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# Approximation by Representative Functions on the Complete Product of $\mathcal{S}_3$

**Rodolfo Toledo**

**Abstract:** This work summarizes some statements with respect to Fourier analysis on the complete product of not necessarily commutative finite groups, achieved recently. In particular we devotes attention to a concrete case: the complete product of the symmetric group on 3 elements. The aim of this work is to emphasize the differences between this noncommutative structure and the commutative cases.

**Keywords:** Fourier analysis, symmetric group, noncommutative structure, Walsh-Paley system.

## 1 Introduction

SEVERAL RESULTS in Fourier analysis with respect to Walsh functions are obtained viewing them as the characters of the dyadic group, i.e., the complete product of the discrete cyclic group of order 2 with the product of topologies and measures. Then we often order the Walsh functions in the Paley's sense writing them as the finite product of the Rademacher functions. It is named the Walsh-Paley system. The above structure was generalized by Vilenkin [1] in 1947 studying the complete product of arbitrary cyclic groups. The construction of the system here is similar, taking the finite product of the characters of the cyclic groups as it Paley did.

In [2] the authors generalize the above structures, taking the complete product of not necessarily commutative finite groups. They use representation theory in order to obtain orthonormal systems, which are named representative product systems. These new structures were introduced in the following way.

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The author is with Inst. of Mathematics and Computer Science, College of Nyíregyháza , P. O. Box 166, 4400 Nyíregyháza, Hungary (e-mail toledo@zeus.nyf.hu).

Denote by **N**, **P**, **C** the set of nonnegative, positive integers and complex numbers, respectively. Let $m := (m_k,\ k \in \mathbf{N})$ be a sequence of positive integers such that $m_k \geq 2$ and $G_k$ a finite group with order $m_k$, $(k \in \mathbf{N})$. Suppose that each group has discrete topology and normalized Haar measure $\mu_k$. Let $G$ be the compact group formed by the complete direct product of $G_k$ with the product of the topologies, operations and measures $(\mu)$. Thus each $x \in G$ consist of sequences $x := (x_0, x_1, ...)$, where $x_k \in G_k$, $(k \in \mathbf{N})$. We call this sequence the *expansion* of $x$. The compact totally disconnected group $G$ is called a *bounded group* if the sequence $m$ is bounded. In order to simplicity we always use the multiplication to denote the group operation and use the symbol $e$ to denote the identity of the groups.

If $M_0 := 1$ and $M_{k+1} := m_k M_k$, $k \in \mathbf{N}$, then every $n \in \mathbf{N}$ can be uniquely expressed as

$$n = \sum_{k=0}^{\infty} n_k M_k, \qquad (0 \leq n_k < m_k,\ n_k \in \mathbf{N}).$$

This allows us to say that the sequence $(n_0, n_1, \dots)$ is the expansion of $n$ with respect to $m$. We often use the following notations: let $|n| := \max\{k \in \mathbf{N} : n_k \neq 0\}$ and

$$n_{(k)} := \sum_{j=0}^{k-1} n_k M_k, \qquad n^{(k)} := \sum_{j=k}^{\infty} n_k M_k.$$

The notation which we used to construct orthonormal systems is similar to the one appeared in [3]. Denote by $\Sigma_k$ the dual object of the finite group $G_k$ $(k \in \mathbf{N})$. Thus each $\sigma \in \Sigma_k$ is a set of continuous irreducible unitary representations of $G_k$ which are equivalent to some fixed representation $U^{(\sigma)}$. Let $d_\sigma$ be the dimension of its representation space and let $\{\zeta_1, \zeta_2, \dots, \zeta_{d_\sigma}\}$ be a fixed but arbitrary orthonormal basis in the representation space. The functions

$$u_{i,j}^{(\sigma)}(x) := \langle U_x^{(\sigma)} \zeta_i, \zeta_j \rangle \qquad (i, j \in \{1, \dots, d_\sigma\}, x \in G_k)$$

are called the coordinate functions for $U^{(\sigma)}$ and the basis $\{\zeta_1, \zeta_2, \dots, \zeta_{d_\sigma}\}$. In this manner for each $\sigma \in \Sigma_k$ we obtain $d_\sigma^2$ number of coordinate functions, in total $m_k$ number of functions for the whole dual object of $G_k$. The $L^2$-norm of these functions is $1/\sqrt{d_\sigma}$.

