Daniela Kuklíková

Approximation of bivariate Markov chains by one-dimensional diffusion processes

Aplikace matematiky, Vol. 23 (1978), No. 4, 267–279

Persistent URL: `http://dml.cz/dmlcz/103752`

APPROXIMATION OF BIVARIATE MARKOV CHAINS BY ONE-DIMENSIONAL DIFFUSION PROCESSES

DANIELA KUKLÍKOVÁ

(Received January 5, 1977)

1. INTRODUCTION

In this paper we deal with several cases of the diffusion approximation. The diffusion approximation simplifies the description of a large number of more-dimensional variables by approximating them in the sense of the weak convergence by processes whose coordinates fulfil stochastic differential equations. It can be used in cases where the knowledge of the characteristics of the limit processes can compensate the ignorance of the approximated variables.

The paper follows up with the work [3] where it is suggested how to control a large number (an aggregate) of quickly unfolding units. In [3] it is assumed that every unit is a Markov chain changing at times $0, \Delta, 2\Delta, \ldots$ and that the aggregate includes only units of a finite number of types. It is assumed that the change of the state of the unit yields a reward and the total reward is a sum of the rewards gained by the units. A stationary control, which is a function of the total reward only, is considered. The behaviour of the total reward as $\Delta \to 0$, may be described by a diffusion process satisfying a stochastic differential equation independent of the states of the units. This fact can be used for finding the optimal control in regard of the criterion, which is not additive with respect to the rewards of the units.

The aim of this paper is to present a general method for solving similar cases and to show more systems where the dimension of the model can be reduced by the suggested method. More precisely, it means to approximate two-dimensional random variables by a one-dimensional diffusion process in the way that one coordinate is replaced by its certain characteristic, e.g. by the stationary expectation. We show on four types of sequences of Markov chains $\{({}^nX_m, {}^nY_m), m = 0, 1, \ldots\}, n = 1, 2, \ldots$ that it is possible to do it when the tendency to the stationary state by $\{{}^nY_m\}$ is greater than that by $\{{}^nX_m\}$.

We define for $t \in \langle 0, 1 \rangle$ processes

$${}^nX_t = {}^nX_m \quad t \in \langle m/n, (m + 1)/n) .$$

In the first part of the work we introduce a theorem where we use general theorems about the weak convergence to diffusion processes to deduce conditions for the convergence of the distributions of processes $\{{}^nX_t\}$ to the probability measure corresponding to a stochastic differential equation. This equation depends on $\{{}^nY_m\}$ only through its stationary distribution. In the second part we introduce several systems and examine them whether they satisfy the assumptions of the convergence theorem.

2. CONVERGENCE THEOREM

Let, for $n = 1, 2, \ldots,$ the double sequence $\{({}^nX_m, {}^nY_m), m = 0, 1, 2, \ldots\}$ be a homogeneous Markov chain with the state space $R^1 \times S$, where S is a Borel subset of R^s. Let the initial distribution fulfil $P(({}^nX_0, {}^nY_0) = (x_0, y_0)) = 1$, where $(x_0, y_0) \in R^1 \times S$. We denote the j-step transition probabilities of $\{{}^nX_m\}$ by ${}^nP^j(x, y, A)$, $n = 1, 2, \ldots$, $j = 1, 2, \ldots, (x, y) \in R^1 \times S$, $A \subset R^1$ a Borel subset. It means that

$$P({}^nX_{m+j} \in A/{}^n\mathscr{F}_m) = {}^nP^j({}^nX_m, {}^nY_m, A),$$

where ${}^n\mathscr{F}_m = \sigma a(({}^nX_0, {}^nY_0), \ldots, ({}^nX_m, {}^nY_m))$. Next we set

$$(1) \qquad n\int(z - x)\, {}^nP^1(x, y, \mathrm{d}z) = a_n(x, y),$$

$$(2) \qquad n\int(z - x - (1/n)\, a_n(x, y))^2\, {}^nP^1(x, y, \mathrm{d}z) = b_n(x, y)^2.$$

