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APPROXIMATION FROM SHIFT-INVARIANT SUBSPACES OF $L_2(\mathbb{R}^d)$

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ABSTRACT. A complete characterization is given of closed shift-invariant subspaces of $L_2(\mathbb{R}^d)$ which provide a specified approximation order. When such a space is principal (i.e., generated by a single function), then this characterization is in terms of the Fourier transform of the generator. As a special case, we obtain the classical Strang-Fix conditions, but without requiring the generating function to decay at infinity. The approximation order of a general closed shift-invariant space is shown to be already realized by a specifiable principal subspace.

1. INTRODUCTION

We are interested in the approximation properties of closed shift-invariant subspaces of $L_2(\mathbb{R}^d)$. We say that a space $\mathscr{S}$ of complex-valued functions defined on $\mathbb{R}^d$ is **shift-invariant** if, for each $f \in \mathscr{S}$, the space $\mathscr{S}$ also contains the **shifts** $f(\cdot+\alpha)$, $\alpha \in \mathbb{Z}^d$. In other words, $\mathscr{S}$ contains all the *integer* translates of f if it contains f. A particularly simple example is provided by the space

$$\mathscr{S}_0(\phi)$$

of all *finite* linear combinations of shifts of a single function ϕ. We call its $L_2(\mathbb{R}^d)$-closure the **principal** shift-invariant space **generated by** ϕ and denote it by

$$\mathscr{S}(\phi).$$

Of course, a closed shift-invariant subspace of $L_2(\mathbb{R}^d)$ need not be principal; it need not even be generated by the shifts of finitely many functions.

Shift-invariant spaces are important in a number of areas of analysis. Many spaces, encountered in approximation theory and in finite element analysis,

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are generated by the shifts of a finite number of functions ϕ on $\mathbb{R}^d$. Shift-invariant spaces also play a key role in the construction of wavelets. In each of these applications, one is interested in how well a general function f can be approximated by the elements of the scaled spaces

$$\mathcal{S}^h := \{s(\cdot/h) : s \in \mathcal{S}\}.$$

We postpone discussion of the literature until we have introduced some additional terminology and stated our main results.

Associated to any closed subspace $\mathcal{S}$ of $L_2(\mathbb{R}^d)$ and any function $f \in L_2(\mathbb{R}^d)$, the approximation error is

$$E(f, \mathcal{S}) := \min\{\|f - s\| : s \in \mathcal{S}\}. \tag{1.1}$$

In this paper, we describe the properties of $\mathcal{S}$ which govern the decay rates of $E(f, \mathcal{S}^h)$. We characterize when the scaled subspaces $\mathcal{S}^h$ are **dense** in the sense that $\lim_{h\to 0} E(f, \mathcal{S}^h) = 0$ for every $f \in L_2(\mathbb{R}^d)$. More generally, we characterize when the spaces $\mathcal{S}^h$ approximate suitably smooth functions to order $O(h^k)$ as $h \to 0$.

Our definitions of approximation orders are in terms of the potential space $W_2^k(\mathbb{R}^d)$, $k \in \mathbb{R}_+$, defined by

$$W_2^k(\mathbb{R}^d) := \{f \in L_2(\mathbb{R}^d) : \|f\|_{W_2^k(\mathbb{R}^d)} := (2\pi)^{-d/2}\|(1 + |\cdot|)^k \hat{f}\| < \infty\}.$$

(Here and later, we use $|x| := (x_1^2 + \cdots + x_d^2)^{1/2}$ to denote the Euclidean norm of a point $x = (x_1, \ldots, x_d)$ in $\mathbb{R}^d$.) When k is a positive integer, these are the usual Sobolev spaces. We say that $\mathcal{S}$ **provides approximation order** k if, for every $f \in W_2^k(\mathbb{R}^d)$,

$$E(f, \mathcal{S}^h) \le \text{const}_{\mathcal{S}}\, h^k \|f\|_{W_2^k(\mathbb{R}^d)}. \tag{1.2}$$

A variant of this problem is to characterize when, for a given $k \ge 0$, we have for each $f \in W_2^k(\mathbb{R}^d)$ (in addition to (1.2)),

$$E(f, \mathcal{S}^h) = o(h^k), \qquad h \to 0. \tag{1.3}$$

When $k = 0$, this is the density problem. For this reason, we say that $\mathcal{S}$ provides **density order** k whenever (1.3) holds.

Our characterizations of density, approximation order, and density order are in terms of Fourier transforms. If $f \in L_1(\mathbb{R}^d)$, its Fourier transform $\hat{f}$ is defined by

$$\hat{f}(y) := \int_{\mathbb{R}^d} f(x) e^{-ix\cdot y}\, dx.$$

Many authors have shown (under various restrictive conditions on ϕ) that the approximation properties of a principal shift-invariant space $\mathcal{S}(\phi)$ are related to the *order of the zeros of the Fourier transform* of ϕ at the integer multiples of 2π. It is therefore not surprising that our characterizations of approximation order involve the behavior near zero of the 2π-periodization of $|\hat{\phi}|^2$, i.e., the $L_2(\mathbb{T}^d)$-function

$$[\hat{\phi}, \hat{\phi}] := \sum_{\beta \in 2\pi\mathbb{Z}^d} |\hat{\phi}(\cdot + \beta)|^2. \tag{1.4}$$

This function enters our considerations as part of the function $\Lambda_\phi \in L_\infty(C)$, defined on the centered cube in $\mathbb{R}^d$ of side length 2π,

$$\Lambda_\phi := \left(1 - \frac{|\hat{\phi}|^2}{[\hat{\phi}, \hat{\phi}]}\right)^{1/2}, \quad \text{on } C := [-\pi..\pi]^d. \tag{1.5}$$

Here (and below without further comment), we identify the space $L_2(\mathbb{T}^d)$ of functions on the d-dimensional torus $\mathbb{T}^d$ with the space $L_2(C)$ of functions on the fundamental domain C. Our characterization of approximation order is in terms of the function $y \mapsto \Lambda_\phi(y)/|y|^k$.

It is the behavior of Λ_ϕ at the origin, or, more precisely, the behavior of the function $y \mapsto |y|^{-k}\Lambda_\phi(y)$, that turns out to be crucial for the approximation order of $\mathcal{S}(\phi)$. Indeed, we shall prove

Theorem 1.6. *The principal shift-invariant subspace $\mathcal{S}(\phi)$ of $L_2(\mathbb{R}^d)$ provides approximation order $k > 0$ if and only if $|\cdot|^{-k}\Lambda_\phi$ is in $L_\infty(C)$.*

The analogue of this result for density orders is

Theorem 1.7. *The principal shift-invariant subspace $\mathcal{S}(\phi)$ of $L_2(\mathbb{R}^d)$ provides density order $k \geq 0$ if and only if $|\cdot|^{-k}\Lambda_\phi$ is in $L_\infty(C)$ and*

$$\lim_{h\to 0} h^{-d} \int_{hC} |y|^{-2k}[\Lambda_\phi(y)]^2\, dy = 0. \tag{1.8}$$

Of course, in the case $k = 0$, (1.8) characterizes when we have density.

It is rather remarkable that these conditions also characterize approximation and density orders for *arbitrary* closed shift-invariant subspaces of $L_2(\mathbb{R}^d)$. Namely, we shall prove:

Theorem 1.9. *A closed shift-invariant subspace $\mathcal{S}$ of $L_2(\mathbb{R}^d)$ provides approximation order $k > 0$ if and only if it contains a function ϕ for which $|\cdot|^{-k}\Lambda_\phi$ is in $L_\infty(C)$. The space $\mathcal{S}$ provides density order $k \geq 0$ if and only if it contains a function ϕ for which $|\cdot|^{-k}\Lambda_\phi \in L_\infty(C)$ and* (1.8) *holds.*

We prove the last theorem by showing in §3 that the case of approximation by arbitrary closed shift-invariant subspaces of $L_2(\mathbb{R}^d)$ can be reduced to the case of principal shift-invariant spaces.

In the case of principal shift-invariant spaces, our method of proof is based on two results which we feel will have other important applications. The first is an explicit formula for the best $L_2(\mathbb{R}^d)$-approximation from $\mathcal{S}(\phi)$. The second is the following characterization

$$\widehat{\mathcal{S}(\phi)} = \{\tau\hat{\phi} \in L_2(\mathbb{R}^d) : \tau \text{ is } 2\pi\text{-periodic}\} \tag{1.10}$$

of the space $\mathcal{S}(\phi)$ in terms of its Fourier transform. Here and later, for a set F of functions, we denote by $\widehat{F} := \{\hat{f} : f \in F\}$ the set of its Fourier transforms.

It turns out that our analysis applies equally well to the more general situation where the h-refinement of the space $\mathcal{S}$ is obtained by means other than scaling. Such cases are known and are of interest in both spline theory (e.g., exponential box splines, cf. [DR]) and radial basis function theory (cf. the detailed discussion in [BR2]). In the nonscaling case, we employ a family $\{\mathcal{S}_h\}_h$ of shift-invariant spaces, and consider the rates of decay of $E(f, \mathcal{S}_h^h)$ as a function of h. The notions of "approximation order k" or "density order k" for the sequence $\{\mathscr{S}_h\}_h$ are obtained by replacing each $E(f, \mathscr{S}^h)$ in the above definitions by $E(f, \mathscr{S}_h^h)$.

