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APPROXIMATION BY INTERPOLATING POLYNOMIALS IN SMIRNOV-ORLICZ CLASS

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ABSTRACT. Let Γ be a bounded rotation (BR) curve without cusps in the complex plane $\mathbb{C}$ and let $G := \operatorname{int}\Gamma$. We prove that the rate of convergence of the interpolating polynomials based on the zeros of the Faber polynomials F_n for $\overline{G}$ to the function of the reflexive Smirnov-Orlicz class $E_M(G)$ is equivalent to the best approximating polynomial rate in $E_M(G)$.

1. Introduction and main results

Let Γ be a closed rectifiable Jordan curve in the complex plane $\mathbb{C}$. The curve Γ separates the plane into two domains $G := \operatorname{int}\Gamma$ and $G^- := \operatorname{ext}\Gamma$. We denote $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}, \mathbb{T} := \partial\mathbb{D}$ and $\mathbb{D}^- := \operatorname{ext}\mathbb{T}$.

Let $w = \phi(z)$ be the conformal map of G^- onto $\mathbb{D}^-$ normalized by the conditions

$$\phi(\infty) = \infty, \quad \lim_{z\to\infty} \frac{\phi(z)}{z} > 0,$$

and let $\psi := \phi^{-1}$ be its inverse mapping.

When $|z|$ is sufficiently large, ϕ has the Laurent expansion

$$\phi(z) = dz + d_0 + \frac{d_1}{z} + \cdots$$

and hence we have

$$[\phi(z)]^n = d^n z^n + \sum_{k=0}^{n-1} d_{n,k} z^k + \sum_{k<0} d_{n,k} z^k.$$

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The polynomial

$$F_n(z) := d^n z^n + \sum_{k=0}^{n-1} d_{n,k} z^k$$

is called n^{th} Faber polynomial with respect to $\overline{G}$.

Note that for every natural number n, F_n is a polynomial of degree n. For further information about the Faber polynomials, it can be seen to monographs [5, Ch. I, Section 6], [14, Ch. II], [15].

By $L^p(\Gamma)$, $1 \leq p < \infty$, we denote the set of all measurable complex valued functions f on Γ such that $|f|^p$ is Lebesgue integrable with respect to arclength.

Let $z = \phi_0(w)$ be the conformal map of $\mathbb{D}$ onto G normalized by the conditions

$$\phi_0(0) = 0, \qquad \phi_0'(0) > 0,$$

and let γ_r be the image of the circle $|w| = r$, $0 < r < 1$, under the mapping ϕ_0.

We say that a function f analytic in G, belongs to the Smirnov class $E^p(G)$, $0 < p < \infty$, if for any $r \in (0,1)$ the inequality

$$\int_{\gamma_r} |f(z)|^p |dz| \leq c < \infty$$

holds.

Every function in $E^p(G)$, $1 < p < \infty$, has nontangential boundary values almost everywhere (a. e.) on Γ and the boundary function belongs to $L^p(\Gamma)$.

For $p > 1$, $E^p(G)$ is a Banach space with respect to the norm

$$\|f\|_{E^p(G)} := \|f\|_{L^p(\Gamma)} := \left(\int_{\Gamma} |f(z)|^p |dz| \right)^{\frac{1}{p}}.$$

A continuous and convex function $M : [0,\infty) \to [0,\infty)$ which satisfies the conditions

$$M(0) = 0, \quad M(x) > 0 \quad \text{for } x > 0,$$

$$\lim_{x\to 0} \frac{M(x)}{x} = 0, \quad \lim_{x\to\infty} \frac{M(x)}{x} = \infty,$$

is called an N-function.

The complementary N-function to M is defined by

$$N(y) := \max_{x>0} (xy - M(x)), \quad y \geq 0.$$

We denote by $L_M(\Gamma)$ the linear space of Lebesgue measurable functions $f:\Gamma\to\mathbb{C}$ satisfying the condition

$$\int_{\Gamma} M\left[\alpha\left|f(z)\right|\right]|dz|<\infty$$

for some $\alpha>0$.

