{N}$ ,  $i \in \mathbb{Z}$ ,*

$$\mathbb{P} \left( \forall j : (i, n) \rightarrow (j, n+1) \in \mathcal{E}(I), \sum_{x \in A_{(j,n)}} \eta_{(n+1)\bar{t}}(x) \geq k \mid \sum_{x \in A_{(i,n)}} \eta_{n\bar{t}}(x) = k \right) > 1 - \varepsilon.$$

*Fact 3 is the same as before and the percolation described in Fact 4 now concerns the graph  $(W, \mathcal{E}(W))$  (instead of  $(I, \mathcal{E}(I)) \times \vec{\mathbb{N}}$  as it was before).*

What we need is to find suitable  $A_i$ s and to prove Fact 2. We choose the initial configuration as  $\delta_o$  ( $o \in X$ ) and we first study the expected value of the number of individuals in one site at some time, that is  $\mathbb{E}^{\delta_o}(\eta_t(x))$ . This is done using the semigroup  $S_t$ , indeed if we define the evaluation maps  $e_x(\eta) := \eta(x)$  for any  $\eta \in \mathbb{N}^X$  and  $x \in X$ , then

$$\mathbb{E}^\eta(\eta_t(x)) = S_t e_x(\eta).$$

By standard theorems ([7], or, since  $\mathbb{N}^X$  is noncompact, [13] and [2]),

$$\frac{d}{dt} S_t e_x \Big|_{t=t_0} = S_{t_0} \mathcal{L} e_x,$$

from which we deduce

$$\frac{d}{dt} \mathbb{E}^\eta(\eta_t(x)) = -\mathbb{E}^\eta(\eta_t(x)) + \lambda \sum_{z \in X} \mu(z, x) \mathbb{E}^\eta(\eta_t(z)) \quad (4.6)$$

It is not difficult to verify that

$$\mathbb{E}^{\delta_{x_0}}(\eta_t(x)) = \sum_{n=0}^{\infty} \mu^{(n)}(x_0, x) \frac{t^n}{n!} e^{-t}. \quad (4.7)$$

The first result is that the expected number of descendants at a fixed site either tends to 0 or to infinity, depending on the value of  $\lambda$ .

**Lemma 4.2.** *Let us fix  $x \in X$ . If  $\lambda < R_\mu$  then  $\lim_{t \rightarrow +\infty} \mathbb{E}^{\delta_{x_0}}(\eta_t(x)) = 0$ ; if  $\lambda > R_\mu$  then  $\limsup_{t \rightarrow +\infty} \mathbb{E}^{\delta_{x_0}}(\eta_t(x)) = +\infty$ .*

*Proof.* Let  $\lambda < R_\mu$ . For all  $\varepsilon > 0$  there exists  $n_0$  such that  $\mu^{(n)}(x_0, x) < 1/(R_\mu - \varepsilon)^n$  for all  $n \geq n_0$ . If  $\varepsilon = (R_\mu - \lambda)/2$  then  $\lambda^n \mu^{(n)}(x_0, x) \leq \left(\frac{2\lambda}{R_\mu + \lambda}\right)^n$  for all  $n \geq n_0$ , hence  $\mathbb{E}^{\delta_{x_0}}(\eta_t(x)) \leq Q(t)e^{-t} + e^{-t(R_\mu - \lambda)/(R_\mu + \lambda)} \rightarrow 0$  as  $t \rightarrow \infty$  ( $Q$  is a polynomial of degree at most  $n_0 - 1$ ).