We close this section with a brief discussion of the connections between the results of this paper and results in the literature. Schoenberg, in his seminal paper [S], was the first to recognize the importance of the Fourier transform for describing approximation properties of principal shift-invariant spaces. For the case $d = 1$, and with ϕ a piecewise continuous function with exponential decay at infinity, Schoenberg showed that all algebraic polynomials of degree $< k$ can be written in the form $\sum_{\alpha\in\mathbb{Z}^d} \phi(\cdot - \alpha)c(\alpha)$ in case

$$(1.11) \qquad \hat{\phi}(0) \neq 0 \text{ and } D^\gamma\hat{\phi} = 0 \text{ on } 2\pi\mathbb{Z}^d\backslash 0 \text{ for all } |\gamma| < k$$

holds (with $d = 1$).

Strang and Fix [SF] have treated the approximation properties of the space

$$\mathscr{S}_*(\phi)$$

of *all* linear combinations $\sum_{\alpha\in\mathbb{Z}^d} \phi(\cdot-\alpha)c(\alpha)$ (finite or not) of the integer shifts of a *compactly supported* function ϕ. There is no problem of convergence of such sums since, for any point $x \in \mathbb{R}^d$, at most finitely many terms of the sum are nonzero at x. Strang and Fix necessarily restricted attention to the subspace

$$\mathscr{S}_2(\phi) := \mathscr{S}_*(\phi) \cap L_2(\mathbb{R}^d).$$

While this space is, in general, not closed in $L_2(\mathbb{R}^d)$, one can show (see Theorem 2.16 below) that its $L_2(\mathbb{R}^d)$-closure is $\mathscr{S}(\phi)$. Strang and Fix proved that $\mathscr{S}_2(\phi)$ provides approximation order k whenever (1.11) holds.

To compare this result with Theorem 1.6 above, note that, for a compactly supported ϕ, $[\hat{\phi}, \hat{\phi}]$ is a trigonometric polynomial, since then

$$(1.12) \qquad [\hat{\phi}, \hat{\phi}] = \sum_{\alpha\in\mathbb{Z}^d} a(\alpha)e_\alpha, \quad \text{with } a(\alpha) := \int_{\mathbb{R}^d} \phi(x-\alpha)\overline{\phi}(x)\,dx.$$

Here and later, we use the abbreviation

$$e_\alpha(y) := e^{i\alpha\cdot y}.$$

If (1.11) holds, then $[\hat{\phi}, \hat{\phi}]$ does not vanish at the origin and Λ_ϕ of (1.5) has a zero of multiplicity k there. Thus, $|\cdot|^{-k}\Lambda_\phi$ is in $L_\infty(C)$ (as we know it must be). However, there are two important points to bear in mind concerning our Theorem 1.6 and the Strang-Fix result. First of all, our theorem does not require that ϕ be compactly supported, nor even that it decay at infinity. Secondly, it applies even when $\hat{\phi}$ vanishes at the origin, a case of practical importance yet not accessible to earlier approaches.

Actually, Strang and Fix proved more than we have just stated since they showed that the approximation order $O(h^k)$ to a given $f \in W_2^k(\mathbb{R}^d)$ by the elements of $\mathscr{S}_2(\phi)^h$ can be achieved with a control on the coefficients of the approximants $s_h \in \mathscr{S}_2(\phi)^h$. Namely, if the approximants are represented with respect to the L_2-normalized functions $\phi(\alpha, h, x) := h^{-d/2}\phi(x/h-\alpha)$ by $s_h = \sum_{\alpha\in\mathbb{Z}^d} c_h(\alpha)\phi(\alpha, h, \cdot)$, then

$$(1.13) \qquad \|c_h\|_{\ell_2(\mathbb{Z}^d)} \leq \text{const}_f.$$

The introduction of such **controlled** approximation is important, since Strang and Fix show that, conversely, if $\mathcal{S}_2(\phi)$ provides controlled approximation order k, then (1.11) holds. In other words, for compactly supported ϕ, $\mathcal{S}_2(\phi)$ provides *controlled* approximation order k if and only if (1.11) holds. Since it can be easily seen that our condition in Theorem 1.6 is weaker than (1.11) (even for compactly supported ϕ), it follows that there are cases when the achievable approximation order cannot be obtained in a controlled manner. In this connection, it is worthwhile to point out (as is done in [SF]) that positive controlled approximation order forces $\hat{\phi}(0) \neq 0$.

There is a rich literature of clarifications and extensions of the Strang-Fix result, including extensions to noncompactly supported ϕ [BH2, J2, DM2, BJ, B1, R, CL, JL, HL, BR2]. In addition, there are many papers studying the approximation order of specific principal (and other) shift-invariant spaces, some of them [Bu1, Bu2, BD, BuD, BH1, BR1, DJLR, DM1, DR, Ja, J1, L, LJ, M, MN1, MN2, Ra, RS] are included in the references; see also the surveys [B2, C, P] and the references therein. By making assumptions on ϕ weaker than those used in any of the above references, we can still translate our conditions on Λ_ϕ into simple conditions on $\hat{\phi}$. For example, we show in §5 the following:

Theorem 1.14. *Assume that $\hat{\phi}$ is bounded on some neighborhood of the origin. If $\mathcal{S}(\phi)$ provides approximation order k, then $\hat{\phi}$ has a zero of order k at every $\beta \in 2\pi\mathbb{Z}^d\backslash 0$. In particular, $D^\gamma\hat{\phi}(\beta) = 0$ for all $|\gamma| < k$ in case $\hat{\phi}$ is k times differentiable (in the classical sense) at such β.*

Note that the boundedness of $\hat{\phi}$ required here holds, for example, if $\hat{\phi}$ is continuous at 0. In particular, it holds for every $\phi \in L_1(\mathbb{R}^d)$.

We also show in §5 the following converse:

Theorem 1.15. *Assume that $1/\hat{\phi}$ is bounded on some neighborhood of the origin and that, for some $\rho > k + d/2$, all derivatives of $\hat{\phi}$ of order $\leq \rho$ are in $L_2(A)$, with $A := B_\varepsilon + (2\pi\mathbb{Z}^d\backslash 0)$ for some open ball B_ε centered at the origin. If $D^\gamma\hat{\phi}(\beta) = 0$ for all $|\gamma| < k$ and all $\beta \in 2\pi\mathbb{Z}^d\backslash 0$, then $\mathcal{S}(\phi)$ provides approximation order k.*

For most of the examples of a noncompactly supported ϕ in the literature (e.g., radial basis functions, see [P]), $\hat{\phi}$ is very smooth on $\mathbb{R}^d\backslash 0$, but has a singularity at the origin. On the other hand, the present standard approach to the derivation of approximation orders (viz., the polynomial reproduction argument) requires ϕ to decay at ∞ (at least) like $O(|\cdot|^{-(k+d)})$, hence requires $\hat{\phi}$ to be *globally* smooth. To circumvent this obstacle, one usually seeks a function $\psi \in \mathcal{S}_0(\phi)$ (or in some superspace of $\mathcal{S}_0(\phi)$) whose Fourier transform $\hat{\psi}$ is smoother than $\hat{\phi}$, since this implies a more favorable decay of ψ at ∞. This "localization" process constitutes the main effort in establishing approximation orders for a noncompactly supported ϕ. Our theorem, though, does not require ϕ to decay at ∞ at any particular rate, thus obviating the search for such ψ. Results (weaker than the above theorem) about $L_\infty(\mathbb{R}^d)$-approximation orders, that apply to functions which decay only mildly at ∞, were derived in [BR2]. The approach there exploits the fact that the exponential functions e_θ, $\theta \in \mathbb{R}^d$, are in the space in which approximation takes place. In contrast, the approach here makes use of the simple and explicit formula for the orthogonal projection onto $\widehat{\mathcal{S}(\phi)}$.

2. The orthogonal projector onto $\mathcal{S}(\phi)$

In this section, we derive two important facts about the principal shift-invariant space $\mathcal{S}(\phi)$ which will be the basis of much of the analysis that follows. The first is a simple formula (given in Theorem 2.9) for the (Fourier transform of the) best L_2-approximation from $\mathcal{S}(\phi)$. The second is the description

$$\widehat{\mathcal{S}(\phi)} = \{\tau\hat{\phi} \in L_2(\mathbb{R}^d) : \tau \text{ is } 2\pi\text{-periodic}\} \tag{2.1}$$

of $\mathcal{S}(\phi)$ in terms of Fourier transforms mentioned in the introduction.