The space $L_M(\Gamma)$ becomes a Banach space with the norm

$$\|f\|_{L_M(\Gamma)}:=\sup\left\{\int_{\Gamma}\left|f(z)\,g(z)\right||dz| : g\in L_N(\Gamma);\ \rho(g;N)\leq 1\right\},$$

where N is the complementary N-function to M and

$$\rho(g;N):=\int_{\Gamma} N\left[\left|g(z)\right|\right]|dz|.$$

This norm is called Orlicz norm and the Banach space $L_M(\Gamma)$ is called Orlicz space.

We note that (see, for example, [12, p.51])

$$L_M(\Gamma)\subset L^1(\Gamma).$$

An N-function M satisfies the Δ_2-condition if

$$\limsup_{x\to\infty}\frac{M(2x)}{M(x)}<\infty.$$

The Orlicz space $L_M(\Gamma)$ is reflexive if and only if the N-function M and its complementary function N both satisfy the Δ_2-condition [12, p.113].

Let Γ_r be the image of the circle $\{w\in\mathbb{C}:|w|=r,\ 0<r<1\}$ under some conformal map of $\mathbb{D}$ onto G and let M be an N-function.

DEFINITION 1. The class of functions which are analytic in G and satisfy the condition

$$\int_{\Gamma_r} M\left[\left|f(z)\right|\right]|dz|<\infty$$

uniformly in r is called the Smirnov-Orlicz class and denoted by $E_M(G)$.

The Smirnov-Orlicz class is a generalization of the familiar Smirnov class $E^p(G)$. In particular, if $M(x):=x^p, 1<p<\infty$, then the Smirnov-Orlicz class $E_M(G)$ determined by M coincides with the Smirnov class $E^p(G)$.

Since (see [10]) $E_M(G) \subset E^1(G)$, every function in the class $E_M(G)$ has the nontangential boundary values a.e. on Γ and the boundary value function belongs to $L_M(\Gamma)$. Hence the $E_M(G)$ norm can be defined as:

$$\|f\|_{E_M(G)} := \|f\|_{L_M(\Gamma)}, \qquad f \in E_M(G). \tag{1}$$

Let γ be an oriented rectifiable curve. For $z \in \gamma$, $\delta > 0$ we denote by $s_+(z,\delta)$ (respectively $s_-(z,\delta)$) the subarc of γ in the positive (respectively negative) orientation of γ with the z starting point and arc length from z to each point less than δ.

If γ is a smooth curve and

$$\lim_{\delta\to 0}\left\{ \int\limits_{s_-(z,\delta)} |d_\varsigma \arg(\varsigma - z)| + \int\limits_{s_+(z,\delta)} |d_\varsigma \arg(\varsigma - z)| \right\} = 0$$

holds uniformly for $z \in \gamma$, then it's said [16] that γ is of vanishing rotation (VR).

As follows from this definition, the VR condition is stronger than smoothness. In [16] L. Zhong and L. Zhu also proved that there exists a smooth curve which is not of VR.

On the other hand, if the angle of inclination $\theta(s)$ of tangent to γ as a function of the arclength s along γ satisfies the condition

$$\int\limits_0^\delta \frac{\omega(t)}{t} dt < \infty,$$

where $\omega(t)$ is the modulus of continuity of $\theta(s)$, then [16] γ is VR.

Approximation properties of the Faber and generalized Faber polynomials in the different functional spaces are well known (see for example: [1]-[2], [4]-[8] and also [5, Chapter 1, pp. 42–57], [15]). In this work we investigate the convergence property of the interpolating polynomials based on the zeros of the Faber polynomials in the reflexive Smirnov-Orlicz class. This problem isn't new. It was studied by several authors. In their work [13] under the assumption $\Gamma \in C(2,\alpha)$, $0 < \alpha < 1$, X. C. Shen and L. Zhong obtain a series of interpolation nodes in G and show that interpolating polynomials and the best approximating polynomial have the same order of convergence in $E^p(G)$, $1 < p < \infty$. In [17] considering $\Gamma \in C(1,\alpha)$ and choosing the interpolation nodes as the zeros of the Faber polynomials L. Y. Zhu obtain similar result.

