

Approximation Algorithms for Perishable Inventory Systems

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We develop the first approximation algorithms with worst-case performance guarantees for periodic-review perishable inventory systems with general product lifetime, for both backlogging and lost-sales models. The demand process can be non-stationary and correlated over time, capturing such features as demand seasonality and forecast updates. The optimal control policy for such systems is notoriously complicated, thus finding effective heuristic policies is of practical importance. In this paper, we construct a computationally efficient inventory control policy, called the *proportional-balancing policy*, for systems with an arbitrarily correlated demand process and show that it has a worst-case performance guarantee less than 3. In addition, when the demands are independent and stochastically non-decreasing over time, we propose another policy, called the *dual-balancing policy*, which admits a worst-case performance guarantee of 2. We demonstrate through an extensive numerical study that both policies perform consistently close to optimal.

Key words: inventory, perishable products, correlated demands, approximation algorithms, cost balancing, worst-case performance guarantee

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1. Introduction and Summary

In this paper, we study the classic periodic-review stochastic inventory systems with perishable products. The product lifetime is known and fixed. Initial interest in these systems was sparked by blood bank applications (see, e.g., Prastacos (1984), Pierskalla (2004), Karaesmen et al. (2011)), but the scope of applications is far greater. For example, perishable products such as food items and pharmaceuticals constitute the majority of sales revenue of grocery retailing industry. Food Market Institute (2012) reported that perishables accounted for 52.63% of the 2011 total supermarket sales

of about \$459 billion¹, and mismanagement of perishable products (such as spoilage and shrinkage) represents a major threat to the profitability of companies in grocery retailing industry. A survey by the National Supermarket Research Group reported an average loss of \$34 million a year due to spoilage in one major 300-store grocery chain.² Thus, finding effective inventory management policies for perishable products has always been of great interest to both practitioners and academic researchers.

We restrict our attention to the first-in-first-out (FIFO) issuing policy which is commonly adopted in the literature (see, e.g., Nahmias (2011) and Karaesmen et al. (2011)), i.e., the oldest inventory is consumed first when demand arrives. This assumption is reasonable for blood inventory systems and online grocery network (e.g., AmazonFresh). It also applies to the retailers who display only the oldest items on the shelves. The demands in different periods in our model can be non-stationary (or time-dependent) and correlated over time, capturing such demand features as seasonality and forecast updates as well as many other demand processes of practical interest. Both backlog model and lost-sales model will be studied.

These systems are fundamental but notoriously hard to analyze in both theory and computation. As seen from our literature review below, the optimal policy is very complex even when the demands are independent and identically distributed. The optimal order quantity depends on both the age distribution of the on-hand inventory and the length till the end of the planning horizon. The computation of the optimal policy using dynamic programming is in general intractable due to the “*curse-of-dimensionality*”. Thus, many researchers turned to seek effective heuristic policies for these problems. To the best of our knowledge, almost all heuristics developed thus far have been focused on the case with independent and identically distributed demands. Moreover, none of the heuristic policies in the literature admits provably worst-case performance guarantees. In this paper, we propose the first approximation algorithm with a worst-case performance guarantee of 2 for these important systems when the demands in different periods are independent and stochastically non-decreasing over time.

The demand processes in practical settings are often seasonal, forecast-based, or driven by the state of the economy (e.g., Markov modulated demand processes). The demands of these processes are correlated over time. For example, many firms employ forecasting methods to learn about the future demands and periodically update their forecasts; and such forecast-based demand processes can often be modeled by the martingale model of forecast evolution (MMFE for short, see, e.g., Graves et al. (1986), and Heath and Jackson (1994)), in which the updated forecast is the original forecast plus an adjustment (or random error) with mean 0. In addition, in practice firms often

receive advance demand information (ADI) from some customers for the future periods, so managers have to periodically incorporate such ADI in the future demands (see, e.g., Gallego and Özer (2001)). These models are practically important, but finding the optimal policies using brute-force dynamic programming is computationally intractable, since the state space of the corresponding dynamic programs is usually large (see, e.g., Lu et al. (2006)). The computational burden is even more severe for perishable inventory systems, given the fact that the age distribution of on-hand inventory has to be tracked too.

To overcome this prohibitive computational challenge, we propose another approximation algorithm for perishable inventory systems with arbitrarily correlated demand processes, and show that it admits a worst-case performance guarantee less than 3. To the best of our knowledge, no effective heuristic policy has ever been developed in the literature for perishable inventory systems with correlated demand processes.

1.1. Main Results and Contributions

The main results and contributions of this paper are summarized as follows.

Algorithms. We develop two approximation algorithms which admit provably worst-case performance guarantees for perishable inventory systems.

Firstly, we develop a proportional-balancing (PB) policy for the perishable inventory systems with product lifetime of m (≥ 2) periods under an arbitrarily non-stationary and correlated demand process. We show that the PB policy has a worst-case performance guarantee of $\left(2 + \frac{(m-2)h}{mh+\theta}\right)$, where $h = \hat{h} + (1 - \alpha)\hat{c}$, $\theta = \hat{\theta} + \alpha\hat{c}$, and \hat{c} , \hat{h} , $\hat{\theta}$, and α are the per-unit ordering, holding, outdated costs, and one-period discount factor, respectively. That is, for any instance of the problem, the expected cost of the PB policy is at most $\left(2 + \frac{(m-2)h}{mh+\theta}\right)$ times the expected cost of an optimal policy. Therefore, the theoretical worst-case performance guarantee is between 2 and 3 and it equals 2 when the product lifetime $m = 2$.

Secondly, when the demand process is independent and stochastically non-decreasing over time, we develop a dual-balancing (DB) policy which has a worst-case performance guarantee of 2, i.e., for any instance of the problem, the expected cost of the DB policy is at most twice the expected cost of an optimal policy.

To the best of our knowledge, our proposed policies are the first set of computationally efficient policies with worst-case performance guarantees in stochastic periodic-review perishable inventory

systems. In contrast, computing the exact optimal policy using dynamic programming suffers from the well-known “curse of dimensionality” and is intractable even with short product lifetimes (e.g., $m = 4$) and under independent and identically distributed demand processes.

Worst-case analysis. In our algorithmic design, we develop a nested marginal cost accounting scheme for perishable inventory systems. This scheme is similar in spirit to that developed in [Levi et al. \(2007\)](#), but has a much more complex and nested structure due to the multi-dimensional inventory state representing the age distribution of on-hand inventory. The main idea of this approach is to associate the costs with each ordering decision instead of each period. However, the techniques developed for our worst-case performance analysis depart significantly from those in the previous studies (e.g., [Levi et al. \(2007\)](#), [Levi et al. \(2008a\)](#)), which heavily rely on the existence of a one-to-one matching between the supply and demand units when the inventory units are consumed in a first-in-first-out manner. That is, when analyzing the performances of the approximation algorithms, all the previous studies “*geometrically*” match product units in a one-to-one manner for the systems operating under two different policies; and the costs for each pair of matched units can be readily compared. However, the perishability of products destroys this matching mechanism, thus the existing techniques developed for non-perishable inventory systems are no longer applicable. To overcome this difficulty, we introduce a key new concept, called the “*trimmed on-hand inventory level*”, defined as the part of on-hand inventory units ordered before a particular time. This key concept enables us to compare the costs of two perishable inventory systems operating under two different policies. Compared with the previous geometric approach, our new approach is purely algebraic and we expect it to be useful in studying other perishable inventory systems.

Empirical performances. Our extensive computational studies show that our proposed policies perform consistently near-optimal for all the tested instances, which are significantly better than the theoretical worst-case performance guarantees. More specifically, for independent and identically distributed demand processes and short product lifetimes (for which the optimal policies can be numerically computed), our numerical results are comparable to those reported in [Nahmias \(1976, 1977b\)](#) and are very close to the optimal (around 0.3% above the optimal cost); for long product lifetimes for which computing optimal policies is intractable, we compare the performance of our methods with [Nahmias \(1976, 1977b\)](#); the overall performance of our policies is also comparable to those of [Nahmias \(1976, 1977b\)](#) and it improves as the frequency of updating increases. For non-stationary and correlated demand processes including Markov modulated demand process, Martingale models of forecast evolution (MMFE), autoregressive (AR) models, and models with

advance demand information (ADI), the proposed algorithms also perform close to optimal for all the problem instances tested – with maximum performance error below 3%, and average error below 1%, of the optimal costs.

1.2. Literature Review

The problem of managing stochastic periodic-review inventory systems with perishable products has attracted the attention of many researchers over the years. The dominant paradigm in the existing literature has been to formulate these models using a dynamic programming framework. Nahmias and Pierskalla (1973) characterized the structure of the optimal ordering policy for a two-period product lifetime problem. Nahmias (1975) and Fries (1975) then, independently, studied the optimal policy for the general lifetime problem with independent and identically distributed (i.i.d.) demands, in a backlogging model and a lost-sales model, respectively. They showed that the optimal ordering quantity depends on both the age distribution of the current inventory and the remaining time until the end of planning horizon. Thus, computing the optimal policy using brute-force dynamic programming is intractable due to the multi-dimensional state space. The complexity of this problem is later reinforced by Cohen (1976) who characterized the stationary distribution of inventory for the two-period lifetime problem, and showed that the optimal policy is a state-dependent policy. Recently, Li and Yu (2014) revisited the structural properties of the optimal inventory policy in perishable inventory systems by employing the “*multimodularity*” concept; and Chen et al. (2014) studied the joint inventory control and pricing problem for perishable inventory systems and characterized the structural properties of the optimal inventory and pricing policies.

Due to the complexity of the optimal policies for perishable inventory systems, a lot of efforts have been dedicated to the design of efficient heuristic policies for both backlogging and lost-sales models. When the demands are i.i.d., Nahmias (1976) constructed a bound on the outdated cost which is a function of only the total on-hand inventory, and developed a base-stock myopic policy using this bound. Nahmias (1977a) employed the same myopic policies to compare two dynamic perishable inventory models developed by Nahmias (1975) and Fries (1975). Subsequently, Nahmias (1977b) used a more refined approximate state transition function which treats the product lifetime as if it were only two periods, thereby resulting in a one-dimensional state variable. The numerical results for problems with product lifetimes of 2 and 3 periods are near-optimal under i.i.d. demands. Nandakumar and Morton (1993) derived upper and lower bounds for the dynamic programming formulation of the lost-sales model, and used a weighted average of the lower and upper bounds to construct an efficient heuristic. The numerical results showed that the heuristic again performs

close to optimal for short product lifetimes and i.i.d. demands. Cooper (2001) derived bounds on the stationary distribution of the number of outdated units in each period, under a fixed critical number order policy. Recently, following the approximation scheme of the outdated costs developed by Nahmias (1976), Chen et al. (2014) proposed two heuristic policies for the joint inventory control and pricing models. Since the future demands depend on the future prices, they approximated the expected demands and prices in the future periods by solving the corresponding deterministic models. Their numerical study showed that the two heuristic policies perform very well when the demands are i.i.d., but the performance error could go up to 15% for the independent but time-varying demands. Li et al. (2009) also designed an effective heuristic following the approximation method of Nahmias (1976) for the joint inventory control and pricing model. Li et al. (2013) proposed two myopic heuristics for perishable inventory systems with last-in-first-out issuing policy and clearance sales. To reduce the state space, the first heuristic treats all on-hand inventories as if they would expire in one period whereas the second heuristic keeps track of the total inventory level and the inventory level of items whose remaining lifetime is one period. Their numerical results showed that these heuristics perform very close to optimal under i.i.d. demands. As seen from the literature above, almost all the existing heuristic policies have been focused on i.i.d. demands, and none of them admits provably worst-case performance guarantees.

There is also a large body of literature discussing other aspects of perishable inventory systems. We partition these studies into the following categories (our list below is by no means exhaustive): (a) continuous-review perishable inventory systems (see, e.g., Weiss (1980), Goh et al. (1993), Liu and Lian (1999), Perry (1999)); (b) perishable inventory systems with multiple products or demands (see, e.g., Nahmias and Pierskalla (1976), Deuermeyer (1979), Ferguson and Koenigsberg (2007), Deniz et al. (2010), Cai and Zhou (2014)); (c) joint inventory and pricing control of perishables (see, e.g., Li et al. (2009), Chen and Sapra (2013), Chen et al. (2014)); (d) perishable inventory systems with depletion or clearance sales (see, e.g., Cai et al. (2009), Xue et al. (2011), Li and Yu (2014), Li et al. (2013)); and (f) blood banks and health-care applications (see, e.g., Prastacos (1984), Haijema et al. (2005, 2007), Zhou et al. (2011)). We also refer interested readers to Nahmias (1982, 2011), Goyal and Giri (2001), and Karaesmen et al. (2011) for comprehensive literature reviews.

Our work is also closely related to the recent stream of literature on approximation algorithms for stochastic periodic-review inventory systems pioneered by Levi et al. (2007). The systems allow for correlated stochastic demand processes, including all of the known approaches to model dynamic demand forecast updates (e.g., Gallego and Özer (2001), Iida and Zipkin (2006), and Lu et al. (2006)). Levi et al. (2007) first introduced the concept of marginal cost accounting which associates

the costs with each decision a particular policy makes. They proposed a 2-approximation policy which admits a worst-case performance guarantee of 2 for the backlogging model with generally correlated demands. Subsequently, [Levi et al. \(2008a\)](#) proposed a 2-approximation algorithm for the lost-sales models under independent demand processes; and [Levi et al. \(2008b\)](#) introduced the concept of forced marginal backlogging cost accounting and proposed a 2-approximation algorithm for the capacitated systems with backlogging. More recently, [Levi and Shi \(2013\)](#) and [Shi et al. \(2014\)](#) proposed two approximation algorithms for the lot-sizing backlogging models without and with capacity constraints, respectively; and [Tao and Zhou \(2014\)](#) proposed a 2-approximation algorithm for inventory systems with remanufacturing. All these previous studies assume that the inventory is non-perishable, and therefore there exists a one-to-one matching between the supply and demand when the inventory issuing policy is FIFO. Perishability, however, destroys this matching mechanism and the existing techniques for the worst-case analysis cannot be applied to perishable inventory systems.

1.3. Structure

The remainder of this paper is organized as follows. In §2, we present the mathematical formulation for the backlogging model. In §3, we design a nested marginal cost accounting scheme for perishable inventory systems. In §4, we construct the proportional-balancing policy and the dual-balancing policy, and provide their worst-case performance guarantees. In §5, we provide the main proofs, while leaving some of the more involved technical details in the supplemental material. In §6, we conduct an extensive numerical study on our proposed policies. Finally, we conclude the paper in §7 with some discussions on extensions and possible future research. Throughout this paper, for any real numbers x and y , we denote $x^+ = \max\{x, 0\}$, $x \vee y = \max\{x, y\}$, and $x \wedge y = \min\{x, y\}$. In addition, for a sequence x_1, x_2, \dots and any integers t and s with $t \leq s$, we denote $x_{[t,s]} = \sum_{j=t}^s x_j$ and $x_{[t,s)} = \sum_{j=t}^{s-1} x_j$. Also, we use “expiration”, “outdating”, and “perishing”, interchangeably.

2. Stochastic Periodic-Review Perishable Inventory Control Problem

In this section, we provide the mathematical formulation of the stochastic periodic-review perishable inventory system over a planning horizon of T (possibly infinite) periods, indexed by $t = 1, \dots, T$. The lifetime of the product is m periods, i.e., a product perishes after staying m periods in stock. Our model allows for a non-stationary and generally correlated demand process. We assume that the order lead time is zero, i.e., an order placed at the beginning of a period can be used in

the same period. This is a common assumption in the perishable inventory literature (see [Karaemen et al. \(2011\)](#)). We shall focus our presentation on the backlogging model but will extend the analysis and results to the lost-sales model in Section 7.

Demand structure. The demands over the planning horizon are random, denoted by D_1, \dots, D_T . The demands in different periods can be non-stationary and correlated over time. At the beginning of each period t , there is an observed *information set* denoted by f_t , which contains all of the information accumulated up to period t . More specifically, the information set f_t consists of the realized demands d_1, \dots, d_{t-1} in the first $t-1$ periods, and possibly some exogenous information denoted by (w_1, \dots, w_t) . The information set f_t in period t is one specific realization in the set of all possible realizations of the random vector $F_t = (D_1, \dots, D_{t-1}, W_1, \dots, W_t)$. The set of all possible realizations is denoted by \mathcal{F}_t . With the information set f_t , the conditional joint distribution of the future demands (D_t, \dots, D_T) is known. We assume that the conditional expectations, given f_t , are well defined. Note that this demand structure is very general, that includes a wide range of demand processes such as Markov modulated demand process (see, e.g. [Song and Zipkin \(1993\)](#) and [Sethi and Cheng \(1997\)](#)), MMFE (see, e.g., [Heath and Jackson \(1994\)](#)), AR(p), ARMA(p, q), ARIMA(p, r, q), (see, e.g., [Mills \(1990\)](#)), and models with advance demand information (ADI) (see, e.g., [Gallego and Özer \(2001\)](#)), among others.