Let $\{\varphi_k^s : 0 \leq s < m_k\}$ be a system of all *normalized coordinate functions* of the group $G_k$. We do not decide now the order of the system $\varphi$, only suppose that $\varphi_k^0$ is always the character 1. Thus for every $0 \leq s < m_k$ there exists a $\sigma \in \Sigma_k$, $i, j \in \{1, ..., d_\sigma\}$ such that

$$\varphi_k^s(x) = \sqrt{d_\sigma}\, u_{i,j}^{(\sigma)}(x) \qquad (x \in G_k).$$

If the finite group $G_k$ is commutative, then $d_\sigma = 1$ for all $\sigma \in \Sigma_k$ and the coordinate functions are characters, that is all of them are continuous complex-valued maps on $G$ which satisfy

$$\varphi_k^s(xy) = \varphi_k^s(x)\varphi_k^s(y) \qquad (x, y \in G_k).$$

and

$$|\varphi_k^s(x)| = 1 \qquad (x \in G_k).$$

In the construction of the Walsh-Paley and Vilenkin systems all of the group $G_k$ are cyclic. For the cyclic group of order 2 ($m_k := 2$) we obtain the concept of *Rademacher functions*

$$\varphi_k^s(x) = (-1)^{sx} \qquad (s \in \{0,1\},\ x \in \mathcal{Z}_2). \tag{1}$$

Moreover, we can generalize the above functions for an arbitrary cyclic groups with order $m_k > 2$ to obtain the concept of *generalized Rademacher functions*

$$\varphi_k^s(x) = \exp(2\pi \imath sx/m_k) \qquad (s \in \{0, \dots m_k - 1\},\ x \in \mathcal{Z}_{m_k},\ \imath^2 = -1). \tag{2}$$

The above equations not only define the systems $\varphi$ for cyclic groups, but also give the order of these systems.

On the other hand, if $G_k$ is a noncommutative finite group, then it has normalized coordinate functions which take the value 0 and with module greater than 1. We can observe this fact in Table 1, where the values of the system $\varphi$ appear for the symmetric group on 3 elements, denoted by $\mathcal{S}_3$. This group has two characters ($\varphi^0$ and $\varphi^1$) and a 2-dimensional representation.

| | $e$ | $(12)$ | $(13)$ | $(23)$ | $(123)$ | $(132)$ | $\lVert\varphi^s\rVert_1$ | $\lVert\varphi^s\rVert_\infty$ |
|---|---|---|---|---|---|---|---|---|
| $\varphi^0$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $\varphi^1$ | $1$ | $-1$ | $-1$ | $-1$ | $1$ | $1$ | $1$ | $1$ |
| $\varphi^2$ | $\sqrt{2}$ | $-\sqrt{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{2}}{2}$ | $\frac{2\sqrt{2}}{3}$ | $\sqrt{2}$ |
| $\varphi^3$ | $\sqrt{2}$ | $\sqrt{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{2}}{2}$ | $\frac{2\sqrt{2}}{3}$ | $\sqrt{2}$ |
| $\varphi^4$ | $0$ | $0$ | $-\frac{\sqrt{6}}{2}$ | $\frac{\sqrt{6}}{2}$ | $\frac{\sqrt{6}}{2}$ | $-\frac{\sqrt{6}}{2}$ | $\frac{\sqrt{6}}{3}$ | $\frac{\sqrt{6}}{2}$ |
| $\varphi^5$ | $0$ | $0$ | $-\frac{\sqrt{6}}{2}$ | $\frac{\sqrt{6}}{2}$ | $-\frac{\sqrt{6}}{2}$ | $\frac{\sqrt{6}}{2}$ | $\frac{\sqrt{6}}{3}$ | $\frac{\sqrt{6}}{2}$ |

Table 1. The system $\varphi$ for $\mathcal{S}_3$

We construct an orthonormal system on $G$ as follows. Let $\psi$ be the product system of $\varphi_k^s$, namely

$$\psi_n(x) := \prod_{k=0}^{\infty} \varphi_k^{n_k}(x_k) \qquad (x \in G),$$

where $n$ is of the form $n = \sum_{k=0}^{\infty} n_k M_k$ and $x = (x_0, x_1, ...)$. Thus we say that $\psi$ is the *representative product system* of $\varphi$. If all of the finite group are cyclic, so the systems $\varphi$ are given by (1), then the system $\psi$ is called the Walsh-Paley system. For the complete product of arbitrary cyclic group with systems $\varphi$ ordered as (2), the system $\psi$ is called a Vilenkin system. The Weyl-Peter's theorem (see [3]) secures that the system $\psi$ is orthonormal and complete on $L^2(G)$.