In the above cited works Γ does not admit corners. Many domains in the complex plane may have corners or cusps. When Γ is a piecewise VR curve without cusps, L. Zhong and L. Zhu [16] showed that the interpolating polynomials based on the zeros of the Faber polynomials converge in the Smirnov class $E^p(G)$, $1 < p < \infty$.

In this work we investigate the convergence property of the interpolating polynomials based on the zeros of the Faber polynomials in the reflexive Smirnov-Orlicz class under the assumption that Γ is a BR curve without cusps.

DEFINITION 2. [6] Let γ be a rectifiable Jordan curve with length L and let $z = z(t)$ be its parametric representation with arclenght $t \in [0, L]$. If $\beta(t) := \arg z'(t)$ can be defined on $[0, L]$ to become a function of bounded variation, then γ is called of bounded rotation ($\gamma \in BR$) and $\int_\Gamma |d\beta(t)|$ is called total rotation of γ.

If $\gamma \in BR$, then there are two half tangents at each point of γ. The class of bounded rotation curves is sufficiently wide. For example, a curve which is made up of finitely many convex arcs (corners are permitted), is bounded rotation [5, p.45]. It is easily seen that every VR curve and also a piecewise VR curve considered in [16] is BR curve. Since a BR curve may have cusps or corners, there exists a BR curve which is not a VR curve (for example, a rectangle in the plane).

In the case that all of the zeros of the n^{th} Faber polynomial $F_n(z)$ are in G, we denote by $L_n(f,z)$ the $(n-1)th$ interpolating polynomial to $f(z) \in E_M(G)$ based on the zeros of the Faber polynomials F_n.

For $f \in E_M(G)$, we denote by

(2)

$$\begin{aligned} E_n^M(f,G) &:= \inf\left\{\|f - p_n\|_{E_M(G)} : p_n \text{ is a polynomial of degree} \le n\right\} \\ &= \inf\left\{\sup\left\{\int_\Gamma |(f(\varsigma) - p_n(\varsigma))\, g(\varsigma)|\, |d\varsigma| \;;\; \rho(g;N) \le 1\right\}\right\} \end{aligned}$$

the minimal error of approximation of f by polynomials of degree at most n.

The main results of this work are the following.

THEOREM. *Let Γ be a BR curve without cusps. Then for sufficiently large natural number n, the roots of the Faber polynomials are in G and for every f which belongs to reflexive Smirnov-Orlicz class $E_M(G)$,*

$$\|f - L_n(f,\cdot)\|_{E_M(G)} \le c \cdot E_{n-1}^M(f,G)$$

with a positive constant c depending only on Γ and M.

In particular, when $M(x) := x^p$, $1 < p < \infty$, we have the following result.

COROLLARY. *Let Γ be a BR curve without cusps. Then for sufficiently large natural number n, the roots of the Faber polynomials are in G and for every f which belongs to Smirnov class $E^p(G)$, $1 < p < \infty$,*

$$\|f - L_n(f,\cdot)\|_{E^p(G)} \le c \cdot E_{n-1}(f,G)_p,$$

where the constant $c > 0$ depend only on p and G.

When Γ is a piecewise VR curve without cups, this corollary was proved in [16].

We use $c, c_1, c_2, \ldots$ to denote constants (which may, in general, differ in different relations) depending only on numbers that are not important for the question of our interest.