Cost structure. In each period t , four types of costs may occur: a unit ordering cost \hat{c} , a unit holding cost \hat{h} for leftover inventory, a unit backlogging cost \hat{b} for unsatisfied demand, and a unit outdated cost $\hat{\theta}$ for expired products. There is also a one-period discount factor α , with $0 < \alpha \leq 1$ when $T < \infty$ and $0 < \alpha < 1$ when $T = \infty$. We assume that $\hat{b} > (1 - \alpha)\hat{c}$ and $\hat{\theta} + \alpha\hat{c} \geq 0$. Thus, $\hat{\theta}$ can be negative, and in this case it can be interpreted as unit salvage value. Following [Nahmias \(1975\)](#) we assume that any remaining inventory at the end of the planning horizon can be salvaged with a return of \hat{c} per unit and unsatisfied demand can be satisfied by an emergency order at a cost of \hat{c} per unit. We note that our analysis can be extended to the case with a unit salvage value \hat{v} for any on-hand inventory and a unit penalty cost \hat{p} for any unsatisfied demand at the end of the planning horizon, as long as $\hat{v} \leq \hat{c}$ and $\hat{b} + \alpha\hat{p} > \hat{c}$. However, our analysis cannot be directly extended to the case with age-dependent salvage values.

System dynamics. For each period t , the sequence of events is as follows. First, at the beginning of period t , the information set $f_t \in \mathcal{F}_t$ and the inventory vector

$$\mathbf{x}_t = (x_{t,1}, \dots, x_{t,m-1}) \tag{1}$$

are observed, where $x_{t,i}$ is the quantity of on-hand products whose remaining lifetime is i periods, $i = 1, \dots, m-2$, and $x_{t,m-1}$ is the quantity of on-hand products whose remaining lifetime is $m -$

1 periods minus the quantity of backlogged demands (if any). Thus, $x_{t,1}, \dots, x_{t,m-2}$ are always nonnegative; while $x_{t,m-1}$ can be positive or negative. For simplicity, we assume that the inventory system is initially empty at the beginning of period 1, i.e., $x_{1,i} = 0$, for all $i = 1, \dots, m-1$; but our analysis and results can be extended to the case with an arbitrary initial state.

Second, an order with quantity q_t is placed, incurring an ordering cost $\hat{c}q_t$. Following the discussions in Levi et al. (2007), we assume that q_t is a continuous decision variable, but it can be extended to the case of integer values. Denote y_t as the total inventory level after receiving the order in period t . Then, $y_t = \sum_{i=1}^{m-1} x_{t,i} + q_t$.

Third, the demand in period t is realized and satisfied as much as possible by the on-hand inventory using the FIFO issuing policy, i.e., the oldest inventory is consumed first when demand arrives. At the end of period t , if $y_t - D_t \geq 0$, then the excess inventory incurs a holding cost $\hat{h}(y_t - D_t)$. Following Nahmias (1975), we assume that all excess inventory (including the inventory which perishes at the end of this period) incurs a holding cost. On the other hand, if $y_t - D_t < 0$, then the system incurs a backlogging cost $\hat{b}(D_t - y_t)$. Furthermore, if the inventory with one period remaining life $x_{t,1} > D_t$, then $e_t := (x_{t,1} - D_t)^+$ units perish and incur an outdated cost $\hat{\theta}e_t$.

Finally, the system proceeds to the subsequent period $t+1$. By the definition of the inventory vector \mathbf{x}_t and the FIFO issuing policy, we obtain the following state transition from \mathbf{x}_t to \mathbf{x}_{t+1} :

$$\begin{aligned} x_{t+1,j} &= \left(x_{t,j+1} - \left(D_t - \sum_{i=1}^j x_{t,i} \right)^+ \right)^+, \text{ for } 1 \leq j \leq m-2, \\ x_{t+1,m-1} &= q_t - \left(D_t - \sum_{i=1}^{m-1} x_{t,i} \right)^+. \end{aligned} \quad (2)$$

We remark that in defining the inventory state \mathbf{x}_t in (1), it is convenient and natural to combine the inventory having $m-1$ periods of remaining life with the number of backlogs in $x_{t,m-1}$. This is because when demand arrives, by the FIFO issuing policy, it is first met by $x_{t,1}$, and when $x_{t,1}$ is consumed then the remaining demand is met by $x_{t,2}$. This process continues and when (and if) $x_{t,m-2}$ also depletes to 0, the remaining demand will be satisfied by $x_{t,m-1}$. Clearly, when the demand is large, this last number will continue to go down after reaching 0, representing the backlog level. We also note that inventory only outdates through the first dimension, $x_{t,1}$, of vector \mathbf{x}_t , while backlogs always stay in the last dimension, $x_{t,m-1}$ (hence backlogs will *not* disappear after m periods). Moreover, if in period t there are backlogs (thus $x_{t,m-1}$ is negative and $x_{t,j} = 0$ for $j = 1, \dots, m-2$), then by (2), in the next period $x_{t+1,j}$ will be equal to 0 for all $j = 1, \dots, m-2$, but $x_{t+1,m-1}$ can be positive or negative, depending on whether q_t is greater or less than $D_t - x_{t,m-1}$.

Objective. For clarity, we often distinguish between a random variable and its realization using a capital letter and a lowercase letter, respectively. Then the expected total discounted cost incurred under a given policy P that orders q_t in period t can be written as

$$\mathcal{C}(P) = \mathbb{E} \left[\sum_{t=1}^T \alpha^{t-1} \left(\hat{c}q_t + \hat{h}(Y_t - D_t)^+ + \hat{b}(D_t - Y_t)^+ + \hat{\theta}e_t \right) - \alpha^T \hat{c} \sum_{i=1}^{m-1} X_{T+1,i} \right]. \quad (3)$$

Note that, the quantities q_t, Y_t, e_t , and X_t all depend on the policy P ; and whenever necessary, we shall make the dependency explicit, i.e., write them as q_t^P, Y_t^P, e_t^P , and X_t^P , respectively.

The objective is to coordinate the sequence of orders to minimize the expected total discounted cost. As discussed in Section 1, it is known that finding the exact optimal policy using dynamic programming is computationally intractable. Thus, our focus in this paper is to design easy-to-compute and near-optimal approximation algorithms.

Approximation policy assessment. To measure the effectiveness of an approximation algorithm P , we define its performance measure by the ratio $\mathcal{C}(P)/\mathcal{C}(OPT)$, where $\mathcal{C}(OPT)$ is the cost under an optimal policy. Clearly, the value of this ratio depends on the problem instance, and is at least 1. If under algorithm P this ratio is always equal to 1 for all problem instances, then P is an optimal policy. Otherwise, if there exists some number $r (> 1)$ such that this ratio is less than or equal to r for any problem instance, then we say that the algorithm admits a worst-case performance guarantee of r , or simply call it an r -approximation algorithm. As mentioned earlier, we will present efficient approximation algorithms for the perishable inventory systems with worst-case performance guarantees of 2 and 3, respectively.

Cost transformation. Next we carry out a cost transformation to obtain an equivalent model with the unit ordering cost equal to 0. This will enable us to assume, without loss of generality, that the unit ordering cost is 0 in the subsequent analysis. The proof of the following proposition is given in the online Appendix.

PROPOSITION 1. *For every perishable inventory system with cost parameters $\hat{c}, \hat{h}, \hat{b}$ and $\hat{\theta}$, there is an equivalent system with non-negative cost parameters $c = 0, h = \hat{h} + (1 - \alpha)\hat{c}, b = \hat{b} - (1 - \alpha)\hat{c}$ and $\theta = \hat{\theta} + \alpha\hat{c}$. And the expected total discounted cost can be rewritten as*

$$\mathcal{C}(P) = \mathbb{E} \left[\sum_{t=1}^T \alpha^{t-1} \left(h(Y_t - D_t)^+ + b(D_t - Y_t)^+ + \theta e_t \right) \right] + \sum_{t=1}^T \alpha^{t-1} \hat{c} \mathbb{E}[D_t]. \quad (4)$$

3. Nested Marginal Cost Accounting Scheme

The traditional cost accounting scheme given in (3) decomposes the total cost by periods. Levi et al. (2007) presented a marginal cost accounting scheme for the classical non-perishable inventory systems. In this section, we develop a marginal cost accounting scheme for perishable inventory systems, similar in spirit to that in Levi et al. (2007). Our marginal cost accounting scheme exhibits a nested structure due to the multi-dimensionality of system state. The main idea underlying this approach is to decompose the total cost in terms of the marginal costs of individual decisions. That is, we associate the decision in period t with its affiliated cost contributions to the system. These marginal costs may include costs (associated with the decision) incurred in both the current and subsequent periods.

Given the inventory vector $\mathbf{x}_t = (x_{t,1}, \dots, x_{t,m-1})$ at the beginning of period t , and that a policy P orders q_t , we aim to compute the marginal cost contributions to the system by these q_t units on the holding, outdated, and backlogging costs. To this end, for $i = 1, \dots, m-1$, we let $B_t(\mathbf{x}_t, i)$ denote the number of outdated units in periods $[t, t+i-1]$ given that the inventory vector at the beginning of period t is \mathbf{x}_t , with the convention that $B_t(\mathbf{x}_t, 0) \equiv 0$. Then, for $1 \leq i \leq m-1$, we have

$$B_t(\mathbf{x}_t, i) = \max \left\{ \sum_{j=1}^i x_{t,j} - D_{[t,t+i-1]}, B_t(\mathbf{x}_t, i-1) \right\}. \quad (5)$$

To see why this is true, note that $\sum_{j=1}^i x_{t,j} - B_t(\mathbf{x}_t, i-1)$ is the number of non-expired units in $x_{t,1}, \dots, x_{t,i}$ that would meet demands in periods $[t, t+i-1]$. These units, if not consumed, will expire at the end of period $t+i-1$. Thus $(\sum_{j=1}^i x_{t,j} - B_t(\mathbf{x}_t, i-1) - D_{[t,t+i-1]})^+$, if positive, would be the number of units that will expire at the end of period $t+i-1$. Adding $B_t(\mathbf{x}_t, i-1)$ to it gives the total number of expired units in $[t, t+i-1]$, which is (5).

The nested structure in the auxiliary function $B_t(\cdot, \cdot)$ follows from the fact that some inventory units reach their expiration date before meeting the demand, and have to be discarded from the on-hand inventory. Using this auxiliary function, the number of outdated units in period $t+i-1$, for $1 \leq i \leq m-1$, is given as

$$e_{t+i-1} = \left(\sum_{j=1}^i x_{t,j} - B_t(\mathbf{x}_t, i-1) - D_{[t,t+i-1]} \right)^+,$$

and the number of outdated units in period $t+m-1$ is

$$e_{t+m-1} = \left(q_t + \sum_{j=1}^{m-1} x_{t,j} - B_t(\mathbf{x}_t, m-1) - D_{[t,t+m-1]} \right)^+.$$

3.1. Nested Marginal Holding Cost Accounting

We first focus on the marginal holding cost accounting of a given policy P . The holding cost for the q_t units ordered in period t may be incurred in any period from t to $t + m - 1$ (after which the remaining ones will perish), or T , whichever is smaller. Let $H_t^P(q_t)$ be the discounted marginal holding cost (to period 1) incurred by these q_t units. Then it follows from the FIFO issuing policy that

$$H_t^P(q_t) := h \sum_{i=t}^{(t+m-1) \wedge T} \alpha^{i-t} \left(q_t - (D_{[t,i]} + B_t(\mathbf{x}_t, i - t) - \sum_{j=1}^{m-1} x_{t,j})^+ \right)^+, \quad (6)$$

where the auxiliary function $B_t(\mathbf{x}_t, i)$ is given recursively via (5). To see why (6) is valid, note that the total number of units in \mathbf{x}_t that do not expire until $t + i$ is $\sum_{j=1}^{m-1} x_{t,j} - B_t(\mathbf{x}_t, i)$, thus the net demand after consuming the units in \mathbf{x}_t is $(D_{[t,t+i]} - (\sum_{j=1}^{m-1} x_{t,j} - B_t(\mathbf{x}_t, i)))^+$. Hence, the number of unconsumed units from q_t at the end of period $t + i$ is $(q_t - (D_{[t,t+i]} + B_t(\mathbf{x}_t, i) - \sum_{j=1}^{m-1} x_{t,j})^+)^+$.

Because the marginal holding cost is computed based on the nested structure of the auxiliary function $B_t(\cdot, \cdot)$, we call it *the nested marginal holding cost accounting*. It is important to note that the marginal holding cost associated with the q_t units ordered in period t is only affected by the future demands but not by the future decisions.

3.2. Nested Marginal Outdating Cost Accounting

Similarly, we can compute the marginal outdating cost associated with the q_t units ordered by policy P in period t using the following nested scheme. For $t = 1, \dots, T - m + 1$,

$$\Theta_t^P(q_t) := \alpha^{t+m-2} \theta e_{t+m-1} = \alpha^{t+m-2} \theta \left(q_t + \sum_{j=1}^{m-1} x_{t,j} - B_t(\mathbf{x}_t, m-1) - D_{[t,t+m-1]} \right)^+, \quad (7)$$

where $B_t(\cdot, \cdot)$ is defined in (5); and for $t = T - m + 2, \dots, T$, we have $\Theta_t^P \equiv 0$ since the ordered units do not expire within the planning horizon.

3.3. Marginal Backlogging Cost Accounting

For each period $t = 1, \dots, T$, the discounted (to period 1) marginal backlogging cost of the q_t units ordered in period t by policy P can be expressed as

$$\Pi_t^P(q_t) := \alpha^{t-1} b \left(D_t - \sum_{i=1}^{m-1} x_{t,i} - q_t \right)^+, \quad (8)$$

which is exactly the same as the traditional backlogging cost using the period-by-period accounting scheme. The intuition is that any negative consequence of under-ordering can be corrected by placing an order in the next period; thus it suffices to only consider the backlogging cost incurred in the current period.

3.4. Total Cost of a Given Policy

Note that the marginal costs defined above, $H_t^P(q_t)$, $\Theta_t^P(q_t)$, and $\Pi_t^P(q_t)$, are random as they depend on the future demands. Since the system is initially empty, the expected total system cost $\mathcal{C}(P)$ of policy P can be obtained by summing (6), (7) and (8) over t from 1 to T , and then taking expectations. Thus, by (4) we have

$$\mathcal{C}(P) = \mathbb{E} \left[\sum_{t=1}^T \left(H_t^P(q_t) + \Pi_t^P(q_t) + \Theta_t^P(q_t) \right) \right] + \sum_{t=1}^T \alpha^{t-1} \hat{c} \mathbb{E}[D_t]. \quad (9)$$

If we ignore the constant terms that are independent of the policy, then we can write the effective cost of a policy P as

$$C(P) = \mathbb{E} \left[\sum_{t=1}^T \left(H_t^P(q_t) + \Pi_t^P(q_t) + \Theta_t^P(q_t) \right) \right]. \quad (10)$$

Clearly, to compare different policies, we only need to compare their effective costs.

4. Balancing Policies and Worst-Case Performance Guarantees

In this section, we propose two efficient cost-balancing algorithms for perishable inventory systems with general product lifetime using the nested cost accounting scheme defined in the previous section. The first one is for arbitrary non-stationary and correlated demand processes; while the second is for independent and stochastically non-decreasing demand processes. These policies will be shown to admit worst-case performance guarantees of 3 and 2, respectively.

4.1. Proportional-Balancing (PB) Policy

For each period $t = 1, \dots, T$, with an observed information set $f_t \in \mathcal{F}_t$, the proportional-balancing (PB) policy orders $q_t^{PB} = q_t$ that balances a proportion of the expected marginal holding and outdating costs with the expected backlogging cost as follows:

$$\frac{mh+\theta}{2(m-1)h+\theta} \mathbb{E}[H_t^{PB}(q_t) + \Theta_t^{PB}(q_t) | f_t] = \mathbb{E}[\Pi_t^{PB}(q_t) | f_t]. \quad (11)$$

It can be verified that the left hand side (LHS) of (11) is an increasing convex function of the order quantity q_t , which equals 0 when $q_t = 0$ and approaches infinity when q_t tends to infinity. On the other hand, the right hand side (RHS) of (11) is a decreasing convex function of the order quantity q_t , which equals a non-negative number when $q_t = 0$ and tends to 0 when q_t goes to infinity. Since q_t can take any non-negative real value and both functions are continuous, q_t in (11) is well defined. Furthermore, since LHS minus RHS of (11) is increasing in q_t , q_t^{PB} can be very efficiently computed using bisection methods. It should be noted that q_t^{PB} is a function of f_t and \mathbf{x}_t , but for simplicity we make this dependency implicit.