The yet to be proven (2.1) suggests that the calculation of integrals and inner products involving functions from $\mathcal{S}(\phi)$ should be taken over the torus $\mathbb{T}^d$. This can be accomplished by periodization. If $g \in L_1(\mathbb{R}^d)$, then

$$\int_{\mathbb{R}^d} g = \sum_{\beta\in 2\pi\mathbb{Z}^d} \int_{C+\beta} g = \int_C g^\circ, \tag{2.2}$$

with

$$g^\circ := \sum_{\beta\in 2\pi\mathbb{Z}^d} g(\cdot + \beta)$$

the (2π)-**periodization** of g. Here, the sum is to be taken in the sense of $L_1(\mathbb{T}^d)$-convergence, which makes sense since, by assumption, $g \in L_1(\mathbb{R}^d)$. In particular, $g^\circ \in L_1(\mathbb{T}^d)$.

Similarly, we have

$$\int_{\mathbb{R}^d} g_0\overline{g}_1 = \int_C [g_0, g_1] \tag{2.3}$$

for the inner product of two functions $g_0, g_1 \in L_2(\mathbb{R}^d)$, with

$$[g_0, g_1] := (g_0\overline{g}_1)^\circ = \sum_{\beta\in 2\pi\mathbb{Z}^d} g_0(\cdot+\beta)\overline{g}_1(\cdot+\beta). \tag{2.4}$$

Note that $[g_0, g_1]$ is in $L_1(\mathbb{T}^d)$ since $g_0\overline{g}_1 \in L_1(\mathbb{R}^d)$. Also, by the Cauchy-Schwarz inequality,

$$|[g_0, g_1]|^2 \le [g_0, g_0][g_1, g_1], \tag{2.5}$$

and the right side of (2.5) is finite a.e. We will most often use (2.3) in the form

$$\int_{\mathbb{R}^d} \tau\hat{f}\overline{\hat{\phi}} = \int_C \tau[\hat{f}, \hat{\phi}] \tag{2.6}$$

which is valid for arbitrary $f, \phi \in L_2(\mathbb{R}^d)$ and arbitrary 2π-periodic τ for which $\tau\hat{f} \in L_2(\mathbb{R}^d)$. We note that (2.6) implies the estimate

$$\|\tau\hat{\phi}\|_{L_2(\mathbb{R}^d)} \le \|\tau\|_{L_2(\mathbb{T}^d)} \|[\hat{\phi}, \hat{\phi}]\|_{L_\infty(\mathbb{T}^d)} \tag{2.7}$$

of use when $[\hat{\phi}, \hat{\phi}]$ is bounded, e.g., when ϕ is compactly supported.

After these brief remarks, let us consider the problem of finding a formula for the projection of $L_2(\mathbb{R}^d)$ onto $\mathcal{S}(\phi)$. Let $P := P_\phi$ denote the orthogonal projector onto $\mathscr{S}(\phi)$. The Pf is the unique best $L_2(\mathbb{R}^d)$-approximation to f from $\mathscr{S}(\phi)$, and is characterized by the fact that it lies in $\mathscr{S}(\phi)$ while its difference from f is orthogonal to $\mathscr{S}(\phi)$. Since the Fourier transform preserves orthogonality, it follows (for example from the uniqueness of best approximation in $L_2(\mathbb{R}^d)$) that the orthogonal projector $\widehat{P}$ onto $\widehat{\mathscr{S}(\phi)}$ satisfies $\widehat{P}\hat{f} = \widehat{Pf}$.

We consider first what it means for a function f to be orthogonal to $\mathscr{S}(\phi)$. Since finite linear combinations of the (integer) shifts $\phi(\cdot + \alpha)$ of ϕ are dense in $\mathscr{S}(\phi)$, $f \in L_2(\mathbb{R}^d)$ is orthogonal to $\mathscr{S}(\phi)$ iff $\hat{f}$ is orthogonal to $e_{-\alpha}\hat{\phi}$ for every $\alpha \in \mathbb{Z}^d$, i.e. (with (2.6)), iff

$$0 = \int_{\mathbb{R}^d} \hat{f} e_\alpha \overline{\hat{\phi}} = \int_C [\hat{f}, \hat{\phi}] e_\alpha \quad \text{for all } \alpha \in \mathbb{Z}^d.$$

This proves

Lemma 2.8. *The orthogonal complement* $\mathscr{S}(\phi)^\perp$ *of* $\mathscr{S}(\phi)$ *in* $L_2(\mathbb{R}^d)$ *consists of exactly those* $f \in L_2(\mathbb{R}^d)$ *for which* $[\hat{f}, \hat{\phi}] = 0$.

From Lemma 2.8, we can easily determine Pf. Suppose, as is suggested by (2.1), that $\widehat{Pf} = \tau\hat{\phi}$, with τ some 2π-periodic function. Then, from Lemma 2.8,

$$[\hat{f}, \hat{\phi}] = [\widehat{Pf}, \hat{\phi}] = [\tau\hat{\phi}, \hat{\phi}] = \tau[\hat{\phi}, \hat{\phi}].$$

This motivates the following:

Theorem 2.9. *For each* $f \in L_2(\mathbb{R}^d)$, $\widehat{P_\phi f} = \tau_f \hat{\phi}$, *with the* 2π*-periodic function* τ_f *defined by*

$$\tau_f := \begin{cases} [\hat{f}, \hat{\phi}]/[\hat{\phi}, \hat{\phi}] & \text{on } \Omega_\phi; \\ 0 & \text{otherwise}, \end{cases} \tag{2.10}$$

and Ω_ϕ *defined up to a null-set by*

$$\Omega_\phi := \operatorname{supp}[\hat{\phi}, \hat{\phi}] := \{\omega \in \mathbb{T}^d : [\hat{\phi}, \hat{\phi}](\omega) \neq 0\}.$$

Proof. It is enough to show that $\widehat{P}\hat{f} = \tau_f\hat{\phi}$ for each $f \in L_2(\mathbb{R}^d)$. We first want to see that $\tau_f\hat{\phi}$ is in $L_2(\mathbb{R}^d)$. By (2.5), $|\tau_f|^2[\hat{\phi}, \hat{\phi}] \le [\hat{f}, \hat{f}]$. With this, two applications of (2.6) give

$$\int_{\mathbb{R}^d} |\tau_f \hat{\phi}|^2 = \int_C |\tau_f|^2 [\hat{\phi}, \hat{\phi}] \le \int_C [\hat{f}, \hat{f}] = \int_{\mathbb{R}^d} |\hat{f}|^2. \tag{2.11}$$

Consequently, $\tau_f\hat{\phi} \in L_2(\mathbb{R}^d)$ and moreover the linear map

$$Q : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d) : \hat{f} \mapsto \tau_f \hat{\phi}$$

is well defined and norm-reducing on $L_2(\mathbb{R}^d)$. We next prove that $Q = \widehat{P}$.

If $\hat{f} \in \widehat{\mathscr{S}(\phi)}^\perp = (\mathscr{S}(\phi)^\perp)^\wedge$, then Lemma 2.8 gives that $\tau_f = 0$, hence $Q\hat{f} = 0$. Thus $Q = \widehat{P}$ on $\widehat{\mathscr{S}(\phi)}^\perp$. On the other hand, on $\Omega_\phi = \operatorname{supp}[\hat{\phi}, \hat{\phi}]$,

$$\tau_{\phi(\cdot+\alpha)} = [e_\alpha\hat{\phi}, \hat{\phi}]/[\hat{\phi}, \hat{\phi}] = e_\alpha, \quad \text{for all } \alpha \in \mathbb{Z}^d.$$

Since $\hat{\phi} = 0$ on the complement of $\Omega_\phi + 2\pi\mathbb{Z}^d$, this implies that Q maps the Fourier transform of each integer shift of ϕ to itself. Since Q is linear and bounded, and coincides with $\widehat{P}$ on a fundamental set for $\widehat{\mathscr{S}(\phi)}$, we have $Q = \widehat{P}$ on $\widehat{\mathscr{S}(\phi)}$. By linearity, $Q = \widehat{P}$ on all of $L_2(\mathbb{R}^d)$. □

Remark. With the convention (which we use throughout this paper) that 0 times any extended number is 0, we are entitled to write

$$\tau_f = [\hat{f}, \hat{\phi}]/[\hat{\phi}, \hat{\phi}] \quad \text{and} \quad \widehat{P_\phi f} = [\hat{f}, \hat{\phi}]\hat{\phi}/[\hat{\phi}, \hat{\phi}]. \tag{2.12}$$

Note that (2.11) supplies the following lemma.

Lemma 2.13. *If $\phi, f \in L_2(\mathbb{R}^d)$, then $\tau_f\hat{\phi} \in L_2(\mathbb{R}^d)$, and $\|\tau_f\hat{\phi}\| \le \|\hat{f}\|$.*

As a consequence, we obtain the characterization (2.1) of the space $\mathscr{S}(\phi)$ in terms of its Fourier transform.

