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APPROXIMATION AND SHAPE PRESERVING PROPERTIES OF THE NONLINEAR FAVARD-SZÁSZ-MIRAKJAN OPERATOR OF MAX-PRODUCT KIND

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Abstract

Starting from the study of the *Shepard nonlinear operator of max-prod type* in [6], [7], in the book [8], Open Problem 5.5.4, pp. 324-326, the *Favard-Szász-Mirakjan max-prod type operator* is introduced and the question of the approximation order by this operator is raised. In the recent paper [1], by using a pretty complicated method to this open question an answer is given by obtaining an upper pointwise estimate of the approximation error of the form $C\omega_1(f;\sqrt{x}/\sqrt{n})$ (with an unexplicit absolute constant $C>0$) and the question of improving the order of approximation $\omega_1(f;\sqrt{x}/\sqrt{n})$ is raised. The first aim of this note is to obtain the same order of approximation but by a simpler method, which in addition presents, at least, two advantages : it produces an explicit constant in front of $\omega_1(f;\sqrt{x}/\sqrt{n})$ and it can easily be extended to other max-prod operators of Bernstein type. Also, we prove by a counterexample that in some sense, in general this type of order of approximation with respect to $\omega_1(f;\cdot)$ cannot be improved. However, for some subclasses of functions, including for example the bounded, nondecreasing concave functions, the essentially better order $\omega_1(f;1/n)$ is obtained. Finally, some shape preserving properties are obtained.

1 Introduction

Starting from the study of the *Shepard nonlinear operator of max-prod type* in [6], [7], by the Open Problem 5.5.4, pp. 324-326 in the recent monograph [8], the following *nonlinear Favard-Szász-Mirakjan max-prod operator* is introduced (here $\bigvee$ means maximum)

$$F_n^{(M)}(f)(x)=\frac{\bigvee_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right)}{\bigvee_{k=0}^{\infty}\frac{(nx)^k}{k!}},$$

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for which by a pretty complicated method in [1], Theorem 8, the order of pointwise approximation $\omega_1(f;\sqrt{x}/\sqrt{n})$ is obtained. Also, by Remark 9, 2) in the same paper [1], the question if this order of approximation could be improved is raised.

The first aim of this note is to obtain the same order of approximation but by a simpler method, which in addition presents, at least, two advantages : it produces an explicit constant in front of $\omega_1(f;\sqrt{x}/\sqrt{n})$ and it can easily be extended to various max-prod operators of Bernstein type, see [2] – [5]. Also, one proves by a counterexample that in some sense, in general this type of order of approximation with respect to $\omega_1(f;\cdot)$ cannot be improved, giving thus a negative answer to a question raised in [1] (see Remark 9, 2) there). However, for some subclasses of functions, including for example the bounded, nondecreasing concave functions, the essentially better order $\omega_1(f;1/n)$ is obtained. This allows us to put in evidence large classes of functions (e.g. bounded, nondecreasing concave polygonal lines on $[0,\infty)$) for which the order of approximation given by the max-product Favard-Szász-Mirakjan operator, is essentially better than the order given by the linear Favard-Szász-Mirakjan operator. Finally, some shape preserving properties are presented.

Section 2 presents some general results on nonlinear operators, in Section 3 we prove several auxiliary lemmas, Section 4 contains the approximation results, while in Section 5 we present some shape preserving properties.

2 Preliminaries

For the proof of the main result we need some general considerations on the so-called nonlinear operators of max-prod kind. Over the set of positive reals, $\mathbb{R}_+$, we consider the operations $\vee$ (maximum) and $\cdot$, product. Then $(\mathbb{R}_+,\vee,\cdot)$ has a semiring structure and we call it as Max-Product algebra.

Let $I\subset\mathbb{R}$ be a bounded or unbounded interval, and

$$CB_+(I)=\{f:I\to\mathbb{R}_+; f \text{ continuous and bounded on } I\}.$$

The general form of $L_n: CB_+(I)\to CB_+(I)$, (called here a discrete max-product type approximation operator) studied in the paper will be

$$L_n(f)(x)=\bigvee_{i=0}^{n}K_n(x,x_i)\cdot f(x_i),$$

or

$$L_n(f)(x)=\bigvee_{i=0}^{\infty}K_n(x,x_i)\cdot f(x_i),$$

where $n\in\mathbb{N}$, $f\in CB_+(I)$, $K_n(\cdot,x_i)\in CB_+(I)$ and $x_i\in I$, for all i. These operators are nonlinear, positive operators and moreover they satisfy a pseudo-linearity condition of the form

$$L_n(\alpha\cdot f\vee\beta\cdot g)(x)=\alpha\cdot L_n(f)(x)\vee\beta\cdot L_n(g)(x), \forall\alpha,\beta\in\mathbb{R}_+, f,g:I\to\mathbb{R}_+.$$

In this section we present some general results on these kinds of operators which will be useful later in the study of the Favard-Szász-Mirakjan max-product kind operator considered in Introduction.

Lemma 2.1. ([1]) *Let $I \subset \mathbb{R}$ be a bounded or unbounded interval,*

$$CB_+(I) = \{f : I \to \mathbb{R}_+; f \text{ continuous and bounded on } I\},$$

and $L_n : CB_+(I) \to CB_+(I)$, $n \in \mathbb{N}$ be a sequence of operators satisfying the following properties :

(i) if $f, g \in CB_+(I)$ satisfy $f \le g$ then $L_n(f) \le L_n(g)$ for all $n \in N$;

(ii) $L_n(f+g) \le L_n(f) + L_n(g)$ for all $f, g \in CB_+(I)$.

Then for all $f, g \in CB_+(I)$, $n \in N$ and $x \in I$ we have

$$|L_n(f)(x) - L_n(g)(x)| \le L_n(|f-g|)(x).$$

Proof. Since is very simple, we reproduce here the proof in [1]. Let $f, g \in CB_+(I)$. We have $f = f - g + g \le |f-g| + g$, which by the conditions $(i)-(ii)$ successively implies $L_n(f)(x) \le L_n(|f-g|)(x) + L_n(g)(x)$, that is $L_n(f)(x) - L_n(g)(x) \le L_n(|f-g|)(x)$.

Writing now $g = g - f + f \le |f-g| + f$ and applying the above reasonings, it follows $L_n(g)(x) - L_n(f)(x) \le L_n(|f-g|)(x)$, which combined with the above inequality gives $|L_n(f)(x) - L_n(g)(x)| \le L_n(|f-g|)(x)$. □

Remarks. 1) It is easy to see that the Favard-Szász-Mirakjan max-product operator satisfy the conditions in Lemma 2.1, (i), (ii). In fact, instead of (i) it satisfies the stronger condition

$$L_n(f \vee g)(x) = L_n(f)(x) \vee L_n(g)(x),\ f, g \in CB_+(I).$$

Indeed, taking in the above equality $f \le g$, $f, g \in CB_+(I)$, it easily follows $L_n(f)(x) \le L_n(g)(x)$.

2) In addition, it is immediate that the Favard-Szász-Mirakjan max-product operator is positive homogenous, that is $L_n(\lambda f) = \lambda L_n(f)$ for all $\lambda \ge 0$.