Let  $\lambda > R_\mu$ . Let  $\{n_i\}_{i \in \mathbb{N}}$  be a sequence of natural numbers such that  $\lim_{i \rightarrow \infty} \sqrt[n_i]{\mu^{(n_i)}(x_0, x)} = 1/R$ . From this it follows

$$\mathbb{E}^{\delta_{x_0}}(\eta_{n_i}(x)) \geq \mu^{(n_i)}(x_0, x) \frac{\lambda^{n_i} n_i^{n_i}}{n_i!} e^{-n_i} \sim \mu^{(n_i)}(x_0, x) \frac{\lambda^{n_i}}{\sqrt{2\pi n_i}},$$

which goes to infinity as  $i \rightarrow \infty$  since  $\lambda > R_\mu$  and eventually  $\mu^{(n_i)}(x_0, x) > (2/(R + \lambda))^{n_i}$ .  $\square$

In the following lemma we prove that if at time 0 we have one individual at each of  $l$  sites  $x_1, \dots, x_l$ , given any choice of  $l$  sites  $y_1, \dots, y_l$ , after some time the expected number of descendants in  $y_i$  of the individual in  $x_i$  exceeds 1 for all  $i = 1, \dots, l$ .

**Lemma 4.3.** *Let us consider a finite set of couples  $\{(x_i, y_i)\}_{i=0}^l$ ; if  $\lambda > R_\mu$  then there exists  $t = t(\lambda) > 0$  such that  $\mathbb{E}^{\delta_{x_i}}(\eta_t(y_i)) > 1$ ,  $\forall i = 0, 1, \dots, l$ .*

*Proof.* Since  $(X, \mu)$  is connected there exist  $\{k_j, q_j\}_{j=1, \dots, l}$  such that, for all  $j = 1, \dots, l$  and  $n \in \mathbb{N}$ ,

$$\mu^{(n+k_j+q_j)}(x_j, y_j) \geq \mu^{(k_j)}(x_j, x_0) \mu^{(n)}(x_0, y_0) \mu^{(q_j)}(y_0, y_j)$$

and  $\mu^{(k_j)}(x_j, x_0) \mu^{(q_j)}(y_0, y_j) > 0$ .

If  $\alpha := \min_{j=1, \dots, l} \{\mu^{(k_j)}(x_j, x_0) \mu^{(q_j)}(y_0, y_j)\}$  and  $\{n_i\}_{i \in \mathbb{N}}$  is such that  $n_i \geq 1$  for all  $i$  and  $\lim_{i \rightarrow +\infty} \sqrt[n_i]{\mu^{(n_i)}(x_0, y_0)} = 1/R_\mu$ , then for all  $j = 1, \dots, l$  and for all  $i$  (consider  $t = n_i$  and the term with  $n = n_i + k_j + q_j$  in the sum (4.7))

$$\begin{aligned} \mathbb{E}^{\delta_{x_j}}(\eta_{n_i}(y_j)) &\geq \mu^{(n_i)}(x_0, y_0) \frac{(\lambda n_i)^{n_i}}{n_i!} e^{-n_i} \frac{\mu^{(k_j)}(x_j, x_0) \mu^{(q_j)}(y_0, y_j)}{(n_i + k_j + q_j)!} n_i! (\lambda n_i)^{k_j+q_j} \\ &\geq \mu^{(n_i)}(x_0, y_0) \frac{(\lambda n_i)^{n_i}}{n_i!} e^{-n_i} \frac{\alpha}{(n_i + k_j + q_j)^{k_j+q_j}} (\lambda n_i)^{k_j+q_j} \\ &\geq \mu^{(n_i)}(x_0, y_0) \frac{(\lambda n_i)^{n_i}}{n_i!} e^{-n_i} \frac{\alpha}{(1 + k_j + q_j)^{k_j+q_j}} \lambda^{k_j+q_j}. \end{aligned}$$

Note that the latter term goes to infinity exponentially as  $i \rightarrow +\infty$ . Since we have a finite number of sequences, there exists  $i_0 \geq 1$  such that  $\mathbb{E}^{\delta_{x_j}}(\eta_{n_i}(y_j)) > 1, \forall j = 0, 1, \dots, l$ . Choose  $t(\lambda) = n_{i_0}$  to conclude.  $\square$

So far we got results on the expected number of individuals, now we show that, given  $k$  particles in a site  $x$  at time 0, “typically” (i.e. with arbitrarily large probability) after some time we will have at least  $k$  individuals in each site of a fixed finite set  $Y$ . Analogously, starting with  $l$  colonies of size  $k$  (in sites  $x_1, \dots, x_l$  respectively), each of them will, after a sufficiently long time, spread at least  $k$  descendants in every site of a corresponding (finite) set of sites  $Y_i$ .