For the special case where the demands in different periods are independent, q_t^{PB} does not depend on the information set f_t , and it becomes a function of only the inventory vector \mathbf{x}_t at the beginning of period t . Several studies in the literature have analyzed the qualitative properties of the optimal order quantity q_t^{OPT} on the starting inventory vector of period t for the case of independent and identically distributed demands (see, e.g., Fries (1975) and Nahmias (1982)). Suppose the inventory vector at the beginning of period t is $\mathbf{x}_t = (x_1, \dots, x_{m-1})$. It has been shown that q_t^{OPT} decreases at a rate less than one when the product inventory of any age group increases, and that it decreases more rapidly in the inventory level of newer product than that of older product. The following result shows that, the order quantity q_t^{PB} under the PB policy satisfies these properties as well. Its proof is provided in the online Appendix.

PROPOSITION 2. *For each period t , the order quantity q_t^{PB} under the PB policy satisfies*

$$-1 \leq \frac{\partial q_t^{PB}}{\partial x_{m-1}} \leq \frac{\partial q_t^{PB}}{\partial x_{m-2}} \leq \dots \leq \frac{\partial q_t^{PB}}{\partial x_1} \leq 0.$$

The more important question is how well the PB policy performs. In what follows, we first provide a theoretical worst-case performance guarantee; then in Section 6, we will provide a comprehensive numerical study to demonstrate its empirical performance.

THEOREM 1. *For an arbitrary non-stationary and correlated demand process, the proportional-balancing policy for the perishable inventory system with $m \geq 2$ periods of product lifetime has a worst-case performance guarantee of $\left(2 + \frac{(m-2)h}{mh+\theta}\right)$, i.e., for any instance of the problem, the expected cost of the proportional-balancing policy is at most $\left(2 + \frac{(m-2)h}{mh+\theta}\right)$ times the expected cost of an optimal policy.*

Theorem 1 shows that, when the product lifetime $m = 2$, the PB policy has a worst-case performance guarantee of 2; while for a general lifetime m , the PB policy has a worst-case performance guarantee between 2 and 3.

We remark that the balancing coefficient on the LHS of (11) is chosen so that the resulting PB policy admits our best provable worst-case performance guarantee. If we select a general positive balancing coefficient β to construct the PB policy, then we can prove that it admits a worst-case performance guarantee of $(\beta + 1)/\min\{\beta, \beta_0\}$, where $\beta_0 = \frac{mh+\theta}{2(m-1)h+\theta}$. Since the worst-case performance guarantee is minimized when $\beta = \beta_0$, we construct the PB policy with this optimized parameter.

4.2. Dual-Balancing (DB) Policy

In this subsection, we propose another approximation policy, referred to as the dual-balancing policy, which has a worst-case performance of 2 for an arbitrary fixed lifetime m when the demands D_1, \dots, D_T are independent and stochastically non-decreasing over time. The random variables D_1, \dots, D_T are said to be stochastically non-decreasing if for any $1 \leq t \leq s \leq T$, D_t is less than D_s in the *usual stochastic order*, or equivalently, $\Pr(D_t > d) \leq \Pr(D_s > d)$ for all d . For more detailed discussions on stochastic orders, we refer interested readers to [Shaked and Shanthikumar \(2007\)](#). In the remainder of this section, we assume that the demands D_1, \dots, D_T are independent and stochastically non-decreasing.

To introduce the dual-balancing policy, we first define the discounted (to period 1) marginal holding cost for period t for an arbitrary policy P by

$$\hat{H}_t^P(q_t) := \alpha^{t-1} h \left(\sum_{i=1}^{m-1} x_{t,i} + q_t - D_t \right)^+.$$

Then, we define the discounted marginal outdated and discounted marginal backlogging costs for a policy P in exactly the same way as those in (7) and (8). In addition, for each period t , let S_t be the solution of y to the equation $h\mathbb{E}[(y - D_t)^+] = b\mathbb{E}[(D_t - y)^+]$, which depends only on the distribution of D_t in period t . Since the demands D_1, \dots, D_T are stochastically non-decreasing, it follows that S_t is non-decreasing in t .

The *dual-balancing* (DB) policy is described as follows: Suppose at the beginning of period t the state is $\mathbf{x}_t = (x_{t,1}, \dots, x_{t,m-1})$, the DB policy orders $q_t^{DB} = q_t$ if $\sum_{i=1}^{m-1} x_{t,i} \leq S_t$, where q_t is the solution of

$$\mathbb{E}[\hat{H}_t^{DB}(q_t) + \Theta_t^{DB}(q_t) \mid \mathbf{x}_t] = \mathbb{E}[\Pi_t^{DB}(q_t) \mid \mathbf{x}_t], \quad (12)$$

and $q_t^{DB} = 0$ otherwise. Note that, since the demands are independent random variables, the information set f_t can be removed, and q_t^{DB} is only a function of the inventory vector \mathbf{x}_t .

The q_t in (12) balances the expected discounted marginal holding and outdated costs with the expected marginal backlogging cost. It can be verified that the LHS of (12) is an increasing convex function of the order quantity q_t , which equals $\alpha^{t-1}hE[(\sum_{i=1}^{m-1} x_{t,i} - D_t)^+]$ when $q_t = 0$ and approaches infinity when q_t goes to infinity. On the other hand, the RHS of (12) is a decreasing convex function of q_t , which equals $\alpha^{t-1}bE[(D_t - \sum_{i=1}^{m-1} x_{t,i})^+]$ when $q_t = 0$ and approaches 0 when q_t goes to infinity. When $\sum_{i=1}^{m-1} x_{t,i} \leq S_t$, the quantity q_t in (12) is well defined with $0 \leq q_t \leq S_t - \sum_{i=1}^{m-1} x_{t,i}$. When $\sum_{i=1}^{m-1} x_{t,i} > S_t$, Equation (12) does not have a nonnegative solution and in this case the DB policy orders $q_t^{DB} = 0$.

It is important to note that since S_t is non-decreasing in t , if for some period t we have $\sum_{i=1}^{m-1} x_{t,i} \leq S_t$, then following the DB policy we have $\sum_{i=1}^{m-1} x_{t',i} \leq S_{t'}$ for all $t' \geq t$ regardless of the demand realizations of D_t, \dots, D_{T} . This is because when $\sum_{i=1}^{m-1} x_{t,i} \leq S_t$, we have

$$\sum_{i=1}^{m-1} x_{t+1,i} = \sum_{i=1}^{m-1} x_{t,i} + q_t - D_t - e_t \leq S_t - D_t - e_t \leq S_t \leq S_{t+1}.$$

This implies that when the demands are independent and stochastically non-decreasing, the DB policy can perfectly balance the expected marginal holding and outdated costs with the expected marginal backlogging cost after placing its first order. If the demands are not independent or stochastically non-decreasing, then S_t will not necessarily be monotonically non-decreasing in t and as a result, the DB policy will not have the above property. This is the reason why we need to assume that the demands are independent and stochastically non-decreasing over time.

The following result shows that, again, the desired properties exhibited by the optimal control policy for the perishable inventory system are inherited by the DB policy. Its proof is very similar to that of Proposition 2 and hence it is omitted.

PROPOSITION 3. *For each period t , the order quantity q_t^{DB} under the DB policy satisfies*

$$-1 \leq \frac{\partial q_t^{DB}}{\partial x_{m-1}} \leq \frac{\partial q_t^{DB}}{\partial x_{m-2}} \leq \dots \leq \frac{\partial q_t^{DB}}{\partial x_1} \leq 0.$$

The following theorem shows that, the DB policy has a worst-case performance guarantee of 2 when the demands are independent and stochastically non-decreasing over time.

THEOREM 2. *For an arbitrary independent and stochastically non-decreasing demand process, the dual-balancing policy for the perishable inventory system with an arbitrary fixed product lifetime has a worst-case performance guarantee of 2, i.e., for any instance of the problem, the expected cost of the dual-balancing policy is at most twice the expected cost of an optimal policy.*

5. Worst-Case Analysis

The arguments used in the literature on proving worst-case performance guarantees for approximation algorithms utilize a “unit-matching” approach (see, e.g., Levi et al. (2007, 2008a,b, 2012), Levi and Shi (2013)). In a sense, the approach is geometric, and it relies on the correspondence of units in the systems operating under different policies throughout the planning horizon, and then it compares the costs incurred by the matched units in different systems. However, the unit-matching approach fails to work for perishable inventory systems because the inventory units can perish and the number of outdated units differs in systems operating under different policies.

To overcome this difficulty, we develop an algebraic approach for comparing different systems. A key concept in our approach is the *trimmed on-hand inventory level*, which is defined as the part of on-hand inventory units ordered before any given particular time. These trimmed inventory levels serve as a generalization of the traditional inventory level, as they provide critical (partial) information on the ages of the products on-hand. Due to the nature of perishable systems, it is impossible to quantify the effect of the decision made in the current period t on future costs only through the traditional total inventory level Y_t . The trimmed inventory levels provide a tractable way to analyze this effect, and also provide the *right* framework for coupling the marginal holding and outdated costs in different systems. More technically, the difference between the trimmed inventory levels of our policy and the optimal policy OPT can be bounded by the difference between the outdated units of the two policies. An essential part of this worst-case analysis presented below is based on this new concept.

The main ideas and arguments for the proofs of our key results are given below. We leave some of the more involved technical analysis in the online Appendix. For simplicity, whenever possible we will abbreviate the marginal costs $H_t^P(q_t)$, $\Theta_t^P(q_t)$ and $\Pi_t^P(q_t)$ by H_t^P , Θ_t^P and Π_t^P , respectively, i.e., we make the ordering quantity q_t in these functions implicit.

In the following, we first study the PB policy and its worst-case performance, and then study the DB policy.

5.1. Analysis of PB Policy

We now compare the PB policy with the optimal policy OPT . To this end, we make the dependency of the relevant quantities on the policy, PB or OPT , explicit. For each realization of demands D_1, \dots, D_T and the exogenous information W_1, \dots, W_T , we compare and analyze the inventory processes of the systems operating under these two policies.

Given a realization $f_T \in \mathcal{F}_T$, let \mathcal{T}_H be the set of periods in which the optimal policy has more total inventory level than the PB policy does. In other words, we denote

$$\mathcal{T}_H = \{t \in [1, T] : Y_t^{OPT} \geq Y_t^{PB}\}.$$

In addition, we let its complement set be denoted by

$$\mathcal{T}_\Pi = \{t \in [1, T] : Y_t^{OPT} < Y_t^{PB}\}.$$

LEMMA 1. For each realization $f_T \in \mathcal{F}_T$, we have

$$\sum_{t \in \mathcal{T}_H} H_t^{PB} \leq \sum_{t=1}^T H_t^{OPT} + (m-2) \frac{h}{\theta} \sum_{t=1}^T \Theta_t^{OPT}.$$

Lemma 1 is one of the key technical results of this paper. Given its complete proof is complicated and lengthy, we leave it in the online Appendix for interested readers. To illustrate the main ideas of our argument, we provide below the proof for the following, weaker, result:

$$\sum_{t \in \mathcal{T}_H} H_t^{PB} \leq \sum_{t=1}^T H_t^{OPT} + (m-1) \frac{h}{\theta} \sum_{t=1}^T \Theta_t^{OPT}. \quad (13)$$

For ease of illustration, we also assume that $\alpha = 1$.

As said earlier, an important concept in proving our main technical result is what we refer to as the *trimmed on-hand inventory level*, denoted by $Y_{t,s}$ for any $s \geq t \geq 1$, which is defined as *the part of on-hand inventory at the beginning of period s which is ordered in period t or earlier*. From the definition of $Y_{t,s}$, it holds that $Y_{t,s} = 0$ when $s \geq t + m$, and

$$Y_{t,s} = (Y_t - D_{[t,s]} - e_{[t,s]})^+, \quad s = t, t+1, \dots, t+m-1. \quad (14)$$

For any period $t = 1, \dots, T$, we define the notation $R(t)$ as follows: if the set $\{s \in \mathcal{T}_H : s > t\}$ is nonempty, then $R(t) := \min\{s \in \mathcal{T}_H : s > t\}$; otherwise, $R(t) := T + 1$. In addition, for any $s \geq 1$, denote \tilde{H}_s as the part of the holding cost incurred in period s associated with the products ordered in periods $\{t : t \in \mathcal{T}_H, t \leq s\}$. Since the lifetime of the products is m , all products ordered in period t or earlier will leave the system by the end of period $t + m - 1$. Then, it follows that for any $t \in \mathcal{T}_H$, $\tilde{H}_s = 0$ when $t + m \leq s \leq R(t) - 1$. Consequently, by the definitions of H_t , \tilde{H}_s , and $R(t)$, we have

$$\sum_{t \in \mathcal{T}_H} H_t = \sum_{t \in \mathcal{T}_H} \sum_{s=t}^{R(t)-1} \tilde{H}_s = \sum_{t \in \mathcal{T}_H} \sum_{s=t}^{(t+m) \wedge R(t)-1} \tilde{H}_s.$$

For each period $s \in [t, R(t) - 1]$, from its definition, \tilde{H}_s is clearly no greater than the part of the holding cost incurred in period s associated with the products ordered in periods $1, \dots, t$, which can be expressed as $h(Y_{t,s} - D_s)^+$. Since $\sum_{t=1}^T H_t = h \sum_{s=1}^T (Y_{s,s} - D_s)^+$ and $Y_{t,s} \geq Y_{s,s}$ for any $t \leq s$, the following inequalities hold for any policy:

$$\sum_{t \in \mathcal{T}_H} H_t \leq h \sum_{t \in \mathcal{T}_H} \sum_{s=t}^{(t+m) \wedge R(t) - 1} (Y_{t,s} - D_s)^+ \leq \sum_{t=1}^T H_t. \quad (15)$$

In particular, we have

$$\sum_{t \in \mathcal{T}_H} H_t^{PB} \leq h \sum_{t \in \mathcal{T}_H} \sum_{s=t}^{(t+m) \wedge R(t) - 1} (Y_{t,s}^{PB} - D_s)^+; \quad (16)$$

$$\sum_{t=1}^T H_t^{OPT} \geq h \sum_{t \in \mathcal{T}_H} \sum_{s=t}^{(t+m) \wedge R(t) - 1} (Y_{t,s}^{OPT} - D_s)^+. \quad (17)$$

Subtracting (17) from (16), we obtain

$$\begin{aligned} \sum_{t \in \mathcal{T}_H} H_t^{PB} - \sum_{t=1}^T H_t^{OPT} &\leq h \sum_{t \in \mathcal{T}_H} \sum_{s=t}^{(t+m) \wedge R(t) - 1} ((Y_{t,s}^{PB} - D_s)^+ - (Y_{t,s}^{OPT} - D_s)^+) \\ &\leq h \sum_{t \in \mathcal{T}_H} \sum_{s=t}^{(t+m) \wedge R(t) - 1} (Y_t^{PB} - Y_t^{OPT} - e_{[t,s]}^{PB} + e_{[t,s]}^{OPT})^+ \\ &\leq h \sum_{t \in \mathcal{T}_H} \sum_{s=t}^{(t+m) \wedge R(t) - 1} (e_{[t,s]}^{OPT} - e_{[t,s]}^{PB})^+, \end{aligned} \quad (18)$$

where the second inequality follows from (14) and $a^+ - b^+ \leq (a - b)^+$ for any real numbers a and b , and the last one holds because $Y_t^{PB} \leq Y_t^{OPT}$ when $t \in \mathcal{T}_H$.

Thus, it follows from $(e_t^{OPT} - e_t^{PB})^+ \leq e_t^{OPT}$ and $e_{[t,t]}^{OPT} = 0$ that

$$\begin{aligned} \sum_{t \in \mathcal{T}_H} H_t^{PB} - \sum_{t=1}^T H_t^{OPT} &\leq h \sum_{t \in \mathcal{T}_H} \sum_{s=t+1}^{(t+m) \wedge R(t) - 1} e_{[t,s]}^{OPT} \leq h \sum_{t \in \mathcal{T}_H} \sum_{s=t+1}^{(t+m) \wedge R(t) - 1} e_{[t, R(t) - 1]}^{OPT} \\ &\leq (m-1)h \sum_{t \in \mathcal{T}_H} \sum_{\ell=t}^{R(t)-2} e_\ell^{OPT} \leq (m-1)h \sum_{t=1}^T e_t^{OPT} \\ &= (m-1)h \sum_{t=m}^T e_t^{OPT} = (m-1) \frac{h}{\theta} \sum_{t=1}^T \Theta_t^{OPT}, \end{aligned}$$

where the last inequality follows from $e_t^{OPT} = 0$ for $t \leq m-1$ as the system is initially empty, $\Theta_t^{OPT} = \theta e_{t+m-1}^{OPT}$ for $1 \leq t \leq T-m+1$, and $\Theta_t^{OPT} = 0$ when $T-m+1 < t \leq T$. This proves a weaker form of Lemma 2, i.e., the result (13).