Corollary 2.2. ([1]) *Let $L_n : CB_+(I) \to CB_+(I)$, $n \in N$ be a sequence of operators satisfying the conditions (i)-(ii) in Lemma 1 and in addition being positive homogenous. Then for all $f \in CB_+(I)$, $n \in N$ and $x \in I$ we have*

$$|f(x) - L_n(f)(x)| \le \left[\frac{1}{\delta} L_n(\varphi_x)(x) + L_n(e_0)(x)\right] \omega_1(f;\delta)_I + f(x) \cdot |L_n(e_0)(x) - 1|,$$

where $\delta > 0$, $e_0(t) = 1$ for all $t \in I$, $\varphi_x(t) = |t-x|$ for all $t \in I$, $x \in I$, $\omega_1(f;\delta)_I = \max\{|f(x) - f(y)|; x, y \in I, |x-y| \le \delta\}$ and if I is unbounded then we suppose that there exists $L_n(\varphi_x)(x) \in \mathbb{R}_+ \bigcup \{+\infty\}$, for any $x \in I$, $n \in \mathbb{N}$.

Proof. The proof is identical with that for positive linear operators and because of its simplicity we reproduce it below. Indeed, from the identity

$$L_n(f)(x) - f(x) = [L_n(f)(x) - f(x) \cdot L_n(e_0)(x)] + f(x)[L_n(e_0)(x) - 1],$$

it follows (by the positive homogeneity and by Lemma 2.1)

$$|f(x) - L_n(f)(x)| \le |L_n(f(x))(x) - L_n(f(t))(x)| + |f(x)| \cdot |L_n(e_0)(x) - 1| \le$$

$$L_n(|f(t) - f(x)|)(x) + |f(x)| \cdot |L_n(e_0)(x) - 1|.$$

Now, since for all $t, x \in I$ we have

$$|f(t) - f(x)| \le \omega_1(f; |t-x|)_I \le \left[\frac{1}{\delta}|t-x| + 1\right] \omega_1(f; \delta)_I,$$

replacing above we immediately obtain the estimate in the statement. □

An immediate consequence of Corollary 2.2 is the following.

Corollary 2.3. ([1]) *Suppose that in addition to the conditions in Corollary 2.2, the sequence $(L_n)_n$ satisfies $L_n(e_0) = e_0$, for all $n \in N$. Then for all $f \in CB_+(I)$, $n \in N$ and $x \in I$ we have*

$$|f(x) - L_n(f)(x)| \le \left[1 + \frac{1}{\delta} L_n(\varphi_x)(x)\right] \omega_1(f; \delta)_I.$$

3 Auxiliary Results

Since it is easy to check that $F_n^{(M)}(f)(0) - f(0) = 0$ for all n, notice that in the notations, proofs and statements of the all approximation results, that is in Lemmas 3.1-3.3, Theorem 4.1, Lemma 4.2, Corollary 4.4, Corollary 4.5, in fact we always may suppose that $x > 0$.

For each $k, j \in \{0, 1, 2, ..., \}$ and $x \in [\frac{j}{n}, \frac{j+1}{n}]$, let us denote $s_{n,k}(x) = \frac{(nx)^k}{k!}$,

$$M_{k,n,j}(x) = \frac{s_{n,k}(x)\left|\frac{k}{n} - x\right|}{s_{n,j}(x)}, m_{k,n,j}(x) = \frac{s_{n,k}(x)}{s_{n,j}(x)}.$$

It is clear that if $k \ge j + 1$ then

$$M_{k,n,j}(x) = \frac{s_{n,k}(x)(\frac{k}{n} - x)}{s_{n,j}(x)}$$

and if $k \le j - 1$ then

$$M_{k,n,j}(x) = \frac{s_{n,k}(x)(x - \frac{k}{n})}{s_{n,j}(x)}.$$

Lemma 3.1. *For all $k, j \in \{0, 1, 2, ..., \}$ and $x \in [\frac{j}{n}, \frac{j+1}{n}]$ we have*

$$m_{k,n,j}(x) \le 1.$$

Proof. We have two cases : 1) $k \ge j$ and 2) $k \le j$.

Case 1). Since clearly the function $h(x)=\frac{1}{x}$ is nonincreasing on $[j/n,(j+1)/n]$, it follows

$$\frac{m_{k,n,j}(x)}{m_{k+1,n,j}(x)}=\frac{k+1}{n}\cdot\frac{1}{x}\geq\frac{k+1}{n}\cdot\frac{n}{j+1}=\frac{k+1}{j+1}\geq 1,$$

which implies $m_{j,n,j}(x)\geq m_{j+1,n,j}(x)\geq m_{j+2,n,j}(x)\geq ....$

Case 2). We get

$$\frac{m_{k,n,j}(x)}{m_{k-1,n,j}(x)}=\frac{nx}{k}\geq\frac{n}{k}\cdot\frac{j}{n}=\frac{j}{k}\geq 1,$$

which immediately implies

$$m_{j,n,j}(x)\geq m_{j-1,n,j}(x)\geq m_{j-2,n,j}(x)\geq ...\geq m_{0,n,j}(x).$$

Since $m_{j,n,j}(x)=1$, the conclusion of the lemma is immediate. □

Lemma 3.2. *Let* $x\in[\frac{j}{n},\frac{j+1}{n}]$.

(i) If $k\in\{j+1,j+3,...,\}$ *is such that* $k-\sqrt{k+1}\geq j$, *then* $M_{k,n,j}(x)\geq M_{k+1,n,j}(x)$.

(ii) If $k\in\{1,2,...j-1\}$ *is such that* $k+\sqrt{k}\leq j$, *then* $M_{k,n,j}(x)\geq M_{k-1,n,j}(x)$.

Proof. (i) We observe that

$$\frac{M_{k,n,j}(x)}{M_{k+1,n,j}(x)}=\frac{k+1}{n}\cdot\frac{1}{x}\cdot\frac{\frac{k}{n}-x}{\frac{k+1}{n}-x}.$$

Since the function $g(x)=\frac{1}{x}\cdot\frac{\frac{k}{n}-x}{\frac{k+1}{n}-x}$ clearly is nonincreasing, it follows that $g(x)\geq g(\frac{j+1}{n})=\frac{n}{j+1}\cdot\frac{k-j-1}{k-j}$ for all $x\in[\frac{j}{n},\frac{j+1}{n}]$.

Then, since the condition $k-\sqrt{k+1}\geq j$ implies $(k+1)(k-j-1)\geq(j+1)(k-j)$, we obtain

$$\frac{M_{k,n,j}(x)}{M_{k+1,n,j}(x)}\geq\frac{k+1}{n}\cdot\frac{n}{j+1}\cdot\frac{k-j-1}{k-j}\geq 1.$$

(ii) We observe that

$$\frac{M_{k,n,j}(x)}{M_{k-1,n,j}(x)}=\frac{n}{k}\cdot x\cdot\frac{x-\frac{k}{n}}{x-\frac{k-1}{n}}.$$

Since the function $h(x)=x\cdot\frac{x-\frac{k}{n}}{x-\frac{k-1}{n}}$ is nondecreasing, it follows that $h(x)\geq h(\frac{j}{n})=\frac{j}{n}\cdot\frac{j-k}{j-k+1}$ for all $x\in[\frac{j}{n},\frac{j+1}{n}]$.

