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Approximation and interpolation of entire functions

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1. Introduction. Let E be a bounded closed set in the complex z-plane C. Denote by $\mathscr{C}(E)$ the algebra of all continuous complex functions on E with the norm

$$\|f\| = \sup_{z\in E} |f(z)| \quad \text{for } f\in\mathscr{C}(E).$$

Let P_n denote the set of the polynomials in z of degree $\leqslant n$. It is known that for every $f\in\mathscr{C}(E)$ there is exactly one polynomial $\pi_n \in P_n$ such that

$$\mathscr{E}_n(f, E) = \inf_{g\in P_n} \|f-g\| = \|f-\pi_n\|.$$

Batyrev [1] proved the following

THEOREM. *A function f, analytic on a bounded closed set E with the positive transfinite diameter d, and with the simply connected $C-E$, can be prolonged to an entire function of order $\varrho(0<\varrho<\infty)$ and of type σ if and only if*

$$\limsup_{n\to\infty} n^{1/\varrho}\big(\mathscr{E}_n(f, E)\big)^{1/n} = d(e\varrho\sigma)^{1/\varrho}.$$

Batyrev has proved this theorem using Faber polynomials. His method cannot be applied in the case where the set $C-E$ is not simply connected or if the function f is not analytic on E.

The main object of this paper is to prove the Batyrev theorem in the case of any compact $E\subset C$ with $d(E)>0$ and of any function f defined and bounded on E. Moreover (see Section 5), we give formulas expressing the type and the order of an entire function f in terms of the coefficients of the expansion of f into a Newton interpolating series. These formulas generalize the classical formulas expressing the order and type of f in terms of the coefficients of type Taylor series of f.

The author wishes to express his gratitude to Professor J. Siciak for suggesting the problem and the way of solving it.

2. Extremal polynomials. Let $\zeta^{(n)} = \{\zeta_{n0}, \zeta_{n1}, \ldots, \zeta_{nn}\}$ be a system of $n+1$ points of a bounded closed set E in the complex plane C. Put

$$V(\zeta^{(n)}) = \prod_{0\leqslant j<k\leqslant n} |\zeta_{nj}-\zeta_{nk}|, \tag{2.0}$$
$$\Delta^{(j)}(\zeta^{(n)}) = \prod_{\substack{k=0\\k\neq j}}^{n} |\zeta_{nj}-\zeta_{nk}|, \quad j = 0, 1, \ldots, n.$$

Definition 2.1. A system of points of E

$$\eta^{(n)} = \{\eta_{n0}, \eta_{n1}, \ldots, \eta_{nn}\} \tag{2.1}$$

satisfying the relations

(a) $V_n = V(\eta^{(n)}) = \sup_{\zeta^{(m)}\subset E} V(\zeta^{(n)})$,

(b) $\Delta^{(0)}(\eta^{(n)}) \leqslant \Delta^{(j)}(\eta^{(n)})$, $j = 1, 2, \ldots, n$

is called the *n-th extremal system of E*. Polynomials

$$L^j(z, \eta^{(n)}) = \prod_{\substack{k=0\\k\neq j}} \frac{z-\eta_{nk}}{\eta_{nj}-\eta_{nk}}, \quad j = 0, 1, \ldots, n \tag{2.2}$$

are called the *n-th Lagrange extremal polynomials*, and the limit

$$d = d(E) = \lim_{n\to\infty} V_n^{2/n(n+1)} = \lim_{n\to\infty} \big(\Delta^{(0)}(\eta^{(n)})\big)^{1/n}$$

is called the *transfinite diameter of E*.

The following theorem is known [2]:

If $d(E) > 0$, then there exists a finite limit

$$\lim_{n\to\infty} |L^0(z, \eta^{(n)})|^{1/n} = L(z) \geqslant 1 \tag{2.3}$$

for every z in the unbounded component E_∞ of $C-E$, the convergence in (2.3) *is uniform on every bounded closed subset of E, and L is a modulus of an analytic function φ in E_∞ which has a univalent branch*

$$\varphi(z) = \gamma z+\gamma_0+\frac{\gamma_1}{z}+\ldots, \quad |\gamma| = \frac{1}{d(E)}, \tag{2.4}$$

in a neighbourhood of infinity. Moreover, $\operatorname{Log} L(z)$ *is the generalized Green function for E_∞ with pole at ∞.*

Let $|E|$ denote the diameter of a compact set E,

$$|E| = \sup\{|z-w|:\ z, w\in E\}.$$

We note that

$$\frac{\operatorname{dist}(z, E)}{|\Delta^{(0)}(\eta^{(n)})|^{1/n}} \leqslant |L^0(z, \eta^{(n)})|^{1/n} \leqslant \frac{\operatorname{dist}(z, E)+|E|}{|\Delta^{(0)}(\eta^{(n)})|^{1/n}} \quad \text{for } zC\ni, n = 1, 2, \ldots$$

In consequence we have

LEMMA 2.1. *If* $d(E) > 0$, *then*

$$\operatorname{dist}(z, E) \leqslant d(E)L(z) \leqslant \operatorname{dist}(z, E) + |E| \quad \textit{for } z \in C - E. \tag{2.5}$$

DEFINITION 2.2. Let z_0 be a fixed point of E. A sequence $\{z_n\}$ of points of E defined by

$$|(z_{n+1}-z_0)\dots(z_{n+1}-z_n)| = \sup_{z\in E}|(z-z_0)\dots(z-z_n)| \quad \text{for } n = 1, 2, \dots,$$

is called the *extremal sequence of E*.