Theorem 2.14. *A function f is in $\mathscr{S}(\phi)$ if and only if $\hat{f} = \tau\hat{\phi}$ for some 2π-periodic τ with $\tau\hat{\phi} \in L_2(\mathbb{R}^d)$. In particular, $\tau\hat{\phi} \in \widehat{\mathscr{S}(\phi)}$ for every bounded τ.*

Proof. If $f \in \mathscr{S}(\phi)$, then $Pf = f$. Hence, by Theorem 2.9, $\hat{f} = \tau_f\hat{\phi}$ with τ_f the 2π-periodic function $[\hat{f}, \hat{\phi}]/[\hat{\phi}, \hat{\phi}]$, and $\tau_f\hat{\phi} \in L_2(\mathbb{R}^d)$ because of Lemma 2.13.

Conversely, if τ is defined on $\mathbb{T}^d$, and $\tau\hat{\phi} \in L_2(\mathbb{R}^d)$, then the inverse transform f of $\tau\hat{\phi}$ is also in $L_2(\mathbb{R}^d)$ and satisfies $\tau_f = [\tau\hat{\phi}, \hat{\phi}]/[\hat{\phi}, \hat{\phi}] = \tau$ on $\Omega_\phi = \operatorname{supp}[\hat{\phi}, \hat{\phi}]$. Since $\hat{\phi}$ vanishes off $\Omega_\phi + 2\pi\mathbb{Z}^d$, this implies with Theorem 2.9 that $\widehat{Pf} = \tau_f\hat{\phi} = \tau\hat{\phi} = \hat{f}$. Consequently, $Pf = f$ and hence $f \in \mathscr{S}(\phi)$. Finally, if τ is bounded, then $\tau\hat{\phi} \in L_2(\mathbb{R}^d)$ since $\hat{\phi} \in L_2(\mathbb{R}^d)$. □

Remark 2.15. Asher Ben-Artzi has pointed out to us that Theorem 2.14 could have been derived from general results (cf. Theorem 8 of [H, p. 59]) concerning closed subspaces of $L_2(\mathbb{T}, l_2)$ which are invariant under multiplication by exponentials. Furthermore, the lemma of [H, p. 58] shows that Theorem 2.14 implies Theorem 2.9.

Remark. The representation $\tau\hat{\phi}$ for $\hat{f} \in \widehat{\mathscr{S}(\phi)}$ is in general not unique. If $\tau_0\hat{\phi} = \tau_1\hat{\phi}$, we can only conclude that $\tau_0 = \tau_1$ a.e. on Ω_ϕ. However, if the shifts of ϕ are an orthonormal basis or, more generally, an $L_2(\mathbb{R}^d)$-stable basis, then, as is well known, $[\hat{\phi}, \hat{\phi}]$ and its reciprocal are both in L_∞ and not only is the representation unique but the function τ is in $L_2(\mathbb{T}^d)$. It is interesting to note further that we have a unique representation even when the shifts of ϕ are not an $L_2(\mathbb{R}^d)$-stable basis provided Ω_ϕ differs from $\mathbb{T}^d$ only by a null-set.

The following consequence of Theorem 2.14 is of importance when comparing our results with related results in the literature.

Theorem 2.16. *If $\phi \in L_2(\mathbb{R}^d)$ has compact support, then $\mathscr{S}(\phi)$ is the $L_2(\mathbb{R}^d)$-closure of $\mathscr{S}_2(\phi) = \mathscr{S}_*(\phi) \cap L_2(\mathbb{R}^d)$.*

Proof. Since $\mathscr{S}(\phi)$ is the $L_2(\mathbb{R}^d)$-closure of $\mathscr{S}_0(\phi)$ and $\mathscr{S}_0(\phi)$ is contained in $\mathscr{S}_2(\phi)$ (since $\phi \in L_2(\mathbb{R}^d)$), we only have to prove that

$$\mathscr{S}_2(\phi) \subset \mathscr{S}(\phi). \tag{2.17}$$

We now prove this by showing that $P_\phi f = f$ for every $f \in \mathscr{S}_2(\phi)$, i.e., with (2.12), that

$$\hat{f} = [\hat{f}, \hat{\phi}]\hat{\phi}/[\hat{\phi}, \hat{\phi}]. \tag{2.18}$$

Since ϕ has compact support, $[\hat\phi, \hat\phi]$ is a trigonometric polynomial (cf. (1.12)), hence (2.18) is equivalent to the equation

$$[\hat\phi, \hat\phi]\hat f = [\hat f, \hat\phi]\hat\phi \quad \text{a.e.,} \tag{2.19}$$

and it is this equation we now verify for any f in $L_2(\mathbb{R}^d)$ of the form

$$\sum_{\beta\in\mathbb{Z}^d} \phi(\cdot - \beta)c(\beta).$$

We do this by showing that both sides of (2.19) are the Fourier transform of the function $\sum_{\alpha\in\mathbb{Z}^d} f(\cdot+\alpha)a(\alpha)$, with $a(\alpha) = \int_{\mathbb{R}^d}\phi(\cdot-\alpha)\overline{\phi}$ the (Fourier) coefficients of the trigonometric polynomial $[\hat\phi, \hat\phi]$, see (1.12). This is immediate for the left side of (2.19) since $(\sum_{\alpha\in\mathbb{Z}^d} f(\cdot+\alpha)a(\alpha))^\wedge = (\sum_{\alpha\in\mathbb{Z}^d} a(\alpha)e_\alpha)\hat f$ for any $f\in L_2(\mathbb{R}^d)$ and any *finite* sequence $(a(\alpha))$, and $[\hat\phi, \hat\phi]$ is indeed a finite sum of exponentials since ϕ is compactly supported. As to the right side of (2.19), $[\hat f, \hat\phi]$ is a 2π-periodic L_2-function (since ϕ is compact supported, thus $\hat\phi$ is bounded), hence the $L_2(\mathbb{T}^d)$-limit of its Fourier series $\sum_{\gamma\in\mathbb{Z}^d} b(\gamma)e_\gamma$, with b given by

$$\begin{aligned} b(\gamma) &:= (2\pi)^{-d}\int_C [\hat f, \hat\phi]e_{-\gamma} = \int_{\mathbb{R}^d} f\overline{\phi}(\cdot+\gamma) \\ &= \sum_{\beta\in\mathbb{Z}^d}\int_{\mathbb{R}^d}\phi(\cdot-\beta)c(\beta)\overline{\phi}(\cdot+\gamma) = \sum_{\beta\in\mathbb{Z}^d} c(\beta)a(\gamma+\beta). \end{aligned}$$

By (2.7), $[\hat f, \hat\phi]\hat\phi$ is the $L_2(\mathbb{R}^d)$-limit of $\sum_{\gamma\in\mathbb{Z}^d} b(\gamma)e_\gamma\hat\phi$, whence $([\hat f, \hat\phi]\hat\phi)^\vee$ is the $L_2(\mathbb{R}^d)$-limit of $\sum_{\gamma\in\mathbb{Z}^d}\phi(\cdot+\gamma)b(\gamma)$. Since this last sum also converges uniformly on compact sets, these two limits must be the same. This implies that the right side of (2.19) is the Fourier transform of

$$\begin{aligned} &\sum_{\gamma\in\mathbb{Z}^d}\phi(\cdot+\gamma)\sum_{\beta\in\mathbb{Z}^d} c(\beta)a(\gamma+\beta) \\ &\qquad = \sum_{\alpha\in\mathbb{Z}^d}\sum_{\beta\in\mathbb{Z}^d}\phi(\cdot+\alpha-\beta)c(\beta)a(\alpha) = \sum_{\alpha\in\mathbb{Z}^d} f(\cdot+\alpha)a(\alpha), \end{aligned}$$

with the rearrangement of the sums justified by the fact that all sums are finite. □

We now turn to our main objective, viz. the error of the best approximation. If $\hat f$ is supported in one of the cubes $\beta + C$, $\beta\in 2\pi\mathbb{Z}^d$, this error takes a very simple form:

Theorem 2.20. *Let* $\phi\in L_2(\mathbb{R}^d)$. *If* $f\in L_2(\mathbb{R}^d)$ *and* $\operatorname{supp}\hat f\subset\beta+C$ *for some* $\beta\in 2\pi\mathbb{Z}^d$, *then*

$$E(f, \mathscr{S}(\phi))^2 = (2\pi)^{-d}E(\hat f, \widehat{\mathscr{S}(\phi)})^2 = (2\pi)^{-d}\int_{\mathbb{R}^d}|\hat f|^2\left(1 - \frac{|\hat\phi|^2}{[\hat\phi, \hat\phi]}\right). \tag{2.21}$$

Proof. Since $\operatorname{supp}\hat f\subset C+\beta$ for some $\beta\in 2\pi\mathbb{Z}^d$, we have $[\hat f, \hat\phi] = \hat f(\cdot+\beta)\overline{\hat\phi}(\cdot+\beta)$ on C. Therefore, with (2.6),

$$\|\tau_f\hat\phi\|^2 = \int_C |\hat f(\cdot+\beta)|^2|\hat\phi(\cdot+\beta)|^2/[\hat\phi, \hat\phi] = \int_{\mathbb{R}^d}|\hat f|^2|\hat\phi|^2/[\hat\phi, \hat\phi].$$

By Theorem 2.9, this shows that

$$\|P_\phi f\|^2 = (2\pi)^{-d} \int_{\mathbb{R}^d} |\hat{f}|^2 |\hat{\phi}|^2 / [\hat{\phi}, \hat{\phi}],$$

and this finishes the proof since $\|f - P_\phi f\|^2 = \|f\|^2 - \|P_\phi f\|^2$. $\square$

3. The reduction to the principal case

The explicit and simple expression, derived in the previous section, for the orthogonal projector onto a *principal* shift-invariant space will also prove to be very useful in the discussion of approximation from a *general* shift-invariant space. For, remarkably, the approximation power of a general shift-invariant space, however large, is already contained in a single (suitably chosen) principal shift-invariant subspace of it. The next proposition provides the algebraic background for this fact. We use repeatedly the simple observation that the best approximation Pf to f from $\mathcal{S}$ is also the best approximation $P_{Pf}f$ to f from $\mathcal{S}(Pf)$, i.e., $P_{Pf}f = Pf$.