Then, since the condition $k+\sqrt{k}\leq j$ implies $j(j-k)\geq k(j-k+1)$, we obtain

$$\frac{M_{k,n,j}(x)}{M_{k-1,n,j}(x)}\geq\frac{n}{k}\cdot\frac{j}{n}\cdot\frac{j-k}{j-k+1}\geq 1,$$

which proves the lemma. □

Also, a key result in the proof of the main result is the following.

Lemma 3.3. *Denoting* $s_{n,k}(x)=\frac{(nx)^k}{k!}$*, we have*

$$\bigvee_{k=0}^{\infty}\frac{(nx)^k}{k!}=s_{n,j}(x),\ \textit{for all } x\in\left[\frac{j}{n},\frac{j+1}{n}\right],\ j=0,1,...,.$$

Proof. First we show that for fixed $n\in\mathbb{N}$ and $0\le k$ we have

$$0\le s_{n,k+1}(x)\le s_{n,k}(x),\ \text{ if and only if } x\in[0,(k+1)/n].$$

Indeed, the inequality one reduces to

$$0\le\frac{(nx)^{k+1}}{(k+1)!}\le\frac{(nx)^k}{k!},$$

which after simplifications is obviously equivalent to

$$0\le x\le\frac{k+1}{n}.$$

By taking $k=0,1,..,$ in the inequality just proved above, we get

$$s_{n,1}(x)\le s_{n,0}(x),\ \text{ if and only if } x\in[0,1/n],$$

$$s_{n,2}(x)\le s_{n,1}(x),\ \text{ if and only if } x\in[0,2/n],$$

$$s_{n,3}(x)\le s_{n,2}(x),\ \text{ if and only if } x\in[0,3/n],$$

so on,

$$s_{n,k+1}(x)\le s_{n,k}(x),\ \text{ if and only if } x\in[0,(k+1)/n],$$

and so on.

From all these inequalities, reasoning by recurrence we easily obtain :

$$\text{if } x\in[0,1/n] \text{ then } s_{n,k}(x)\le s_{n,0}(x),\ \text{ for all } k=0,1,...,$$

$$\text{if } x\in[1/n,2/n] \text{ then } s_{n,k}(x)\le s_{n,1}(x),\ \text{ for all } k=0,1,...,$$

$$\text{if } x\in[2/n,3/n] \text{ then } s_{n,k}(x)\le s_{n,2}(x),\ \text{ for all } k=0,1,...,$$

and so on, in general

$$\text{if } x\in[j/n,(j+1)/n] \text{ then } s_{n,k}(x)\le s_{n,j}(x),\ \text{ for all } k=0,1,...,$$

which proves the lemma. □

4 Approximation Results

If $F_n^{(M)}(f)(x)$ represents the nonlinear Favard-Szász-Mirakjan operator of max-product type defined in Introduction, then the main result is the following.

Theorem 4.1. *Let $f : [0, \infty) \to \mathbb{R}_+$ be bounded and continuous on $[0, \infty)$. Then we have the estimate*

$$|F_n^{(M)}(f)(x) - f(x)| \leq 8\omega_1\left(f, \frac{\sqrt{x}}{\sqrt{n}}\right), \textit{for all } n \in \mathbb{N}, x \in [0, \infty),$$

where

$$\omega_1(f, \delta) = \sup\{|f(x) - f(y)|; x, y \in [0, \infty), |x - y| \leq \delta\}.$$

Proof. It is easy to check that the max-product Favard-Szász-Mirakjan operators fulfil the conditions in Corollary 2.3 and we have

$$|F_n^{(M)}(f)(x) - f(x)| \leq \left(1 + \frac{1}{\delta_n} F_n^{(M)}(\varphi_x)(x)\right)\omega_1(f, \delta_n), \tag{1}$$

where $\varphi_x(t) = |t - x|$. So, it is enough to estimate

$$E_n(x) := F_n^{(M)}(\varphi_x)(x) = \frac{\bigvee_{k=0}^{\infty} \frac{(nx)^k}{k!}\left|\frac{k}{n} - x\right|}{\bigvee_{k=0}^{\infty} \frac{(nx)^k}{k!}}, x \in [0, \infty).$$

Let $x \in [j/n, (j+1)/n]$, where $j \in \{0, 1, ..., \}$ is fixed, arbitrary. By Lemma 3.3 we easily obtain

$$E_n(x) = \max_{k=0,1,...,}\{M_{k,n,j}(x)\}, x \in [j/n, (j+1)/n].$$

In all what follows we may suppose that $j \in \{1, 2, ..., \}$, because for $j = 0$ we get $E_n(x) \leq \frac{\sqrt{x}}{\sqrt{n}}$, for all $x \in [0, 1/n]$. Indeed, in this case we obtain $M_{k,n,0}(x) = \frac{(nx)^k}{k!}\left|\frac{k}{n} - x\right|$, which for $k = 0$ gives $M_{k,n,0}(x) = x = \sqrt{x} \cdot \sqrt{x} \leq \sqrt{x} \cdot \frac{1}{\sqrt{n}}$. Also, for any $k \geq 1$ we have $\frac{1}{n} \leq \frac{k}{n}$ and we obtain

$$M_{k,n,0}(x) \leq \frac{(nx)^k}{k!} \cdot \frac{k}{n} = \sqrt{x} \cdot \frac{n^{k-1}x^{k-1/2}}{(k-1)!} \leq \sqrt{x} \cdot \frac{n^{k-1}}{(k-1)!n^{k-1/2}} \leq \frac{\sqrt{x}}{\sqrt{n}}.$$

So it remains to obtain an upper estimate for each $M_{k,n,j}(x)$ when $j = 1, 2, ...,$ is fixed, $x \in [j/n, (j+1)/n]$ and $k = 0, 1, ...,$. In fact we will prove that

$$M_{k,n,j}(x) \leq \frac{4\sqrt{x}}{\sqrt{n}}, \text{ for all } x \in [j/n, (j+1)/n], k = 0, 1, ..., \tag{2}$$

which immediately will imply that

$$E_n(x) \leq \frac{4\sqrt{x}}{\sqrt{n}}, \text{ for all } x \in [0, \infty), n \in \mathbb{N},$$

and taking $\delta_n = \frac{4\sqrt{x}}{\sqrt{n}}$ in (1) we immediately obtain the estimate in the statement.

In order to prove (2) we distinguish the following cases :

1) $k = j$; 2) $k \geq j+1$ and 3) $k \leq j-1$.

Case 1). If $k = j$ then $M_{j,n,j}(x) = \left|\frac{j}{n} - x\right|$. Since $x \in [\frac{j}{n}, \frac{j+1}{n}]$, it easily follows that $M_{j,n,j}(x) \leq \frac{1}{n}$. Now, since $j \geq 1$ we get $x \geq \frac{1}{n}$, which implies $\frac{1}{n} = \frac{1}{\sqrt{n}} \cdot \frac{1}{\sqrt{n}} \leq \sqrt{x} \cdot \frac{1}{\sqrt{n}}$.