LEMMA 2. For each realization $f_T \in \mathcal{F}_T$, we have $\sum_{t \in \mathcal{T}_H} \Theta_t^{PB} \leq \sum_{t=1}^T \Theta_t^{OPT}$.

Proof. For brevity, we only prove below the result for the special case when $\alpha = 1$. The complete proof for a general $\alpha \in [0, 1]$ is provided in the online Appendix. Since any products ordered after period $T - m + 1$ do not perish within the planning horizon, we only need to consider periods $t = 1, \dots, T - m + 1$. For each realization f_T and the resulting \mathcal{T}_H , we partition the periods $\{1, \dots, T - m + 1\}$ as follows: First, start in period $T - m + 1$ and search backward for the latest period $t \in \mathcal{T}_H$ such that $\Theta_t^{PB} > \Theta_t^{OPT}$. If no such period exists, then we terminate the partition process. Otherwise, let t' be that period and mark the periods $t', t' - 1, \dots, (t' - m)^+ + 1$. Next, repeat the above procedure over periods $1, \dots, (t' - m)^+$ until the remaining set of periods is empty. As a result, this procedure partitions the periods $\{1, \dots, T - m + 1\}$ into marked and unmarked periods. Let \mathcal{T}_M denote the set of marked periods.

We first consider any period $t \in \mathcal{T}_H \setminus \mathcal{T}_M$. Then, it follows from the definition of \mathcal{T}_M that $\Theta_t^{PB} \leq \Theta_t^{OPT}$. Consequently

$$\sum_{t \in \mathcal{T}_H \setminus \mathcal{T}_M} \Theta_t^{PB} \leq \sum_{t \in \mathcal{T}_H \setminus \mathcal{T}_M} \Theta_t^{OPT}. \quad (19)$$

Since the set \mathcal{T}_M is made up of disjoint intervals, we consider a representative interval with its largest period being t , i.e., this interval consists of periods $(t - m)^+ + 1, \dots, t - 1, t$. Then, by the construction of \mathcal{T}_M , $t \in \mathcal{T}_H$ and $\Theta_t^{PB} > \Theta_t^{OPT}$. Since $\Theta_t = \theta e_{t+m-1}$, we have $e_{t+m-1}^{PB} > e_{t+m-1}^{OPT} \geq 0$. Note that e_t is the number of perished products in period t and it satisfies the following identity for any feasible policy:

$$e_{t+m-1} = (Y_t - D_{[t, t+m-1]} - e_{[t, t+m-1]})^+. \quad (20)$$

Thus it follows from (20) and $e_{t+m-1}^{PB} > 0$ that

$$\begin{aligned} e_{[t, t+m-1]}^{PB} &= (Y_t^{PB} - D_{[t, t+m-1]} - e_{[t, t+m-1]}^{PB})^+ + e_{[t, t+m-1]}^{PB} \\ &= Y_t^{PB} - D_{[t, t+m-1]}. \end{aligned} \quad (21)$$

On the other hand, for the OPT policy we have

$$\begin{aligned} e_{[t, t+m-1]}^{OPT} &= (Y_t^{OPT} - D_{[t, t+m-1]} - e_{[t, t+m-1]}^{OPT})^+ + e_{[t, t+m-1]}^{OPT} \\ &\geq Y_t^{OPT} - D_{[t, t+m-1]}. \end{aligned} \quad (22)$$

Subtracting (22) from (21) yields

$$e_{[t \vee m, t+m-1]}^{PB} - e_{[t \vee m, t+m-1]}^{OPT} = e_{[t, t+m-1]}^{PB} - e_{[t, t+m-1]}^{OPT} \leq Y_t^{PB} - Y_t^{OPT} \leq 0,$$

where the equality holds since $e_t^{PB} = e_t^{OPT}$ for $1 \leq t \leq m-1$ and the last inequality follows from $t \in \mathcal{T}_H$. This proves, by $\Theta_s = \theta e_{s+m-1}$ for any period s , that for all t ,

$$\Theta_{[(t-m+1) \vee 1, t]}^{PB} \leq \Theta_{[(t-m+1) \vee 1, t]}^{OPT}.$$

As the above result holds for any of the disjoint intervals of \mathcal{T}_M , adding them up yields

$$\sum_{t \in \mathcal{T}_M} \Theta_t^{PB} \leq \sum_{t \in \mathcal{T}_M} \Theta_t^{OPT}. \quad (23)$$

Finally, since

$$\mathcal{T}_H \subset (\mathcal{T}_H \setminus \mathcal{T}_M) \cup \mathcal{T}_M \subset \{1, 2, \dots, T\},$$

we obtain, using (19) and (23), that

$$\sum_{t \in \mathcal{T}_H} \Theta_t^{PB} \leq \sum_{t \in \mathcal{T}_H \setminus \mathcal{T}_M} \Theta_t^{PB} + \sum_{t \in \mathcal{T}_M} \Theta_t^{PB} \leq \sum_{t \in \mathcal{T}_H \setminus \mathcal{T}_M} \Theta_t^{OPT} + \sum_{t \in \mathcal{T}_M} \Theta_t^{OPT} \leq \sum_{t=1}^T \Theta_t^{OPT}.$$

This completes the proof of Lemma 2 when $\alpha = 1$.

Q.E.D.

Note that for each perished unit ordered in periods $1, \dots, T$, it must stay in the system for exactly m periods. Thus, for any policy, we have the following inequality

$$mh \sum_{t=1}^T \Theta_t \leq \theta \sum_{t=1}^T H_t. \quad (24)$$

Combining this inequality with Lemmas 1 and 2 leads to the following result.

COROLLARY 1. *For each realization $f_T \in \mathcal{F}_T$, we have*

$$\sum_{t \in \mathcal{T}_H} (H_t^{PB} + \Theta_t^{PB}) \leq \left(1 + \frac{(m-2)h}{mh+\theta}\right) \sum_{t=1}^T (H_t^{OPT} + \Theta_t^{OPT}). \quad (25)$$

Proof. We apply Lemmas 1 and 2, and (24) to obtain

$$\begin{aligned} \sum_{t \in \mathcal{T}_H} (H_t^{PB} + \Theta_t^{PB}) &\leq \sum_{t=1}^T H_t^{OPT} + \frac{(m-2)h}{\theta} \sum_{t=1}^T \Theta_t^{OPT} + \sum_{t=1}^T \Theta_t^{OPT} \\ &= \sum_{t=1}^T (H_t^{OPT} + \Theta_t^{OPT}) + \frac{(m-2)h}{mh+\theta} \left(1 + \frac{mh}{\theta}\right) \sum_{t=1}^T \Theta_t^{OPT} \\ &\leq \left(1 + \frac{(m-2)h}{mh+\theta}\right) \sum_{t=1}^T (H_t^{OPT} + \Theta_t^{OPT}), \end{aligned}$$

thereby proving the corollary.

Q.E.D.

LEMMA 3. For each realization $f_T \in \mathcal{F}_T$, we have $\sum_{t \in \mathcal{T}_\Pi} \Pi_t^{PB} \leq \sum_{t=1}^T \Pi_t^{OPT}$.

Proof. From the definition of Π_t and \mathcal{T}_Π , we have

$$\sum_{t \in \mathcal{T}_\Pi} \Pi_t^{PB} = b \sum_{t \in \mathcal{T}_\Pi} \alpha^{t-1} (D_t - Y_t^{PB})^+ \leq b \sum_{t \in \mathcal{T}_\Pi} \alpha^{t-1} (D_t - Y_t^{OPT})^+ \leq \sum_{t=1}^T \Pi_t^{OPT},$$

where the first inequality holds since $Y_t^{OPT} < Y_t^{PB}$ when $t \in \mathcal{T}_\Pi$. **Q.E.D.**

With the preparations above, we are now ready to prove our first main result, i.e., Theorem 1.

Proof of Theorem 1. For each period $t = 1, \dots, T$, denote Z_t^{PB} as the conditional expected balanced cost by the PB policy in period t . That is,

$$Z_t^{PB} = \frac{mh+\theta}{2(m-1)h+\theta} \mathbb{E}[H_t^{PB} + \Theta_t^{PB} | F_t] = \mathbb{E}[\Pi_t^{PB} | F_t].$$

Note that Z_t^{PB} is a random variable before period t ; and in period t , $F_t = f_t$ is realized and its value is the expected balanced cost conditional on the observed information set f_t . Using the marginal cost accounting scheme and a standard argument of conditional expectations, we have

$$\begin{aligned} C(PB) &= \sum_{t=1}^T \mathbb{E}[H_t^{PB} + \Theta_t^{PB} + \Pi_t^{PB}] = \sum_{t=1}^T \mathbb{E}[\mathbb{E}[H_t^{PB} + \Theta_t^{PB} + \Pi_t^{PB} | F_t]] \\ &= \left(2 + \frac{(m-2)h}{mh+\theta}\right) \sum_{t=1}^T \mathbb{E}[Z_t^{PB}]. \end{aligned} \quad (26)$$

Applying Corollary 1, Lemma 3, and the fact that $\{t \in \mathcal{T}_H\}$ and $\{t \in \mathcal{T}_\Pi\}$ are completely determined by F_t , we obtain

$$\begin{aligned} C(OPT) &= \mathbb{E} \left[\sum_{t=1}^T (H_t^{OPT} + \Theta_t^{OPT}) + \sum_{t=1}^T \Pi_t^{OPT} \right] \\ &\geq \mathbb{E} \left[\frac{1}{1 + \frac{(m-2)h}{mh+\theta}} \sum_{t \in \mathcal{T}_H} (H_t^{PB} + \Theta_t^{PB}) + \sum_{t \in \mathcal{T}_\Pi} \Pi_t^{PB} \right] \\ &= \sum_{t=1}^T \mathbb{E} \left[\frac{1}{1 + \frac{(m-2)h}{mh+\theta}} \mathbf{1}(t \in \mathcal{T}_H) (H_t^{PB} + \Theta_t^{PB}) + \mathbf{1}(t \in \mathcal{T}_\Pi) \Pi_t^{PB} \right] \\ &= \sum_{t=1}^T \mathbb{E} \left[\mathbb{E} \left[\frac{1}{1 + \frac{(m-2)h}{mh+\theta}} \mathbf{1}(t \in \mathcal{T}_H) (H_t^{PB} + \Theta_t^{PB}) + \mathbf{1}(t \in \mathcal{T}_\Pi) \Pi_t^{PB} \mid F_t \right] \right] \\ &= \sum_{t=1}^T \mathbb{E} \left[\frac{1}{1 + \frac{(m-2)h}{mh+\theta}} \mathbf{1}(t \in \mathcal{T}_H) \mathbb{E}[H_t^{PB} + \Theta_t^{PB} \mid F_t] + \mathbf{1}(t \in \mathcal{T}_\Pi) \mathbb{E}[\Pi_t^{PB} \mid F_t] \right] \\ &= \sum_{t=1}^T \mathbb{E}[(\mathbf{1}(t \in \mathcal{T}_H) + \mathbf{1}(t \in \mathcal{T}_\Pi)) Z_t^{PB}] = \sum_{t=1}^T \mathbb{E}[Z_t^{PB}]. \end{aligned}$$

Thus, it follows from (26) that $C(PB) \leq \left(2 + \frac{(m-2)h}{mh+\theta}\right) C(OPT)$. The proof is complete. **Q.E.D.**

5.2. Analysis of DB Policy

We next analyze the DB policy for the case of independent and stochastically non-decreasing demand processes. Similar to the analysis for the PB policy, for each realization of D_1, \dots, D_T and W_1, \dots, W_T , we denote $\mathcal{T}_H = \{t \in [1, T] : Y_t^{OPT} \geq Y_t^{DB}\}$ and $\mathcal{T}_\Pi = \{t \in [1, T] : Y_t^{OPT} < Y_t^{DB}\}$. Then, we have the following result.

LEMMA 4. For each realization $f_T \in \mathcal{F}_T$, we have

$$\sum_{t \in \mathcal{T}_H} \left(\hat{H}_t^{DB} + \Theta_t^{DB} \right) + \sum_{t \in \mathcal{T}_\Pi} \Pi_t^{DB} \leq \sum_{t=1}^T \left(\hat{H}_t^{OPT} + \Theta_t^{OPT} + \Pi_t^{OPT} \right).$$

Proof. By the definition of \hat{H}_t and \mathcal{T}_H , we have

$$\sum_{t \in \mathcal{T}_H} \hat{H}_t^{DB} = h \sum_{t \in \mathcal{T}_H} \alpha^{t-1} (Y_t^{DB} - D_t)^+ \leq h \sum_{t \in \mathcal{T}_H} \alpha^{t-1} (Y_t^{OPT} - D_t)^+ \leq \sum_{t=1}^T \hat{H}_t^{OPT},$$

where the first inequality follows from $Y_t^{OPT} \geq Y_t^{DB}$ for $t \in \mathcal{T}_H$. Combining the above inequality with Lemmas 2 and 3, which can be shown to continue to hold under the DB policy, we obtain the desired result. **Q.E.D.**

LEMMA 5. Suppose D_1, \dots, D_T are independent and stochastically non-decreasing. For each period t and realization $f_t \in \mathcal{F}_t$, if $t \in \mathcal{T}_\Pi$, then $\mathbb{E}[\hat{H}_t^{DB} + \Theta_t^{DB} | f_t] = \mathbb{E}[\Pi_t^{DB} | f_t]$.

Proof. From the definition of the DB policy, to prove the lemma, we only need to prove that $\sum_{i=1}^{m-1} x_{t,i}^{DB} \leq S_t$. When the demands are independent and stochastically non-decreasing, based on our discussions in Section 4.2, it suffices to show that there exists one period $t' (\leq t)$ such that $\sum_{i=1}^{m-1} x_{t',i}^{DB} \leq S_{t'}$. This is clearly true if $t \in \mathcal{T}_\Pi$ (or equivalently $Y_t^{OPT} < Y_t^{DB}$), because otherwise by definition the DB policy will not place any order in periods $1, \dots, t$ and consequently we must have $Y_t^{OPT} \geq Y_t^{DB}$, leading to a contradiction. The proof is complete. **Q.E.D.**

We are now ready to prove our second main result, i.e., Theorem 2.

Proof of Theorem 2. For any policy P , define $\hat{C}(P) := \sum_{t=1}^T \mathbb{E}[\hat{H}_t^P + \Theta_t^P + \Pi_t^P]$. Then, it follows from (4) and (9) that $\mathcal{C}(P) = \hat{C}(P) + \sum_{t=1}^T \alpha^{t-1} \hat{c} \mathbb{E}[D_t]$. Thus, to prove $\mathcal{C}(DB) \leq 2\mathcal{C}(OPT)$, it suffices to show that $\hat{C}(DB) \leq 2\hat{C}(OPT)$. Using the marginal cost accounting scheme and a standard argument of conditional expectations, we have

$$\hat{C}(DB) = \sum_{t=1}^T \mathbb{E} \left[\hat{H}_t^{DB} + \Theta_t^{DB} + \Pi_t^{DB} \right]$$

$$\begin{aligned}
&= \sum_{t=1}^T \mathbb{E} \left[(\mathbf{1}(t \in \mathcal{T}_H) + \mathbf{1}(t \in \mathcal{T}_\Pi)) (\hat{H}_t^{DB} + \Theta_t^{DB} + \Pi_t^{DB}) \right] \\
&\leq \hat{C}(OPT) + \sum_{t=1}^T \mathbb{E} \left[\mathbf{1}(t \in \mathcal{T}_\Pi) (\hat{H}_t^{DB} + \Theta_t^{DB}) + \mathbf{1}(t \in \mathcal{T}_H) \Pi_t^{DB} \right] \\
&= \hat{C}(OPT) + \sum_{t=1}^T \mathbb{E} \left[\mathbb{E} \left[\mathbf{1}(t \in \mathcal{T}_\Pi) (\hat{H}_t^{DB} + \Theta_t^{DB}) + \mathbf{1}(t \in \mathcal{T}_H) \Pi_t^{DB} \mid F_t \right] \right] \\
&\leq \hat{C}(OPT) + \sum_{t=1}^T \mathbb{E} \left[\mathbb{E} \left[\mathbf{1}(t \in \mathcal{T}_\Pi) (\hat{H}_t^{DB} + \Theta_t^{DB}) + \mathbf{1}(t \in \mathcal{T}_H) (\hat{H}_t^{DB} + \Theta_t^{DB}) \mid F_t \right] \right] \\
&= \hat{C}(OPT) + \sum_{t=1}^T \mathbb{E} \left[\mathbb{E} \left[\mathbf{1}(t \in \mathcal{T}_\Pi) \Pi_t^{DB} + \mathbf{1}(t \in \mathcal{T}_H) (\hat{H}_t^{DB} + \Theta_t^{DB}) \mid F_t \right] \right] \\
&\leq 2\hat{C}(OPT),
\end{aligned}$$

where the first and last inequalities follow from Lemma 4, the second inequality holds since $\mathbb{E}[\Pi_t^{DB} \mid F_t] \leq \mathbb{E}[\hat{H}_t^{DB} + \Theta_t^{DB} \mid F_t]$ for each period t and any realization of F_t under the DB policy, and the fourth equality follows from Lemma 5. The proof of Theorem 2 is thus complete. **Q.E.D.**

6. Numerical Experiments

To test the empirical performance of our proposed policies, we have conducted an extensive numerical study. The numerical results show that our proposed policies perform consistently close to optimal for a large set of demand and parameter instances.