We shall assume

$$(\mathrm{i}) \qquad |a_n(x, y)| + |b_n(x, y)| \leqq K(1 + |x|).$$

Here and in the following text K denotes an arbitrary positive constant. In the theorems about the convergence to a diffusion process it is assumed that (1), (2) have limits $a(x, y)$, $b(x, y)$. We want to deal only with such cases where the coefficients in the limit equation do not depend on y. We cannot assume this for the limits of $a_n(x, y)$ and $b_n(x, y)$ corresponding to one-step transition probabilities. It can be, however, assumed for the coefficients corresponding to more-step transition probabilities where the dependence on y vanishes. Let us consider an integer function $\varphi(n) > 0$, $\varphi(n) = o(n)$ as $n \to \infty$, and let us denote

$$(n/\varphi(n))\int(z - x)\, {}^nP^{\varphi(n)}(x, y, \mathrm{d}z) = a'_n(x, y),$$

$$(n/\varphi(n))\int(z - x - (\varphi(n)/n)\, a'_n(x, y))^2\, {}^nP^{\varphi(n)}(x, y, \mathrm{d}z) = b'_n(x, y)^2.$$

We shall suppose that

(ii) $$\sup_{x,y} \frac{|a'_n(x, y) - a(x)| + |b'_n(x, y) - b(x)|}{1 + |x|} \underset{n\to\infty}{\to} 0$$

$$(b'_n = \sqrt{b'^2_n}),$$

$a(x)$ and $b(x)$ are functions which satisfy

(iii) $$|a(x) - a(z)| + |b(x) - b(z)| \leqq K|x - z|,$$

(iv) $$|a(x)| + |b(x)| \leqq K(1 + |x|).$$

Further, let us define $\{\Delta\, {}^n\omega_m,\ m = 0, 1, 2, \ldots\}$, $n = 1, 2, \ldots$, by

$$ {}^nX_{(m+1)\varphi(n)} = {}^nX_{m\varphi(n)} + (\varphi(n)/n)\, a'_n({}^nX_{m\varphi(n)}, {}^nY_{m\varphi(n)}) + b'_n({}^nX_{m\varphi(n)}, {}^nY_{m\varphi(n)})\, \Delta\, {}^n\omega_m .$$

$\{\Delta\, {}^n\omega_m\}$ are martingale differences with respect to ${}^n\mathscr{F}_{m\varphi(n)}$, and $\{{}^n\omega_t\}$ are processes defined in the following way:

$$ {}^n\omega_t = {}^n\omega_m = \sum_{j=0}^{m-1} \Delta\, {}^n\omega_j \quad \text{for} \quad t \in \langle\, m\varphi(n)/n\,,\quad (m + 1)\, \varphi(n)/n) .$$

We shall introduce a further condition:

(v) The finite-dimensional distributions of the processes $\{{}^n\omega_t\}$ converge to the finite-dimensional distributions of the Wiener process.

Finally, let $(D, \mathscr{D})$ be the Skorochod space of functions on $< 0, 1 >$ that are right continuous and have left-sided limits. $W = \{W_t, t \in < 0, 1 >\}$ denotes the Wiener process.

Theorem 1. *Let the assumption (i) be satisfied. Let an integer function $\varphi(n) > 0$, $\varphi(n) = o(n)$ as $n \to \infty$, exists such that (ii) – (v) hold. Then the probabilities $\{Q_n, n = 1, 2, \ldots\}$ induced on $(D, \mathscr{D})$ by the processes $\{{}^nX_t\}$ converge weakly to the probability Q induced by the solution of the stochastic differential equation*

$$\mathrm{d}X_t = a(X_t)\,\mathrm{d}t + b(X_t)\,\mathrm{d}W_t\,, \quad t \in \langle 0, 1\rangle\,, \quad X_0 = x_0 .$$

Remark 1. The property (i) is required only to guarantee the relative compactness of the sequence $\{Q_n\}$. If it is satisfied, then

(3) $$E\{\sup_{0\leqq m\leqq n} {}^nX_m^2/{}^n\mathscr{F}_0\} \leqq K(1 + {}^nX_0^2),$$

(4) $$E\{\sup_{s\leqq m\leqq r} ({}^nX_m - {}^nX_s)^2/{}^n\mathscr{F}_s\} \leqq K(1 + {}^nX_s^2)\,(r - s)/n$$

(see [1], Lemma 1, § 3, Chapter II) and therefore $\{Q_n\}$ is relatively compact (see [1], Theorem 2, § 3, Chapter II).