Case 2). Subcase a). Suppose first that $k - \sqrt{k+1} < j$. We get

$$M_{k,n,j}(x) = m_{k,n,j}(x)(\frac{k}{n} - x) \leq \frac{k}{n} - x \leq \frac{k}{n} - \frac{j}{n} \leq$$

$$\frac{k}{n} - \frac{k - \sqrt{k+1}}{n} = \frac{\sqrt{k+1}}{n}.$$

But we necessarily have $k \leq 3j$. Indeed, if we suppose that $k > 3j$, then because $g(x) = x - \sqrt{x+1}$ is nondecreasing, it follows $j > k - \sqrt{k+1} \geq 3j - \sqrt{3j+1}$, which implies the obvious contradiction $j > 3j - \sqrt{3j+1}$.

In conclusion, we obtain

$$M_{k,n,j}(x) \leq \frac{\sqrt{k+1}}{n} \leq \frac{\sqrt{3j+1}}{n} \leq 2\frac{\sqrt{j}}{n} \leq 2\frac{\sqrt{x}}{\sqrt{n}},$$

taking into account that $\sqrt{x} \geq \frac{\sqrt{j}}{\sqrt{n}}$.

Subcase b). Suppose now that $k - \sqrt{k+1} \geq j$. Since the function $g(x) = x - \sqrt{x+1}$ is nondecreasing on the interval $[0, \infty)$ it follows that there exists $\overline{k} \in \{1, 2, ..., \}$, of maximum value, such that $\overline{k} - \sqrt{\overline{k}+1} < j$. Then for $k_1 = \overline{k} + 1$ we get $k_1 - \sqrt{k_1+1} \geq j$ and

$$\begin{aligned} M_{\overline{k}+1,n,j}(x) &= m_{\overline{k}+1,n,j}(x)(\frac{\overline{k}+1}{n} - x) \leq \frac{\overline{k}+1}{n} - x \leq \frac{\overline{k}+1}{n} - \frac{j}{n} \\ &\leq \frac{\overline{k}+1}{n} - \frac{\overline{k} - \sqrt{\overline{k}+1}}{n} = \frac{\sqrt{\overline{k}+1}+1}{n} \leq 3\frac{\sqrt{x}}{\sqrt{n}}. \end{aligned}$$

The last above inequality follows from the fact that $\overline{k} - \sqrt{\overline{k}+1} < j$ necessarily implies $\overline{k} \leq 3j$ (see the similar reasonings in in the above subcase a)). Also, we have $k_1 \geq j+1$. Indeed, this is a consequence of the fact that g is nondecreasing and because is easy to see that $g(j) < j$.

By Lemma 3.2, (i) it follows that $M_{\overline{k}+1,n,j}(x) \geq M_{\overline{k}+2,n,j}(x) \geq ....$ We thus obtain $M_{k,n,j}(x) \leq 3\frac{\sqrt{x}}{\sqrt{n}}$ for any $k \in \{\overline{k}+1, \overline{k}+2, ..., \}$.

Case 3). Subcase a). Suppose first that $k + \sqrt{k} > j$. Then we obtain

$$\begin{aligned} M_{k,n,j}(x) &= m_{k,n,j}(x)(x - \frac{k}{n}) \leq \frac{j+1}{n} - \frac{k}{n} \leq \frac{k + \sqrt{k} + 1}{n} - \frac{k}{n} \\ &= \frac{\sqrt{k}+1}{n} \leq \frac{\sqrt{j-2}+1}{n} = \frac{1}{\sqrt{n}} \cdot \frac{\sqrt{j-2}+1}{\sqrt{n}} \leq 2\frac{\sqrt{x}}{\sqrt{n}}, \end{aligned}$$

taking into account that $\frac{\sqrt{j-2}+1}{\sqrt{n}} \leq 2\frac{\sqrt{j}}{\sqrt{n}} \leq 2\sqrt{x}$.

Subcase b). Suppose now that $k+\sqrt{k} \leq j$. Let $\widetilde{k} \in \{0, 1, 2, ..., \}$ be the minimum value such that $\widetilde{k} + \sqrt{\widetilde{k}} > j$. Then $k_2 = \widetilde{k} - 1$ satisfies $k_2 + \sqrt{k_2} \leq j$ and

$$\begin{aligned} M_{\widetilde{k}-1,n,j}(x) &= m_{\widetilde{k}-1,n,j}(x)(x - \frac{\widetilde{k}-1}{n}) \leq \frac{j+1}{n} - \frac{\widetilde{k}-1}{n} \\ &\leq \frac{\widetilde{k}+\sqrt{\widetilde{k}}+1}{n} - \frac{\widetilde{k}-1}{n} = \frac{\sqrt{\widetilde{k}}+2}{n} \leq 4\frac{\sqrt{x}}{\sqrt{n}}. \end{aligned}$$

For the last inequality we used the obvious relationship $\widetilde{k} - 1 = k_2 \leq k_2 + \sqrt{k_2} \leq j$, which implies $\widetilde{k} \leq j+1$ and $\sqrt{\widetilde{k}} + 2 \leq \sqrt{j+1} + 2 \leq 4\sqrt{j}$. Also, because $j \geq 1$ it is immediate that $k_2 \leq j - 1$.

By Lemma 3.2, (ii) it follows that $M_{\widetilde{k}-1,n,j}(x) \geq M_{\widetilde{k}-2,n,j}(x) \geq ... \geq M_{0,n,j}(x)$. We thus obtain $M_{k,n,j}(x) \leq 4\frac{\sqrt{x}}{\sqrt{n}}$ for any $k \leq j-2$ and $x \in [\frac{j}{n}, \frac{j+1}{n}]$.

Collecting all the above estimates we get (2), which completes the proof. $\square$

Remark. It is clear that on each compact subinterval $[0, a]$, with arbitrary $a > 0$, the order of approximation in Theorem 4.1 is $\mathcal{O}(1/\sqrt{n})$. In what follows, we will prove that this order cannot be improved. In this sense, first we observe that

$$\begin{aligned} M_{k,n,j}(x) &= (nx)^{k-j}\frac{j!}{k!}\left|\frac{k}{n} - x\right| = (nx)^{k-j}\frac{1}{(j+1)(j+2)...k}\left|\frac{k}{n} - x\right| \\ &\geq (nx)^{k-j}\frac{1}{k^{k-j}}\left|\frac{k}{n} - x\right| = \left(\frac{nx}{k}\right)^{k-j}\left|\frac{k}{n} - x\right| \end{aligned}$$

for any $k > j$.

Now, for $n \in \mathbb{N}$ and $a > 0$, let us denote $j_n = [na]$, $k_n = [na] + [\sqrt{n}]$, $x_n = \frac{[na]}{n}$. Then

$$\begin{aligned} M_{k_n,n,j_n}(x_n) &\geq \left(\frac{[na]}{[na]+[\sqrt{n}]}\right)^{[\sqrt{n}]}\frac{[\sqrt{n}]}{n} > \left(\frac{na-1}{na+\sqrt{n}}\right)^{\sqrt{n}}\frac{\sqrt{n}-1}{n} \\ &\geq \left(\frac{na-1}{na+\sqrt{n}}\right)^{\sqrt{n}}\frac{1}{2\sqrt{n}} \end{aligned}$$

for any $n \geq \max\{4, 1/a\}$. Because $\lim_{n\to\infty}\left(\frac{na-1}{na+\sqrt{n}}\right)^{\sqrt{n}} = e^{-1/a}$ it follows that there exists $n_0 \in \mathbb{N}$, $n_0 \geq \max\{4, 1/a\}$, such that

$$\left(\frac{na-1}{na+\sqrt{n}}\right)^{\sqrt{n}} \geq e^{-1-1/a},$$

for any $n \geq n_0$. Then we get

$$M_{k_n,n,j_n}(x_n) \geq \frac{1}{2}e^{-1-1/a}\frac{1}{\sqrt{n}}.$$

Since $x_n \le a$ and $\lim_{n\to\infty} x_n = a$, we get $x_n \in [0,a]$ for any $n \in \mathbb{N}$, and combining that with the relationship (2) in the proof of Theorem 4.1, it easily implies that $\frac{1}{\sqrt{n}}$, the order of $\max_{x\in[0,a]}\{E_n(x)\}$, cannot be made smaller. Finally, this implies that the order of approximation $\omega_1(f;1/\sqrt{n})$ on $[0,a]$ obtained by the statement of Theorem 4.1, cannot be improved.