**Lemma 4.4.**

- (1) Suppose that  $\lambda > R_\mu$ . Let us fix  $x \in X$ ,  $Y$  a finite subset of  $X$  and  $\varepsilon > 0$ . Then there exists  $t = t(\lambda, x) > 0$  (independent of  $\varepsilon$ ),  $k(\varepsilon, x, \lambda)$  such that, for all  $k \geq k(\varepsilon, x, \lambda)$ ,

$$\mathbb{P} \left( \bigcap_{y \in Y} (\eta_t(y) \geq k) \mid \eta_0(x) = k \right) > 1 - \varepsilon.$$

- (2) Let us consider a finite set of vertices  $\{x_i\}_{i=1, \dots, m}$  and a collection of finite sets  $\{Y_i\}_{i=1, \dots, l}$  of vertices of  $X$ . Suppose that  $\lambda > R_\mu$  and let us fix  $\varepsilon > 0$ . Then there exists  $t = t(\lambda, \{x_i\}, \{Y_i\})$  (independent of  $\varepsilon$ ),  $k(\varepsilon, \{x_i\}, \{Y_i\}, \lambda)$  such that, for all  $i = 1, \dots, l$  and  $k \geq k(\varepsilon, \{x_i\}, \{Y_i\}, \lambda)$ ,

$$\mathbb{P} \left( \bigcap_{y \in Y_i} (\eta_t(y) \geq k) \mid \eta_0(x_i) = k \right) > 1 - \varepsilon.$$

*Proof.*

- (1) If we denote by  $\{\xi_t\}_t$  the branching process starting from  $\xi_0 = \delta_x$  then, by Lemma 4.3, we can choose  $t$  such that  $\mathbb{E}^{\delta_x}(\xi_t(y)) > 1$  for all  $y \in Y$ . We can write  $\eta_t(y) = \sum_{j=1}^k \xi_{t,j}(y)$  where  $\xi_{t,j}(y)$  denotes the number of descendants in  $y$  of the  $j$ -th initial particle; note that  $\{\xi_{t,j}(y)\}_{j \in \mathbb{N}}$  is an iid family with  $\mathbb{E}(\xi_{t,j}(y)) = \mathbb{E}^{\delta_x}(\xi_t(y))$  and  $\text{Var}(\xi_{t,j}(y)) = \sigma_{t,y}^2$ . Since  $\xi_{t,j}$  is stochastically dominated by a continuous time branching process with birth rate  $\lambda$ , it is clear that  $\sigma_{t,y}^2 < +\infty$ . Thus by the CLT, given any  $\delta > 0$ , if  $k$  is sufficiently large,

$$\delta \geq \left| \mathbb{P} \left( \sum_{j=1}^k \xi_{t,j}(y) \geq z \right) - 1 + \Phi \left( \frac{z - k \mathbb{E}^{\delta_x}(\xi_t(y))}{\sqrt{k} \sigma_{t,y}} \right) \right|$$

uniformly with respect to  $z \in \mathbb{R}$ . Whence there exists  $k(x, y, \delta)$  such that, for all  $k \geq k(x, y, \delta)$ ,

$$\mathbb{P}(\eta_t(y) \geq k) \geq 1 - \Phi \left( \sqrt{k} \frac{1 - \mathbb{E}^{\delta_x}(\eta_t(y))}{\sigma_{t,y}} \right) - \delta \geq 1 - 2\delta,$$

since  $\sqrt{k}(1 - \mathbb{E}^{\delta_x}(\eta_t(y)))/\sigma_{t,y} \rightarrow -\infty$  as  $k \rightarrow +\infty$ . Take  $k_x = \max_{y \in Y} k_{x,y} < +\infty$ , and let  $D$  be the cardinality of  $Y$ . Hence, for all  $k \geq k_x$ ,

$$\mathbb{P}^{k\delta_x} \left( \bigcap_{y \in Y} (\eta_t(y) \geq k) \right) \geq 1 - 2D\delta.$$