Remark 2. Condition (v) is satisfied for example if there exists $\delta > 0$ such that

$$(5) \qquad E \sum_{m=0}^{n/\varphi(n)} |\Delta\, {}^n\omega_m|^{2+\delta} \xrightarrow[n\to\infty]{} 0$$

(see [1], § 3, Chapter II).

Proof of Theorem 1. Let us construct the processes $\{{}^n v_t\}$, $n = 1, 2, \ldots$; ${}^n v_t = {}^n X_{k\varphi(n)}$ if $t \in \langle k\, \varphi(n)/n, (k+1)\, \varphi(n)/n)$. The sets of probabilities on $(D, \mathscr{D})$ induced by the processes $\{{}^n v_t\}$ and $\{{}^n X_t\}$, respectively, are relatively compact. The proof is analogous to the already mentioned proof of Lemma 1, § 3, Chapter II in [1]. The finite-dimensional distributions of the processes $\{{}^n v_t\}$ converge to the finite-dimensional distributions of the process $\{X_t\}$, where $\{X_t\}$ is the solution of the equation

$$\mathrm{d}X_t = a(X_t)\, \mathrm{d}t + b(X_t)\, \mathrm{d}W_t, \; t \in \langle 0, 1\rangle, \quad X_0 = x_0$$

(see [1], Theorem 13, § 3, Chapter II). If we use (4), we obtain for fixed n, arbitrary $k = 0, \ldots, n/\varphi(n)$, $m = 0, \ldots, \varphi(n)$, the inequality

$$E({}^n X_{k\varphi(n)+m} - {}^n X_{k\varphi(n)})^2 \leqq K\, \varphi(n)/n\,.$$

This implies that the limits of the finite-dimensional distribution of $\{{}^n X_t\}$ coincide with those of $\{{}^n v_t\}$, and hence, the finite-dimensional distributions of $\{{}^n X_t\}$ converge to the finite-dimensional distributions of the process $\{X_t\}$.

3. APPLICATIONS

Now we shall present examples of how to apply Theorem 1.

In the first case we assume that the coordinate ${}^n X_m$ fulfils the Itô difference equation where the drift coefficient changes as a Markov chain. Equations of this type appear as models of production processes.

The Itô Difference Equation

For $n = 1, 2, \ldots$ we consider a sequence of random variables $\{{}^n X_m\}$ fulfilling

$${}^n X_{m+1} = {}^n X_m + (1/n) f({}^n X_m, {}^n Y_m) + (1/\sqrt{n})\, \varepsilon_m\,, \quad {}^n X_0 = x_0, {}^n Y_0 = j_0\,,$$

where $\{\varepsilon_m\}$ are independent, identically distributed, and $E\varepsilon_0 = 0$, $E\varepsilon_0^2 = 1$, $E\varepsilon_0^4 < \infty$. Furthermore (for $m = 0, 1, \ldots$) ε_m are independent of ${}^n\mathscr{F}_m$. The sequence $\{{}^n Y_m\}$ is for $n = 1, 2, \ldots$ a Markov chain with a countable state space I controlled by $\{{}^n X_m\}$, with the transition probability $p(x, j, i)$, i.e.,

$$P({}^n Y_{m+1} = i / {}^n\mathscr{F}_m) = p({}^n X_m, {}^n Y_m, i)\,.$$

Let the matrix $\|p(x, i, j)\|_{i,j \in I}$ involve only one closed class of recurrent states for $x \in R^1$. We suppose that the system of equations for unknowns $\{w(x, i)\ i \in I\}$ and $\mu(x)$

$$(6) \qquad f(x, j) + \sum_i p(x, j, i)\, w(x, i) - w(x, j) - \mu(x) = 0, j \in I,$$

has for every x a bounded solution. Then the functions $\{w(x, i)\}$ (except for an additive constant) and $\mu(x)$ are determined uniquely. System (6) is well-known from the dynamic programming in Markov chains. Our method makes use of [2]. If the stationary distribution $\pi(x, j)$ exists, then $\mu(x) = \sum f(x, j)\, \pi(x, j)$.