In what follows we will prove that for some subclasses of functions f, the order of approximation $\omega_1(f;\sqrt{x}/\sqrt{n})$ in Theorem 4.1 can essentially be improved to $\omega_1(f;1/n)$.

For this purpose, for any $k,j \in \{0,1,...,\}$, let us define the functions $f_{k,n,j} : [\frac{j}{n},\frac{j+1}{n}] \to \mathbb{R}$,

$$f_{k,n,j}(x) = m_{k,n,j}(x) f\left(\frac{k}{n}\right) = \frac{s_{n,k}(x)}{s_{n,j}(x)} f\left(\frac{k}{n}\right) = \frac{j!}{k!}\cdot (nx)^{k-j} f\left(\frac{k}{n}\right).$$

Then it is clear that for any $j \in \{0,1,...,\}$ and $x \in [\frac{j}{n},\frac{j+1}{n}]$ we can write

$$F_n^{(M)}(f)(x) = \bigvee_{k=0}^{\infty} f_{k,n,j}(x).$$

Also, we need the following auxiliary lemmas.

Lemma 4.2. *Let $f : [0,\infty) \to [0,\infty)$ be bounded and such that*

$$F_n^{(M)}(f)(x) = \max\{f_{j,n,j}(x), f_{j+1,n,j}(x)\} \text{ for all } x \in [j/n,(j+1)/n].$$

Then

$$\left|F_n^{(M)}(f)(x) - f(x)\right| \le \omega_1\left(f;\frac{1}{n}\right), \text{ for all } x \in [j/n,(j+1)/n],$$

where $\omega_1(f;\delta) = \max\{|f(x)-f(y)|; x,y \in [0,\infty), |x-y| \le \delta\} < \infty$.

Proof. We distinguish two cases :

Case (i). Let $x \in [j/n,(j+1)/n]$ be fixed such that $F_n^{(M)}(f)(x) = f_{j,n,j}(x)$. Because by simple calculation we have $0 \le x - \frac{j}{n} \le \frac{1}{n}$ and $f_{j,n,j}(x) = f(\frac{j}{n})$, it follows that

$$\left|F_n^{(M)}(f)(x) - f(x)\right| \le \omega_1\left(f;\frac{1}{n}\right).$$

Case (ii). Let $x \in [j/n,(j+1)/n]$ be such that $F_n^{(M)}(f)(x) = f_{j+1,n,j}(x)$. We have two subcases :

(ii_a) $F_n^{(M)}(f)(x) \le f(x)$, when evidently $f_{j,n,j}(x) \le f_{j+1,n,j}(x) \le f(x)$ and we immediately get

$$\begin{aligned}\left|F_n^{(M)}(f)(x) - f(x)\right| &= |f_{j+1,n,j}(x) - f(x)| \\ &= f(x) - f_{j+1,n,j}(x) \le f(x) - f(j/n) \le \omega_1\left(f;\frac{1}{n}\right).\end{aligned}$$

(ii_b) $F_n^{(M)}(f)(x) > f(x)$, when

$$\left|F_n^{(M)}(f)(x) - f(x)\right| = f_{j+1,n,j}(x) - f(x) = m_{j+1,n,j}(x) f\left(\frac{j+1}{n}\right) - f(x)$$

$$\leq f(\frac{j+1}{n}) - f(x).$$

Because $0 \leq \frac{j+1}{n} - x \leq \frac{1}{n}$ it follows $f(\frac{j+1}{n}) - f(x) \leq \omega_1\left(f; \frac{1}{n}\right)$, which proves the lemma. □

Lemma 4.3. *If the function $f : [0, \infty) \to [0, \infty)$ is concave, then the function $g : (0, \infty) \to [0, \infty), g(x) = \frac{f(x)}{x}$ is nonincreasing.*

Proof. Let $x, y \in (0, \infty)$ be with $x \leq y$. Then

$$f(x) = f\left(\frac{x}{y}y + \frac{y-x}{y}0\right) \geq \frac{x}{y}f(y) + \frac{y-x}{y}f(0) \geq \frac{x}{y}f(y),$$

which implies $\frac{f(x)}{x} \geq \frac{f(y)}{y}$. □

Corollary 4.4. *If $f : [0, \infty) \to [0, \infty)$ is bounded, nondecreasing and such that the function $g : (0, \infty) \to [0, \infty), g(x) = \frac{f(x)}{x}$ is nonincreasing, then*

$$\left|F_n^{(M)}(f)(x) - f(x)\right| \leq \omega_1\left(f; \frac{1}{n}\right), \text{ for all } x \in [0, \infty).$$

Proof. Since f is nondecreasing it follows (see the proof of Theorem 5.4 in the next section)

$$F_n^{(M)}(f)(x) = \bigvee_{k \geq j}^{\infty} f_{k,n,j}(x), \text{ for all } x \in [j/n, (j+1)/n].$$

Let $x \in [0, \infty)$ and $j \in \{0, 1, ..., \}$ such that $x \in [\frac{j}{n}, \frac{j+1}{n}]$. Let $k \in \{0, 1, ..., \}$ be with $k \geq j$. Then

$$f_{k+1,n,j}(x) = \frac{j!}{(k+1)!}(nx)^{k+1-j}f(\frac{k+1}{n}) = \frac{(nx)j!}{(k+1)!}(nx)^{k-j}f(\frac{k+1}{n}).$$

Since $g(x)$ is nonincreasing we get $\frac{f(\frac{k+1}{n})}{\frac{k+1}{n}} \leq \frac{f(\frac{k}{n})}{\frac{k}{n}}$ that is $f(\frac{k+1}{n}) \leq \frac{k+1}{k}f(\frac{k}{n})$. From $x \leq \frac{j+1}{n}$ it follows

$$f_{k+1,n,j}(x) \leq \frac{(j+1)!}{(k+1)!}(nx)^{k-j} \cdot \frac{k+1}{k}f(\frac{k}{n}) = f_{k,n,j}(x)\frac{j+1}{k}.$$

It is immediate that for $k \geq j+1$ we have $f_{k,n,j}(x) \geq f_{k+1,n,j}(x)$. Thus we obtain

$$f_{j+1,n,j}(x) \geq f_{j+2,n,j}(x) \geq ... \geq f_{n,j,n}(x) \geq ...$$

that is

$$F_n^{(M)}(f)(x) = \max\{f_{j,n,j}(x), f_{j+1,n,j}(x)\}, \text{ for all } x \in [j/n, (j+1)/n],$$

and from Lemma 4.2 we obtain

$$\left|F_n^{(M)}(f)(x) - f(x)\right| \leq \omega_1\left(f; \frac{1}{n}\right).$$