2. Auxiliary results

Let Γ be a BR curve without cusps. Then (see, for example, Pommerenke [11])

$$F_n(z) = \frac{1}{\pi}\int\limits_{\Gamma} [\phi(\varsigma)]^n \, d_\varsigma \arg(\varsigma - z), \qquad z \in \Gamma,$$

where the jump of $\arg(\varsigma - z)$ at $\varsigma = z$ equals to the exterior angle $\alpha_z \pi$. Hence we have

$$F_n(z) - [\phi(z)]^n = \frac{1}{\pi}\int\limits_{\Gamma\setminus\{z\}} [\phi(\varsigma)]^n \, d_\varsigma \arg(\varsigma - z) + (\alpha_z - 1)[\phi(z)]^n, \tag{3}$$

and

$$0 \le \max_{z\in\Gamma} |\alpha_z - 1| < 1. \tag{4}$$

LEMMA 1. [3] *Let Γ be a BR curve. For any $\epsilon > 0$ and θ , there exists a $\delta > 0$ such that*

(5)

$$\int\limits_{\theta-\delta}^{\theta-} \left| d_t \arg\left(\psi\left(e^{it}\right) - \psi\left(e^{i\theta}\right)\right)\right| + \int\limits_{\theta+}^{\theta+\delta} \left| d_t \arg\left(\psi\left(e^{it}\right) - \psi\left(e^{i\theta}\right)\right)\right| < \epsilon,$$

and for any $\eta \in (\theta - \delta, \theta + \delta)$ different from θ

$$\int\limits_{\eta-\delta}^{\eta} \left| d_t \arg\left(\psi\left(e^{it}\right) - \psi\left(e^{i\eta}\right)\right)\right| + \int\limits_{\eta}^{\eta+\delta} \left| d_t \arg\left(\psi\left(e^{it}\right) - \psi\left(e^{i\eta}\right)\right)\right|$$

$$< \epsilon + |\alpha_z - 1| \pi,$$

where $\alpha_z \pi$ *is the external angle to* Γ *at* $z = \psi(e^{i\theta})$.

LEMMA 2. *If* Γ *is a BR curve, then for every* $\epsilon > 0$, *there exists a* $\delta > 0$ *such that*

$$(6) \quad \int\limits_{s_-(z,\delta)\setminus\{z\}} |d_\varsigma \arg(\varsigma - z)| + \int\limits_{s_+(z,\delta)\setminus\{z\}} |d_\varsigma \arg(\varsigma - z)| < \epsilon, \qquad z \in \Gamma.$$

Proof. We take arbitrary $z \in \Gamma$ and fix it. By the change of variable $\varsigma = \psi(e^{it})$ we get

$$\int\limits_{s_-(z,\delta)\setminus\{z\}} |d_\varsigma \arg(\varsigma - z)| = \int\limits_{\theta-\delta}^{\theta-} \left| d_{\psi(e^{it})} \arg\left(\psi(e^{it}) - \psi\left(e^{i\theta}\right)\right)\right|$$

$$= \int\limits_{\theta-\delta}^{\theta-} \left| d_t \arg\left(\psi(e^{it}) - \psi\left(e^{i\theta}\right)\right)\right|$$

and similarly

$$\int\limits_{s_+(z,\delta)\setminus\{z\}} |d_\varsigma \arg(\varsigma - z)| = \int\limits_{\theta+}^{\theta+\delta} \left| d_{\psi(e^{it})} \arg\left(\psi(e^{it}) - \psi\left(e^{i\theta}\right)\right)\right|$$

$$= \int\limits_{\theta+}^{\theta+\delta} \left| d_t \arg\left(\psi(e^{it}) - \psi\left(e^{i\theta}\right)\right)\right|.$$

For any $\epsilon > 0$, using (5) we have (6). □

For any $\delta > 0$, $\theta \in [0, 2\pi]$, we denote by $I_{\theta,\delta}$, the image of the set $\left\{s_-\left(\psi\left(e^{i\theta}\right), \delta\right) \cup s_+\left(\psi\left(e^{i\theta}\right), \delta\right)\right\}$ under ϕ and let

$$v(t, \theta; \delta) := \begin{cases} \frac{e^{it}\psi'\left(e^{it}\right)}{\psi(e^{it}) - \psi\left(e^{i\theta}\right)} & e^{it} \notin I_{\theta,\delta}, \\ 0 & e^{it} \in I_{\theta,\delta}. \end{cases}$$

The following lemma was proved by L. Zhong and L. Zhu [16] in the case of a domain bounded by a piecewise VR curve without cusps. When the boundary of the domain is a BR curve, the proof goes similarly.