## 2 Representation on the Interval [0,1]

In [4] the author establishes a natural relation between the Haar integration on the complete direct product of finite discrete topological groups and the Lebesgue integration on the interval $[0,1]$. With this intention, order the elements of all $G_k$ $(k \in \mathbf{N})$ groups in some way such that the first is always their identity. In fact, the ordering is a bijection between $G_k$ and $\{0, 1, \ldots, m_k - 1\}$ which give to every $x \in G_k$ the integer $0 \le \overline{x} < m_k$ ($\overline{e} = 0$). Define

$$|x| := \sum_{k=0}^{\infty} \frac{\overline{x_k}}{M_{k+1}} \qquad (x \in G).$$

It is easy to see that $|.|$ is a norm and the proceeded metric $d(x,y) := |xy^{-1}|$ induces the topology of $G$. In addition, $0 \le |x| \le 1$ for all $x \in G$. Using this fact we represent the group $G$ in the interval $[0,1]$.

Any $x \in [0,1]$ can be written

$$x := \sum_{k=0}^{\infty} \frac{\overline{x_k}}{M_{k+1}} \qquad (0 \le \overline{x_k} \le m_k - 1),$$

but there are numbers with two expressions of this form. They are all numbers in the set

$$\mathbf{Q} := \left\{ \frac{p}{M_n} : 0 \le p < M_n,\ n, p \in \mathbf{N} \right\}$$

called *m-adic rational numbers* (Note that 1 is not an $m$-adic rational number). The other numbers have only one expression. The $m$-adic rational numbers have an expression terminates in 0's and other terminates in $m_k - 1$'s. We choose the first one to make an unique relation for all numbers in the interval $[0,1]$ with their expression, named de *m-adic expansion* of the number. In this manner we assign to a number in the interval $[0,1]$ having an $m$-adic expansion $(\overline{x_0}, \overline{x_1}, \ldots)$ an element of $G$ with expansion $(x_0, x_1, \ldots)$ denoting this relation by $\rho$. $\rho$ is called the *Fine's map*. Using Fine's map we introduce a new operation on the interval $[0,1[$:

$$x \odot y := |\rho(x)\rho(y)| \qquad (x, y \in [0,1[).$$

Let $L^0(G)$ denote the set of all measurable functions on $G$ which are a.e. finite. In some way denote by $L^0$ the set of all Lebesgue measurable functions on $[0,1]$ which are a.e. finite. The following theorem shows the relation between the Haar integration on $G$ and the Lebesgue integration on the interval $[0,1]$.

**Theorem 1** (see [4])**.** *Let $\rho$ denote the Fine's map.*

*(a) If $f \in L^0(G)$ then $f \circ \rho \in L^0$. Conversely, if $g \in L^0$ and*

$$f(x) := g(|x|) \qquad (x \in G) \tag{3}$$

*then $f \in L^0(G)$.*

*(b) If $f$ is integrable on $G$ then $f \circ \rho$ is Lebesgue integrable and*

$$\int_G f\,d\mu = \int_0^1 (f \circ \rho)(x)\,dx.$$

*Conversely, if $g$ is Lebesgue integrable and $f$ is defined by (3) then $f$ is integrable on $G$ and*

$$\int_0^1 g(x)\,dx = \int_G f\,d\mu.$$

According to Theorem 1, we can represent the system $\psi$ on the interval $[0,1]$ substituting it by the

$$\upsilon_n := \psi_n \circ \rho \qquad (n \in \mathbf{N})$$

system. In Figure 1 we plot the corresponding values of $\psi_{12}$ and $\psi_{23}$ with respect to the complete product of $\mathcal{S}_3$. These graphs show two properties of the system $\psi$ which are different to the commutative cases and difficult the study of the noncommutative cases: the system $\psi$ is not uniformly bounded and can take the value 0.