One can prove [3] that if $\{z_n\}$ is an extremal sequence of E, then

$$\lim_{n\to\infty}\big(\sup_{z\in E}|(z-z_0)\dots(z-z_{n-1})|\big)^{1/n+1} = d(E) \tag{2.6}$$

and if $d(E) > 0$, then

$$\lim_{n\to\infty}|(z-z_0)\dots(z-z_{n-1})|^{1/n+1} = dL(z) \quad \text{for } z \in E_\infty. \tag{2.7}$$

Equation (2.6) remits from the following [6]

THEOREM 2.1. *Let* $\{\zeta_{n1}, \dots, \zeta_{nn}\}$ *be a system of n points of a bounded closed set E with a positive transfinite diameter d. If we introduce the notation*

$$M_n = \max_{z\in E}|(z-\zeta_{n1})\dots(z-\zeta_{nn})|, \tag{2.8}$$

then

$$\lim_{n\to\infty} M_n^{1/n} = d$$

is a sufficient condition that

$$\lim_{n\to\infty}|(z-\zeta_{n1})\dots(z-\zeta_{nn})|^{1/n} = dL(z) \tag{2.9}$$

should hold uniformly on any closed limited point set interior to E_∞.

3. Order and type of an entire function. Let f be an entire function and let

$$M(r) = \sup_{|z|=r}|f(z)|, \qquad r > 0, \tag{3.1}$$

be its supremum modulus.

DEFINITION 3.1. We call $\varrho = \varrho(f)$ the *order of f* if

$$\varrho(f) = \limsup_{r\to\infty}\frac{\ln\ln M(r)}{\ln r}. \tag{3.2}$$

We have $0 \leqslant \varrho(f) \leqslant \infty$. If $0 < \varrho(f) < \infty$, we say that f is of type $\sigma = \sigma(f)$ if

$$\sigma(f) = \limsup_{r\to\infty}\frac{\ln M(r)}{r^\varrho}. \tag{3.3}$$

Replacing in Definition 3.1 the circle $|z| = r$ by the curve

$$E_r = \{z \in C\colon d(E)L(z) = r\} \quad \text{for } r > 1, \tag{3.4}$$

and $M(r)$ by

$$\tilde{M}(r) = \sup_{z \in E_r} |f(z)| \quad \text{for } r > 1, \tag{3.5}$$

we shall prove the following

LEMMA 3.1. *If f is an entire function of order ϱ and type σ, and $d(E) > 0$, then*

$$\varrho = \limsup \frac{\ln\ln \tilde{M}(r)}{\ln r}.$$

If $0 < \varrho < \infty$, then

$$\sigma = \limsup \frac{\ln \tilde{M}(r)}{r^\varrho}.$$

Proof. Let z_0 be a fixed point of the set E, and $r > 1$. For every point $z \in E_r$ there exists a $z_1 = z_1(z) \in E$ such that

$$|z - z_1| = \operatorname{dist}(z, E).$$

By the triangle inequality and by (2.5) we have

$$|z - z_0| \leqslant |z - z_1| + |z_1 - z_0| \leqslant r + |E| \quad \text{for } z \in E_r,\ r > 1,$$

and

$$r - |E| \leqslant |z - z_1|, \quad |E| \geqslant |z_1 - z_0|.$$

Observe that

$$r - 2|E| - |z_0| \leqslant |z| \leqslant r + |E| + |z_0| \quad \text{for } z \in E_r,\ r > 1.$$

Let $R > 1$ be such that

$$r - 2|E| - |z_0| \geqslant \tfrac{1}{2} r \quad \text{and} \quad r + |E| + |z_0| \leqslant 2r \quad \text{for } r > R.$$

Hence for $r > R$ we have

$$\frac{\ln\ln M(\frac{1}{2} r)}{\ln r} \leqslant \frac{\ln\ln \tilde{M}(r)}{\ln r} \leqslant \frac{\ln\ln M(2r)}{\ln r}$$

and, if $0 < \varrho < \infty$,

$$\frac{\ln M(r - a)}{r^\varrho} \leqslant \frac{\ln \tilde{M}(r)}{r^\varrho} \leqslant \frac{\ln M(r + b)}{r^\varrho},$$

where

$$a = 2|E| + |z_0|, \quad b = |E| + |z_0|.$$

After passing to the upper limit the proof of Lemma 3.1 is completed.