□

Corollary 4.5. *Let $f : [0, \infty) \to [0, \infty)$ be a bounded, nondecreasing concave function. Then*

$$\left| F_n^{(M)}(f)(x) - f(x) \right| \le \omega_1 \left(f; \frac{1}{n} \right), \text{ for all } x \in [0, \infty).$$

Proof. The proof is immediate by Lemma 4.3 and Corollary 4.4. □

Remarks. 1) If we suppose, for example, that in addition to the hypothesis in Corollary 4.5, $f : [0, \infty) \to [0, \infty)$ is a Lipschitz function, that is there exists $M > 0$ such that $|f(x) - f(y)| \le M|x - y|$, for all $x, y \in [0, \infty)$, then it follows that the order of uniform approximation on $[0, \infty)$ by $F_n^{(M)}(f)(x)$ is $\frac{1}{n}$, which is essentialy better than the order $\frac{a}{\sqrt{n}}$ obtained from Theorem 4.1 on each compact subinterval $[0, a]$ for f Lipschitz function on $[0, \infty)$.

2) It is known that for the linear Favard-Szász-Mirakjan operator given by

$$F_n(f)(x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f(k/n),$$

the best possible uniform approximation result is given by the equivalence (see [10]), $\|F_n(f) - f\| \sim \omega_2^{\varphi}(f; 1/\sqrt{n})$, where $\|f\| = \sup\{|f(x)|; x \in [0, \infty)\}$ and $\omega_2^{\varphi}(f; \delta)$ is the Ditzian-Totik second order modulus of smoothness on $[0, \infty)$ given by

$$\omega_2^{\varphi}(f; \delta) = \sup\{\sup\{|f(x + h\varphi(x)) - 2f(x) + f(x - h\varphi(x))|; x \in [h^2, \infty)\}, h \in [0, \delta]\},$$

with $\varphi(x) = \sqrt{x}$, $\delta \le 1$.

Now, if f is, for example, a nondecreasing concave polygonal line on $[0, \infty)$, constant on an interval $[a, \infty)$, then by simple reasonings we get that $\omega_2^{\varphi}(f; \delta) \sim \delta$ for $\delta \le 1$, which shows that the order of approximation obtained in this case by the linear Favard-Szász-Mirakjan operator is exactly $\frac{1}{\sqrt{n}}$. On the other hand, since such of function f obviously is a Lipschitz function on $[0, \infty)$ (as having bounded all the derivative numbers) by Corollary 4.5 we get that the order of approximation by the max-product Favard-Szász-Mirakjan operator is less than $\frac{1}{n}$, which is essentially better than $\frac{1}{\sqrt{n}}$. In a similar manner, by Corollary 4.4 we can produce many subclasses of functions for which the order of approximation given by the max-product Favard-Szász-Mirakjan operator is essentially better than the order of approximation given by the linear Favard-Szász-Mirakjan operator. Intuitively, the max-product Favard-Szász-Mirakjan operator has better approximation properties than its linear counterpart, for non-differentiable functions in a finite number of points (with the graphs having some "corners"), as for example for functions defined as a maximum of a finite number of continuous functions on $[0, \infty)$.

3) Since it is clear that a bounded nonincreasing concave function on $[0, \infty)$ necessarily one reduces to a constant function, the approximation of such functions is not of interest.

5 Shape Preserving Properties

In this section we will present some shape preserving properties. First we have the following simple result.

Lemma 5.1. *For any arbitrary bounded function $f : [0,\infty) \to \mathbb{R}_+$, the max-product operator $F_n^{(M)}(f)(x)$ is positive, bounded, continuous on $[0,\infty)$ and satisfies $F_n^{(M)}(f)(0) = f(0)$.*

Proof. The positivity of $F_n^{(M)}(f)(x)$ is immediate. Also, if $f(x) \leq K$ for all $x \in [0,\infty)$ it is immediate that $F_n^{(M)}(f)(x) \leq K$, for all $x \in [0,\infty)$.

From Lemma 3.3, taking into account that $s_{n,j}((j+1)/n) = s_{n,j+1}((j+1)/n)$, we immediately obtain that the denominator is a continuous function on $(0,\infty)$. Also, since $s_{n,k}(x) > 0$ for all $x \in (0,\infty)$, $n \in \mathbb{N}$, $k \in \{0, 1, ...,\}$, it follows that the denominator $\bigvee_{k=0}^{\infty} s_{n,k}(x) > 0$ for all $x \in (0,\infty)$ and $n \in \mathbb{N}$.

To prove the continuity on $[0,\infty)$ of the numerator, let us denote $h(x) = \bigvee_{k=0}^{\infty} s_{n,k}(x) f(k/n)$, and for each $m \in \mathbb{N}$, $h_m(x) = \bigvee_{k=0}^{m} s_{n,k}(x) f(k/n)$. It is clear that for each $m \in \mathbb{N}$, the function $h_m(x)$ is continuous on $[0,\infty)$, as a maximum of finite number of continuous functions. Also, fix $a > 0$ arbitrary and consider $x \in [0,a]$. First, since

$$0 \leq h(x) = \max\left\{\bigvee_{k=0}^{m} s_{n,k}(x) f(k/n), \bigvee_{k=m+1}^{\infty} s_{n,k}(x) f(k/n)\right\} \leq$$

$$\bigvee_{k=0}^{m} s_{n,k}(x) f(k/n) + \bigvee_{k=m+1}^{\infty} s_{n,k}(x) f(k/n),$$

it follows that for all $m \in \mathbb{N}$ we have

$$0 \leq h(x) - h_m(x) \leq \bigvee_{k=m+1}^{\infty} s_{n,k}(x) f(k/n) \leq \bigvee_{k=m+1}^{\infty} \frac{(na)^k}{k!} K, \text{ for all } x \in [0,a],$$

where $0 \leq f(x) \leq K$ for all $x \in [0,\infty)$.

Now, fix $\varepsilon > 0$. Since $\frac{s_{n,k+1}(a)}{s_{n,k}(a)} = \frac{na}{k+1}$, there exists an index $k_0 > 0$ (independent of x), such that $\frac{na}{k+1} < \varepsilon$, for all $k \geq k_0$. Choose now $m = k_0$. It is immediate that $\bigvee_{k=m+1}^{\infty} \frac{(na)^k}{k!} K < \varepsilon \cdot \frac{K(na)^{k_0}}{k_0!}$, which implies that

$$0 \leq h(x) - h_m(x) < \varepsilon \cdot \frac{K(na)^{k_0}}{k_0!}, \text{ for all } x \in [0,a] \text{ and } m \geq k_0.$$

This implies that the numerator $h(x)$ is the uniform limit (as $m \to \infty$) of a sequence of continuous functions on $[0,a]$, $h_m(x), m \in \mathbb{N}$, which implies the continuity of $h(x)$ on $[0,a]$. Because $a > 0$ was chosen arbitrary, it follows the continuity of $h(x)$ on $[0,\infty)$.

As a first conclusion, we get the continuity of $F_n^{(M)}(f)(x)$ on $(0,\infty)$.