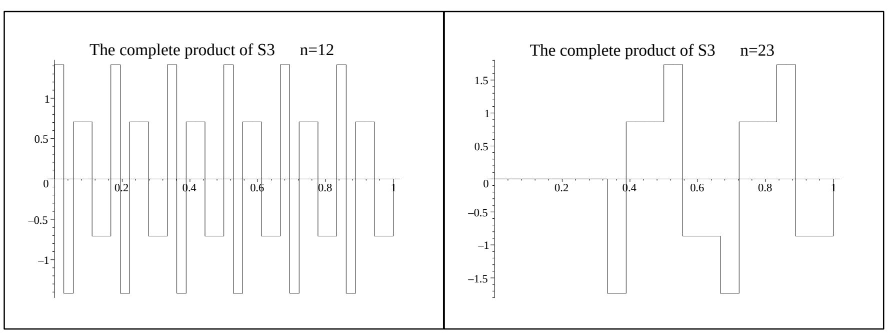


Fig. 1. $\psi_{12}$ and $\psi_{23}$ with respect to the complete product of $\mathcal{S}_3$.

## 3 Properties of Dirichlet Kernels

For an integrable complex function $f$ defined in $G$ we define the Fourier coefficients and partial sums by

$$\widehat{f}_k := \int_{G_m} f\overline{\psi}_k \, d\mu \quad (k \in \mathbf{N}), \qquad S_n f := \sum_{k=0}^{n-1} \widehat{f}_k \psi_k \quad (n \in \mathbf{N}).$$

The *Dirichlet kernels* are defined as follows:

$$D_n(x,y) := \sum_{k=0}^{n-1} \psi_k(x)\overline{\psi}_k(y) \qquad (n \in \mathbf{N}).$$

It is easy to see that

$$S_n f(x) = \int_G f(y) D_n(x,y) d\mu(y), \tag{4}$$

which shows the importance of the Dirichlet kernels in the study of the convergence of Fourier series.

Define $I_0(x) := G$,

$$I_n(x) := \{y \in G : y_k = x_k, \text{ for } 0 \leq k < n\} \quad (x \in G, n \in \mathbf{P}).$$

We say that every set $I_n(x)$ is an *interval*. The set of intervals $I_n$ is a countable neighborhood base at the identity of the product topology on $G$.

The following lemma is known by Paley's lemma for commutative cases. It can be also stated for representative product systems in general.

**Lemma 1** (Paley's lemma)**.** *If $n \in \mathbf{N}$ and $x,y \in G$, then*

$$D_{M_k}(x,y) = \begin{cases} M_k & \text{for} \quad x \in I_k(y), \\ 0 & \text{for} \quad x \notin I_k(y) \end{cases}$$

The Paley lemma is used to prove that the $S_{M_n} f$ partial sequence of Fourier sums converge to $f$ in $L^p$-norm and a.e., if $f \in L^p(G)$, $p \geq 1$. So we can also state this proposition for the complete product of $\mathcal{S}_3$. However, the other values of $D_n$ are more different when $n \neq M_k$. To illustrate this statement define by

$$\mathcal{D}_n := \sup_{x,y \in G} |D_n(x,y)| \qquad (n \in \mathbf{P})$$

the maximal value of the Dirichlet kernel. For commutative cases $\mathcal{D}_n = n$ for all $n \in \mathbf{P}$, but the general case is a bit more different.

**Theorem 2.** *Let $G$ be the complete product of $\mathcal{S}_3$ and $\psi$ be the representative product system with respect to the system $\varphi$ of Table 1. If $n \in \mathbf{P}$ and $A := \max\{k \in \mathbf{N} : n_k \neq 0\}$, then*

$$n \le D_n \le 6^{A+1}.$$

*Proof.* By the inequality of Cauchy-Bunyakovszki we have

$$\begin{aligned}\left|\sum_{k=0}^{n-1} \psi_k(x)\overline{\psi}_k(y)\right|^2 &\le \sum_{k=0}^{n-1} |\psi_k(x)|^2 \sum_{k=0}^{n-1} |\psi_k(y)|^2 \\ &\le \max{}^2 \left\{\sum_{k=0}^{n-1} |\psi_k(x)|^2, \sum_{k=0}^{n-1} |\psi_k(y)|^2\right\} \qquad (x, y \in G).\end{aligned}$$

Consequently,

$$D_n = \sup_{x\in G} D_n(x,x) = \sup_{x\in G} \sum_{k=0}^{n-1} |\psi_k(x)|^2 \qquad (n \in \mathbf{P}) \tag{5}$$

from which we have that $D_n$ is monotone increasing sequence.
The unitary property of the representations implies (see [5])

$$\sum_{s=0}^{j} |\varphi_k^s(x_k)|^2 \le 6 \qquad (j < 6, x_k \in \mathcal{S}_3). \tag{6}$$