LEMMA 3.2. *If a function f is defined and bounded on a compact set $E \subset C$ containing infinitely many points, then*

$$\mathscr{E}_n(f, E) \leqslant \|f - L_n\| \leqslant (n+2)\,\mathscr{E}_n(f, E)$$

and

$$\|L_n - L_{n-1}\| \leqslant 2(n+2)\,\mathscr{E}_{n-1}(f, E) \quad \textit{for } n = 2, 3, \ldots,$$

where

$$L_n(z) = \sum_{j=0}^{n} L^{(j)}(z, \eta^{(n)}) f(\eta_{nj})$$

is the n-th Lagrange interpolation polynomial for f with nodes at extremal points η_{nj} $(j = 0, \ldots, n)$ of E defined by (2.1).

Proof. Let π_n and P_n $(n = 1, 2, \ldots)$ be polynomials of degree $\leqslant n$ such that

$$\mathscr{E}_n(f, E) = \|f - \pi_n\|$$

and

$$f(\eta_{nj}) - \pi_n(\eta_{nj}) = P_n(\eta_{nj}) \quad \text{for } n = 1, 2, \ldots,\ j = 0, 1, \ldots, n.$$

Since

$$L_n(\eta_{nj}) = \pi_n(\eta_{nj}) + P_n(\eta_{nj}) \quad \text{for } j = 0, 1, \ldots, n,$$

we have

$$L_n(z) = \pi_n(z) + P_n(z) \quad \text{for } z \in C.$$

Therefore

$$\|f - L_n\| \leqslant \|f - \pi_n\| + \|P_n\| \leqslant \mathscr{E}_n(f, E) + \mathscr{E}_n(f, E) \sum_{j=0}^{n} \|L^{(j)}\| \leqslant (n+2)\,\mathscr{E}_n(f, E).$$

Hence and the sequence $\{\mathscr{E}_n(f; E)\}$ being decreasing, we have

$$\|L_n - L_{n-1}\| \leqslant \|f - L_n\| + \|f - L_{n-1}\| \leqslant 2(n+2)\,\mathscr{E}_{n-1}(f, E) \quad \text{for } n = 2, 3, \ldots$$

This concludes the proof of Lemma 3.2.

LEMMA 3.3. *Let $\{P_n\}$ be a sequence of polynomials in z of the respective degrees $\leqslant n$. If the transfinite diameter of the compact $E \subset C$ is positive, and if there exist numbers K, μ, $N > 0$ such that*

$$\|P_n\| \leqslant d^n \left(\frac{eK\mu}{n}\right)^{n/\mu} \quad \textit{for } n \geqslant N,$$

then

$$f(z) = \sum_{n=1}^{\infty} P_n(z) \quad \textit{for } z \in C$$

is an entire function, and for all $\varepsilon > 0$ there exists an $R = R(\varepsilon)$ such that

$$\tilde{M}(r) \leqslant \exp\big((K+\varepsilon) r^{\mu}\big) \quad \textit{for } r \geqslant R.$$

Proof. By the Bernstein-Walsh inequality (see Theorem 2.2 and Corollary 9.1 in [5]) we have

$$|P_n(z)| \leqslant \|P_n\| L^n(z) \quad \text{for } z \in C,\ n = 1, 2, \ldots$$

Hence

$$\tilde{M}(r) \leqslant \sum_{n=1}^{\infty} \frac{\|P_n\|}{d^n} r^n \quad \text{for } r > 1.$$

Take $r_0 > 1$ in such a way that

$$n(r) = 2^{\mu}(eK\mu)r^{\mu} > N \quad \text{for } r \geqslant r_0.$$

Then

$$\left(\frac{eK\mu}{n}\right)^{n/\mu} r^n \leqslant \frac{1}{2^n} \quad \text{for } n \geqslant n(r)$$

and

$$\tilde{M}(r) \leqslant \sum_{n=1}^{N} \frac{\|P_n\|}{d^n} r^n + \sum_{n=N+1}^{n(r)} \left(\frac{eK\mu}{n}\right)^{n/\mu} r^n + 1 \quad \text{for } r \geqslant r_0.$$

It is clear that for a fixed r the maximum value of the expression $\left(\frac{eK\mu}{n}\right)^{n/\mu} r^n$ is reached for $r = \mu K r^{\mu}$ and is equal to $\exp(Kr^{\mu})$. It follows that if $r \geqslant r_0$, then

$$\begin{aligned}
\tilde{M}(r) &< r^N \sum_{n=1}^{N} \frac{\|P_n\|}{d^n} + [n(r)+1-N]e^{Kr^{\mu}} \\
&< r^N \sum_{n=1}^{N} \frac{\|P_n\|}{d^n} + (2^{\mu} e\mu K r^{\mu} + 1 - N)e^{Kr^{\mu}} \\
&= e^{Kr^{\mu}}\left(2^{\mu} e\mu K r^{\mu} + 1 - N + r^N e^{-Kr^{\mu}} \sum_{n=1}^{N} \frac{\|P_n\|}{d^n}\right).
\end{aligned}$$

Hence, by a standard argument, for every $\varepsilon > 0$ there exists an $R > r_0$ such that for all $r > R$

$$2^{\mu} e\mu K r^{\mu} + 1 - N + r^N e^{-Kr^{\mu}} \sum_{n=1}^{N} \frac{\|P_n\|}{d^n} < e^{\varepsilon r^{\mu}}.$$

The proof is completed.