Proposition 3.1. *Let P be the orthogonal projector onto the closed shift-invariant subspace $\mathcal{S}$ of $L_2(\mathbb{R}^d)$ and denote by $\widehat{P}$ the corresponding orthogonal projector onto $\widehat{\mathcal{S}}$. Then $\widehat{P}(\tau\hat{f}) = \tau\widehat{P}(\hat{f})$ for any $f \in L_2(\mathbb{R}^d)$ and any 2π-periodic τ for which $\tau\hat{f} \in L_2(\mathbb{R}^d)$.*

Proof. If $\mathcal{S}$ is principal, then the conclusion follows directly from (2.12). For the general case, the assumptions on τ and $\hat{f}$ imply with Theorem 2.14 that $\tau\hat{f} \in \widehat{\mathcal{S}(f)}$. Since $\mathcal{S}(f)$ is, by definition, the $L_2(\mathbb{R}^d)$-closure of $\mathcal{S}_0(f)$, and $\widehat{\mathcal{S}_0(f)} = \{\tau_n\hat{f} : \tau_n \text{ a trig.polynomial}\}$, it follows that $\tau\hat{f}$ is the $L_2(\mathbb{R}^d)$-limit of $\tau_n\hat{f}$ for some sequence (τ_n) of trigonometric polynomials. The shift-invariance of $\mathcal{S}$ and the uniqueness of the best L_2-approximation imply that $P(f(\cdot + \alpha)) = (Pf)(\cdot + \alpha)$ for every $f \in L_2(\mathbb{R}^d)$ and every $\alpha \in \mathbb{Z}^d$. Hence, taking finite linear combinations of Fourier transforms, $\widehat{P}(\tau_n\hat{f}) = \tau_n\widehat{Pf}$, and so, by the continuity of $\widehat{P}$,

$$\widehat{P}(\tau\hat{f}) = \lim_{n\to\infty} \widehat{P}(\tau_n\hat{f}) = \lim_{n\to\infty} \tau_n\widehat{Pf}.$$

Each $\tau_n\widehat{Pf}$ is in the closed space $\mathcal{S}(\widehat{Pf})$, therefore also $\widehat{P}(\tau\hat{f})$ lies in $\mathcal{S}(\widehat{Pf})$. Thus, projecting $\tau\hat{f}$ onto $\widehat{\mathcal{S}}$ is the same as projecting it onto the subspace $\mathcal{S}(\widehat{Pf})$ of $\widehat{\mathcal{S}}$. Since we already know that $\widehat{P_\phi}(\tau\hat{f}) = \tau\widehat{P_\phi f}$ for any $\phi, f \in L_2(\mathbb{R}^d)$, this means that we obtain

$$\widehat{P}(\tau\hat{f}) = \widehat{P}_{Pf}(\tau\hat{f}) = \tau\widehat{P}_{Pf}(\hat{f}) = \tau\widehat{Pf},$$

the last equality since $P_{Pf}f = Pf$. $\square$

Corollary 3.2. *If P is the orthogonal projector onto some shift-invariant subspace of $L_2(\mathbb{R}^d)$ and $g \in L_2(\mathbb{R}^d)$, then $PP_g = P_{Pg}P_g$.*

Proof. If $f \in L_2(\mathbb{R}^d)$, then $\widehat{P_g f} = \tau\hat{g}$ for some 2π-periodic τ and therefore by Proposition 3.1, $\widehat{P}(\widehat{P_g f}) = \tau\widehat{Pg}$. On the other hand, $\widehat{P_{Pg}}(\widehat{P_g f}) = \widehat{P_{Pg}}(\tau\hat{g}) = \tau\widehat{P_{Pg}}\hat{g} = \tau\widehat{Pg}$. $\square$

Theorem 3.3. *For any closed shift-invariant subspace $\mathscr{S}$ of $L_2(\mathbb{R}^d)$ and any $f, g \in L_2(\mathbb{R}^d)$,*

$$E(f, \mathscr{S}) \le E(f, \mathscr{S}(Pg)) \le E(f, \mathscr{S}) + 2E(f, \mathscr{S}(g)), \tag{3.4}$$

with $P = P_{\mathscr{S}}$ the orthogonal projector onto $\mathscr{S}$.

Proof. Only the second inequality needs proof. By Corollary 3.2,

$$f - P_{Pg}f = f - Pf + Pf - PP_g f + P_{Pg}P_g f - P_{Pg} f,$$

and therefore

$$\|f - P_{Pg}f\| \le \|f - Pf\| + \|f - P_g f\| + \|P_g f - f\|. \quad \square \tag{3.5}$$

This theorem shows that the approximation order of the particular principal subspace $\mathscr{S}(Pg)$ of $\mathscr{S}$ is the same as that of all of $\mathscr{S}$, provided that the approximation order of the principal space $\mathscr{S}(g)$ is at least as good as that of $\mathscr{S}$. This suggests the use of a special function g^* for which $\mathscr{S}(g^*)$ has arbitrarily high approximation order. We can take g^* to be the inverse Fourier transform of the characteristic function of the cube $C = [-\pi..\pi]^d$, i.e., $g^* := (\chi_C)^\vee$. We note that, by (2.12), $\widehat{P_{g^*}f} = [\hat{f}, \chi_C]/[\chi_C, \chi_C]\chi_C = \chi_C \hat{f}$. Hence,

$$E(f, \mathscr{S}(g^*)) = (2\pi)^{-d/2}\|(1 - \chi_C)\hat{f}\|. \tag{3.6}$$

This allows us to show easily that the space $\mathscr{S}(g^*)$ provides approximation and density order k for all $k \ge 0$. For this, we follow the example of [BR2] and consider, equivalently, the approximation of the scaled function

$$f_h := f(h\cdot)$$

from the fixed space $\mathscr{S}$ instead of the approximation of the function f from the scaled space $\mathscr{S}^h$. For,

$$E(f, \mathscr{S}^h) = h^{d/2}E(f_h, \mathscr{S}), \tag{3.7}$$

as is easily established by a change of variables.

Lemma 3.8. *Let $f \in W_2^k(\mathbb{R}^d)$, $k \ge 0$, $h > 0$. Then*

$$E(f, \mathscr{S}(g^*)^h) \le \varepsilon_f(h) h^k \|f\|_{W_2^k(\mathbb{R}^d)},$$

with the (nonnegative) function ε_f defined by

$$\varepsilon_f(h)^2 := \frac{\int_{(\mathbb{R}^d \backslash C)/h} (1 + |\cdot|)^{2k}|\hat{f}|^2}{\int_{\mathbb{R}^d}(1 + |\cdot|)^{2k}|\hat{f}|^2}, \tag{3.9}$$

hence $\varepsilon_f(h) \le 1$, and $\varepsilon_f(0+) = 0$.

Proof. Since $f \in W_2^k(\mathbb{R}^d)$, the function $\nu := (1 + |\cdot|)^k \hat{f}$ is in $L_2(\mathbb{R}^d)$, and $\|f\|_{W_2^k(\mathbb{R}^d)} = (2\pi)^{-d/2}\|\nu\|$. Since $\widehat{f_h} = h^{-d}\hat{f}(\cdot/h)$,(3.7) and (3.6) imply that

$$\begin{aligned}(2\pi)^d E(f, \mathscr{S}(g^*)^h)^2 &= (2\pi)^d h^d E(f_h, \mathscr{S}(g^*))^2 \\ &= h^d \|(1-\chi_C)\widehat{f_h}\|^2 = h^d \int_{\mathbb{R}^d\setminus C} |\widehat{f_h}(y)|^2\,dy \\ &= h^{-d}\int_{\mathbb{R}^d\setminus C} |\hat{f}(y/h)|^2\,dy = h^{2k-d}\int_{\mathbb{R}^d\setminus C} \frac{|\nu(y/h)|^2}{(h+|y|)^{2k}}\,dy \\ &\le h^{2k-d}\int_{\mathbb{R}^d\setminus C} |\nu(y/h)|^2\,dy \\ &= h^{2k}\int_{(\mathbb{R}^d\setminus C)/h} |\nu|^2 = (2\pi)^d h^{2k}\varepsilon_f(h)^2 \|f\|^2_{W_2^k(\mathbb{R}^d)}. \quad \square\end{aligned}$$

We note for later reference the following useful result established during the proof of Lemma 3.8.