We further assume

$$|f(x, j)| + |w(x, j)| + |\mu(x)| \leqq K(1 + |x|)\,, \quad x \in R^1, j \in I\,,$$
$$|w(x, j) - w(x', j)| \quad + |\mu(x) - \mu(x')| \leqq$$
$$\leqq K|x - x'|\,, \quad x, x' \in R^1\,, \quad j \in I\,.$$

Proposition 1. *The probabilities $\{Q_n\}$ on $(D, \mathscr{D})$ induced by the processes $\{{}^nX_t\}$ converge weakly to the probability Q corresponding to the stochastic differential equation*

$$\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \mathrm{d}W_t,\ t \in <0,1>,\ X_0 = x_0.$$

Proof. We shall verify the assumptions of Theorem 1. Since $a_n(x, j) = f(x, j)$, $b_n(x, j) = 1$, $a(x) = \mu(x)$ and $b(x) = 1$, the assumptions (i), (iii), (iv) are satisfied. Let us choose $\varphi(n) > 0$, $\varphi(n) = o(n)$ and let us begin to verify the assumption (ii). In the estimates that we make for fixed n we write (X_m, Y_m) instead of $({}^nX_m, {}^nY_m)$.

$$a_n'(x, j) = (n/\varphi(n))\, E_{xj}({}^nX_{\varphi(n)} - {}^nX_0) = (1/\varphi(n))\, E_{xj} \sum_{m=0}^{\varphi(n)-1} f({}^nX_m, {}^nY_m)\,,$$
$$b_n'(x, j)^2 = (n/\varphi(n))\, E_{xj}({}^nX_{\varphi(n)} - {}^nX_0)^2 - (\varphi(n)/n)\, a_n'(x, j)^2\,.$$

For arbitrary $x \in R^1$, $j \in I$ the following estimates hold:

$$|a_n'(x, j) - \mu(x)| = (1/\varphi(n))|\, E_{xj} \sum_{m=0}^{\varphi(n)-1} (f(X_m, Y_m) - \mu(X_0))| =$$
$$= (1/\varphi(n))\, |E_{xj} \sum_{m=0}^{\varphi(n)-1} E\{w(X_m, Y_m) - w(X_m, Y_{m+1}) + \mu(X_m) - \mu(X_0)/{}^n\mathscr{F}_m\}| \leqq$$
$$\leqq (1/\varphi(n))\, E_{xj} | \sum_{m=0}^{\varphi(n)-1} w(X_m, Y_m) - w(X_m, Y_{m+1})| + (1/\varphi(n))\, E_{xj} | \sum_{m=1}^{\varphi(n)-1} (\mu(X_m) - \mu(X_0))| \leqq$$
$$\leqq (1/\varphi(n))\, E_{xj}|w(X_0, Y_0) - w(X_{\varphi(n)}, Y_{\varphi(n)})| +$$
$$+ (1/\varphi(n))\, E_{xj}| \sum_{m=1}^{\varphi(n)} w(X_m, Y_m) - w(X_{m-1}, Y_m)| + (1/\varphi(n))\, E_{xj}| \sum_{m=1}^{\varphi(n)-1} (\mu(X_m) - \mu(X_0))| \leqq$$

$$\leqq (1/\varphi(n))\, E_{xj}\, K(1 + |X_0| + |X_{\varphi(n)}|) + (1/\varphi(n))\, E_{xj} \sum_{m=1}^{\varphi(n)} K|X_m - X_{m-1}| +$$

$$+ (1/\varphi(n))\, E_{xj} \sum_{m=1}^{\varphi(n)-1} K|X_m - X_0| \leqq$$

$$\leqq K(1 + |x|)\,((1/\varphi(n)) + (1/\sqrt{n}) + (\sqrt{\varphi(n)}/\sqrt{n}))\,.$$

To estimate the function $a_n'(x, j)$ we have used the properties of the functions $w(x, j)$ and $\mu(x)$ and the validity of (3), (4) in Remark 1.