4. Generalizations of the Batyrev theorem. Let E be a bounded closed set in the complex plane C with the positive transfinite diameter d, and let f be an entire function of order ϱ and type σ. If we preserve the notation of the former section and set

$$(4.0)\qquad \lambda_n^{(1)} = \mathcal{E}_n(f, E), \quad \lambda_n^{(2)} = \|L_n - L_{n-1}\|, \quad \lambda_n^{(3)} = \|f - L_n\| \quad \text{for } n = 1, 2, \ldots,$$

then we can prove the following

THEOREM 4.1. *If* $0 < \varrho < \infty$, *then*

$$\limsup_{n\to\infty} n^{1/\varrho}(\lambda_n^{(j)})^{1/n} = d(e\varrho\sigma)^{1/\varrho} \quad \textit{for } j = 1, 2, 3.$$

If $\varrho = 0$, *then for every* $a > 0$

$$\limsup_{n\to\infty} n^{a}(\lambda_n^{(j)})^{1/n} = 0 \quad \textit{for } j = 1, 2, 3.$$

Proof. Let $R > 1$ be so large that the function φ defined by (2.4) is univalent in $|z| \geqslant R$. Then $g(z) = d\varphi(z)$ is also univalent in $|z| \geqslant R$, $g'(\infty) = 1$ and $dL(z) = |g(z)|$. Hence

$$E_r = \{z \in C\colon\ |g(z)| = r\}, \quad r > 1,$$

and there exists an $m_0 > 0$ such that

$$g'(z) \geqslant m_0 \quad \text{for } |z| \geqslant R.$$

Therefore

$$(4.1)\qquad \int_{E_r} |d\zeta| = \int_{|z|=r} \frac{|dz|}{|g'(z)|} \leqslant \frac{2\pi r}{m_0} \quad \text{for } r \geqslant R.$$

Put

$$w_n(z) = (z - \eta_{n1}) \ldots (z - \eta_{nn}), \quad n = 1, 2, \ldots,$$

where $\eta^{(n)} = \{\eta_{n0}, \eta_{n1}, \ldots, \eta_{nn}\}$ is of n-th extremal points system of E. Put

$$M_n = \sup_{z\in E} |w_n(z)| = \big(\Delta^{(0)}(\eta^{(n)})\big)^{1/n}.$$

One can easily check that

$$(4.2)\qquad \lim_{\varrho\to\infty} \frac{|w_n(\zeta)|}{|g(\zeta)|^n} = 1 \quad \text{for } n = 1, 2, \ldots$$

Since $\lim M_n^{1/n} = d$, and $d > 0$, so by Theorem 2.1 for any fixed $r_0 > 1$ there exists a finite limit

$$\lim_{n\to\infty} |w_n(\zeta)|^{1/n} = r_0 \quad \text{for } \zeta \in E_{r_0}$$

and convergence is uniform on E_{r_0}. Hence for every $\varepsilon > 0$ there exists an $n_0 = n_0(\varepsilon, r_0)$ such that

$$(4.3)\qquad \left(\frac{|w_n(\zeta)|}{M_n |g(\zeta)|^n}\right)^{1/n} \geqslant \frac{e^{-\varepsilon}}{d} \quad \text{for } n \geqslant n_0,\ \zeta \in E_{r_0}.$$

From (4.2) and by the extremum principle applied to the harmonic function

$$\frac{1}{n}\ln\frac{|w_n(\zeta)|}{M_n|g(\zeta)|^n},$$

inequality (4.3) holds true for $\zeta\in E_r$, $r\geqslant r_0$. Hence for $r\geqslant r_0$ and $n\geqslant n_0$ we get

$$\left(\frac{|w_n(\zeta)|}{M_n}\right)^{1/n}\geqslant\frac{r}{d}e^{-\varepsilon}\quad\text{for }\zeta\in E_r. \tag{4.4}$$

If r is sufficiently large, say $r>r_1$, then E_r is a union of a finite number of mutually disjoint analytic Jordan curves; therefore (see [6]),

$$f(z)-L_n(z)=\frac{1}{2\pi i}\int_{E_r}\frac{w_n(z)(z-\eta_{n0})}{w_n(\zeta)(\zeta-\eta_{n0})}\,\frac{f(\zeta)}{\zeta-z}\,d\zeta,\qquad z\in E.$$

Hence and from (4.1), (4.4) and (2.5) we get

$$\|f-L_n\|\leqslant\lambda\frac{\tilde{M}(r)}{r^n}(de^{\varepsilon})^n, \tag{4.5}$$

where

$$\lambda=\frac{2r_0|E|}{m_0(r_0-|E|)},\qquad r>R_1=\max(r_0,r_1,|E|),\qquad n\geqslant n_0.$$