Moreover, let $n \in \mathbf{N}$, $x, y \in G$. Thus, we have

$$\begin{aligned}D_n(x,y) =& D_{6^A}(x,y) \left(\sum_{s=0}^{n_A-1} \varphi_A^s(x_A)\overline{\varphi}_A^s(y_A)\right) \\ &+ \varphi_A^{n_A}(x_A)\overline{\varphi}_A^{n_A}(y_A) D_{n_{(A)}}(x,y).\end{aligned}$$

By Paley lemma and (5) we obtain

$$D_n(x,x) = 6^A \left(\sum_{s=0}^{n_A-1} |\varphi_A^s(x_A)|^2\right) + |\varphi_A^{n_A}(x_A)|^2 D_{n_{(A)}}(x,x). \tag{7}$$

Observe, $n = \sum_{k=0}^{A} n_k 6^k$ and $n_{(A)} = \sum_{k=0}^{A-1} n_k 6^k$. By induction on $A$ we prove $D_n(x,x) \le 6^{A+1}$ for all $x \in G$. Indeed, by (6)

$$D_{n_{(1)}}(x,x) = \sum_{s=0}^{n_0-1} |\varphi_0^s(x_0)|^2 \le 6$$

and supposing $D_{n_{(A)}}(x,x) \le 6^A$, by (6) and (7) we have

$$D_{n_{(A+1)}}(x,x) = D_n(x,x) \le 6^A \left( \sum_{s=0}^{n_A-1} |\varphi_A^s(x_A)|^2 \right) + |\varphi_A^{n_A}(x_A)|^2 6^A$$
$$= 6^A \left( \sum_{s=0}^{n_A} |\varphi_A^s(x_A)|^2 \right) \le 6^A \cdot 6 = 6^{A+1},$$

from which the inequality $D_n \le 6^{|n|+1}$ follows.

On the other hand, by the orthonormality of the system $\psi$ we have

$$\int_G \sum_{k=0}^{n-1} |\psi_k|^2 \, d\mu = \sum_{k=0}^{n-1} \int_G |\psi_k|^2 \, d\mu = n$$

and by the $\mu(G) = 1$ propety

$$\int_G \sum_{k=0}^{n-1} |\psi_k|^2 \, d\mu \le \sup_{x \in G} \sum_{k=0}^{n-1} |\psi_k|^2 = D_n \tag{8}$$

from which we obtain the property $D_n \ge n$. This completes the proof of the lemma. $\square$

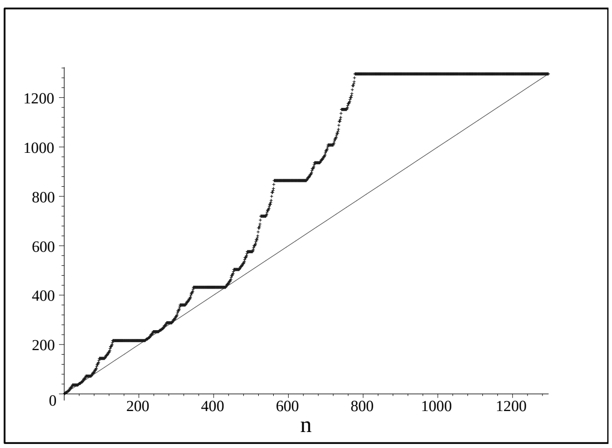


Fig. 2. $D_n$ ($n \le 6^4$) on the complete product of $\mathcal{S}_3$

Figure 2 illustrates the statements of Theorem 2 with respect to the system $\varphi$ on $\mathcal{S}_3$ appeared in Table 1.

## 4 Convergence in $L^p$-Norm

An essential difference is revealed when we study the convergence of Fourier series. Paley proved the fact that the partial sums of Fourier series are uniformly bounded, from $L^p$ into itself, where $1 < p < \infty$. It is equivalent to the convergence of these operators in $L^p$-norm. This statement is known as the Paley's theorem. Paley's theorem was shown independently for arbitrary Vilenkin systems by Young [6], Schipp [7] and Simon [8].

**Theorem 3** (Paley's theorem)**.** *Let $G$ be a Vilenkin group and $f \in L^p(G)$, $1 < p < \infty$. Then the partial sums of Fourier series of the function $f$, denoted by $S_n f$, converge to the function $f$ in $L^p$-norm.*

Unfortunately, we can not extend this statement for the complete product of $\mathcal{S}_3$.