- (2) By Lemma 4.3 we can choose  $t$  such that  $\mathbb{E}^{\delta_{x_i}}(\xi_t(y)) > 1$  for all  $y \in Y_i$  and for all  $i = 1, \dots, l$ . According to (1) above we fix  $k_i$  such that, for all  $k \geq k_i$ ,

$$\mathbb{P} \left( \bigcap_{y \in Y_i} (\eta_t(y) \geq k) \mid \eta_0(x_i) = k \right) \geq 1 - \varepsilon.$$

Take  $k \geq \max_{i=1, \dots, l} k_i$  to conclude. □

### Theorem 4.5.

If at least one of the following conditions holds

- (1)  $(X, \mu)$  is quasi-transitive;
- (2)  $(X, \mu)$  is connected and there exists  $\gamma$  bijection on  $X$  such that
  - (a)  $\mu$  is  $\gamma$ -invariant
  - (b) for some  $x_0 \in X$  we have  $x_0 = \gamma^n x_0$  if and only if  $n = 0$ ;

then

$$\lim_{m \rightarrow +\infty} \lambda_s^m = \lambda_s \geq \lim_{m \rightarrow +\infty} \lambda_w^m \geq \lambda_w.$$

Moreover if  $\lambda_s = \lambda_w$  then  $\lambda_w^m \downarrow_{m \rightarrow +\infty} \lambda_w$ .

*Proof.* Remember that  $\lambda_s = R_\mu$ .

- (1) Let us collect one vertex from each orbit into the (finite) set  $\{x_i\}_{i=1, \dots, l}$  and let  $Y_i := \{y \in X : x_i \rightarrow y\}$ . Fix any  $\lambda > R_\mu$ ; Lemma 4.4 implies Fact 2. In this case  $I = X$ ,  $\mathcal{E}(I) = \{(x, y) : (x, y) \in \mathcal{E}(X) \text{ or } (y, x) \in \mathcal{E}(X)\}$  and  $A_x = \{x\}$ . Note that  $(I, \mathcal{E}(I))$  coincides with  $(X, \mathcal{E}(X))$  if the latter is nonoriented. To prove Fact 4 we note that the existence of the supercritical phase for the Bernoulli percolation on  $X \times \vec{\mathbb{N}}$  follows from the fact that the graph  $\mathbb{N}$  is a subgraph of  $X$  and in the supercritical Bernoulli percolation on  $\mathbb{N} \times \vec{\mathbb{N}}$  with positive probability the infinite open cluster contains  $(0, 0)$  and intersects the  $y$ -axis infinitely often. Hence we have that  $\lambda_s^m \leq \lambda$  and this yields the result.
- (2) Lemma 4.3 allows us to fix  $t$  such that  $\mathbb{E}^{\delta_x}(\eta_t(\gamma x)) > 1$  and  $\mathbb{E}^{\delta_{\gamma x}}(\eta_t(x)) > 1$  whence, by Lemma 4.4,

$$\mathbb{P} \left( \eta_t(\gamma x) \geq k \mid \eta_0(x) = k \right) > 1 - \varepsilon \quad \text{and} \quad \mathbb{P} \left( \eta_t(x) \geq k \mid \eta_0(\gamma x) = k \right) > 1 - \varepsilon.$$

This implies

$$\mathbb{P} \left( \eta_t(\gamma^n x) \geq k \mid \eta_0(\gamma^{n-1} x) = k \right) > 1 - \varepsilon \quad \text{and} \quad \mathbb{P} \left( \eta_t(\gamma^{n-1} x) \geq k \mid \eta_0(\gamma^n x) = k \right) > 1 - \varepsilon$$

for all  $n \in \mathbb{Z}$  since  $\mu$  is  $\gamma$ -invariant; one more time, Facts 3 and 4 yield  $\lambda_s^m \leq \lambda$  and the claim (here  $I = \mathbb{Z}$  and  $A_i = \{\gamma^i x_0\}$ ).