Let $K>\sigma$. By the definition of the type of f there exists an $R_2=R_2(K)>R_1$ such that

$$\tilde{M}(r)\leqslant e^{Kr^{\varrho}}\quad\text{for }r\geqslant R_2.$$

Let $N>n_0$ be so large that

$$\left(\frac{n}{K\varrho}\right)^{1/\varrho}>R_2\quad\text{for }n\geqslant N.$$

Putting $r=\left(\frac{n}{K\varrho}\right)^{1/\varrho}$ in (4.5) gives

$$\|f-L_n\|\leqslant\lambda(de^{\varepsilon})^n\left(\frac{eK\varrho}{n}\right)^{n/\varrho}\quad\text{for }n\geqslant N. \tag{4.6}$$

Hence

$$\limsup_{n\to\infty}n^{1/\varrho}\|f-L_n\|^{1/n}\leqslant d(eK\varrho)^{1/\varrho};$$

therefore, by Lemma 3.2, we have

$$\limsup_{n\to\infty}n^{1/\varrho}(\lambda_n^{(j)})^{1/n}\leqslant d(e\sigma\varrho)^{1/\varrho}\quad\text{for }j=1,2,3. \tag{4.7}$$

Suppose that

$$\limsup_{n\to\infty} n^{1/\varrho}(\lambda_n^{(2)})^{1/n} < d(e\sigma\varrho)^{1/\varrho}.$$

Then, by Lemma 3.4, the type of f is smaller than σ. We have got a contradiction which shows that

$$\limsup_{n\to\infty} n^{1/\varrho}(\lambda_n^{(2)})^{1/n} = d(e\sigma\varrho)^{1/\varrho}.$$

Hence by Lemma 3.2 and from (4.7) we get

$$\limsup_{n\to\infty} n^{1/\varrho}(\lambda_n^{(j)})^{1/n} = d(e\sigma\varrho)^{1/\varrho} \quad \text{for } j = 1, 2, 3.$$

The proof of the first part of Theorem 4.1 is completed.

If $\varrho = 0$, then by the definition of the order of f for every $\varepsilon > 0$ there exists an $R = R(\varepsilon)$ such that

$$\tilde{M}(r) \leqslant e^{r^\varepsilon} \quad \text{for all } r \geqslant R.$$

In the same way as in the proof of (4.6) we have

$$\lambda_n^{(3)} \leqslant (de^\varepsilon)^n \left(\frac{e\varepsilon}{n}\right)^{n/\varepsilon} \quad \text{for } n \geqslant N_1 = N_1(\varepsilon).$$

Hence, and by Lemma 3.2

$$\limsup_{n\to\infty} n^{1/\varepsilon}(\lambda_n^{(j)})^{1/n} < \infty \quad \text{for every } \varepsilon > 0,\ j = 1, 2, 3. \tag{4.8}$$

So, if there existed $a > 0$ such that

$$\limsup_{n\to\infty} n^{a}(\lambda_n^{(3)})^{1/n} > 0,$$

we would have

$$\limsup_{n\to\infty} n^{a+1}(\lambda_n^{(3)})^{1/n} = \infty.$$

This is impossible because of (4.8). The proof is completed.

THEOREM 4.2. *If $d(E) > 0$, then the order of f is given by*

$$\varrho = \limsup_{n\to\infty} \frac{\ln n}{\ln \dfrac{d}{(\lambda_j^{(n)})^{1/n}}} \quad \textit{for } j = 1, 2, 3. \tag{4.9}$$

Proof. We denote the right-hand member of (4.9) by γ_j ($j = 1, 2, 3$) and we start by showing that $\varrho \leqslant \gamma_j$ ($j = 1, 2, 3$). If γ_2 is finite and $\varepsilon > 0$ is fixed, there exists an n_ε such that

$$\frac{\ln n}{\ln \dfrac{d}{(\lambda_n^{(j)})^{1/n}}} \leqslant \gamma_2 + \varepsilon, \quad n > n_\varepsilon.$$

From this we get

$$n^{1/\mu}(\lambda_n^{(2)})^{1/n} \leqslant d\left(e\frac{1}{e\mu}\mu\right)^{1/\mu}, \quad \text{where } \mu = \gamma_2+\varepsilon,\ n > n_\varepsilon.$$

Then, because of Lemma 3.4, there exists an $R > 0$ such that

$$\tilde{M}(r) \leqslant e^{\left(\frac{1}{e\mu}+\varepsilon\right)r^\mu} \quad \text{for } r > R.$$

Therefore $\varrho \leqslant \gamma_2+\varepsilon$ for every $\varepsilon > 0$. Hence $\varrho \leqslant \gamma_2$ provided γ_2 is finite. But the inequality is trivially true if $\gamma_2 = \infty$, so by Lemma 3.2 we have $\varrho \leqslant \gamma_j$ $(j = 1, 2, 3)$, regardless of the value of γ_j.

Suppose that ϱ is finite. Then for any fixed $\varepsilon > 0$ there is a finite $R_1(\varepsilon)$ such that

$$\tilde{M}(r) \leqslant e^{r^{\varrho+\varepsilon}} \quad \text{for } r > R_1(\varepsilon).$$

Then, because of (4.6), we get

$$\lambda_n^{(3)} \leqslant \lambda(de^\varepsilon)^n\left(\frac{ea}{n}\right)^{n/a} \quad \text{for } n \geqslant N(\varepsilon),$$

where $a = \varrho+\varepsilon$.