Parameterized policies. Similar to Levi and Shi (2013), we can slightly improve the performance of the approximation algorithms by employing an instance-dependent balancing parameter if the system parameters and demand process are stationary over time, and the planning horizon is long. The parameterized policies involve a balancing parameter β . Specifically, the parameterized proportional-balancing policy (PPB) computes the balancing quantity q_t^{PPB} that solves $\beta \mathbb{E}[H_t^{PPB}(q_t^{PPB}) + \Theta_t^{PPB}(q_t^{PPB}) \mid f_t] = \mathbb{E}[\Pi_t^{PPB}(q_t^{PPB}) \mid f_t]$. Similarly, the parameterized dual-balancing policy (PDB) computes the balancing quantity q_t^{PDB} that solves $\hat{\beta} \mathbb{E}[\hat{H}_t^{PDB}(q_t^{PDB}) + \Theta_t^{PDB}(q_t^{PDB})] = \mathbb{E}[\Pi_t^{PDB}(q_t^{PDB})]$. In addition to the PB and DB policies, we also report the empirical performance of the PPB and PDB policies when comparing with optimal policies.

Design of experiments. In our numerical experiments, we consider five demand settings (one independent and four correlated demand settings).

- (a) Independent and identically distributed (i.i.d.) demands;
- (b) ADI demands with two periods of advance demand information;

- (c) Autoregressive demands AR(1);
- (d) MMFE demands with two periods of forecast evolution;
- (e) Markov modulated demands with three states of the economy.

For the i.i.d. demand setting (a), we consider the lifetime $m = 2, 3, 4$ and 6 (see, e.g., [Hajjema et al. \(2005\)](#) for blood bank applications where platelet pools are the most expensive and most perishable blood product having a shelf life of four to six days). When $m = 2$ and 3, computing the exact optimal policies using dynamic programming is tractable. Thus, we compare the performance of our proposed policies directly with that of the optimal policies. In addition, we adopt the same set of numerical parameters as that in [Nahmias \(1976, 1977b\)](#), and also compare the performance of our proposed policies with their heuristics. When $m = 4$ and 6, computing the exact optimal policies becomes intractable (even for this i.i.d. demand case); thus we compare our policies with two other effective policies in [Nahmias \(1976, 1977b\)](#). (The key idea behind these heuristics in [Nahmias \(1976, 1977b\)](#) is to collapse the state space into a single scalar, which has also been used in [Li et al. \(2009\)](#) and [Chen et al. \(2014\)](#) for other perishable inventory systems.) For correlated demand settings (b) to (e), we are not aware of any heuristic policies in the literature, thus we only consider the lifetime $m = 2$ and 3. Following the numerical studies in the literature on perishable inventory systems, we assume for all testing instances that the system starts empty in period 1, the unit holding cost \hat{h} is normalized to 1, and the discounted factor is $\alpha = 0.95$.

Performance metrics. We use two types of performance metrics in our numerical study. First, when the product lifetime $m = 2$ or 3, we are able to compare the performance of our proposed policies with that of an optimal policy. From (9) and (10), the cost ratio is

$$\frac{\mathcal{C}(P)}{\mathcal{C}(OPT)} = \frac{C(P) + \sum_{t=1}^T \alpha^{t-1} \hat{c} \mathbf{E}[D_t]}{C(OPT) + \sum_{t=1}^T \alpha^{t-1} \hat{c} \mathbf{E}[D_t]}.$$

We define the *performance error* of an approximation policy P as the percentage of increase in the total cost of this policy over the planning horizon compared to the optimal total cost, i.e.,

$$\% \text{ error} = \left(\frac{\mathcal{C}(P)}{\mathcal{C}(OPT)} - 1 \right) \times 100\%.$$

Second, when the product lifetime $m = 4$ or 6, since computing the exact optimal solution using dynamic programming is intractable, we compare the performance of our policies against those of [Nahmias \(1976, 1977b\)](#). Denote the heuristic algorithms in [Nahmias \(1976\)](#) and [Nahmias \(1977b\)](#)

by N_1 and N_2 , respectively. We define the *performance ratio* of an approximation policy P as the ratio of Nahmias' minimum cost to the cost of P , i.e.,

$$r(P) = \frac{\min\{\mathcal{C}(N_1), \mathcal{C}(N_2)\}}{\mathcal{C}(P)}.$$

Demand setting (a). When the product lifetime $m = 2$ and 3 , we adopt the same set of parameters as that in Nahmias (1976, 1977b) for our numerical test and directly use the optimal costs reported in those papers. More specifically, the planning horizon is $T = 50$ periods, the ordering cost $\hat{c} \in \{0, 5, 10\}$, the backlogging cost $\hat{b} \in \{5, 10\}$, and the outdating cost $\hat{\theta} \in \{0, 5, 10\}$. Same as Nahmias (1976, 1977b), we also test two demand distributions, i.e., exponential distribution and Erlang-2 distribution, both with mean 10. The numerical results are summarized in Table EC.1 in the online Appendix. The empirical performance error of the DB policy does not exceed 1.41% in all test cases, with an average error of 0.84% (resp., 0.80%) under exponential demands and $m = 2$ (resp., $m = 3$), and with an average error of 0.25% (resp., 0.48%) under Erlang-2 demands and $m = 2$ (resp., $m = 3$). Hence, the average performance error is uniformly within 1%. Similar to the numerical results in Nahmias (1976, 1977b), the approximation algorithms perform better under Erlang-2 demands than exponential demands due to a smaller coefficient of variation. Furthermore, if the balancing parameter is optimized (we search for β and $\hat{\beta}$ over $\{0.5, 0.6, 0.7, \dots, 1.8, 1.9, 2\}$), then the performance error of the PPB policy does not exceed 0.92%, with an average performance error of 0.28%; and the performance error of the PDB policy does not exceed 0.60%, with an average performance error of 0.20%.

When the product lifetime $m = 4$ and 6 , computing the exact optimal policies is intractable. Thus, we compare the performance of our proposed policies with that of Nahmias (1976, 1977b), and report the performance ratios $r(\text{DB})$ and $r(\text{PB})$. We consider the problem with $T = 50$, $\hat{c} = 0$, $\hat{b} \in \{5, 10, 15\}$ and $\hat{\theta} \in \{10, 50, 100\}$. We test Erlang-2, exponential, and hyper-exponential demands with mean 10. The performance ratios are summarized in Table EC.2 in the online Appendix. Our numerical results show that the PB policy outperforms Nahmias' policies in 54% of the cases, and the DB policy outperforms Nahmias' policies in 65% of the cases. The PB and DB policies perform 1.09% and 1.34% better on average than Nahmias' policies, respectively. From the numerical results, we can also see that the PB and DB policies have similar performance ratios, which improve as the outdating cost becomes more significant. Our explanation for this finding is as follows: When the frequency of outdating is low, the problem almost reduces to a non-perishable inventory model, for which a myopic policy is optimal in the case of i.i.d. demand. Nahmias' policies, using modified single period cost, yield near-optimal solution when outdating frequency is low.

In summary, under i.i.d. demands, for short product lifetime $m = 2$ and 3 , the numerical results of our proposed policies are comparable to those reported in Nahmias (1976, 1977b), as both ours and their policies are very close to optimal. For longer product lifetime $m = 4$ and 6 , the overall performance of our proposed policies is also comparable with those of Nahmias (1976, 1977b) and it improves as the frequency of outdated increases.

Since we are more interested in the performances of our policies under correlated demand processes, we conduct more comprehensive studies for this case. As mentioned earlier, when the demand process is correlated over time, the computation of exact optimal solution is intractable for reasonable problem sizes. Thus, in order to compare with the optimal cost under demand settings (b) to (e), we consider a planning horizon $T = 20$ periods and product lifetime $m = 2$ or 3 periods (the same as the majority of the literature under i.i.d. demands). More specifically, we consider $m = 2$ and 3 for the setting (e) and $m = 2$ for the other settings. The cost parameters for each demand class are $\hat{c} \in \{0, 5, 10\}$, $\hat{b} \in \{5, 10, 15\}$, and $\hat{\theta} \in \{0, 5, 10, 15\}$. For these instances, the optimal costs are computed using dynamic programming via backward induction.

Demand setting (b). For demand processes with *advance demand information* (ADI), we adopt a model proposed in Gallego and Özer (2001). We assume that customers can place orders two periods ahead. Thus, in each period t , a demand vector $(D_{t,t}, D_{t,t+1}, D_{t,t+2})$ is received, where $D_{t,s}$ is the order placed in period t for period $s \geq t$. The total demand for period t is $D_t = D_{t-2,t} + D_{t-1,t} + D_{t,t}$. We tested the cases for which each entry $D_{t,s}$ follows an exponential distribution, an Erlang-2 distribution, or a truncated normal distribution with coefficient of variation (cv) being 0.1, 0.3, or 0.5. The mean value for each $D_{t,s}$ is 3, thus the average demand for each period is 9.

To report the numerical results under ADI, we group the instances as follows. The ordering costs are L ($\hat{c} = 0$), M ($\hat{c} = 5$), and H ($\hat{c} = 10$); the outdated costs are L ($\hat{\theta} \in \{0, 5\}$), and H ($\hat{\theta} \in \{10, 15\}$). The first five rows of Table EC.3 in the online Appendix report the numerical results under ADI. The row corresponds to the demand processes (one for exponential demands, one for Erlang-2 demands, and three for truncated normal demands with different cv's). For each pair $(\hat{c}, \hat{\theta})$, we choose between one value of \hat{c} , two values of $\hat{\theta}$, and three values of $\hat{b} \in \{5, 10, 15\}$, giving 6 combinations for each pair $(\hat{b}, \hat{\theta})$. The maximum and the average performance errors for both PB and PPB policies are reported in Table EC.3. Among all the test instances under ADI, the average error of the PB (resp., PPB) policy is 0.45% (resp., 0.32%), and the maximum performance error of the PB (resp., PPB) policy is 2.65% (resp., 1.79%).

Demand setting (c). For the *autoregressive* demand model, we consider an AR(1) process $D_t = D_{t-1} + \epsilon_t$ with $D_0 = 10$, where the perturbation term ϵ_t follows a normal distribution with mean

0 and variance 1. The numerical results are reported in the sixth row of Table EC.3 in the online Appendix. The average performance error of the PB (resp., PPB) policy is 0.66% (resp., 0.40%) while the maximum performance error of the PB (resp., PPB) policy among all test instances is 2.21% (resp., 1.88%).

Demand setting (d). For demand processes of *Martingale model of forecast evolution* (MMFE), we assume that the system in each period t updates its forecast for the next-two-period demands $(D_{t,t+1}, D_{t,t+2})$. The true demand in period t is given by

$$D_t = D_{t-1,t} + \epsilon_{t,1} = D_{t-2,t} + \epsilon_{t-1,2} + \epsilon_{t-1,1},$$

where $\epsilon_{t,i}$ follows a normal distribution with mean 0 and variance 1 for all t and $i = 1, 2$, and $D_{t-2,t}$ follows a normal distribution with mean 10 and variance 10 for all t . The numerical results are reported in the last row of Table EC.3 in the online Appendix. The average performance error of the PB (resp., PPB) policy is 0.61% (resp., 0.34%); and the maximum performance error of the PB (resp., PPB) policy for all the test instances is 2.39% (resp., 1.50%).

Demand setting (e). The *Markov modulated demand process* (MMDP) is governed by the state of the economy: poor (1), fair (2), and good (3). If the state of the economy in period t is i ($i = 1, 2, 3$), then the demand in period t is iD_t , where D_t has mean 10 and follows one of the following distributions: exponential, Erlang-2, and truncated normal with coefficient of variation being 0.1, 0.3, or 0.5. We assume that the state of the economy follows a Markov chain with transition probabilities

$$p_{11} = 0.6, p_{12} = 0.3, p_{13} = 0.1, p_{21} = 0.4, p_{22} = 0.2, p_{23} = 0.4, p_{31} = 0.1, p_{32} = 0.3, \text{ and } p_{33} = 0.6.$$

Note that this Markov chain is stochastically monotone, i.e., the state of the economy in the next period is stochastically non-decreasing in the state of the economy in the current period.

The performance of our proposed policies under MMDP is reported in Table EC.4 in the online Appendix. The first column specifies the product lifetimes and demand processes. Similar to Table EC.3, \hat{c} takes three possible values, denoted by L, M and H, and $\hat{\theta}$ is divided into two groups, with L standing for $\{0, 5\}$ and H standing for $\{10, 15\}$. Thus, for each demand process and each pair $(\hat{c}, \hat{\theta})$, there are 6 combinations of $\hat{\theta}$ and \hat{b} . The numerical results for both PB and PPB policies are reported in Table EC.4. The average performance error of the PB (resp., PPB) policy among all test instances is 0.71% (resp., 0.44%), and the maximum performance error of the PB (resp., PPB) policy is 2.94% (resp., 1.88%).

Computation time. For the instances with product lifetime $m = 2$ or 3 , the optimal policies were computed using dynamic programming. For ADI and AR(1) demand settings, the average computation times for one instance are 784 seconds and 152 seconds, respectively. For MMFE, the average computation time for one instance is 1148 seconds. For MMDP, the average computation time for one instance is 94 seconds for $m = 2$ and 435 seconds for $m = 3$.

In contrast, our proposed policies (including PB, DB, PPB, and PDB policies) do not require any recursive computation and the ordering decisions can be computed in an *online* manner. In period t , the main computation effort lies in the computation of expectation $E[H_t(q_t) + \Theta_t(q_t) | f_t]$ via (6) and (7). The PB policy takes on average 0.05 second to find a decision for $m = 2$ and 0.6 second for $m = 3$. For longer lifetimes, the PB policy takes on average 1.52 seconds to make a decision for $m = 4$ and 1.94 seconds for $m = 6$. We note that the heuristics proposed in Nahmias (1976, 1977b) are also very efficient as they ignore most of the inventory and future demand information when making a decision. To compute mathematical expectations, both in our procedure and in that of Nahmias (1976, 1977b), we use Monte Carlo simulation with 10000 sample paths. All computations were done using Matlab R2013a on a desktop computer with an Inter Core I7-3770 3.40GHz CPU.

7. Conclusion and Discussions

It is well known that the optimal control policy for perishable inventory systems is complicated and computationally challenging. In this paper, we develop two approximation algorithms for perishable inventory systems with worst-case performance guarantees. For systems with independent and stochastically non-decreasing demand processes, we propose a dual-balancing (DB) policy that admits a worst-case performance guarantee of 2; and for systems with arbitrarily correlated demand processes, we propose a proportional-balancing (PB) policy that admits a worst-case performance guarantee between 2 and 3 (2 when the product lifetime is 2). Both policies are easy to compute and implement. More importantly, our numerical study shows that they perform consistently close to optimal for all the tested instances for which we are able to compute their optimal policies: the maximum performance error for the PB policy among all tested instances is below 3%, while the maximum error for the parameterized proportional-balancing (PPB) policy among all tested instances is below 2%. In addition, the average performance errors of the PB policy and the PPB policy among all the correlated demand processes tested, including ADI, AR(1), MMFE, and MMDP, are 0.625% and 0.397%, respectively. For the instances where we are unable to compute the optimal policies, we compare the performances of our approximation algorithms with those of the heuristic policies reported in the literature (Nahmias (1976, 1977b)), and our numerical results

show that the performance of our policies is comparable with those of Nahmias (1976, 1977b) and it improves as the frequency of outdateding increases.