□

## 5. APPROXIMATION OF $\lambda_w$ BY $m$ -PARTICLE BRWS

From now on we set  $\mu(x, y) = p(x, y)$  where  $P$  is a stochastic matrix. We stress that in this case  $\lambda_w = 1$ . We are concerned with the question whether  $\lambda_w^m \downarrow \lambda_w = 1$  or not. Under the hypotheses of Theorem 4.5, this is the case when the BRW has no pure weak phase (i.e.  $R = 1$ ). The interesting case is  $R > 1$ . Most natural examples are drifting random walks on  $\mathbb{Z}^d$  and the simple random walk on homogeneous trees. In both cases we show that  $\lambda_k^m \xrightarrow{m \rightarrow \infty} \lambda_w$ .

**Theorem 5.1.** *Let  $P$  be a random walk on  $\mathbb{Z}$  such that  $p(i, i+1) = p$ ,  $p(i, i-1) = q$  and  $p(i, i) = 1 - p - q$  for all  $i \in \mathbb{Z}$ . Then  $\lim_{k \rightarrow +\infty} \lambda_w^k = 1 = \lambda_w$ .*

*Proof.* We consider  $\alpha, \beta \in (0, 1)$ ,  $\alpha \leq \beta \leq (1 + \alpha)/2$  and write

$$\begin{aligned} p^{(n)}(0, \alpha n) &= \sum_{i=\alpha n}^{(1+\alpha)n/2} \binom{n}{i, i-\alpha n, n-2i+\alpha n} p^i q^{i-\alpha n} (1-p-q)^{n-2i+\alpha n} \\ &\geq \binom{n}{\beta n, (\beta-\alpha)n, (1-2\beta+\alpha)n} p^{\beta n} q^{(\beta-\alpha)n} (1-p-q)^{(1-2\beta+\alpha)n} \\ &\stackrel{n \rightarrow \infty}{\sim} \frac{1}{2\pi n \sqrt{\beta(\beta-\alpha)(1-2\beta+\alpha)}} \left( \frac{p^\beta q^{\beta-\alpha} (1-p-q)^{1-2\beta+\alpha}}{\beta^\beta (\beta-\alpha)^{\beta-\alpha} (1-2\beta+\alpha)^{1-2\beta+\alpha}} \right)^n. \end{aligned}$$

Thus if  $\lambda > 1$ ,  $\mathbb{E}^{\delta_0}(\eta_n(\alpha n))$  is bounded below by a quantity which is asymptotic to

$$\frac{1}{(2\pi n)^{3/2} \sqrt{\beta(\beta-\alpha)(1-2\beta+\alpha)}} (g_\lambda(\alpha, \beta))^n.$$

where

$$g_\lambda(\alpha, \beta) = \frac{\lambda p^\beta q^{\beta-\alpha} (1-p-q)^{1-2\beta+\alpha}}{\beta^\beta (\beta-\alpha)^{\beta-\alpha} (1-2\beta+\alpha)^{1-2\beta+\alpha}}.$$

Note that  $g_\lambda(p-q, p) = \lambda$ , thus we may find  $\alpha_1 < \alpha_2 \leq \beta_1 < \beta_2$  (with  $\beta_i \leq (1 + \alpha_i)/2$ ,  $i = 1, 2$ ) such that  $g_\lambda(x, y) > 1$ , for all  $(x, y) \in [\alpha_1, \alpha_2] \times [\beta_1, \beta_2]$ . By taking  $n = \bar{n}$  sufficiently large one can find three distinct integers  $d_1, d_2$  and  $d_3$  such that  $\alpha_1 n \leq d_1 < d_2 \leq \alpha_2 n$ ,  $\beta_1 n \leq d_3 \leq \beta_2 n$  and  $g_\lambda(d_1/n, d_3/n) > 1$ ,  $l = 1, 2$ .