Therefore by Lemma 3.2 we have

$$\limsup_{n\to\infty} n^{1/a}(\lambda_n^{(j)})^{1/n} \leqslant d(ea)^{1/a} \quad \text{for } j = 1, 2, 3.$$

So there exists an N_1 such that

$$n^{1/a}(\lambda_n^{(j)})^{1/n} \leqslant d(ea+\varepsilon)^{1/a} \quad \text{for } n \geqslant N_1,\ j = 1, 2, 3.$$

Hence

$$a \geqslant \frac{\ln(ea+\varepsilon)^{-1}}{\ln\frac{d}{(\lambda_n^{(j)})^{1/n}}} + \frac{\ln n}{\ln\frac{d}{(\lambda_n^{(j)})^{1/n}}}, \quad n \geqslant N,\ j = 1, 2, 3. \tag{4.10}$$

Since (4.10) holds for all $a > \varrho$, we have $\varrho \geqslant \gamma_j$ $(j = 1, 2, 3)$. This is obviously true if $\varrho = \infty$. Since the opposite inequality holds, we conclude that $\varrho = \gamma_j$ $(j = 1, 2, 3)$ as asserted.

Let f be a function defined and bounded on a compact set E with the positive transfinite diameter d.

Denote by

$$\Gamma(f, E) = \left\{\mu > 0\colon \limsup_{n\to\infty} n^{1/\mu}\big(\mathscr{E}_n(f, E)\big)^{1/n} < \infty\right\}.$$

THEOREM 4.3. *If $\Gamma(f, E) \neq \emptyset$, then the function*

$$\tilde{f}(z) = \pi_1(z) + \sum_{n=2}^{\infty}\big(\pi_n(z)-\pi_{n-1}(z)\big), \quad z \in C$$

is entire and

1° $\tilde{f}(z) = f(z)$ *for* $z \in E$,

2° *order* $\tilde{f} = \varrho$, *where* $\varrho = \inf \Gamma(f, E)$,

3° *if* $0 < \varrho < \infty$, *then*

$$\text{type } \tilde{f} = \begin{cases} \dfrac{\beta^\varrho}{d^\varrho e \varrho}, & \text{when } \beta < \infty, \\ \infty, & \text{when } \beta = \infty, \end{cases}$$

where

$$\beta = \limsup_{n\to\infty} n^{1/\varrho} \big(\mathscr{E}_n(f, E)\big)^{1/n}.$$

Proof. Since $\Gamma(f, E) \neq \emptyset$; then $\lim_{n\to\infty} \big(\mathscr{E}_n(f, E)\big)^{1/n} = 0$. Hence and by the Bernstein-Walsh inequality the function $\tilde{f}$ is entire. Moreover, by Lemma 3.3 for any fixed $\gamma \in \Gamma(f, E)$ and $\varepsilon > 0$ there exists an r_0 such that

$$\tilde{M}(r) \leqslant e^{r^{\gamma+\varepsilon}} \quad \text{for } r \geqslant r_0.$$

Therefore $\tilde{f}$ is an entire function of order $\varrho' \leqslant \varrho$. By Theorem 4.2 we have

$$\varrho' = \limsup_{n\to\infty} \frac{\ln n}{\ln \dfrac{d}{\big(\mathscr{E}_n(f, E)\big)^{1/n}}}.$$

Then for every fixed $\mu > \varrho'$ there is a finite N_μ such that

$$n^{1/\mu} \big(\mathscr{E}_n(f, E)\big)^{1/n} < d.$$

Hence and by the definition of the set $\Gamma(f, E)$ we see that $\mu \in \Gamma(f, E)$. Therefore $\varrho' \geqslant \varrho$. Since the opposite inequality holds, we conclude that $\varrho' = \varrho$ as asserted.

Let $0 < \varrho < \infty$, and $\beta < \infty$. Write β in the form

$$\beta = d(e\sigma\varrho)^{1/\varrho}, \quad \text{where } \sigma = \frac{\beta^\varrho}{d^\varrho e \varrho}.$$

Since

$$\|\pi_n - \pi_{n-1}\| \leqslant \|f - \pi_n\| + \|f - \pi_{n-1}\| \leqslant 2\,\mathscr{E}_{n-1}(f, E),$$

so by Lemma 3.3 the type σ' of the function $\tilde{f}$ is finite and $\sigma' \leqslant \sigma$. Moreover, by Theorem 4.1 we get

$$\limsup_{n\to\infty} n^{1/\varrho} \big(\mathscr{E}_n(f, E)\big)^{1/n} = d(e\sigma'\varrho)^{1/\varrho}.$$

So finally $\sigma = \sigma'$. The proof is completed.