Our analysis and results can be extended along two important directions. First, all of our results continue to hold under the lost-sales models (where b would be interpreted as the unit lost-sales cost). When unsatisfied demand is lost, $x_{t,m-1}$ would represent the on-hand inventory level of products whose remaining lifetime is $m-1$ periods and it is nonnegative. Hence, for the lost-sales models we need to modify the state transition on $x_{t+1,m-1}$ in (2) to

$$x_{t+1,m-1} = \left(q_t - \left(D_t - \sum_{i=1}^{m-1} x_{t,i} \right)^+ \right)^+,$$

and $(D_t - \sum_{i=1}^{m-1} x_{t,i} - q_t)^+$ is the number of lost sales in period t . Interestingly, this difference does not affect our analysis, and all of our results on the PB and DB policies continue to hold under the lost-sales setting. We note that Nandakumar and Morton (1993) proposed a myopic heuristic policy for perishable inventory systems with lost sales.

Second, we can also extend our analysis and results to the models with random non-crossover lifetimes, i.e., the product lifetimes are random but the inventories outdate in the same order in which they enter the system. Nahmias (1977c) first studied perishable inventory systems with random non-crossover lifetimes; and we refer the interested reader to this article for more details. For $t = 1, \dots, T$, suppose the lifetime of an order placed in period t is an integer random variable M_t with support $[\underline{m}, \bar{m}]$. Then, the non-crossover property requires that $M_t + t \leq M_s + s$ for any $t \leq s$. For these models, we can similarly define the nested marginal cost accounting scheme and then the PB and DB policies. The marginal holding and outdateding costs under random product lifetimes, however, have to be defined as the expected marginal holding and outdateding costs over all possible realizations of random lifetimes. The marginal backlogging cost remains essentially the same as that for the fixed product lifetime case, i.e., the one given in (8) but with m replaced by \bar{m} . Following similar analysis to that for the fixed product lifetime model, we can show that the PB policy has a worst-case performance guarantee of $\left(2 + \frac{(\bar{m}-2)h}{mh+\theta} \right)$ for systems with a general demand process and the DB policy has a worst-case performance guarantee of 2 for systems with an independent and stochastically non-decreasing demand process. The main differences in the analysis are twofold. First, the outdated units e_t in period t may include products that were ordered in different periods. However, since successive orders outdate in the same sequence, the identity $Y_{t,s} = (Y_t - D_{[t,s]} - e_{[t,s]})^+$ continues to hold for any $t \leq s$. Second, we need to introduce another notation $e_{t,s}$ for any $t \leq s$, recording the part of outdated products in period s that are ordered

in period t or earlier, and make use of the new identity $e_{t,t+M_t-1} = (Y_t - D_{[t,t+M_t-1]} - e_{[t,t+M_t-1]})^+$ which is implied by the non-crossover property.

There are several interesting directions for future research. First, we assume in this paper that the order lead time is zero. This assumption may not hold in practice; hence it would be interesting to investigate the models with a positive order lead time. We note that [Levi et al. \(2008a\)](#) develop an approximation algorithm for non-perishable lost-sales inventory systems with a positive lead time, which admits a worst-case performance guarantee under some strong conditions on demand processes. Hence, it is conceivable that it is more challenging to develop approximation algorithms for perishable inventory systems with a positive lead time. The second direction is to consider capacitated systems. In this paper we assume that there is no constraint on the order quantities. In many applications, however, there is a capacity constraint and the order quantity cannot exceed the capacity in each period. For non-perishable products, [Levi et al. \(2008b\)](#) develop an approximation algorithm for capacitated systems using the idea of forced backlogging cost. It would be interesting to investigate whether a similar idea can be applied to study capacitated perishable inventory systems. Third, it is often observed in practice that there are multiple types of demands. For example, in blood bank applications, it is known that there are different types of demands, some have stronger requirements on the freshness of blood than the others (see, e.g., [Karaesmen et al. \(2011\)](#)). Hence, developing approximation algorithms for perishable inventory systems with multiple types of demands is also an important research topic. Finally, in this paper we assume that the product can perish but cannot be depleted. In some applications of fresh foods, depletion can also be a decision, i.e., the system can intentionally discard inventories before they perish. The methodology developed in this paper does not seem to work under this setting, since the marginal costs of a decision made in one period are affected by decisions in the future periods. Therefore, an interesting open question is how to develop other types of approximation algorithms with worst-case performance guarantees for such models.

Supplemental Material

An electronic companion to this paper is available at <http://or.journal.informs.org/>.

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Endnotes

1. <http://www.fmi.org/docs/facts-figures/grocerydept.pdf>, accessed on March 19, 2015.
2. http://www.power-id.com/data/pdf/powertmp_white_paper.pdf, accessed on March 19, 2015.

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Electronic Companion to “Approximation Algorithms for Perishable Inventory Systems”

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This e-companion contains two sections. Section EC.1 provides the technical proofs of Proposition 1, Proposition 2, Lemma 1, and Lemma 2; and Section EC. 2 provides the computational complexity of PB and DB policies and the numerical tables.

EC.1. Technical Proofs

EC.1.1. Proof of Proposition 1

Recall that the amount of outdated products in period t is

$$e_t := (x_{t,1} - D_t)^+.$$

The starting inventory level in period $t + 1$ is equal to the ending inventory level in period t minus the demand and also the outdated units in period t , i.e.,

$$\sum_{i=1}^{m-1} x_{t+1,i} = Y_t - D_t - e_t. \tag{EC.1}$$

Hence using the relationship

$$q_t = Y_t - \sum_{i=1}^{m-1} x_{t,i} = (Y_t - D_t)^+ - (D_t - Y_t)^+ + D_t - \sum_{i=1}^{m-1} x_{t,i}, \tag{EC.2}$$

we can rewrite the cost $\mathcal{C}(P)$ in (3) as

$$\begin{aligned}
\mathcal{C}(P) &= \mathbb{E} \left[\sum_{t=1}^T \alpha^{t-1} \left(\hat{c}q_t + \hat{h}(Y_t - D_t)^+ + \hat{b}(D_t - Y_t)^+ + \hat{\theta}e_t \right) - \alpha^T \hat{c} \sum_{i=1}^{m-1} X_{T+1,i} \right] \\
&= \mathbb{E} \left[\sum_{t=1}^T \alpha^{t-1} \left(\hat{c}D_t - \hat{c} \sum_{i=1}^{m-1} x_{t,i} + (\hat{h} + \hat{c})(Y_t - D_t)^+ + (\hat{b} - \hat{c})(D_t - Y_t)^+ + \hat{\theta}e_t \right) - \alpha^T \hat{c} \sum_{i=1}^{m-1} X_{T+1,i} \right] \\
&= \mathbb{E} \left[\sum_{t=1}^T \alpha^{t-1} \left(-\alpha \hat{c} \sum_{i=1}^{m-1} x_{t+1,i} + (\hat{h} + \hat{c})(Y_t - D_t)^+ + (\hat{b} - \hat{c})(D_t - Y_t)^+ + \hat{\theta}e_t \right) \right] + R \\
&= \mathbb{E} \left[\sum_{t=1}^T \alpha^{t-1} \left(-\alpha \hat{c}(Y_t - D_t) + (\hat{h} + \hat{c})(Y_t - D_t)^+ + (\hat{b} - \hat{c})(D_t - Y_t)^+ + (\hat{\theta} + \alpha \hat{c})e_t \right) \right] + R \\
&= \mathbb{E} \left[\sum_{t=1}^T \alpha^{t-1} \left((\hat{h} + \hat{c} - \alpha \hat{c})(Y_t - D_t)^+ + (\hat{b} - \hat{c} + \alpha \hat{c})(D_t - Y_t)^+ + (\hat{\theta} + \alpha \hat{c})e_t \right) \right] + R \\
&= \mathbb{E} \left[\sum_{t=1}^T \alpha^{t-1} \left(h(Y_t - D_t)^+ + b(D_t - Y_t)^+ + \theta e_t \right) \right] + R,
\end{aligned}$$

where the second equality follows from (EC.2), the fourth equality follows from (EC.1), $h = \hat{h} + (1 - \alpha)\hat{c}$, $b = \hat{b} - (1 - \alpha)\hat{c}$, $\theta = \hat{\theta} + \alpha\hat{c}$, and

$$R = -\hat{c} \sum_{i=1}^{m-1} x_{1,i} + \sum_{t=1}^T \alpha^{t-1} \hat{c} \mathbb{E}[D_t] = \sum_{t=1}^T \alpha^{t-1} \hat{c} \mathbb{E}[D_t].$$

Note that we have used the assumption that the inventory system is initially empty, i.e., $x_{1,i} = 0$, for all $i = 1, \dots, m-1$. The proof is complete. **Q.E.D.**

EC.1.2. Proof of Proposition 2

The marginal costs are functions of \mathbf{x} and q , thus in this proof we write them as $H_t^{PB}(\mathbf{x}, q)$, $\Theta_t^{PB}(\mathbf{x}, q)$, and $\Pi_t^{PB}(\mathbf{x}, q)$. Define

$$h_t(\mathbf{x}, q) = \frac{mh + \theta}{2(m-1)h + \theta} \mathbb{E}[H_t^{PB}(\mathbf{x}, q) + \Theta_t^{PB}(\mathbf{x}, q) \mid f_t] - \mathbb{E}[\Pi_t^{PB}(\mathbf{x}, q) \mid f_t].$$

Then, q_t^{PB} is the solution of the equation $h_t(\mathbf{x}, q) = 0$. First, one can easily verify that $h_t(\mathbf{x}, q)$ is increasing in \mathbf{x} and in q . Thus, it follows that q_t^{PB} is decreasing in x_i , $i = 1, \dots, m-1$. Next, we argue that to complete the proof it suffices to show that $h_t(x_1, x_2 - x_1, \dots, x_{m-1} - x_{m-2}, \hat{q} - x_{m-1})$ is decreasing in (x_1, \dots, x_{m-1}) while increasing in \hat{q} . Now suppose these results are true. Define $\hat{q}^*(x_1, \dots, x_{m-1})$ as the solution of the equation $h_t(x_1, x_2 - x_1, \dots, x_{m-1} - x_{m-2}, \hat{q} - x_{m-1}) = 0$.

Then, it follows from our assumption that $\hat{q}^*(x_1, \dots, x_{m-1})$ is increasing in (x_1, \dots, x_{m-1}) . From the definition of q_t^{PB} , we must have, if we write the dependency on state variables explicit,

$$q_t^{PB}(x_1, x_2 - x_1, \dots, x_{m-1} - x_{m-2}) = \hat{q}^*(x_1, \dots, x_{m-1}) - x_{m-1}.$$

After taking derivative with respect to x_i , $i = 1, \dots, m-1$, we obtain

$$-1 \leq \frac{\partial q_t^{PB}}{\partial x_{m-1}} \leq \frac{\partial q_t^{PB}}{\partial x_{m-2}} \leq \dots \leq \frac{\partial q_t^{PB}}{\partial x_1}.$$

Now we prove that $h_t(x_1, x_2 - x_1, \dots, x_{m-1} - x_{m-2}, \hat{q} - x_{m-1})$ is decreasing in (x_1, \dots, x_{m-1}) while increasing in \hat{q} . To this end, it suffices to show that, for any realizations of demands D_t, \dots, D_T , $H_t^{PB}(x_1, x_2 - x_1, \dots, x_{m-1} - x_{m-2}, \hat{q} - x_{m-1})$, $\Theta_t^{PB}(x_1, x_2 - x_1, \dots, x_{m-1} - x_{m-2}, \hat{q} - x_{m-1})$ and $-\Pi_t^{PB}(x_1, x_2 - x_1, \dots, x_{m-1} - x_{m-2}, \hat{q} - x_{m-1})$ all satisfy the above properties. First, notice that

$$-\Pi_t^{PB}(x_1, x_2 - x_1, \dots, x_{m-1} - x_{m-2}, \hat{q} - x_{m-1}) = -\alpha^{t-1}b(D_t - \hat{q})^+.$$

Thus the desired results are trivially true.

To show that $H_t^{PB}(\cdot)$ and $\Theta_t^{PB}(\cdot)$ also satisfy the desired results, we first prove by induction that $B_t(x_1, x_2 - x_1, \dots, x_{m-1} - x_{m-2}, i)$ is increasing in (x_1, \dots, x_i) and independent in $(x_{i+1}, \dots, x_{m-1})$, $i = 1, \dots, m-1$. When $i = 1$, the results are obviously true. Now suppose the results hold for i . For $i + 1$, we have

$$\begin{aligned} & B_t(x_1, x_2 - x_1, \dots, x_{m-1} - x_{m-2}, i + 1) \\ &= \max \{ x_{i+1} - D_{[t, t+i-1]}, B_t(x_1, x_2 - x_1, \dots, x_{m-1} - x_{m-2}, i) \}. \end{aligned}$$

Then, it follows from the above expression that $B_t(x_1, x_2 - x_1, \dots, x_{m-1} - x_{m-2}, i + 1)$ is increasing in (x_1, \dots, x_{i+1}) and independent in $(x_{i+2}, \dots, x_{m-1})$. Hence, by induction, the desired results on $B_t(\cdot, i)$ hold for $i = 1, \dots, m-1$.

Now consider $\Theta_t^{PB}(\cdot)$. Since

$$\begin{aligned} & \Theta_t^{PB}(x_1, x_2 - x_1, \dots, x_{m-1} - x_{m-2}, \hat{q} - x_{m-1}) \\ &= \alpha^{t+m-2} \theta (\hat{q} - B_t(x_1, x_2 - x_1, \dots, x_{m-1} - x_{m-2}, m-1) - D_{[t, t+m-1]})^+, \end{aligned}$$

then $\Theta_t^{PB}(x_1, x_2 - x_1, \dots, x_{m-1} - x_{m-2}, \hat{q} - x_{m-1})$ is decreasing in (x_1, \dots, x_{m-1}) while increasing in \hat{q} . Similarly, we can prove that the desired results hold for $H_t^{PB}(x_1, x_2 - x_1, \dots, x_{m-1} - x_{m-2}, \hat{q} - x_{m-1})$.

The proof is complete. **Q.E.D.**

EC.1.3. Proof of Lemma 1

For brevity, we only prove the result for the case when the discount factor $\alpha = 1$. The general case with $\alpha \in [0, 1]$ can be proved similarly. For any $t = 1, \dots, T$, denote e_t as the amount of perished products in period t . Since the lifetime of the products is m , we have $\Theta_t = \theta e_{t+m-1}$. In addition, for any t and $s \geq t \geq 0$, denote $Y_{t,s}$ as the part of on-hand inventory at the beginning of period s which is ordered in period t or earlier. Note that $Y_{t,s} - Y_{0,s}$ is the part of on-hand inventory at the beginning of period s which is ordered in between periods 1 and t (in the case of 0 initial inventory level, $Y_{0,s} \equiv 0$). Thus, $Y_{t,s} \geq Y_{0,s}$. For convenience, we denote $D_0 = e_0 = 0$. From the definition of $Y_{t,s}$, it is readily verified that

$$Y_{t,s} = (Y_t - D_{[t,s]} - e_{[t,s]})^+, \quad s = t, t+1, \dots, t+m-1, \quad (\text{EC.3})$$

and $Y_{t,s} = 0$ when $s \geq t+m$.