LEMMA 3. *For any* $\epsilon > 0$, $\delta > 0$, *there exists a natural number* k *such that for* $\theta \in [0, 2\pi]$ *there exists a trigonometric polynomial* $T_\theta(t)$

of t with degree at most k satisfying

$$\int_0^{2\pi} |v(t,\theta;\delta) - T_\theta(t)|\, dt < \epsilon. \tag{7}$$

LEMMA 4. *Let Γ be a BR curve without cusps. Then for arbitrary $\epsilon > 0$, there exists a positive integer n_0 such that*

$$|F_n(z) - [\phi(z)]^n| < |\alpha_z - 1| + \epsilon, \qquad z \in \Gamma \tag{8}$$

holds for $n > n_0$.

Proof. For any $\epsilon > 0$, there exists a $\delta > 0$ such that (6) holds. Let $s(z) := \{s_-(z,\delta) \cup s_+(z,\delta)\}$, $z \in \Gamma$. Hence by Lemma 3, for given ϵ and δ there is a positive integer n_0 such that (7) is valid. By (3) for $z = \psi\left(e^{i\theta}\right)$ we have

$$\begin{aligned}
&F_n(z) - [\phi(z)]^n \\
&= \frac{1}{\pi}\left\{ \int\limits_{s_+(z,\delta)\setminus\{z\}} + \int\limits_{s_-(z,\delta)\setminus\{z\}} \right\} [\phi(\varsigma)]^n\, d_\varsigma \arg(\varsigma - z) \\
&\quad + \frac{1}{\pi} \int\limits_{\Gamma\setminus s(z)} [\phi(\varsigma)]^n\, d_\varsigma \arg(\varsigma - z) + (\alpha_z - 1)\, e^{in\theta} \\
&= \frac{1}{\pi}\left\{ \int\limits_{s_+(z,\delta)\setminus\{z\}} + \int\limits_{s_-(z,\delta)\setminus\{z\}} \right\} [\phi(\varsigma)]^n\, d_\varsigma \arg(\varsigma - z) \\
&\quad + \frac{1}{\pi} \int\limits_{e^{it}\notin I_{\theta,\delta}} e^{\mathrm{int}} d_t \arg\left(\psi\left(e^{it}\right) - \psi\left(e^{i\theta}\right)\right) + (\alpha_z - 1)\, e^{in\theta} \\
&= \frac{1}{\pi}\left\{ \int\limits_{s_+(z,\delta)\setminus\{z\}} + \int\limits_{s_-(z,\delta)\setminus\{z\}} \right\} [\phi(\varsigma)]^n\, d_\varsigma \arg(\varsigma - z) \\
&\quad + \frac{1}{\pi} \int_0^{2\pi} e^{\mathrm{int}}\, \mathrm{Im}\left[iv(t,\theta;\delta)\right] dt + (\alpha_z - 1)\, e^{in\theta}.
\end{aligned}$$