**Theorem 4** (see [9])**.** *Let $G$ be the complete product of $\mathcal{S}_3$ and $\psi$ be the representative product system with respect to the system $\varphi$ of Table 1. If $1 < p < \infty$ and $p \neq 2$, then there exists an $f \in L^p(G)$ such that $S_n f$ does not converge to the function $f$ in $L^p$-norm.*

For this reason it is very interesting the fact that the Fejér means of an arbitrary function belongs to $L^p(G)$ $(1 \leq p < \infty)$ converge to the function in $L^p$-norm. This statement was proved in [2]. We define the Fejér means of Fourier series of the function $f$ by

$$\sigma_n f = \frac{1}{n} \sum_{k=1}^{n} S_k f \qquad (n \in \mathbf{P}).$$

Thus, we have

**Theorem 5** (see [2])**.** *Let $G$ be the complete product of $\mathcal{S}_3$ and $\psi$ be the representative product system with respect to the system $\varphi$ of Table 1. If $f \in L^p(G)$, $1 \leq p < \infty$, then $\sigma_n f$ converge to the function $f$ in $L^p$-norm.*

The last theorem can be extended for Cesàro means of order $\alpha$ $(0 < \alpha < 1)$ of the Fourier series, but only for certain values of $\alpha$. The *Cesàro numbers of order $\alpha$* are given by the formula

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)\dots(\alpha+n)}{n!} \qquad (n \in \mathbf{N})$$

where $\alpha$ is a real number. We summarize the main properties of this numbers as follows (see [10]).

- $A_n^\alpha = \sum_{k=0}^{n} A_{n-k}^{\alpha-1}$,

- $A_n^\alpha - A_{n-1}^\alpha = A_n^{\alpha-1}$,
- $\lim\limits_{n\to\infty} \dfrac{A_n^\alpha}{n^\alpha} = \dfrac{1}{\Gamma(\alpha+1)} \qquad (\alpha \neq -1,-2,\ldots)$,
- the numbers $A_n^\alpha$ are positive if $\alpha > -1$, and $A_n^\alpha < 1$ if $-1 < \alpha < 0$,
- the sequence $A_n^\alpha$ increasing for $\alpha > 0$ and decreasing for $-1 < \alpha < 0$.

Using the notation above we denote *the Cesàro means of order $\alpha$ of Fourier series* or simply $(C,\alpha)$ *means* by

$$\sigma_n^\alpha f := \frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} S_k f \qquad (n \in \mathbf{P}). \tag{9}$$

We define the number $\alpha_0$ as follows

$$\alpha_0 := \limsup_{k\to\infty} \log_{m_k} \left( \max_{0 \le s < m_k} \|\varphi_k^s\|_1 \|\varphi_k^s\|_\infty \right).$$

$\alpha_0$ is the infimum of all $0 < \alpha < 1$ such that

$$\|\varphi_k^s\|_1 \|\varphi_k^s\|_\infty < m_k^\alpha \qquad (0 \le s < m_k)$$

holds except finite numbers of $k \in \mathbf{N}$. We remark that the number $\alpha_0$ exists and it less than $\frac{1}{2}$ since $\|\varphi_k^s\|_\infty^2 < m_k$ and $\|\varphi_k^s\|_1 \le 1$ for all $k \in \mathbf{N}$. For the commutative cases it is obvious that $\alpha_0 = 0$. For the system $\varphi$ of Table 1 it is easy to see that $\alpha_0 = \log_6 \frac{4}{3}$. Thus we have

**Theorem 6** (see [11])**.** *Let $G$ be the complete product of $\mathcal{S}_3$ and $\psi$ be the representative product system with respect to the system $\varphi$ of Table 1. If $f \in L^p(G)$, $1 \le p < \infty$ and $\alpha > \log_6 \frac{4}{3}$, then $\sigma_n^\alpha f$ converge to the function $f$ in $L^p$-norm.*

For values of $\alpha$ less than $\log_6 \frac{4}{3}$ we obtain divergence in $L^1$-norm.

**Theorem 7** (see [11])**.** *Let $G$ be the complete product of $\mathcal{S}_3$ and $\psi$ be the representative product system with respect to the system $\varphi$ of Table 1. If $\alpha < \log_6 \frac{4}{3}$, then there exists an $f \in L^1(G)$ such that $\sigma_n^\alpha f$ does not converge to the function $f$ in $L^1$-norm.*

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