For any period $t = 1, \dots, T$, we define the notation $R(t)$ as follows: if the set $\{s \in \mathcal{T}_H : s > t\}$ is not empty, then $R(t) := \min\{s \in \mathcal{T}_H : s > t\}$; otherwise, $R(t) := T+1$. In addition, for any $s \geq 1$, denote \tilde{H}_s as the part of holding cost incurred in period s which is associated with the products ordered in periods $\{t : t \in \mathcal{T}_H, t \leq s\}$. Since the lifetime of the products is m , all products ordered in period t or earlier will leave the system at the end of period $t+m-1$. Then, it follows that for any $t \in \mathcal{T}_H$, $\tilde{H}_s = 0$ when $t+m \leq s \leq R(t)-1$. Consequently, by the definitions of H_t , \tilde{H}_s , and $R(t)$, we have

$$\sum_{t \in \mathcal{T}_H} H_t = \sum_{t \in \mathcal{T}_H} \sum_{s=t}^{R(t)-1} \tilde{H}_s = \sum_{t \in \mathcal{T}_H} \sum_{s=t}^{(t+m) \wedge R(t)-1} \tilde{H}_s. \quad (\text{EC.4})$$

For each period $s \in [t, R(t)-1]$, the amount of leftover inventories (after satisfying the demand D_s but before product outdating) associated with the orders in periods $1, \dots, t$ can be expressed as $(Y_{t,s} - Y_{0,s} - (D_s - Y_{0,s})^+)^+$, which also equals $(Y_{t,s} - D_s)^+ - (Y_{0,s} - D_s)^+$. It is clear that these leftover inventories consist of two parts: 1) the leftover inventories associated with the orders in periods $\{t : t \in \mathcal{T}_H, t \leq s\}$; and 2) the leftover inventories associated with the orders in periods $\{t : t \in \mathcal{T}_H, t \leq s\}$. Note that the outdating products during periods $s, \dots, t+m-1$ only come from the above leftover inventories. Thus, the total outdating products in periods $\{t' : t' = s, \dots, t+m-1, t' - m + 1 \in \mathcal{T}_H\}$ are less than or equal to the leftover inventories associated with the orders in periods $\{t : t \in \mathcal{T}_H, t \leq s\}$. For convenience, for $t = m, \dots, T$, we denote

$$e_t^\Pi = \begin{cases} e_t, & \text{if } t - m + 1 \in \mathcal{T}_H; \\ 0, & \text{otherwise.} \end{cases}$$

In addition, we denote $e_t^H = e_t - e_t^\Pi$. Then, we have

$$\tilde{H}_s \leq h(Y_{t,s} - D_s)^+ - h(Y_{0,s} - D_s)^+ - h e_{[s,t+m-1]}^\Pi. \quad (\text{EC.5})$$

Combining (EC.4) and (EC.5), we obtain

$$\sum_{t \in \mathcal{T}_H} H_t \leq h \sum_{t \in \mathcal{T}_H} \sum_{s=t}^{(t+m) \wedge R(t)-1} ((Y_{t,s} - D_s)^+ - (Y_{0,s} - D_s)^+ - e_{[s,t+m-1]}^\Pi). \quad (\text{EC.6})$$

On the other hand, for any $t < s$, since $Y_s \geq Y_{t,s} + q_s$, it is clear that $(Y_s - D_s)^+ \geq (Y_{t,s} - D_s)^+ + e_{s+m-1}$. Therefore, we have

$$\begin{aligned} \sum_{t=1}^T H_t &= h \sum_{t=1}^T ((Y_t - D_t)^+ - (Y_{0,t} - D_t)^+) \geq h \sum_{t \in \mathcal{T}_H} \sum_{s=t}^{R(t)-1} ((Y_{t,s} - D_s)^+ - (Y_{0,s} - D_s)^+) + h \sum_{s \in \mathcal{T}_\Pi} e_{s+m-1} \\ &\geq h \sum_{t \in \mathcal{T}_H} \sum_{s=t}^{(t+m) \wedge R(t)-1} ((Y_{t,s} - D_s)^+ - (Y_{0,s} - D_s)^+) + h \sum_{t=m}^T e_t^\Pi. \end{aligned} \quad (\text{EC.7})$$

Note that $Y_{0,s}$ is the part of on-hand inventory at the beginning of period s which is ordered before period 1, it follows from the FIFO issuing policy that $Y_{0,s}$ is independent of the ordering policy. Hence,

$$\sum_{t \in \mathcal{T}_H} \sum_{s=t}^{(t+m) \wedge R(t)-1} (Y_{0,s}^{PB} - D_s)^+ = \sum_{t \in \mathcal{T}_H} \sum_{s=t}^{(t+m) \wedge R(t)-1} (Y_{0,s}^{OPT} - D_s)^+. \quad (\text{EC.8})$$

Combining (EC.6), (EC.7), and (EC.8), we obtain

$$\begin{aligned} \sum_{t \in \mathcal{T}_H} H_t^{PB} - \sum_{t=1}^T H_t^{OPT} &\leq h \sum_{t \in \mathcal{T}_H} \sum_{s=t}^{(t+m) \wedge R(t)-1} \left((Y_{t,s}^{PB} - D_s)^+ - (Y_{t,s}^{OPT} - D_s)^+ - e_{[s,t+m-1]}^{PB,\Pi} \right) - h \sum_{t=m}^T e_t^{OPT,\Pi} \\ &\leq h \sum_{t \in \mathcal{T}_H} \left((Y_t^{PB} - D_t)^+ - (Y_t^{OPT} - D_t)^+ - e_{[t,t+m-1]}^{PB,\Pi} \right) - h \sum_{t=m}^T e_t^{OPT,\Pi} \\ &\quad + h \sum_{t \in \mathcal{T}_H} \sum_{s=t+1}^{(t+m) \wedge R(t)-1} (e_{[t,s]}^{OPT} - e_{[t,s]}^{PB})^+, \end{aligned} \quad (\text{EC.9})$$

where the second inequality holds because

$$(Y_{t,s}^{PB} - D_s)^+ - (Y_{t,s}^{OPT} - D_s)^+ \leq (Y_{t,s}^{PB} - Y_{t,s}^{OPT})^+ = (Y_t^{PB} - Y_t^{OPT} - e_{[t,s]}^{PB} + e_{[t,s]}^{OPT})^+ \leq (e_{[t,s]}^{OPT} - e_{[t,s]}^{PB})^+,$$

where the equality is from (14) and the second inequality holds since $Y_t^{PB} \leq Y_t^{OPT}$ when $t \in \mathcal{T}_H$.

For convenience, we denote \hat{e}_t as follows: when $1 \leq t \leq m-1$, $\hat{e}_t := 0$; and when $m \leq t \leq T$, $\hat{e}_t := e_t$. Since inventories are consumed in a first-in-first-out manner, it is seen that $e_t^{PB} = e_t^{OPT}$ for $1 \leq t \leq m-1$ for any realization of the demands. Then, for $1 \leq t \leq T$, $e_t^{OPT} - e_t^{PB} = \hat{e}_t^{OPT} - \hat{e}_t^{PB}$.

Thus, we have

$$\begin{aligned}
& \sum_{t \in \mathcal{T}_H} \sum_{s=t+1}^{(t+m) \wedge R(t)-1} (e_{[t,s]}^{OPT} - e_{[t,s]}^{PB})^+ = \sum_{t \in \mathcal{T}_H} \sum_{s=t+1}^{(t+m) \wedge R(t)-1} (\hat{e}_{[t,s]}^{OPT} - \hat{e}_{[t,s]}^{PB})^+ \\
& \leq \sum_{\substack{t \in \mathcal{T}_H, t \geq m \\ R(t) \geq t+m}} (e_t^{OPT} - e_t^{PB})^+ + \sum_{\substack{t \in \mathcal{T}_H \\ R(t) \geq t+m}} \sum_{s=t+2}^{t+m-1} \hat{e}_{[t,s]}^{OPT} + \sum_{\substack{t \in \mathcal{T}_H \\ R(t) \leq t+m-1}} \sum_{s=t+1}^{R(t)-1} \hat{e}_{[t,s]}^{OPT} \\
& \leq \sum_{\substack{t \in \mathcal{T}_H, t \geq m \\ R(t) \geq t+m}} (e_t^{OPT} - e_t^{PB})^+ + (m-2) \sum_{\substack{t \in \mathcal{T}_H \\ R(t) \geq t+m}} \hat{e}_{[t,R(t)]}^{OPT} + (m-3) \sum_{\substack{t \in \mathcal{T}_H \\ R(t) \leq t+m-1}} \hat{e}_{[t,R(t)]}^{OPT} + \sum_{\substack{t \in \mathcal{T}_H \\ R(t) = t+m-1}} \hat{e}_t^{OPT} \\
& = \sum_{\substack{t \in \mathcal{T}_H, t \geq m \\ R(t) \geq t+m}} (e_t^{OPT} - e_t^{PB})^+ + (m-2) \sum_{t \in \mathcal{T}_H} \hat{e}_{[t,R(t)]}^{OPT} - \sum_{\substack{t \in \mathcal{T}_H \\ R(t) \leq t+m-2}} \hat{e}_{[t,R(t)]}^{OPT} - \sum_{\substack{t \in \mathcal{T}_H \\ R(t) = t+m-1}} \hat{e}_{[(t+1),R(t)]}^{OPT}.
\end{aligned} \tag{EC.10}$$

Note that

$$\sum_{t \in \mathcal{T}_H} \hat{e}_{[t,R(t)]}^{OPT} \leq \sum_{t=1}^T \hat{e}_t^{OPT} = \sum_{t=m}^T e_t^{OPT} = \frac{1}{\theta} \sum_{t=1}^T \Theta_t^{OPT}. \tag{EC.11}$$

Hence, according to (EC.9)-(EC.11), to prove the lemma, it suffices to show

$$\begin{aligned}
\sum_{\substack{t \in \mathcal{T}_H, t \geq m \\ R(t) \geq t+m}} (e_t^{OPT} - e_t^{PB})^+ & \leq \sum_{t \in \mathcal{T}_H} \left((Y_t^{OPT} - D_t)^+ - (Y_t^{PB} - D_t)^+ + e_{[t,t+m-1]}^{PB,\Pi} \right) + \sum_{t=m}^T e_t^{OPT,\Pi} \\
& + \sum_{\substack{t \in \mathcal{T}_H \\ R(t) \leq t+m-2}} \hat{e}_{[t,R(t)]}^{OPT} + \sum_{\substack{t \in \mathcal{T}_H \\ R(t) = t+m-1}} \hat{e}_{[t+1,R(t)]}^{OPT}.
\end{aligned} \tag{EC.12}$$

Denote $\hat{\mathcal{T}}_H = \{t : t \in \mathcal{T}_H, R(t) \geq t+m\}$, and define $\hat{L}(t)$ as follows: if the set $\{s \in \hat{\mathcal{T}}_H : s < t\}$ is not empty, then $\hat{L}(t) := \max\{s \in \hat{\mathcal{T}}_H : s < t\}$; otherwise, $\hat{L}(t) := 0$. Then, to prove (EC.12), it suffices to show that, for any $s \geq m$ and $s \in \hat{\mathcal{T}}_H$,

$$\begin{aligned}
(e_s^{OPT} - e_s^{PB})^+ & \leq \sum_{t \in \mathcal{T}_H, \hat{L}(s) < t \leq s} \left((Y_t^{OPT} - D_t)^+ - (Y_t^{PB} - D_t)^+ + e_{[t,t+m-1]}^{PB,\Pi} \right) + \sum_{t=\hat{L}(s) \vee m}^s e_t^{OPT,\Pi} \\
& + \sum_{\substack{t \in \mathcal{T}_H, \hat{L}(s) < t \leq s \\ R(t) \leq t+m-2}} \hat{e}_{[t,R(t)]}^{OPT} + \sum_{\substack{t \in \mathcal{T}_H, \hat{L}(s) < t \leq s \\ R(t) = t+m-1}} \hat{e}_{[t+1,R(t)]}^{OPT}.
\end{aligned} \tag{EC.13}$$

Furthermore, to prove (EC.13), it is sufficient to show that, for any $s \geq m$,

$$\begin{aligned} (e_s^{OPT} - e_s^{PB})^+ &\leq \mathbf{1}_{\{s-m+1 \in \mathcal{T}_H\}} \left((Y_{s-m+1}^{OPT} - D_{s-m+1})^+ - (Y_{s-m+1}^{PB} - D_{s-m+1})^+ + e_{[s-m+1, s]}^{PB, \Pi} \right) \\ &\quad + e_s^{OPT, \Pi} + \mathbf{1}_{\{R(s-2m+2) < s-m+1 \in \mathcal{T}_H\}} \hat{e}_{[R(s-2m+1), s-m+1]}^{OPT} \\ &\quad + \mathbf{1}_{\{s-2m+2 \in \mathcal{T}_H, R(s-2m+2) = s-m+1\}} \left((e_{s-2m+2}^{OPT} - e_{s-2m+2}^{PB})^+ + \hat{e}_{[s-2m+3, s-m+1]}^{OPT} \right). \end{aligned} \quad (\text{EC.14})$$

In what follows, we prove that (EC.14) is indeed true for any $s \geq m$ and $s \in \hat{\mathcal{T}}_H$, which then completes the proof of Lemma 1.

Note that $(e_s^{OPT} - e_s^{PB})^+ = 0$ when $e_s^{OPT} \leq e_s^{PB}$, and $(e_s^{OPT} - e_s^{PB})^+ \leq e_s^{OPT, \Pi}$ when $s-m+1 \in \mathcal{T}_\Pi$. (EC.14) is obviously true in both cases. In the following, we assume $e_s^{OPT} > e_s^{PB}$ and $s-m+1 \in \mathcal{T}_H$.

From the definition of Y_s and e_s , the following identity holds for any policy:

$$e_s = \left((Y_{s-m+1} - D_{s-m+1})^+ - D_{[s-m+2, t]} - e_{[s-m+1, s-1]} \right)^+. \quad (\text{EC.15})$$

Since $e_t^{OPT} > e_t^{PB}$, with some simple algebra, it follows from the above identity that

$$\begin{aligned} e_s^{OPT} - e_s^{PB} &\leq (Y_{s-m+1}^{OPT} - D_{s-m+1})^+ - (Y_{s-m+1}^{PB} - D_{s-m+1})^+ + e_{[s-m+1, s-1]}^{PB} - e_{[s-m+1, s-1]}^{OPT} \\ &\leq (Y_{s-m+1}^{OPT} - D_{s-m+1})^+ - (Y_{s-m+1}^{PB} - D_{s-m+1})^+ + e_{[s-m+1, t]}^{PB, \Pi} + e_{[s-m+1, s-1]}^{PB, H} - e_{[s-m+1, s-1]}^{OPT}. \end{aligned}$$

If $e_{[s-m+1, s-1]}^{PB, H} \leq e_{[s-m+1, s-1]}^{OPT}$, then (EC.14) is proved. Now suppose $e_{[s-m+1, s-1]}^{PB, H} > e_{[s-m+1, s-1]}^{OPT}$. In this case, there must exist a period $w(s) \in [s-m+1, s-1]$ such that $e_{w(s)}^{PB, H} > e_{w(s)}^{OPT}$, and $e_{(w(s), s-1]}^{PB, H} \leq e_{(w(s), s-1]}^{OPT}$. From the definition of $e_{w(s)}^{PB, H}$, we also have $w(s) - m + 1 \in \mathcal{T}_H$. Now applying the identity (EC.15) and the fact that $Y_{w(s)-m+1}^{OPT} \geq Y_{w(s)-m+1}^{PB}$, we obtain

$$\begin{aligned} e_{[s-m+1, t-1]}^{PB, H} - e_{[s-m+1, t-1]}^{OPT} &\leq e_{[s-m+1, w(s)]}^{PB} - e_{[s-m+1, w(s)]}^{OPT} \leq e_{[w(s)-m+1, s-m+1]}^{OPT} - e_{[w(s)-m+1, t+m+1]}^{PB} \\ &= \hat{e}_{[w(s)-m+1, s-m+1]}^{OPT} - \hat{e}_{[w(s)-m+1, t+m+1]}^{PB}. \end{aligned} \quad (\text{EC.16})$$

Since both $w(s) - m + 1$ and $s - m + 1$ belong to \mathcal{T}_H , $R(s - 2m + 1) \leq w(s) - m + 1$, and $s - w(s) \leq m - 1$, it can be verified that

$$\begin{aligned} \hat{e}_{[w(s)-m+1, s-m+1]}^{OPT} - \hat{e}_{[w(s)-m+1, s+m+1]}^{PB} &\leq \mathbf{1}_{\{R(s-2m+2) < s-m+1 \in \mathcal{T}_H\}} \hat{e}_{[R(s-2m+1), s-m+1]}^{OPT} \\ &\quad + \mathbf{1}_{\{s-2m+2 \in \mathcal{T}_H, R(s-2m+2) = s-m+1\}} \left((e_{s-2m+2}^{OPT} - e_{s-2m+2}^{PB})^+ + \hat{e}_{[s-2m+3, s-m+1]}^{OPT} \right). \end{aligned} \quad (\text{EC.17})$$

Hence, (EC.14) is proved, and the proof of Lemma 1 is complete.

Q.E.D.

EC.1.4. Proof of Lemma 2

We first establish a preliminary result: For any period $t \in \mathcal{T}_H$, if $\Theta_t^{PB} > \Theta_t^{OPT}$, then there exists a period w_t such that $(t - m + 1) \vee 1 \leq w_t < t$ and

$$\sum_{s=w_t}^t \Theta_s^{PB} \leq \sum_{s=w_t}^t \Theta_s^{OPT}. \quad (\text{EC.18})$$

Suppose $t \in \mathcal{T}_H$ and $\Theta_t^{PB} > \Theta_t^{OPT}$. Since $\Theta_t = \theta \alpha^{t+m-2} e_{t+m-1}$, we have $e_{t+m-1}^{PB} > e_{t+m-1}^{OPT} \geq 0$. Recall that e_t is the number of perished products in period t , and it satisfies the following identity under any policy:

$$e_{t+m-1} = (Y_t - D_{[t,t+m-1]} - e_{[t,t+m-1]})^+.$$

Thus it follows from $e_{t+m-1}^{PB} > 0$ that

$$\begin{aligned} e_{[t,t+m-1]}^{PB} &= (Y_t^{PB} - D_{[t,t+m-1]} - e_{[t,t+m-1]}^{PB})^+ + e_{[t,t+m-1]}^{PB} \\ &= Y_t^{PB} - D_{[t,t+m-1]}. \end{aligned} \quad (\text{EC.19})$$

On the other hand, for the OPT policy we have

$$\begin{aligned} e_{[t,t+m-1]}^{OPT} &= (Y_t^{OPT} - D_{[t,t+m-1]} - e_{[t,t+m-1]}^{OPT})^+ + e_{[t,t+m-1]}^{OPT} \\ &\geq Y_t^{OPT} - D_{[t,t+m-1]}, \end{aligned} \quad (\text{EC.20})$$

Subtracting (EC.20) from (EC.19) yields

$$e_{[t \vee m, t+m-1]}^{PB} - e_{[t \vee m, t+m-1]}^{OPT} = e_{[t, t+m-1]}^{PB} - e_{[t, t+m-1]}^{OPT} \leq Y_t^{PB} - Y_t^{OPT} \leq 0, \quad (\text{EC.21})$$

where the equality holds since $e_t^{PB} = e_t^{OPT}$ for $1 \leq t \leq m - 1$ and the last inequality follows from $t \in \mathcal{T}_H$. This proves $e_{[t \vee m, t+m-1]}^{PB} < e_{[t \vee m, t+m-1]}^{OPT}$.