Since e^{int} is orthogonal to $T_\theta(t)$ as $n > n_0$, we get

$$F_n(z) - [\phi(z)]^n = \frac{1}{\pi}\left\{ \int\limits_{s_+(z,\delta)\setminus\{z\}} + \int\limits_{s_-(z,\delta)\setminus\{z\}} \right\} [\phi(\varsigma)]^n\, d_\varsigma \arg(\varsigma - z)$$

$$+\frac{1}{\pi}\int_{0}^{2\pi} e^{\mathrm{int}}\,\mathrm{Im}\left[iv\left(t,\theta;\delta\right)-iT_{\theta}\left(t\right)\right]dt+\left(\alpha_{z}-1\right)e^{in\theta},$$

and hence

$$\left|F_{n}\left(z\right)-\left[\phi\left(z\right)\right]^{n}\right|\leq\frac{1}{\pi}\left\{\int\limits_{s_{+}(z,\delta)\setminus\{z\}}+\int\limits_{s_{-}(z,\delta)\setminus\{z\}}\right\}\left|d_{\varsigma}\arg\left(\varsigma-z\right)\right|+\left|\alpha_{z}-1\right|$$

$$+\frac{1}{\pi}\int_{0}^{2\pi}\left|v\left(t,\theta;\delta\right)-T_{\theta}\left(t\right)\right|dt.$$

If z is not a corner of Γ, then $\left|\alpha_{z}-1\right|=0$ and by (6) and (7) our assumption follows. If z is a corner of Γ, then $0\leq\left|\alpha_{z}-1\right|<1$ and hence by (6) and (7) we have (8) again. □

For $z\in\Gamma$ and $\epsilon>0$ let $\Gamma\left(z,\epsilon\right)$ denote the portion of Γ which is inside the open disk of radius ϵ centered at z, i.e., $\Gamma\left(z,\epsilon\right):=\left\{t\in\Gamma:\left|t-z\right|<\epsilon\right\}$. Further, let $\left|\Gamma\left(z,\epsilon\right)\right|$ denote the length of $\Gamma\left(z,\epsilon\right)$. A rectifiable Jordan curve Γ is called a Carleson curve if

$$\sup_{\epsilon>0}\sup_{z\in\Gamma}\frac{1}{\epsilon}\left|\Gamma\left(z,\epsilon\right)\right|<\infty.$$

We consider the Cauchy-type integral

$$\left(\mathcal{H}f\right)\left(z\right):=\frac{1}{2\pi i}\int_{\Gamma}\frac{f\left(\varsigma\right)}{\varsigma-z}d\varsigma,\qquad z\in G$$

and Cauchy's singular integral of $f\in L^{1}\left(\Gamma\right)$ defined as

$$S_{\Gamma}f\left(z\right):=\lim_{\varepsilon\to0}\frac{1}{2\pi i}\int\limits_{\Gamma\setminus\Gamma(z,\epsilon)}\frac{f\left(\varsigma\right)}{\varsigma-z}d\varsigma,\qquad z\in\Gamma.$$

The linear operator $S_{\Gamma}:f\to S_{\Gamma}f$ is called the Cauchy singular operator.

LEMMA 5. [9] *Let Γ be a rectifiable Jordan curve and let $L_{M}\left(\Gamma\right)$ be a reflexive Orlicz space on Γ. Then the singular operator S_{Γ} is bounded on $L_{M}\left(\Gamma\right)$,* i.e.,

$$\left\|S_{\Gamma}f\right\|_{L_{M}(\Gamma)}\leq c_{1}\left\|f\right\|_{L_{M}(\Gamma)},\quad f\in L_{M}\left(\Gamma\right),$$

for some constant $c_{1}>0$ if and only if Γ is a Carleson curve.

3. Proof of Theorem

We proof firstly that for sufficiently large n, all zeros of the Faber polynomials F_n are in G. Let

$$\kappa := \max_{z\in\Gamma} |\alpha_z - 1|, \quad z \in \Gamma.$$

Then by (4) we have $0 \le \kappa < 1$. Setting $\epsilon := \frac{1-\kappa}{2}$ in Lemma 4, for sufficiently large n we get

$$|F_n(z) - [\phi(z)]^n| < \frac{1+\kappa}{2}, \quad z \in \Gamma. \tag{9}$$

Since $F_n(z) - [\phi(z)]^n$ is analytic on $C\overline{G} := \overline{\mathbb{C}}\backslash\overline{G}$, by the maximum principle we have

$$|F_n(z) - [\phi(z)]^n| < \frac{1+\kappa}{2}, \quad z \in CG,$$

and therefore

$$|F_n(z)| \ge |\phi(z)|^n - \frac{1+\kappa}{2} \ge \frac{1-\kappa}{2} > 0, \quad z \in CG.$$

This gives to us the first part of the theorem.