Corollary 3.10. *For each* $f \in W_2^k(\mathbb{R}^d)$,

$$h^{d/2}\|(1-\chi_C)\widehat{f_h}\| \le (2\pi)^{d/2}\varepsilon_f(h)h^k\|f\|_{W_2^k(\mathbb{R}^d)},$$

with ε_f *given by* (3.9).

Now let $\mathscr{S}$ be an arbitrary closed shift-invariant subspace of $L_2(\mathbb{R}^d)$ and let $\phi^* := Pg^*$ be the best $L_2(\mathbb{R}^d)$-approximation to g^* from $\mathscr{S}$. Using (3.7) and Lemma 3.8 in (3.4), we obtain

$$(3.11)\qquad E(f, \mathscr{S}^h) \le E(f, \mathscr{S}(\phi^*)^h) \le E(f, \mathscr{S}^h) + 2\varepsilon_f(h)h^k\|f\|_{W_2^k(\mathbb{R}^d)},$$

with $\varepsilon_f(h)$ given by (3.9). This means that $\mathscr{S}$ provides approximation order $k > 0$ or density order $k \ge 0$ if and only if its principal shift-invariant subspace $\mathscr{S}(\phi^*)$ does. More than that, since $\varepsilon_f(h)$ does not depend on $\mathscr{S}$, it proves the following:

Theorem 3.12. *The sequence* $\{\mathscr{S}_h\}_h$ *of closed shift-invariant subspaces of* $L_2(\mathbb{R}^d)$ *provides approximation order* $k > 0$ *or density order* $k \ge 0$ *if and only if the corresponding sequence* $\{\mathscr{S}(\phi_h^*)\}_h$ *of principal shift-invariant subspaces* (*with* $\phi_h^* := P_{\mathscr{S}_h}(g^*)$ *and* $g^* = \chi_C^\vee$) *does.*

4. Approximation orders and density orders

In this section we give a complete characterization of approximation orders and density orders from the sequence $\{\mathscr{S}_h\}_h$ of shift-invariant spaces. In view of Theorem 3.12, we need only to consider the special case when each $\mathscr{S}_h$ is principal. For $\phi \in L_2(\mathbb{R}^d)$, we let $\Lambda_\phi \in L_\infty(C)$ be defined as in the introduction

$$\Lambda_\phi := \left(1 - \frac{|\hat{\phi}|^2}{[\hat{\phi}, \hat{\phi}]}\right)^{1/2}, \quad \text{on } C.$$

In terms of this Λ_ϕ, (2.21) gives that

$$(4.1)\qquad E(f, \mathscr{S}(\phi)) = (2\pi)^{-d/2}\|\hat{f}\Lambda_\phi\| \quad \text{if } \operatorname{supp}\hat{f} \subset C.$$

For $f \in L_2(\mathbb{R}^d)$ with $\hat{f}$ not just supported in C, we estimate $E(f, \mathscr{S}(\phi)) = (2\pi)^{-d/2} E(\hat{f}, \widehat{\mathscr{S}(\phi)})$ with the aid of Corollary 3.10 and the simple observation that

$$|E(\hat{f}, \widehat{\mathscr{S}}) - E(\chi_C \hat{f}, \widehat{\mathscr{S}})| \leq \|(1-\chi_C)\hat{f}\|$$

for an arbitrary subspace $\mathscr{S}$ of $L_2(\mathbb{R}^d)$. Indeed, with the aid of (3.7), this estimate implies that

$$\begin{aligned}
&|E(f, \mathscr{S}^h) - (h/2\pi)^{d/2} E(\chi_C \widehat{f_h}, \widehat{\mathscr{S}})| \\
&\quad = |h^{d/2} E(f_h, \mathscr{S}) - (h/2\pi)^{d/2} E(\chi_C \widehat{f_h}, \widehat{\mathscr{S}})| \\
&\quad = (h/2\pi)^{d/2} |E(\widehat{f_h}, \widehat{\mathscr{S}}) - E(\chi_C \widehat{f_h}, \widehat{\mathscr{S}})| \leq (h/2\pi)^{d/2} \|(1-\chi_C)\widehat{f_h}\|.
\end{aligned}$$

Therefore, Corollary 3.10 establishes

$$|E(f, \mathscr{S}^h) - (h/2\pi)^{d/2} E(\chi_C \widehat{f_h}, \widehat{\mathscr{S}})| \leq \varepsilon_f(h) h^k \|f\|_{W_2^k(\mathbb{R}^d)}. \tag{4.2}$$

Theorem 4.3. *For $\{\phi_h\}_h \subset L_2(\mathbb{R}^d)$, the sequence $\{\mathscr{S}(\phi_h)\}_h$ provides approximation order k if and only if*

$$\left\{ \frac{\Lambda_{\phi_h}}{(h + |\cdot|)^k} \right\}_h$$

is bounded in $L_\infty(C)$.

Remark. Since each Λ_{ϕ_h} is nonnegative and bounded above by 1, and since each $(h+|\cdot|)^k$ is bounded below by h^k, it is clear that each $\Lambda_{\phi_h}/(h+|\cdot|)^k$ is an element of $L_\infty(C)$. So it is the *uniform* boundedness of $\Lambda_{\phi_h}/(h+|\cdot|)^k$ as $h \to 0$ that characterizes the approximation order k.

Proof. In view of (4.2), $\{\mathscr{S}(\phi_h)\}_h$ provides approximation order k if and only if there exists some constant c such that for every $f \in W_2^k(\mathbb{R}^d)$ and every $h > 0$,

$$h^{d/2} E(\chi_C \widehat{f_h}, \widehat{\mathscr{S}(\phi_h)}) \leq c h^k \|f\|_{W_2^k(\mathbb{R}^d)}. \tag{4.4}$$

Since $\chi_C \widehat{f_h}$ is supported in C, we may appeal to (4.1) (i.e., to Theorem 2.20) to conclude that

$$\begin{aligned}
h^d E(\chi_C \widehat{f_h}, \widehat{\mathscr{S}(\phi_h)})^2 &= h^d \int_C |\widehat{f_h}|^2 \Lambda_{\phi_h}^2 \\
&= h^{-d} \int_C |\hat{f}(\cdot/h)|^2 \Lambda_{\phi_h}^2 = \int_{C/h} |\hat{f}|^2 \Lambda_{\phi_h}(h\cdot)^2.
\end{aligned} \tag{4.5}$$

For $f \in W_2^k(\mathbb{R}^d)$, the function $\nu := (1+|\cdot|)^k \hat{f}$ is in $L_2(\mathbb{R}^d)$, and $\|f\|_{W_2^k(\mathbb{R}^d)} = (2\pi)^{-d/2}\|\nu\|$. With the aid of ν, the last expression in (4.5) can be rewritten as

$$\int_{C/h} |\nu|^2 \frac{\Lambda_{\phi_h}(h\cdot)^2}{(1+|\cdot|)^{2k}}.$$

Further, when f varies over all of $W_2^k(\mathbb{R}^d)$, ν varies over all of $L_2(\mathbb{R}^d)$, i.e., $g := |\nu|^2$ varies over all nonnegative functions in $L_1(\mathbb{R}^d)$. This means that the k-approximation order requirement is equivalent to the existence of $c > 0$ such that

$$(4.6) \qquad \int_{\mathbb{R}^d} |g| \chi_{C/h} \frac{\Lambda_{\phi_h}(h\cdot)^2}{(1+|\cdot|)^{2k}} \le ch^{2k} \|g\|_{L_1(\mathbb{R}^d)}, \quad \forall h > 0, \ \forall g \in L_1(\mathbb{R}^d).$$

Fixing h, the last condition states that $\chi_{C/h}\frac{\Lambda_{\phi_h}(h\cdot)^2}{(1+|\cdot|)^{2k}}$, considered as a linear functional on $L_1(\mathbb{R}^d)$, is bounded by ch^{2k}. Consequently, having $\{\mathscr{S}(\phi_h)\}_h$ provide approximation order k is equivalent to the existence of $c > 0$ such that

$$\left\| \frac{\Lambda_{\phi_h}(h\cdot)}{(1+|\cdot|)^k} \right\|_{L_\infty(C/h)} \le ch^k.$$

The proof is thus completed, since upon rescaling the last condition becomes

$$(4.7) \qquad \left\| \frac{\Lambda_{\phi_h}}{(h+|\cdot|)^k} \right\|_{L_\infty(C)} \le c. \quad \square$$

Proof of Theorem 1.6. In the case of this theorem, $\phi_h = \phi$ for all $h > 0$. Using this in (4.7) and letting $h \to 0$, we get that (4.7) is equivalent to $|\cdot|^{-k}\Lambda_\phi \in L_\infty(C)$. $\square$

Remark. Note that the cube C that appears in the characterization of approximation orders is entirely incidental. Since, for every h, Λ_{ϕ_h} is bounded by 1, and also $(h+|\cdot|)^{-k}$ is bounded, independently of h, in any complement of a neighborhood of the origin, the cube C can be replaced by *any* neighborhood of the origin.