To prove now the continuity of $F_n^{(M)}(f)(x)$ at $x=0$, we observe that $s_{n,k}(0)=0$ for all $k\in\{1,2,...,\}$ and $s_{n,k}(0)=1$ for $k=0$, which implies that $\bigvee_{k=0}^{\infty} s_{n,k}(x)=1$ in the case of $x=0$. The fact that $F_n^{(M)}(f)(x)$ coincides with $f(x)$ at $x=0$ immediately follows from the above considerations, proving the theorem. □

Remark. Note that because of the continuity of $F_n^{(M)}(f)(x)$ on $[0,\infty)$, it will suffice to prove the shape properties of $F_n^{(M)}(f)(x)$ on $(0,\infty)$ only. As a consequence, in the notations and proofs below we always may suppose that $x>0$.

As in Section 4, for any $k,j\in\{0,1,...,\}$, let us consider the functions $f_{k,n,j}:[\frac{j}{n},\frac{j+1}{n}]\to\mathbb{R}$,

$$f_{k,n,j}(x)=m_{k,n,j}(x)f\left(\frac{k}{n}\right)=\frac{s_{n,k}(x)}{s_{n,j}(x)}f\left(\frac{k}{n}\right)=\frac{j!}{k!}\cdot(nx)^{k-j}f\left(\frac{k}{n}\right).$$

For any $j\in\{0,1,...,\}$ and $x\in[\frac{j}{n},\frac{j+1}{n}]$ we can write

$$F_n^{(M)}(f)(x)=\bigvee_{k=0}^{\infty} f_{k,n,j}(x).$$

Lemma 5.2. *If $f:[0,\infty)\to\mathbb{R}_+$ is a nondecreasing function then for any $k,j\in\{0,1,...,\}$ with $k\le j$ and $x\in[\frac{j}{n},\frac{j+1}{n}]$ we have $f_{k,n,j}(x)\ge f_{k-1,n,j}(x)$.*

Proof. Because $k\le j$, by the proof of Lemma 3.1, case 2), it follows that $m_{k,n,j}(x)\ge m_{k-1,n,j}(x)$. From the monotonicity of f we get $f\left(\frac{k}{n}\right)\ge f\left(\frac{k-1}{n}\right)$. Thus we obtain

$$m_{k,n,j}(x)f\left(\frac{k}{n}\right)\ge m_{k-1,n,j}(x)f\left(\frac{k-1}{n}\right),$$

which proves the lemma. □

Corollary 5.3. *If $f:[0,\infty)\to\mathbb{R}_+$ is nonincreasing then $f_{k,n,j}(x)\ge f_{k+1,n,j}(x)$ for any $k,j\in\{0,1,...,\infty\}$ with $k\ge j$ and $x\in[\frac{j}{n},\frac{j+1}{n}]$.*

Proof. Because $k\ge j$, by the proof of Lemma 3.1, case 1), it follows that $m_{k,n,j}(x)\ge m_{k+1,n,j}(x)$. From the monotonicity of f we get $f\left(\frac{k}{n}\right)\ge f\left(\frac{k+1}{n}\right)$. Thus we obtain

$$m_{k,n,j}(x)f\left(\frac{k}{n}\right)\ge m_{k+1,n,j}(x)f\left(\frac{k+1}{n}\right),$$

which proves the corollary. □

Theorem 5.4. *If $f:[0,\infty)\to\mathbb{R}_+$ is nondecreasing and bounded on $[0,\infty)$ then $F_n^{(M)}(f)$ is nondecreasing (and bounded).*

Proof. Because $F_n^{(M)}(f)$ is continuous (and bounded) on $[0,\infty)$, it suffices to prove that on each subinterval of the form $[\frac{j}{n},\frac{j+1}{n}]$, with $j\in\{0,1,...,\}$, $F_n^{(M)}(f)$ is nondecreasing.

So let $j \in \{0, 1, ..., \}$ and $x \in [\frac{j}{n}, \frac{j+1}{n}]$. Because f is nondecreasing, from Lemma 5.2 it follows that

$$f_{j,n,j}(x) \geq f_{j-1,n,j}(x) \geq f_{j-2,n,j}(x) \geq ... \geq f_{0,n,j}(x).$$

But then it is immediate that

$$F_n^{(M)}(f)(x) = \bigvee_{k \geq j}^{\infty} f_{k,n,j}(x),$$

for all $x \in [\frac{j}{n}, \frac{j+1}{n}]$. Clearly that for $k \geq j$ the function $f_{k,n,j}$ is nondecreasing and since $F_n^{(M)}(f)$ is defined as supremum of nondecreasing functions, it follows that it is nondecreasing. □

Corollary 5.5. *If $f : [0, \infty) \to \mathbb{R}_+$ is nonincreasing then $F_n^{(M)}(f)$ is nonincreasing.*

Proof. By hypothesis, f implicitly is bounded on $[0, \infty)$. Because $F_n^{(M)}(f)$ is continuous and bounded on $[0, \infty)$, it suffices to prove that on each subinterval of the form $[\frac{j}{n}, \frac{j+1}{n}]$, with $j \in \{0, 1, ..., \}$, $F_n^{(M)}(f)$ is nonincreasing.

So let $j \in \{0, 1, ..., \}$ and $x \in [\frac{j}{n}, \frac{j+1}{n}]$. Because f is nonincreasing, from Corollary 5.3 it follows that

$$f_{j,n,j}(x) \geq f_{j+1,n,j}(x) \geq f_{j+2,n,j}(x) \geq ...$$

But then it is immediate that

$$F_n^{(M)}(f)(x) = \bigvee_{k \geq 0}^{j} f_{k,n,j}(x),$$

for all $x \in [\frac{j}{n}, \frac{j+1}{n}]$. Clearly that for $k \leq j$ the function $f_{k,n,j}$ is nonincreasing and since $F_n^{(M)}(f)$ is defined as the maximum of nonincreasing functions, it follows that it is nonincreasing. □

In what follows, let us consider the following concept generalizing the monotonicity and convexity.

Definition 5.6. Let $f : [0, \infty) \to \mathbb{R}$ be continuous on $[0, \infty)$. One says that f is quasi-convex on $[0, \infty)$ if it satisfies the inequality

$$f(\lambda x + (1 - \lambda)y) \leq \max\{f(x), f(y)\}, \text{ for all } x, y \in [0, \infty) \text{ and } \lambda \in [0, 1].$$

(see e.g. the book [8], p. 4, (iv)).

Remark. By [9], the continuous function f is quasi-convex on the bounded interval $[0, a]$, equivalently means that there exists a point $c \in [0, a]$ such that f is nonincreasing on $[0, c]$ and nondecreasing on $[c, a]$. But this property easily can be extended to continuous quasiconvex functions on $[0, \infty)$, in the sense that there exists $c \in [0, \infty]$ ($c = \infty$ by convention for nonincreasing functions on $[0, \infty)$) such that f is nonincreasing on $[0, c]$ and nondecreasing on $[c, \infty)$. This easily follows from the fact that the quasiconvexity of f on $[0, \infty)$ means the quasiconvexity of f on any bounded interval $[0, a]$, with arbitrary large $a > 0$.

The class of quasi-convex functions includes the both classes of nondecreasing functions and of nonincreasing functions (obtained from the class of quasi-convex functions by taking $c = 0$ and $c = \infty$, respectively). Also, it obviously includes the class of convex functions on $[0, \infty)$.