By reasoning as in Lemma 4.4 we have that, for all  $i \in \mathbb{Z}$  and  $\varepsilon > 0$ , there exists  $t = \bar{n}$ ,  $k = k(\varepsilon, \lambda)$  such that

$$\mathbb{P}(\eta_{\bar{n}}(i+j) \geq k, j = d_1, d_2 | \eta_0(i) = k) > 1 - \varepsilon.$$

Since  $k$  and  $\bar{t} := \bar{n}$  are independent of  $i$  we have proven the slightly more general version of Fact 2 as stated in Remark 4.1 (where  $W = \{a(d_1, 1) + b(d_2, 1) : a, b \in \mathbb{N}\}$ ,  $A_{(i,n)} = A_i$  and  $(i, n) \rightarrow (j, n+1)$  if and only if  $j - i = d_1$  or  $j - i = d_2$ ).

□

In view of Corollary 5.2 and Theorem 5.4 it is useful to introduce the concept of local isomorphism which allows to extend some results from  $\mathbb{Z}^d$  to more general graphs. Given two weighted graphs  $(X, \mu)$  and  $(I, \nu)$ , we say that a map  $f : X \rightarrow I$  is a *local isomorphism* if for any  $x \in X$  and  $i \in I$  we have  $\sum_{z \in f^{-1}(i)} \mu(x, z) = \nu(f(x), i)$ .

In this case it is clear that, if we consider the partition of  $X$  given by  $\{A_i\}_{i \in I}$  where  $A_i := f^{-1}(i)$ , we can easily compute the expected number of particles alive at time  $t$  in  $A_i$  starting from a single particle alive in  $x$  at time 0

$$\sum_{z \in A_i} \mathbb{E}_\mu^{\delta_x}(\eta_t(z)) = \mathbb{E}_\nu^{\delta_{f(x)}}(\xi_t(i)) \quad (5.8)$$

(where  $\{\xi_t\}_{t \geq 0}$  is a branching random walk on  $(I, \nu)$ ) since  $\sum_{z \in f^{-1}(i)} \mu^{(n)}(x, z) = \nu^{(n)}(f(x), i)$ , for all  $n \in \mathbb{N}$ . We note that the latter depends only on  $f(x)$  and  $i$ . As a consequence  $R_\mu \geq R_\nu$ .

**Corollary 5.2.** *If  $P$  is an adapted and translation invariant random walk on  $\mathbb{Z}^d$  then  $\lim_{k \rightarrow +\infty} \lambda_w^k = 1 = \lambda_w$ .*

*Proof.* Let  $\{Z_n\}_{n \in \mathbb{N}}$  be a realization of the random walk and  $A_i = \{x \in \mathbb{Z}^d : x(1) = i\}$ . Note that

$$\mathbb{P}(Z_{n+1} \in A_j | Z_n = w) = \tilde{p}(i, j), \quad \forall w \in A_i,$$

where  $\tilde{P}$  is a random walk on  $\mathbb{Z}$  with  $p = p(0, e_1)$ ,  $q = p(0, -e_1)$ . Using equation 5.8 and reasoning as in the proof of the previous theorem, we conclude.  $\square$

**Remark 5.3.** *The argument of the previous corollary may be applied to a more general case: let  $(Y, Q)$  be a random walk and  $(\mathbb{Z}, P)$  as in Theorem 5.1 and consider  $Y \times \mathbb{Z}$  with transition matrix  $\alpha \mathbb{I}^Y \times P + (1-\alpha)Q \times \mathbb{I}^{\mathbb{Z}}$ , where  $\alpha \in (0, 1)$  and by  $\mathbb{I}$  we denote the identity matrix (on the superscripted space). Using the projection on the second coordinate one proves that  $\lim_{k \rightarrow +\infty} \lambda_w^k = 1 = \lambda_w$ .*