$$|b_n'(x, j)^2 - 1| \leqq (n/\varphi(n))\, E_{xj}((1/n) \sum_{m=0}^{\varphi(n)-1} f(X_m, Y_m))^2 +$$

$$+ (n/\varphi(n))\, 2|E_{xj}(((1/n) \sum_{m=0}^{\varphi(n)-1} f(X_m, Y_m))\,((1/\sqrt{n}) \sum_{i=0}^{\varphi(n)-1} \varepsilon_i))| +$$

$$+ |(n/\varphi(n))\, E_{xj}((1/\sqrt{n}) \sum_{i=0}^{\varphi(n)-1} \varepsilon_i)^2 - 1| + (\varphi(n)/n)\, K(1 + x^2) \leqq$$

$$\leqq K(1 + x^2)\,((\varphi(n)/n) + (\sqrt{\varphi(n)}/\sqrt{n}))\,.$$

The proof of condition (v) is omitted here due to its length. An auxiliary sequence $\{{}^n\bar{X}_m, {}^n\bar{Y}_m\}$ used there fulfils ${}^n\bar{X}_{m+1} = {}^n\bar{X}_m + \bar{a}_n({}^n\bar{X}_m, {}^n\bar{Y}_m)\,(1/n) + (1/\sqrt{n})\,\varepsilon_m$, $m = 0, 1, \ldots$, ${}^n\bar{X}_0 = x_0$, ${}^n\bar{Y}_0 = j_0$, where $\bar{a}_n = a_n - [(a_n - \varkappa(n))^+ + (a_n + \varkappa(n))^-]$, and $\varkappa(n) > 0$ is chosen so that $(\sqrt{\varphi(n)}/\sqrt{n})\,\varkappa(n) \to 0$, $\varkappa(n) \to \infty$ as $n \to \infty$. The sequences $\{\Delta\, {}^n\varpi_m\}$, $n = 1, 2, \ldots$ corresponding to $\{{}^n\bar{X}_m, {}^n\bar{Y}_m\}$ fulfil (5), and $\lim_{n\to\infty} P(\sup_{t\in\langle 0,1\rangle} |{}^n\varpi_t - {}^n\omega_t| > \varepsilon) = 0$ for arbitrary $\varepsilon > 0$.

The System of Itô Difference Equations with Different Time-Scales

We consider, for $n = 1, 2, \ldots$, two sequences of random variables $\{{}^nX_m\}$, $\{{}^nY_m\}$ defined by the recurrent relations

$$\begin{aligned} {}^nX_0 &= x_0\,, \quad {}^nY_0 = y_0\,, \\ {}^nX_{m+1} &= {}^nX_m + (1/n)\, f({}^nX_m, {}^nY_m) + (1/\sqrt{n})\,\varepsilon_m\,, \\ {}^nY_{m+1} &= {}^nY_m + (1/\psi(n))\, g({}^nX_m, {}^nY_m) + (1/\sqrt{\psi(n)})\,\delta_m\,. \end{aligned}$$

The random variables $\{\varepsilon_m\}$ are identically distributed with $E\varepsilon_0 = 0$, $E\varepsilon_0^2 = 1$, $E\varepsilon_0^4 < \infty$. The random variables $\{\delta_m\}$ are identically distributed and $E\delta_0 = 0$, $E\delta_0^2 = 1$, $E|\delta_0|^3 < \infty$. The sequences $\{\varepsilon_m\}$ and $\{\delta_m\}$ are mutually independent. The function $\psi(n) > 0$ is an integer function such that $\psi(n) = o(\sqrt{n})$. We suppose that the functions f and g satisfy

$$\sup_y |f(x, y)| \leqq K(1 + |x|)\,, \quad x \in R^1\,,$$

$$\sup_{x,y} |g(x, y)| \leqq K\,.$$

Further, let us consider the equation

$$(7)\qquad (1/2)\frac{\partial^2}{\partial y^2} w(x, y) + g(x, y)\frac{\partial}{\partial y} w(x, y) + f(x, y) - \mu(x) = 0 .$$