Another remark concerns the case $k = 0$ which will soon be considered in the context of density orders. We have not discussed approximation order 0 simply because of lack of any mathematical interest: the requirement in this case is vacuous. This is in agreement with Theorem 4.3, for the boundedness of $\{\Lambda_{\phi_h}/(1+|\cdot|)^0\}_h$ is also a vacuous condition, since each Λ_{ϕ_h} is uniformly bounded by 1. This means that the statement of Theorem 4.3 is valid also for $k = 0$.

With Theorem 4.3 in hand, we turn our attention to the characterization of density orders. Our result concerning density orders is as follows.

Theorem 4.8. *For $\{\phi_h\}_h \subset L_2(\mathbb{R}^d)$, the sequence $\{\mathscr{S}(\phi_h)\}_h$ provides density order k if and only if $\{\Lambda_{\phi_h}/(h+|\cdot|)^k\}_h$ is bounded in $L_\infty(C)$, and*

$$(4.9) \qquad \lim_{h\to 0} h^{-d} \int_{haC} \frac{\Lambda_{\phi_h}^2}{(h+|\cdot|)^{2k}} = 0, \qquad \forall a > 0.$$

Proof. In view of Theorem 4.3 and the definition of density orders, the theorem here is proved as soon as we show that, under the assumption that $\{\Lambda_{\phi_h}/(h+|\cdot|)^k\}_h$ is bounded, the condition

$$(4.10) \qquad \lim_{h\to 0} h^{d/2-k} E(f_h, \mathscr{S}(\phi_h)) = 0, \qquad \forall f \in W_2^k(\mathbb{R}^d)$$

is equivalent to (4.9). For this we can follow the proof of Theorem 4.3 up to (4.6) to conclude that (4.10) is equivalent to the condition that

$$(4.11) \qquad \lim_{h\to 0} h^{-2k} \int_{\mathbb{R}^d} |g| \chi_{C/h} \frac{\Lambda_{\phi_h}(h\cdot)^2}{(1+|\cdot|)^{2k}} = 0, \qquad \forall g \in L_1(\mathbb{R}^d).$$

Choosing $g := \chi_{aC}$ in (4.11) and rescaling, we obtain (4.9), so that the *necessity* of (4.9) for k-density order is proved.

To prove the sufficiency, we define

$$\lambda_h := h^{-2k}\chi_{C/h}\frac{\Lambda_{\phi_h}(h\cdot)^2}{(1+|\cdot|)^{2k}}, \qquad h>0.$$

We view the λ_h as elements of $L_1(\mathbb{R}^d)^*$. We want to show that (4.11) holds, namely that $\{\lambda_h\}_h$ converges weak-$*$ to 0. We know that $\{\lambda_h\}_h$ are positive, uniformly bounded, and by (4.9), $\lambda_h(\chi_{aC}) \to 0$ for every $a>0$. This latter condition implies that $\lambda_h(\chi_K) \to 0$ for any compact K. By linearity, $\lambda_h(g)$ tends to 0 for each compactly supported simple function g. Since such functions are dense in $L_1(\mathbb{R}^d)$, we obtain (4.11). □

Proof of Theorem 1.7. Since $(h+|\cdot|)^{-2k} \le |\cdot|^{-2k}$, (1.8) implies that

$$\lim_{h\to 0} h^{-d}\int_{hC}\frac{\Lambda^2_{\phi_h}}{(h+|\cdot|)^{2k}} = 0,$$

which is the case $a=1$ in (4.9), and implies the rest of (4.9), since here $\phi_h = \phi$ for all h, hence Λ_ϕ does not change with h. Thus, Theorem 4.8 implies the sufficiency of (1.8).

On the other hand, if $\mathscr{S}(\phi)$ provides density order k, then (4.9) holds (with $\Lambda_{\phi_h} = \Lambda_\phi$, all h). Since $|y|^{-2k} \le c(h+|y|)^{-2k}$ for $y \in hC\backslash(hC/2)$ and some absolute constant c, we obtain from (4.9) (with $a=1$)

$$\int_{hC\backslash(hC/2)}\frac{\Lambda_\phi(y)^2}{|y|^{2k}} \le \varepsilon(h)h^d \tag{4.12}$$

where $\lim_{h\to 0}\varepsilon(h) = 0$. Summing these estimates gives

$$\int_{hC}\frac{\Lambda_\phi(y)^2}{|y|^{2k}} \le \sum_{j\ge 0}\varepsilon(2^{-j}h)2^{-jd}h^d \le 2\max_{0<u\le h}\varepsilon(u)h^d. \tag{4.13}$$

Since the right side of (4.13) is $o(h^d)$, we obtain the necessity of (1.8). □

Combining the two last theorems with Theorem 3.12, we obtain

Theorem 4.14. *Let $\{\mathscr{S}_h\}$ be a sequence of shift-invariant spaces. For each h, let ϕ_h be the best approximation from $\mathscr{S}_h$ to $g^* = \chi_C^\vee$. Then, $\{\mathscr{S}_h\}_h$ provides approximation order k if and only if $\{\Lambda_{\phi_h}/(h+|\cdot|)^k\}_h$ is bounded in $L_\infty(C)$, and $\{\mathscr{S}_h\}_h$ are kth-order dense if and only if, in addition to the above,*

$$\lim_{h\to 0} h^{-d}\int_{haC}\frac{\Lambda^2_{\phi_h}}{(h+|\cdot|)^{2k}} = 0, \qquad \forall a>0. \tag{4.15}$$

Proof of Theorem 1.9. This follows from Theorem 1.6, Theorem 1.7, and the reduction to the principal shift-invariant case given by Theorem 3.12 (with $\phi_h^* = \phi^* = P_{\mathscr{S}}g^*$ for all h). □

5. The Strang-Fix conditions

As mentioned in the introduction, approximation orders from the scaled spaces $\{\mathscr{S}^h\}_h$ were characterized in [SF] under the assumptions that (a) the space $\mathscr{S}^h$ is obtained as the h-dilate of the *same principal* shift-invariant space $\mathscr{S}(\phi)$; (b) the generator ϕ of $\mathscr{S}(\phi)$ is compactly supported; and (c) the approximation order is realized in a controlled manner. The controlled approximation assumption, in turn, forces the condition $\hat{\phi}(0) \neq 0$.

In order to compare these conditions to the characterization of approximation orders for principal shift-invariant spaces that we obtain in the present paper, we assume in this section that we have in hand a sequence $\{\mathscr{S}(\phi_h)\}_h$ of principal shift-invariant spaces which satisfy one or both of the following conditions, in which Ω is some neighborhood of the origin, and η and μ are positive constants

$$\exists \Omega, \mu, h_0 \ \text{ s.t. } \ |\hat{\phi}_h(x)| \leq \mu \ \text{ a.e. on } \Omega, \ \forall 0 < h < h_0; \tag{5.1}$$

$$\exists \Omega, \eta, h_0 \ \text{ s.t. } \ \eta \leq |\hat{\phi}_h(x)| \ \text{ a.e. on } \Omega, \ \forall 0 < h < h_0. \tag{5.2}$$

Note that, in case ϕ_h does not change with h (i.e., when assumption (a) above holds), and $\hat{\phi}$ is continuous at the origin (e.g., ϕ is compactly supported, as in assumption (b) above), (5.1) is satisfied automatically and (5.2) is reduced to the mere condition

$$\hat{\phi}(0) \neq 0. \tag{5.3}$$

We recall (see the remark after the proof of Theorem 1.6) that the uniform boundedness required in Theorem 4.3 for k-approximation order can be checked in any neighborhood Ω of the origin, hence we can replace the cube C in the theorem by Ω. As the next results show, Λ_{ϕ_h} can often be replaced by

$$M_h := \left(\sum_{\beta \in 2\pi\mathbb{Z}^d \setminus 0} |\hat{\phi}_h(\cdot + \beta)|^2 \right)^{1/2} = ([\hat{\phi}_h, \hat{\phi}_h] - |\hat{\phi}_h|^2)^{1/2}. \tag{5.4}$$

Lemma 5.5. *If* (5.1) *holds and the sequence* $\{\mathscr{S}(\phi_h)\}_h$ *provides approximation order* k, *then*

$$\left\{ \frac{M_h}{(h + |\cdot|)^k} \right\}_{h < h_0'} \tag{5.6}$$

is bounded in $L_\infty(\Omega')$ *for some* 0*-neighborhood* Ω' *and some* $h_0' > 0$. *On the other hand, if* (5.2) *holds and* (5.6) *is bounded in* $L_\infty(\Omega')$ *for some* 0*-neighborhood* Ω *and some* h_0', *then* $\{\mathscr{S}(\phi_h)\}_h$ *provides approximation order* k.