We argue that (EC.21) implies the existence of $w_t \in [(t - m + 1) \vee 1, t)$ that satisfies (EC.18). To this end, we apply Abel's lemma (see e.g., [Chow and Teicher \(2012\)](#)) and the identity $\Theta_s = \theta \alpha^{s+m-2} e_{s+m-1}$ to obtain, under any policy, that

$$\begin{aligned} \theta e_{[t \vee m, t+m-1]} &= \sum_{s=(t-m+1) \vee 1}^t \alpha^{-s-m+2} \Theta_s \\ &= \alpha^{-t \vee m+1} \left[\sum_{s=(t-m+1) \vee 1}^t \Theta_s \right] + \sum_{t'=(t-m+1) \vee 1+1}^t \alpha^{-t'-m+2} (1-\alpha) \left[\sum_{s=t'}^t \Theta_s \right]. \end{aligned}$$

Therefore, $e_{[t \vee m, t+m-1]}^{PB} < e_{[t \vee m, t+m-1]}^{OPT}$ together with the condition that $\Theta_t^{PB} > \Theta_t^{OPT}$ imply the existence of at least one $w_t \in [(t-m+1) \vee 1, t)$, such that (EC.18) holds.

We are now ready to prove Lemma 2. Since any products ordered after period $T-m+1$ do not perish within the planning horizon, we only need to consider periods $t = 1, \dots, T-m+1$. For each realization of f_T and the resulting \mathcal{T}_H , we partition the periods $\{1, \dots, T-m+1\}$ as follows: First, start in period $T-m+1$ and search backward for the latest period $t \in \mathcal{T}_H$ such that $\Theta_t^{PB} > \Theta_t^{OPT}$. If no such period exists, then we terminate the partition process. Otherwise, let t' be that period, and by the preliminary result, there exists a $w_{t'} \in [(t'-m+1) \vee 1, t')$ that satisfies $\sum_{s=w_{t'}}^{t'} \Theta_s^{PB} \leq \sum_{s=w_{t'}}^{t'} \Theta_s^{OPT}$. Mark the periods $w_{t'}, w_{t'}+1, \dots, t'$. Next, repeat the above procedure over periods $1, \dots, w_{t'}-1$ until the remaining set of periods is empty. As a result, the above procedure partitions the periods $\{1, \dots, T-m+1\}$ into marked and unmarked periods. Let \mathcal{T}_M denote the set of marked periods.

Consider any period $t \in \mathcal{T}_H \setminus \mathcal{T}_M$. By the definition of \mathcal{T}_M , we have $\Theta_t^{PB} \leq \Theta_t^{OPT}$. Thus,

$$\sum_{t \in \mathcal{T}_H \setminus \mathcal{T}_M} \Theta_t^{PB} \leq \sum_{t \in \mathcal{T}_H \setminus \mathcal{T}_M} \Theta_t^{OPT}. \quad (\text{EC.22})$$

On the other hand, if a period $t \in \mathcal{T}_H$ is also in $t \in \mathcal{T}_M$, then by the construction of \mathcal{T}_M it belongs to an interval of the form $[w_{t'}, \dots, t']$, and the preliminary result above has shown $\sum_{s=w_{t'}}^{t'} \Theta_s^{PB} \leq \sum_{s=w_{t'}}^{t'} \Theta_s^{OPT}$. This proves

$$\sum_{t \in \mathcal{T}_M} \Theta_t^{PB} \leq \sum_{t \in \mathcal{T}_M} \Theta_t^{OPT}. \quad (\text{EC.23})$$

Since

$$\mathcal{T}_H \subset (\mathcal{T}_H \setminus \mathcal{T}_M) \cup \mathcal{T}_M \subset \{1, 2, \dots, T\},$$

we obtain, using (EC.22) and (EC.23), that

$$\sum_{t \in \mathcal{T}_H} \Theta_t^{PB} \leq \sum_{t \in \mathcal{T}_H \setminus \mathcal{T}_M} \Theta_t^{PB} + \sum_{t \in \mathcal{T}_M} \Theta_t^{PB} \leq \sum_{t \in \mathcal{T}_H \setminus \mathcal{T}_M} \Theta_t^{OPT} + \sum_{t \in \mathcal{T}_M} \Theta_t^{OPT} \leq \sum_{t=1}^T \Theta_t^{OPT}.$$

This completes the proof of Lemma 2.

Q.E.D.

EC.2. Numerical Results

EC.2.1. Computational complexity of PB and DB policies

The PB and DB policies are computationally very efficient, and can be computed in an *online* manner. The computational complexities of PB and DB algorithms are both $O(K^m T)$ for some positive constant K , where m is product lifetime and T is the length of the planning horizon.

To see this, for each period $t = 1, \dots, T$, the complexity for evaluating the marginal holding and outdating costs is $O(K^m)$. This is unavoidable as the complexity is the same as computing the single-period outdating costs. We would like to note that even myopic policies (that minimize single period costs of ordering, holding, shortage, and outdating) have to incur this computational overhead. For practical purposes, for long lifetime products (e.g., $m \geq 4$), evaluating the expected marginal costs using Monte Carlo simulations can cut down the computational time dramatically (we use 10000 samples in our numerical experiments). Since the complexity for carrying out bisection search is $O(\log U)$ (where U is an upper bound on the balancing quantities), the algorithms run in time $O(K^m T \log U) \approx O(K^m T)$. In contrast, computing the exact optimal policy using dynamic programming is exponential in the length of the planning horizon T .

EC.2.2. Tables

Table EC.1 Performance errors of heuristics for i.i.d. demands (% Errors) for $m = 2$ and $m = 3$

	$m = 2$								$m = 3$							
	Exponential Demand				Erlang-2 Demand				Exponential Demand				Erlang-2 Demand			
$\hat{c}, \hat{b}, \hat{\theta}$	PB	PPB	DB	PDB	PB	PPB	DB	PDB	PB	PPB	DB	PDB	PB	PPB	DB	PDB
0,5,5	0.89	0.21	0.82	0.18	0.44	0.35	0.27	0.11	0.95	0.36	0.63	0.27	0.80	0.59	0.65	0.28
0,10,5	0.76	0.36	0.92	0.11	0.63	0.13	0.22	0.21	0.93	0.48	0.74	0.12	1.05	0.82	0.66	0.16
0,5,10	1.37	0.81	1.41	0.42	0.73	0.13	0.59	0.27	1.12	0.88	1.40	0.52	1.63	0.59	0.56	0.47
5,10,5	0.63	0.09	0.96	0.05	0.12	0.07	0.09	0.06	1.06	0.92	1.25	0.15	0.22	0.13	0.57	0.22
5,5,10	0.49	0.36	0.69	0.34	0.35	0.03	0.47	0.28	0.57	0.31	0.62	0.29	0.17	0.12	0.22	0.13
5,5,5	0.32	0.15	0.64	0.13	0.15	0.06	0.11	0.10	0.68	0.38	0.58	0.35	0.16	0.12	0.24	0.11
5,10,0	0.28	0.10	0.18	0.11	0.36	0.11	0.15	0.11	0.58	0.39	0.56	0.21	0.45	0.18	0.89	0.24
10,10,5	0.52	0.07	0.78	0.06	0.27	0.08	0.25	0.18	0.79	0.56	1.28	0.60	0.19	0.11	0.62	0.42
10,10,10	0.80	0.18	1.38	0.21	0.09	0.08	0.24	0.15	0.92	0.74	1.07	0.15	0.06	0.04	0.17	0.04
10,5,5	0.77	0.28	0.92	0.22	0.13	0.11	0.19	0.12	0.74	0.46	0.24	0.05	0.10	0.06	0.26	0.10
10,10,0	0.08	0.06	0.54	0.14	0.07	0.05	0.19	0.03	0.44	0.25	0.47	0.13	0.14	0.09	0.46	0.17
max	1.37	0.81	1.41	0.42	0.73	0.35	0.59	0.28	1.12	0.92	1.40	0.60	1.63	0.82	0.89	0.47
mean	0.63	0.24	0.84	0.18	0.30	0.11	0.25	0.15	0.80	0.52	0.80	0.26	0.45	0.26	0.48	0.21

Table EC.2 Performance ratios of heuristics for i.i.d. demands (r) for $m = 4$ and $m = 6$

$\hat{b}, \hat{\theta}$	m=4						
	Erlang-2		Exponential		Hyper-exponential		
	PB	DB	PB	DB	PB	DB	
5,10	99.14%	99.52%	99.73%	101.11%	99.09%	100.56%	
5,50	100.27%	100.43%	100.43%	100.96%	99.96%	100.70%	
5,100	99.78%	100.29%	100.22%	100.02%	100.88%	100.59%	
10,10	98.66%	99.01%	101.18%	102.43%	101.49%	102.43%	
10,50	101.67%	101.80%	107.57%	106.69%	106.71%	107.43%	
10,100	100.89%	101.70%	100.05%	101.07%	102.02%	102.84%	
15,10	100.06%	100.28%	102.17%	103.85%	102.81%	103.35%	
15,50	102.71%	102.81%	116.69%	111.59%	110.86%	111.55%	
15,100	100.46%	101.06%	110.21%	108.46%	115.87%	116.65%	
$\hat{b}, \hat{\theta}$	m=6						
	5,10	98.78%	97.80%	98.76%	98.93%	99.13%	99.23%
	5,50	98.79%	98.93%	98.38%	100.06%	100.00%	100.40%
	5,100	98.48%	98.79%	100.04%	100.06%	100.57%	100.60%
	10,10	97.58%	97.38%	98.39%	97.97%	98.30%	98.15%
	10,50	98.41%	98.84%	99.07%	99.35%	99.05%	100.31%
	10,100	98.67%	98.71%	100.79%	100.66%	101.14%	101.57%
	15,10	98.28%	98.19%	99.65%	99.50%	98.71%	98.60%
	15,50	97.05%	97.24%	99.53%	99.79%	100.03%	100.17%
	15,100	97.37%	97.57%	101.38%	102.50%	100.94%	102.08%

Table EC.3 Performance errors of heuristics for ADI, AR(1), and MMFE demands (% Errors)

demand	\hat{c} $\hat{\theta}$ Policy	L				M				H				All	
		mean	max	mean	max	mean	max	mean	max	mean	max	mean	max		
ADI cv=0.1	PB	0.56	1.04	0.61	0.94	0.34	0.55	0.33	0.48	0.14	0.24	0.10	0.13	0.33	1.04
	PPB	0.36	0.69	0.40	0.69	0.30	0.51	0.28	0.46	0.12	0.21	0.08	0.12	0.25	0.69
ADI cv=0.3	PB	1.02	1.45	0.77	1.62	0.32	0.99	0.12	0.19	0.10	0.19	0.06	0.10	0.34	1.62
	PPB	0.74	1.39	0.60	1.30	0.21	0.67	0.07	0.11	0.07	0.13	0.05	0.07	0.25	1.39
ADI cv=0.5	PB	0.33	0.53	1.13	2.04	0.51	0.68	0.42	0.56	0.40	0.56	0.17	0.27	0.51	2.04
	PPB	0.24	0.38	0.87	1.58	0.33	0.53	0.32	0.42	0.29	0.48	0.07	0.17	0.36	1.58
ADI Erlang-2	PB	0.75	1.08	1.10	1.89	0.45	0.98	0.40	0.64	0.33	0.55	0.23	0.50	0.52	1.89
	PPB	0.61	0.91	0.79	1.49	0.19	0.55	0.19	0.33	0.15	0.32	0.15	0.28	0.32	1.49
ADI Exp.	PB	1.35	2.65	0.86	1.57	0.38	0.65	0.49	0.83	0.30	0.54	0.32	0.58	0.55	2.65
	PPB	0.97	1.79	0.74	1.49	0.31	0.50	0.37	0.73	0.22	0.46	0.17	0.33	0.42	1.79
AR(1)	PB	1.51	1.95	1.68	2.21	0.36	0.43	0.42	0.76	0.22	0.35	0.22	0.34	0.66	2.21
	PPB	1.06	1.42	1.19	1.88	0.18	0.26	0.14	0.32	0.11	0.20	0.07	0.13	0.40	1.88
MMFE	PB	1.60	2.26	1.49	2.39	0.35	0.58	0.38	0.50	0.15	0.21	0.19	0.42	0.61	2.39
	PPB	0.84	1.26	0.93	1.50	0.14	0.18	0.17	0.23	0.09	0.15	0.09	0.19	0.34	1.50

Table EC.4 Performance errors of heuristics for MMDP demands (% Errors)

case	\hat{c}	L				M				H				All	
	$\hat{\theta}$	L		H		L		H		L		H			
	Policy	mean	max	mean	max	mean	max	mean	max	mean	max	mean	max	mean	max
m=2	PB	1.57	2.00	2.44	2.94	0.31	0.49	0.41	0.67	0.30	0.59	0.17	0.27	0.80	2.94
cv=0.1	PPB	1.05	1.41	1.22	1.57	0.17	0.33	0.24	0.53	0.16	0.24	0.10	0.19	0.44	1.57
m=2	PB	0.91	1.94	1.79	2.47	0.31	0.67	0.41	0.91	0.30	0.54	0.30	0.65	0.65	2.47
cv=0.3	PPB	0.57	1.27	1.09	1.74	0.16	0.52	0.21	0.38	0.19	0.48	0.14	0.24	0.38	1.74
m=2	PB	0.39	0.75	1.30	2.18	0.29	0.52	0.49	0.77	0.33	0.61	0.31	0.49	0.53	2.18
cv=0.5	PPB	0.29	0.56	0.76	1.44	0.15	0.23	0.25	0.60	0.19	0.40	0.15	0.23	0.30	1.44
m=2	PB	0.85	1.51	1.55	2.69	0.69	1.06	0.85	1.12	0.66	0.93	0.58	0.77	0.86	2.69
Erlang-2	PPB	0.70	1.20	1.00	1.51	0.44	0.68	0.54	0.74	0.42	0.76	0.34	0.57	0.56	1.51
m=2	PB	0.58	1.32	1.61	2.82	0.95	1.60	0.98	1.44	0.64	1.03	0.68	0.97	0.94	2.82
Exp.	PPB	0.45	1.03	0.89	1.59	0.51	0.91	0.64	1.15	0.27	0.42	0.46	0.77	0.55	1.59
m=3	PB	1.32	1.62	1.90	2.43	0.74	1.34	0.47	0.56	0.41	0.80	0.24	0.46	0.80	2.43
cv=0.1	PPB	0.96	1.13	1.34	1.88	0.44	1.18	0.27	0.47	0.25	0.37	0.14	0.35	0.53	1.88
m=3	PB	1.21	1.28	1.69	1.95	0.45	0.97	0.47	0.59	0.17	0.26	0.14	0.19	0.64	1.95
cv=0.3	PPB	0.61	0.89	1.11	1.62	0.32	0.75	0.40	0.54	0.10	0.17	0.10	0.15	0.43	1.62
m=3	PB	1.15	1.65	0.83	1.51	0.33	0.67	0.38	0.82	0.24	0.38	0.30	0.38	0.48	1.65
cv=0.5	PPB	0.95	1.41	0.57	1.03	0.21	0.44	0.22	0.74	0.13	0.25	0.14	0.23	0.32	1.41
m=3	PB	1.31	1.94	1.41	1.78	0.53	0.85	0.59	0.79	0.40	0.62	0.44	0.64	0.73	1.94
Erlang-2	PPB	0.73	1.04	0.98	1.58	0.38	0.84	0.38	0.51	0.27	0.42	0.27	0.43	0.48	1.58
m=3	PB	0.98	1.41	1.18	1.81	0.53	0.96	0.69	1.05	0.53	0.66	0.35	0.75	0.69	1.81
Exp.	PPB	0.61	0.69	0.68	1.51	0.35	0.88	0.43	0.64	0.32	0.52	0.22	0.56	0.42	1.51