Remark. In Theorem 4.3 $\Gamma(f, E)$ may be replaced by

$$\Gamma_j(f, E) = \{\mu > 0 : \limsup_{n\to\infty} n^{1/\mu} (\lambda_n^{(j)})^{1/n} < \infty\} \quad \text{for } j = 1, 2, 3.$$

5. Coefficients of the Newton series of an entire function. Let $\{z_n\}$ be a bounded sequence of points of C. Put

$$(5.0) \qquad w_n(z) = (z-z_0)\dots(z-z_n) \quad \text{for } n = 0, 1, \dots,$$
$$w_{-1}(z) = 1 \quad \text{for } z \in C.$$

LEMMA 5.1. *If* $f(z) = \sum_{n=0}^{\infty} a_n w_{n-1}(z)$ *is of order* ϱ, $0 < \varrho < \infty$, *and of type* σ, *then*

$$\limsup_{n\to\infty} n^{1/\varrho}|a_n|^{1/n} \leqslant (e\sigma\varrho)^{1/\varrho}.$$

Proof. Let $L = \sup_{n\geqslant 0}|z_n|$. Since

$$\left|\frac{w_n(z)}{z^{n+1}}\right| = \left|\left(1-\frac{z_0}{z}\right)\dots\left(1-\frac{z_n}{z}\right)\right| \geqslant \left(1-\frac{L}{z}\right)^{n+1} \quad \text{for } n = 0, 1, \dots;\ |z| > L,$$

we have

$$\inf_{|z|=r}|w_n(z)| \geqslant \left(1-\frac{L}{r}\right)^{n+1} \quad \text{for } n = 0, 1, \dots;\ r > L.$$

Applying these inequalities and the formula

$$a_n = \frac{1}{2\pi i}\int_{|z|=r}\frac{f(\zeta)}{w_n(\zeta)}d\zeta \quad (r > M),$$

we get

$$(5.1) \qquad |a_n| \leqslant \left(\frac{r}{r-L}\right)^{n+1}\frac{M(r)}{r^n} \quad \text{for } n \geqslant 1,\ r > L,$$

where $M(r)$ is defined by (3.1). By the definition of the type σ for any fixed $K > \sigma$ there exists a finite r_K such that

$$M(r) \leqslant e^{Kr^\varrho} \quad \text{for all } r \geqslant r_K.$$

Let n_0 be such that $\left(\frac{n}{K\varrho}\right)^{1/\varrho} > L$ for all $n \geqslant n_0$. Now, for any fixed n the minimum value $\left(\frac{eK\varrho}{n}\right)^{n/\varrho}$ of the expression $r^{-n}\exp(Kr^\varrho)$ is reached for $r = \left(\frac{n}{K\varrho}\right)^{1/\varrho}$. It follows that

$$|a_n| \leqslant \left(\frac{n^{1/\varrho}}{n^{1/\varrho}-(K\varrho)^{1/\varrho}L}\right)^{n+1}\left(\frac{eK\varrho}{n}\right)^{1/\varrho} \quad \text{for } n \geqslant n_0.$$

Hence

$$\limsup_{n\to\infty} n^{1/\varrho}|a_n|^{1/n} \leqslant (eK\varrho)^{1/\varrho}.$$

Thus

$$\limsup_{n\to\infty} n^{1/\varrho}|a_n|^{1/n} \leqslant (e\sigma\varrho)^{1/\varrho}.$$

The proof is completed.

LEMMA 5.2. *If* $\limsup_{n\to\infty} n^{1/\varrho}|a_n|^{1/n} \leqslant (e\sigma\varrho)^{1/\varrho}$, *then*

$$f(z) = \sum_{n=0}^{\infty} a_n w_{n-1}(z)$$

is an entire function and for every $\varepsilon > 0$ *there exists an* R_ε *such that*

$$M(r) \leqslant e^{(\sigma+\varepsilon)r^\varrho} \quad \textit{for all } r > R_\varepsilon.$$

Proof. Let $\varepsilon > 0$ and $K = \sigma+\varepsilon/2$. Taking N_ε and $R > 1$ sufficiently large, we have

$$|a_n| \leqslant \left(\frac{eK\varrho}{n}\right)^{n/\varrho} \quad \text{for } n \geqslant N_\varepsilon,$$

and

$$\left|1-\frac{z_j}{z}\right| \leqslant e^{\varepsilon} \quad \text{for } |z| \geqslant R,\ j = 1, 2, \ldots$$

Hence

$$M(r) \leqslant \sum_{n=0}^{N_\varepsilon} |a_n| e^{n\varepsilon} r^n + \sum_{n=N_\varepsilon+1}^{\infty} e^{n\varepsilon}\left(\frac{eK\varrho}{n}\right)^{n/\varrho} r^n \quad \text{for } r > R.$$

Therefore, repeating the proof of Lemma 3.3, we get the proof of Lemma 5.2.

If c_n $(n = 0, 1, \ldots)$ is the n-th coefficient of the Taylor series of an entire function f of order ϱ, $0 < \varrho < \infty$, and type σ, then [4]

$$\limsup_{n\to\infty} n^{1/\varrho}|c_n|^{1/n} = (e\sigma\varrho)^{1/\varrho}.$$

This formula may be generalized as follows.