**Theorem 5.4.** *If  $(X, P)$  is the simple random walk on the homogeneous tree of degree  $r$  then  $\lim_{k \rightarrow +\infty} \lambda_w^k = 1 = \lambda_w$ .*

*Proof.* Fix an end  $\tau$  in  $X$  and a root  $o \in X$  and define the map  $h : X \rightarrow \mathbb{Z}$  as the usual height (see [23] page 129). Define  $A_k = h^{-1}(k)$ ,  $k \in \mathbb{Z}$  (these sets are usually referred to as horocycles). The projection of the simple random walk on  $X$  onto  $\mathbb{Z}$  is a random walk with transition matrix  $Q$  where  $q(a, a+1) = 1 - 1/r$  and  $q(a, a-1) = 1/r$ . Note that for all  $x \in X$

$$\sum_{y \in A_k} p^{(n)}(x, y) = q^{(n)}(h(x), k).$$

By using equation 5.8 and reasoning as in Lemma 4.4 and Theorem 5.1 we have that, for all  $i \in \mathbb{N}$ ,

$$\mathbb{P} \left( \sum_{x \in A_{i+j}} \eta_{\bar{n}}(x) \geq k, j = d_1, d_2 \mid \sum_{x \in A_i} \eta_0(x) = k \right) > 1 - \varepsilon.$$

The claim follows as in Theorem 5.1.  $\square$

## 6. BRANCHING RANDOM WALKS IN RANDOM ENVIRONMENT

We use the results of Section 3 to prove some properties of the BRW in random environment.

Let  $(X, \mu)$  be a non-oriented weighted graph. We consider any subgraph  $(Y, \mathcal{E}(Y))$  of  $(X, \mathcal{E}(X))$  as a weighted subgraph with weight function  $\mathbb{1}_{\mathcal{E}(Y)}\mu$ .

Given any  $p \in [0, 1]$  we consider the Bernoulli bond percolation on  $(X, \mathcal{E}(X))$  and we define the random weighted subgraphs  $(Y^a, \mathcal{E}(Y^a))$  where  $Y^a = X$  and  $\mathcal{E}(Y^a)$  is the random set of edges resulting from the percolation process. We define  $\lambda_s(Y^a) := \inf_{A \in \mathcal{A}} \lambda_s(A)$  where  $\mathcal{A}$  is the random collection of all the connected components of  $Y^a$ . This corresponds to the critical (strong) parameter of a BRW where the initial state is one particle alive at time 0 in every connected component of  $Y^a$ .

On the other hand, if there exists a nontrivial critical parameter  $p_c$  for the Bernoulli percolation on  $X$  then, for all  $p > p_c$  we denote by  $(Y^c, \mathcal{E}(Y^c))$  the infinite cluster and we consider the critical (strong) parameter  $\lambda_s(Y^c)$ .

If we have a sequence  $\{p_n\}_{n \in \mathbb{N}}$  such that  $p_n \in [0, 1]$  for all  $n \in \mathbb{N}$ , then we consider the sequences  $\{Y_n^a\}_{n \in \mathbb{N}}$  and  $\{Y_n^c\}_{n \in \mathbb{N}}$  as the results of independent Bernoulli percolation processes on  $X$  with parameters  $\{p_n\}_{n \in \mathbb{N}}$ .

Here is the main result; we note that, even when  $X = \mathbb{Z}^d$ , we do not require  $\mu$  to be the simple random walk.