Equation (7) is analogue of (6) for the case of the continuous state space (see [4]). We introduce $G(x, y) = 2\int_0^y g(x, s)\,\mathrm{d}s$ and we suppose $\int_{-\infty}^{\infty} e^{G(x,y)}\mathrm{d}y < \infty$. Let us denote $\gamma(x, y) = e^{G(x,y)}/\int e^{G(x,s)}\,\mathrm{d}s$. If we look for $\mu(x)$ and $w(x, y)$ among the solutions fulfilling $\lim_{y\to\pm\infty} e^{G(x,y)}(\partial/\partial y)w(x, y) = 0$ for each $x \in R^1$, then such $\mu(x)$ is unique and

$$\mu(x) = \int_{-\infty}^{\infty} f(x, y)\,\gamma(x, y)\,\mathrm{d}y .$$

The function $w(x, y)$ is also unique (except for an additive constant) and

$$w(x, y) = \int_0^y e^{-G(x,u)} \int_u^{\infty} (f(x, s) - \mu(x))\, 2e^{G(x,s)}\,\mathrm{d}s\,\mathrm{d}u .$$

We consider $\mu(x)$ and $w(x, y)$ so defined and assume $\sup_y (|w(x, y)| + |(\partial^2/\partial y^2)\, w(x, y)|) \leqq K(1 + |x|)$ for arbitrary $x \in R^1$, $(\partial^2/\partial y^2)\, w(x, y)$ is Hölder continuous (with exponent α) in $y \in R^1$ uniformly in the variable $x \in R^1$.

$$|w(x, y) - w(x', y)| + |\mu(x) - \mu(x')| \leqq K|x - x'| , \quad x, x', y \in R^1 .$$

Proposition 2. *The probabilities $\{Q_n\}$ induced on $(D, \mathscr{D})$ by the processes $\{{}^nX_t\}$ converge weakly to the probability Q corresponding to the stochastic differential equation*

$$\mathrm{d}X_t = \int \gamma(X_t, y) f(X_t, y)\,\mathrm{d}y\,\mathrm{d}t + \mathrm{d}W_t,\ t \in \langle 0, 1 \rangle,\ X_0 = x_0 .$$

Proof. Assumptions (i), (iii), (iv) are satisfied because $a_n(x, y) = f(x, y)$, $b_n(x, y) = 1$, $a(x) = \mu(x) = \int \gamma(x, y) f(x, y)\,\mathrm{d}y$, $b(x) = 1$. We choose $\varphi(n)$ such that $\varphi(n) = o(n)$, and simultaneously $\psi(n)/\varphi(n) \to 0$ as $n \to \infty$. We shall verify assumption (ii). For fixed n we shall write (X_m, Y_m) instead of $({}^nX_m, {}^nY_m)$.

$$a_n'(x, y) = (1/\varphi(n))\, E_{xy} \sum_{m=0}^{\varphi(n)-1} f(X_m, Y_m) = (1/\varphi(n))\, E_{xy} \Big(\sum_{m=0}^{\varphi(n)-1} E\Big\{\mu(X_m) - g(X_m, Y_m)\frac{\partial}{\partial y} w(X_m, Y_m) - (1/2)\frac{\partial^2}{\partial y^2} w(X_m, Y_m)/{}^n\mathscr{F}_m\Big\}\Big) .$$

First we shall make an auxiliary estimation. Using Taylor's series and equation (7) we get

$$E_{xy}\, \psi(n) \sum_{m=0}^{\varphi(n)-1} E\{w(X_m, Y_{m+1}) - w(X_m, Y_m)/\mathscr{F}_m\} =$$

$$= E_{xy}\,\psi(n)\sum_{m=0}^{\varphi(n)-1} E\{w(X_m, Y_m + (1/\psi(n))\,g(X_m, Y_m) + (1/\sqrt{\psi(n)})\,\delta_m) -$$

$$- w(X_m, Y_m)/\mathcal{F}_m\} = E_{xy}\,\psi(n)\sum_{m=0}^{\varphi(n)-1} E\left\{\frac{\partial}{\partial y} w(X_m, Y_m)\,g(X_m, Y_m)\,(1/\psi(n)) +\right.$$

$$+ (1/2)\frac{\partial^2}{\partial y^2} w(X_m, Y_m)\,(1/\psi(n)) + (1/2)\left(\frac{\partial^2}{\partial y^2} w(X_m, Y_m)\right) g(X_m, Y_m)^2\,(1/\psi(n))^2 -$$

$$- ((1/2)\frac{\partial^2}{\partial y^2} w(X_m, Y_m) - (1/2)\frac{\partial^2}{\partial y^2} w(X_m, \tilde{Y}_m))\,(g(X_m, Y_m)/\psi(n) +$$