Corollary 5.7. *If $f : [0, \infty) \to \mathbb{R}_+$ is continuous, bounded and quasi-convex on $[0, \infty)$ then for all $n \in \mathbb{N}$, $F_n^{(M)}(f)$ is quasi-convex on $[0, \infty)$.*

Proof. If f is nonincreasing (or nondecreasing) on $[0, \infty)$ (that is the point $c = \infty$ (or $c = 0$) in the above Remark) then by the Corollary 5.5 (or Theorem 5.4, respectively) it follows that for all $n \in \mathbb{N}$, $F_n^{(M)}(f)$ is nonincreasing (or nondecreasing) on $[0, \infty)$.

Suppose now that there exists $c \in (0, \infty)$, such that f is nonincreasing on $[0, c]$ and nondecreasing on $[c, \infty)$. Define the functions $F, G : [0, \infty) \to \mathbb{R}_+$ by $F(x) = f(x)$ for all $x \in [0, c]$, $F(x) = f(c)$ for all $x \in [c, \infty)$ and $G(x) = f(c)$ for all $x \in [0, c]$, $G(x) = f(x)$ for all $x \in [c, \infty)$.

It is clear that F is nonincreasing and continuous on $[0, \infty)$, G is nondecreasing and continuous on $[0, \infty)$ and that $f(x) = \max\{F(x), G(x)\}$, for all $x \in [0, \infty)$.

But it is easy to show (see also Remark 1 after the proof of Lemma 2.1) that

$$F_n^{(M)}(f)(x) = \max\{F_n^{(M)}(F)(x), F_n^{(M)}(G)(x)\}, \text{ for all } x \in [0, \infty),$$

where by the Corollary 5.5 and Theorem 5.4 , $F_n^{(M)}(F)(x)$ is nonincreasing and continuous on $[0, \infty)$ and $F_n^{(M)}(G)(x)$ is nondecreasing and continuous on $[0, \infty)$. We have two cases : 1) $F_n^{(M)}(F)(x)$ and $F_n^{(M)}(G)(x)$ do not intersect each other ; 2) $F_n^{(M)}(F)(x)$ and $F_n^{(M)}(G)(x)$ intersect each other.

Case 1). We have $\max\{F_n^{(M)}(F)(x), F_n^{(M)}(G)(x)\} = F_n^{(M)}(F)(x)$ for all $x \in [0, \infty)$ or $\max\{F_n^{(M)}(F)(x), F_n^{(M)}(G)(x)\} = F_n^{(M)}(G)(x)$ for all $x \in [0, \infty)$, which obviously proves that $F_n^{(M)}(f)(x)$ is quasi-convex on $[0, \infty)$.

Case 2). In this case it is clear that there exists a point $c' \in [0, \infty)$ such that $F_n^{(M)}(f)(x)$ is nonincreasing on $[0, c']$ and nondecreasing on $[c', \infty)$, which by the considerations in the above Remark implies that $F_n^{(M)}(f)(x)$ is quasiconvex on $[0, \infty)$ and proves the corollary. □

It is of interest to exactly calculate $F_n^{(M)}(f)$ for $f(x) = e_0(x) = 1$ and for $f(x) = e_1(x) = x$. In this sense we can state the following.

Lemma 5.8. *For all $x \in [0, \infty)$ and $n \in \mathbb{N}$ we have $F_n^{(M)}(e_0)(x) = 1$ and $F_n^{(M)}(e_1)(x) = x$.*

Proof. The formula $F_n(e_0)(x) = 1$ is immediate by the definition of $F_n^{(M)}(f)(x)$. To find the formula for $F_n^{(M)}(e_1)(x)$, we observe that

$$\bigvee_{k=0}^{\infty} s_{n,k}(x)\frac{k}{n} = \bigvee_{k=1}^{\infty} s_{n,k}(x)\frac{k}{n} = x \cdot \bigvee_{k=1}^{\infty} s_{n,k-1}(x) = x \bigvee_{j=0}^{\infty} s_{n,j}(x),$$

which implies

$$F_n^{(M)}(e_1)(x) = x \cdot \frac{\bigvee_{j=0}^{\infty} s_{n,j}(x)}{\bigvee_{k=0}^{\infty} s_{n,k}(x)} = x.$$

□

Also, we can prove the interesting property that for any arbitrary function f, the max-product Bernstein operator $F_n^{(M)}(f)$ is piecewise convex on $[0,\infty)$. In this sense the following result holds.

Theorem 5.9. *For any function $f : [0,\infty) \to [0,\infty)$, $F_n^{(M)}(f)$ is convex on any interval of the form $[\frac{j}{n}, \frac{j+1}{n}]$, $j = 0, 1, ...,$.*

Proof. For any $k, j \in \{0, 1, ..., \}$ let us consider the functions $f_{k,n,j} : [\frac{j}{n}, \frac{j+1}{n}] \to \mathbb{R}$,

$$f_{k,n,j}(x) = m_{k,n,j}(x) f(\frac{k}{n}) = \frac{j!(nx)^{k-j}}{k!} f(\frac{k}{n}).$$

Clearly we have

$$F_n^{(M)}(f)(x) = \bigvee_{k=0}^{\infty} f_{k,n,j}(x),$$

for any $j \in \{0, 1, ..., \}$ and $x \in [\frac{j}{n}, \frac{j+1}{n}]$.

We will prove that for any fixed j, each function $f_{k,n,j}(x)$ is convex on $[\frac{j}{n}, \frac{j+1}{n}]$, which will imply that $F_n^{(M)}(f)$ can be written as a supremum of some convex functions on $[\frac{j}{n}, \frac{j+1}{n}]$.

Since $f \geq 0$ and $f_{k,n,j}(x) = \frac{j! \cdot n^{k-j}}{k!} \cdot x^{k-j} \cdot f(k/n)$, it suffices to prove that the functions $g_{k,j} : [0,1] \to \mathbb{R}_+$, $g_{k,j}(x) = x^{k-j}$ are convex on $[\frac{j}{n}, \frac{j+1}{n}]$.

For $k = j$, $g_{j,j}$ is constant so is convex.

For $k = j + 1$ we get $g_{j+1,j}(x) = x$ for any $x \in [\frac{j}{n}, \frac{j+1}{n}]$, which obviously is convex.

For $k = j - 1$ it follows $g_{j-1,j}(x) = \frac{1}{x}$ for any $x \in [\frac{j}{n}, \frac{j+1}{n}]$. Then $g''_{j-1,j}(x) = \frac{2}{x^3} > 0$ for any $x \in [\frac{j}{n}, \frac{j+1}{n}]$.

If $k \geq j + 2$ then $g''_{k,j}(x) = (k - j)(k - j - 1)x^{k-j-2} > 0$ for any $x \in [\frac{j}{n}, \frac{j+1}{n}]$.

If $k \leq j - 2$ then $g''_{k,j}(x) = (k - j)(k - j - 1)x^{k-j-2} > 0$, for any $x \in [\frac{j}{n}, \frac{j+1}{n}]$.

Since all the functions $g_{k,j}$ are convex on $[\frac{j}{n}, \frac{j+1}{n}]$, we get that $F_n^{(M)}(f)$ is convex on $[\frac{j}{n}, \frac{j+1}{n}]$ as maximum of these functions, proving the theorem. □

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