THEOREM 5.1. *Let* $\{z_n\}$ *be a bounded sequence of complex numbers. If* $f(z) = \sum_{n=0}^{\infty} a_n w_{n-1}(z)$ *is an entire function of order* ϱ $(0 < \varrho < \infty)$ *and of type* σ, *then*

$$\limsup_{n\to\infty} n^{1/\varrho}|a_n|^{1/n} = (e\sigma\varrho)^{1/\varrho}.$$

Proof. Let $\sigma < \infty$. By Lemma 5.1 we get

$$\limsup_{n\to\infty} n^{1/\varrho}|a_n|^{1/n} = \beta \leqslant (e\sigma\varrho)^{1/\varrho}. \tag{5.2}$$

Write β in the form

$$\beta = (e\sigma'\varrho)^{1/\varrho}, \quad \text{where } \sigma' = \frac{\beta^\varrho}{e\varrho}.$$

Now, by Lemma 5.2, for any fixed $K > \sigma'$ there exists a finite $A = A(K)$ such that

$$M(r) \leqslant e^{Kr^\varrho} \quad \text{for } r \geqslant A.$$

Thus, by the definition of type of f we have $\sigma' \geqslant \sigma$.

Therefore we must have

$$\beta = (e\varrho\sigma')^{1/\varrho} \geqslant (e\varrho\sigma)^{1/\varrho}.$$

Since the opposite inequality holds, we conclude that $\sigma' = \sigma$, as asserted.

Let $\sigma = \infty$. If

$$\limsup_{n\to\infty} n^{1/\varrho}|a_n|^{1/n} < \infty,$$

then Lemma 5.2 would imply that the type σ of the function f is finite. This contradiction concludes the proof.

Applying the method of proving Theorems 4.2 and 4.3, respectively, one can get the folloving two theorems.

THEOREM 5.2. *If f is of order ϱ, then*

$$\varrho = \limsup_{n\to\infty} \frac{\ln n}{\ln \dfrac{1}{|a_n|^{1/n}}}.$$

THEOREM 5.3. *If the set Γ defined by*

$$\Gamma = \{\mu > 0 \colon \limsup_{n\to\infty} n^{1/\mu}|a_n|^{1/n} < \infty\}$$

in non-empty, then

$$f(z) = \sum_{n=0}^{\infty} a_n w_{n-1}(z)$$

is an entire function of order ϱ, where $\varrho = \inf\Gamma$, and if $0 < \varrho < \infty$ the type of f is given by

$$\sigma = \frac{1}{e\varrho} \limsup_{n\to\infty} n|a_n|^{\varrho/n}.$$

Let $\{z_n\}$ be an extremal sequence of a compact set $E \subset C$ and let f be an entire function of order ϱ, $0 < \varrho < \infty$, and type σ, $\sigma < \infty$. Then

$$f(z) = \sum_{n=0}^{\infty} a_n w_{n-1}(z) \quad \text{for } z \in C,$$

where $w_n(z)$ is defined by (5.0).

Writing

$$p_n(z) = a_0 + a_1 w_0(z) + \ldots + a_n w_{n-1}(z) \quad \text{for } n = 1, 2, \ldots$$

we have

$$\mathscr{E}_n(f, E) \leqslant \|f - p_n\| \leqslant \sum_{\nu=n+1}^{\infty} |a_\nu| \|w_{\nu-1}(z_\nu)\| \quad \text{for } n = 1, 2, \ldots$$

Hence by Theorem 5.1 and by (2.6) for every $K > \sigma$ and any fixed $\varepsilon > 0$ there exists an $n_0 = n_0(K, \varepsilon)$ such that

$$\mathscr{E}_n(f, E) \leqslant \sum_{\nu=n}^{\infty} \left(\frac{eK\varrho}{\nu}\right)^{\nu/\varrho} (d+\varepsilon)^\nu \quad \text{for } n \geqslant n_0,$$

where d is the transfinite diameter of the compact E. Therefore

$$\mathscr{E}_n(f, E) \leqslant \left(\frac{eK\varrho}{n}\right)^{n/\varrho} (d+\varepsilon)^n \left[1 + \sum_{\nu=1}^{\infty} \left(\frac{n}{n+\nu}\right)^{n/\varrho} \left(\frac{eK\varrho}{n+\nu}\right)^{\nu/\varrho} (d+\varepsilon)^\nu\right]$$

$$\text{for } n \geqslant n_0.$$

It is clear that the series

$$\beta_n = \sum_{\nu=1}^{\infty} \left(\frac{n}{n+\nu}\right)^{n/\varrho} \left(\frac{eK\varrho}{n+\nu}\right)^{\nu/\varrho} (d+\varepsilon)^\nu, \quad n = 1, 2, \ldots$$

is convergent and there exists an $M < \infty$ such that $\beta_n < M$ for all n. Hence and by Lemma 3.2 we get

$$\limsup_{n \to \infty} n^{1/\varrho} (\lambda_n^{(j)})^{1/n} \leqslant d(e\varrho\sigma)^{1/\varrho} \quad \text{for } j = 1, 2, 3, \tag{5.3}$$

where $\lambda_n^{(j)}$ is defined by (4.0).

We note that inequality (5.3) holds also in the case where the transfinite diameter of E is equal to zero. Thus, we can drop in both parts of Theorem 4.1 the assumption that $d(E) > 0$.

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