Proof. If $\{\mathscr{S}(\phi_h)\}_h$ provides approximation order k, then, by Theorem 4.3, $\{(h+|\cdot|)^{-k}\Lambda_{\phi_h}\}_h$ is bounded, say by c, on Ω. This, together with (5.1), implies that

$$(h + |\cdot|)^{-2k} M_h^2 \leq c(M_h^2 + |\hat{\phi}_h|^2) \leq c(M_h^2 + \mu^2), \tag{5.7}$$

and therefore, $((h + |\cdot|)^{-2k} - c)M_h^2 \leq c\mu^2$. Thus, for sufficiently small h and some neighborhood $\Omega' \subset \Omega$ of the origin, the leftmost term in (5.7) does not exceed $2c\mu^2$.

Conversely, (5.2) implies that, on Ω,

$$\Lambda_{\phi_h}^2 = 1 - \frac{|\hat{\phi}_h|^2}{M_h^2 + |\hat{\phi}_h|^2} \leq \frac{M_h^2}{|\hat{\phi}_h|^2} \leq \eta^{-2} M_h^2.$$

Therefore, by Theorem 4.3, the boundedness of (5.6) implies that $\{\mathscr{S}(\phi_h)\}_h$ provides approximation order k. $\square$

We now consider in more detail necessary conditions for approximation order which follow from our characterization of approximation order. Since $|\hat{\phi}_h(\cdot+\beta)| \le M_h$ for all $\beta \in 2\pi\mathbb{Z}^d\backslash 0$, the next theorem is a direct consequence of the last lemma:

Theorem 5.8. *If* (5.1) *holds and* $\{\mathscr{S}(\phi_h)\}_h$ *provides approximation order* k, *then, for all* $0 < h < h_0$ *and for all* $\beta \in 2\pi\mathbb{Z}^d\backslash 0$, *and in some* 0*-neighborhood,*

$$|\hat{\phi}_h(\cdot+\beta)| \le c(h+|\cdot|)^k,$$

for some c *independent of* β *and* h.

In case $\hat{\phi}$ does not change with h, we may let $h \to 0$ in the last display and so obtain Theorem 1.14. This shows that the *necessity* of the Strang-Fix conditions (1.11) for k-approximation order holds in a very general setting. This is remarkable, since this implication is considered to be the "harder" one. An analogous L_∞-result has been obtained in [BR2] by other means.

We now consider in more detail sufficient conditions for approximation order. There is no reason to believe that (upon assuming (5.2)) the assumptions

$$D^\gamma \hat{\phi} = 0 \quad \text{on } 2\pi\mathbb{Z}^d\backslash 0 \quad \text{for all } |\gamma| < k \tag{5.9}$$

would suffice for approximation order k since from Lemma 5.5 we only can deduce the following:

Corollary 5.10. *If* $0 < \eta \le \hat{\phi}$ *a.e. on some neighborhood* Ω *of the origin, and if*

$$\sum_{\beta \in 2\pi\mathbb{Z}^d\backslash 0} |\hat{\phi}(\cdot+\beta)|^2 \le c|\cdot|^{2k}, \quad \text{a.e. on } \Omega, \tag{5.11}$$

then $\mathscr{S}(\phi)$ *provides approximation order* k.

However, assumptions like (5.9) can only imply that, for each individual $\beta \in 2\pi\mathbb{Z}^d\backslash 0$,

$$|\hat{\phi}(\cdot+\beta)|^2 \le c_\beta|\cdot|^{2k},$$

hence will not in general yield (5.11). On the other hand, there are several results in the literature which show that, under additional assumptions on ϕ, (5.9) does imply that $\mathscr{S}(\phi)$ provides approximation order k. For example, standard polynomial reproduction/quasi-interpolation arguments show that if

$$|\phi(x)| = O(|x|^{-k-d-\varepsilon}), \quad \text{as } x \to \infty, \tag{5.12}$$

and if $\hat{\phi}(0) \neq 0$, then (5.9) implies that $\mathscr{S}(\phi)$ provides approximation order k (cf. e.g., Proposition 1.1 and Corollary 1.2 in [DJLR]). Unfortunately, the decay conditions (5.12) fail to hold for many functions ϕ of interest (primarily radial basis functions, and usually because $\hat{\phi}$ is not smooth enough at 0), and in such a case, the polynomial reproduction argument either fails, or is not easily converted into approximation orders. Circumventing the polynomial reproduction argument was actually the major objective of [BR2]. In our context, Theorem 1.6 leads to a remarkable result, which allows (5.12) to be replaced by a much weaker condition, and which we now describe.

For this result, we need a local version $W_2^\rho(\Omega)$ of the potential spaces $W_2^\rho(\mathbb{R}^d)$. If ρ is an integer, then this space is simply the Sobolev space of all functions whose (weak) derivatives up to order ρ (inclusive) are in $L_2(\Omega)$. In this case, if $\{\Omega_\beta\}_{\beta\in I}$ is a disjoint collection of open subsets of $\mathbb{R}^d$, we have $\sum_{\beta\in I}\|f\|^2_{W_2^\rho(\Omega_\beta)} = \|f\|^2_{W_2^\rho(\cup_\beta\Omega_\beta)}$. As is well known, there are several equivalent extensions of the definition of $W_2^\rho(\Omega)$ to the case of a fractional ρ (see, e.g., [A, Chapter 7]). For fractional ρ, we have the following subadditivity property:

$$\sum_{\beta\in I}\|f\|^2_{W_2^\rho(\Omega_\beta)} \le c\|f\|^2_{W_2^\rho(\cup_{\beta\in I}\Omega_\beta)}, \tag{5.13}$$

whenever, say, $\{\Omega_\beta\}_\beta$ is a disjoint collection of cubes; (cf. [A, p. 225]). Our result is as follows:

Theorem 5.14. *Assume that* $0 < \eta \le \hat\phi$ *a.e. on some cube* Ω *centered at the origin. Let* $A := \bigcup_{\beta\in 2\pi\mathbb{Z}^d\backslash 0}(\Omega+\beta)$. *If* $\hat\phi \in W_2^\rho(A)$ *for some* $\rho > k + d/2$, *and if* (5.9) *holds, then* $\mathscr{S}(\phi)$ *provides approximation order* k.

The virtue of this theorem is that we can take Ω to be so small that A does not contain the origin. This is important since in many cases of interest $\hat\phi$ is smooth on $\mathbb{R}^d\backslash 0$ but has some singularity at the origin (this happens, e.g., when ϕ is obtained by the application of a difference operator to a fundamental solution of an elliptic equation). But, if ϕ satisfies (5.12), then $\hat\phi$ is *globally* smooth, since we obtain from (5.12) that $\hat\phi \in W_2^\rho(\mathbb{R}^d)$ for $\rho = k + d/2 + \varepsilon/2$ as well as $\hat\phi \in C^k(\mathbb{R}^d)$. Thus, Theorem 5.14 and Theorem 1.14 together imply the following result.

Corollary 5.15. *If* ϕ *satisfies* (5.12) *and* $\hat\phi(0) \ne 0$, *then* $\mathscr{S}(\phi)$ *provides approximation order* k *if and only if* (5.9) *holds.*

Proof of Theorem 5.14. It follows from (5.9) that, for every $\beta \in 2\pi\mathbb{Z}^d\backslash 0$, and with $\Omega_\beta := \Omega + \beta$,

$$|\hat\phi(x+\beta)| \le c|x|^k \max_{|\gamma|=k}\|D^\gamma\hat\phi\|_{L_\infty(\Omega_\beta)}, \quad \text{for } x \in \Omega. \tag{5.16}$$

Since $\rho > k + d/2$, the Sobolev embedding theorem (cf. [A, p. 217]) implies that $W_2^\rho(\Omega_\beta)$ is continuously embedded in the Sobolev space $W_\infty^k(\Omega_\beta)$. Thus,

$$\max_{0\le|\gamma|\le k}\|D^\gamma\hat\phi\|_{L_\infty(\Omega_\beta)} \le c_1\|\hat\phi\|_{W_2^\rho(\Omega_\beta)},$$

with c_1 *independent of* β (since all the Ω_β are translates of each other). Substituting this into (5.16) we obtain that

$$|\hat\phi(x+\beta)| \le c_2|x|^k\|\hat\phi\|_{W_2^\rho(\Omega_\beta)}, \quad x\in\Omega,\ \beta\in 2\pi\mathbb{Z}^d\backslash 0.$$

Squaring the last inequality and summing over $\beta \in 2\pi\mathbb{Z}^d\backslash 0$, we obtain, in view of (5.13), that

$$\sum_{\beta\in 2\pi\mathbb{Z}^d\backslash 0}|\hat\phi(x+\beta)|^2 \le c_3|x|^{2k}\|\hat\phi\|^2_{W_2^\rho(A)}.$$

Lemma 5.5 now supplies the conclusion that $\mathscr{S}(\phi)$ provides approximation order k. □

In applications, it might be convenient to take ρ to be the least integer that satisfies $\rho > k + d/2$. For this case, Theorem 5.14 reduces to Theorem 1.15.

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