Let $P_{n-1}(z)$ be the $(n-1)\,th$ best approximating polynomial to f in $E_M(G)$. Then

$$\begin{aligned}\|f - L_n(f,\cdot)\|_{E_M(G)} &= \|f - P_{n-1} - L_n(f - P_{n-1},\cdot)\|_{E_M(G)} \\ &\le (1 + \|L_n\|)\,\|f - P_{n-1}\|_{E_M(G)}\end{aligned}$$

because $L_n(f,z)$ is a linear interpolating polynomial operator. Now we only need to show that, for large values of n, $L_n(f,z)$ is uniformly bounded in the reflexive Smirnov-Orlicz class $E_M(G)$.

Choosing the interpolation nodes as the zeros of the Faber polynomials we have for $z' \in G$

$$\begin{aligned} f(z') - L_n(f,z') &= \frac{F_n(z')}{2\pi i}\int_\Gamma \frac{f(\varsigma)}{F_n(\varsigma)(\varsigma - z')}d\varsigma \\ &= F_n(z')\left(\mathcal{H}\left[\frac{f}{F_n}\right]\right)(z').\end{aligned}$$

Taking the limit $z' \to z \in \Gamma$ along all nontangential paths inside of Γ we get by (1)

$$\begin{aligned} \|f - L_n(f,\cdot)\|_{E_M(G)} &= \left\| F_n(z) \cdot \left(\mathcal{H}\left[\frac{f}{F_n}\right]\right)(z) \right\|_{L_M(\Gamma)} \\ &\leq \left\{ \max_{z\in\Gamma} |F_n(z)| \right\} \cdot \left\| \mathcal{H}\left[\frac{f}{F_n}\right] \right\|_{L_M(\Gamma)}, \end{aligned}$$

and later by Lemma 5,

$$\begin{aligned} \|f - L_n(f,\cdot)\|_{E_M(G)} &\leq c_1 \cdot \left\{ \max_{z\in\Gamma} |F_n(z)| \right\} \cdot \left\| \frac{f}{F_n} \right\|_{L_M(\Gamma)} \\ &\leq c_1 \cdot \left\{ \max_{z,\varsigma\in\Gamma} \left| \frac{F_n(z)}{F_n(\varsigma)} \right| \right\} \cdot \|f\|_{L_M(\Gamma)}. \end{aligned}$$

From (9),

$$\frac{1-\kappa}{2} < |F_n(z)| < \frac{3+\kappa}{2}, \qquad z \in \Gamma$$

and hence

$$\|f - L_n(f,\cdot)\|_{E_M(G)} \leq c_1 \cdot \frac{3+\kappa}{1-\kappa} \cdot \|f\|_{L_M(\Gamma)}.$$

Since

$$\begin{aligned} \|L_n(f,\cdot)\|_{E_M(G)} &\leq \|f\|_{E_M(G)} + \|f - L_n(f,\cdot)\|_{E_M(G)} \\ &\leq \left(1 + c_1 \cdot \frac{3+\kappa}{1-\kappa}\right) \cdot \|f\|_{L_M(\Gamma)}, \end{aligned}$$

by choosing $c_2 := 1 + c_1 \cdot \frac{3+\kappa}{1-\kappa}$ we obtain that $\|L_n\| \leq c_2$ and therefore we conclude by (2)

$$\begin{aligned} \|f - L_n(f,\cdot)\|_{E_M(G)} &\leq (1+c_2)\|f - P_{n-1}\|_{E_M(G)} \\ &= c_3 \cdot E^M_{n-1}(f,G), \end{aligned}$$

where $c_3 := 1 + c_2$. □

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