### Theorem 6.1.

- (1) If  $\sum_n (1 - p_n) < +\infty$  then  $\lambda_s(Y_n^a) \rightarrow \lambda_s(X)$  a.s.
- (2) If  $(X, \mu)$  is quasi-transitive then  $\lambda_s(Y^a) = \lambda_s(X)$  a.s.
- (3) If  $X = \mathbb{Z}^d$  and  $\sum_n (1 - p_n) < +\infty$ , then  $\lambda_s(Y_n^c) \rightarrow \lambda_s(\mathbb{Z}^d)$  a.s.
- (4) If  $X = \mathbb{Z}^d$ ,  $\mu$  is translation invariant and  $p > p_c$  then  $\lambda_s(Y^c) = \lambda_s(\mathbb{Z}^d)$  a.s.

*Proof.*

- (1) By using the Borel-Cantelli Lemma, we have that any finite connected subgraph of  $X$  is eventually contained in a (random) connected component of  $Y_n^a$  almost surely. Theorem 3.2 yields the conclusion.
- (2) In this case if we take an infinite orbit  $X_0$  then, by Borel-Cantelli Lemma, for any  $m \in \mathbb{N}$ , with probability 1,  $Y^a$  contains a ball  $B_m$ , centered on a vertex  $z \in X_0$  and of radius  $m$ , with all open edges. Since the critical parameter of a ball  $\lambda_s(B_m)$  does not depends on how we choose its center in  $X_0$ , then using Theorem 3.2, we have that  $\lambda_s(X) \leq \lambda_s(Y^a) \leq \lambda_s(B_m) \rightarrow \lambda_s(X)$  as  $m \rightarrow +\infty$ .

- (3) Note that  $p_n > p_c$  eventually, hence  $\lambda_s(Y_n^c)$  is well-defined for all sufficiently large  $n$ . We already know that  $\mu_n(x, y) \rightarrow \mu(x, y)$  almost surely, hence what we need to prove is that, almost surely, any edge is eventually connected to the infinite cluster. To this aim we apply the FKG inequality obtaining that the probability of the event “the edge  $(x, y)$  is open and connected to the infinite cluster  $Y_n^c$ ” is bigger than  $p_n\theta(p_n)$ . According to Theorem 8.92 of [8],  $\theta$  is a differentiable function on  $[0, 1]$  hence  $1 - p\theta(p) \sim (1 - p)(1 + \theta'(1))$  and this implies  $\sum_n(1 - p_n\theta(p_n)) < +\infty$ . The Borel-Cantelli Lemma yields the conclusion.
- (4) It is tedious but essentially straightforward to prove that, for any  $m \in \mathbb{N}$ , with probability 1,  $Y^c$  contains an hypercube  $Q_m$  of side-length  $m$  with all open edges; as before, Theorem 3.2 yields the result.

□

## 7. FINAL REMARKS

At this point the theory of spatial approximation (see Section 3) is quite complete as far as we are concerned with the basic questions on the convergence of the critical parameters. Indeed we proved results in this direction (see Theorem 3.2) for the strong parameter under reasonable assumptions, while the question on the weak critical parameter, in the pure spatial approximation by finite subsets, is uninteresting (see Remark 3.3). It is possible to further investigate the convergence of the sequence of weak critical parameters under the hypotheses of Theorem 3.2 by using the characterization  $\lambda_w(X) = 1/\limsup_n \sqrt[n]{\sum_{y \in X} \mu^{(n)}(x, y)}$  which holds in many cases (see [3] for details).

As for the approximation of the BRW by  $\text{BRW}_{ms}$ , we proved that, on quasi-transitive or “self-similar” graphs (in the sense of Theorem 4.5 (2)),  $\lambda_s^m \downarrow \lambda_s$  as  $m \rightarrow \infty$  and, if there is no weak phase, on  $\mathbb{Z}^d$  or on regular trees,  $\lambda_w^m \downarrow \lambda_w$  as  $m \rightarrow \infty$ . Here are some natural questions which, as far as we know, are still open:

- can we get rid of the hypothesis of quasi-transitivity or self-similarity in the case concerning the strong critical parameter?
- when  $\lambda_s > \lambda_w$ , is it still true that  $\lambda_w^m \downarrow_{m \rightarrow \infty} \lambda_w$ , at least for Cayley graphs or on quasi transitive graphs?

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