$$\left. + \delta_m/\sqrt{\psi(n)})^2/\mathcal{F}_m\right\},$$

where $|\tilde{Y}_m - Y_m| \leqq |g(X_m, Y_m)/\psi(n) + \delta_m/\sqrt{\psi(n)}|$ *a.e.* Further, we shall use the assumptions imposed on the function w and the preceding equation. For arbitrary $x, y \in R^1$ we have

$$\left|E_{xy}\,\psi(n)\sum_{m=0}^{\varphi(n)-1} E\left\{(w(X_m, Y_m) - w(X_m, Y_{m+1})) +\right.\right.$$

$$\left.\left. + \left(\frac{\partial}{\partial y} w(X_m, Y_m)\right) g(X_m, Y_m)\,(1/\psi(n)) + (1/2)\frac{\partial^2}{\partial y^2} w(X_m, Y_m)\,(1/\psi(n))/\mathcal{F}_m\right\}\right| \leqq$$

$$\leqq K((1 + |x|)\,\varphi(n)/\psi(n) + \varphi(n)/\psi(n)^{\alpha/2} + \varphi(n)/\psi(n)^{\alpha+1}).$$

Using the preceding inequality we shall estimate $a_n'(x, y)$.

$$|a_n'(x, y) - \mu(x)| \leqq \left|E_{xy}(1/\varphi(n)\sum_{m=0}^{\varphi(n)-1} (\mu(X_m) - \mu(X_0) +\right.$$

$$\left. + \psi(n)\,E\{w(X_m, Y_m) - w(X_m, Y_{m+1})/\mathcal{F}_m\})\right| + K((1 + |x|)\,(1/\psi(n)) +$$

$$+ 1/\psi(n)^{\alpha/2} + 1/\psi(n)^{\alpha+1}) \leqq (1/\varphi(n))\,(E_{xy}\sum_{m=0}^{\varphi(n)-1} |\mu(X_m) - \mu(X_0)| +$$

$$+ E_{xy}\,\psi(n)\,\left|\sum_{m=0}^{\varphi(n)-1} w(X_m, Y_m) - w(X_m, Y_{m+1})\right|) + K((1 + |x|)\,(1/\psi(n)) +$$

$$+ 1/\psi(n)^{\alpha/2} + 1/\psi(n)^{\alpha+1}) \leqq K(1 + |x|)\,(\sqrt{\varphi(n)}/\sqrt{n} + 1/\psi(n) +$$

$$+ \psi(n)/\varphi(n) + \psi(n)/\sqrt{n}) + K(1/\psi(n)^{\alpha/2} + 1/\psi(n)^{\alpha+1}).$$

We have derived the last inequality in the same way as in System 1. In just the same way as in System 1

$$\sup_{xy} |b_n'(x, y) - 1|/(1 + |x|) \xrightarrow[n\to\infty]{} 0,$$

and the validity of assumption (v) can be proved.

The third system involves an application which is a simplification of [3]. The application was mentioned in the introduction.

The Difference Equation with Noise Created by the Deviation from the Expectation

Let sequences of random variables $\{{}^nX_m\}$, $n = 1, 2, \ldots$ satisfy the difference equations

$$(8) \qquad {}^nX_{m+1} = {}^nX_m + (1/n) f({}^nX_m, {}^nY_m) + (1/\sqrt{n}) \big(h({}^nX_m, {}^nY_m, {}^nY_{m+1}) - - f({}^nX_m, {}^nY_m)\big),$$

$${}^nX_0 = x_0, \quad {}^nY_0 = j_0,$$

and let the sequences $\{{}^nY_m\}$ be Markov chains controlled by $\{{}^nX_m\}$, with a countable state space I. The transition probability matrix of $\{{}^nY_m\}$, $\|p(x, i, j)\|_{i,j\in I}$ involves only one closed class of recurrent states. The functions f and h fulfil

$$\sum_i p(x, j, i)\, h(x, j, i) = f(x, j),$$

$$\sup_{x,j,i} |h(x, j, i)| \leqq K.$$

Let the systems of equations

$$f(x, j) + \sum_i p(x, j, i)\, w_1(x, i) - w_1(x, j) - \mu(x) = 0, \quad j \in I,$$

$$\sum_i p(x, j, i)\, h(x, j, i)^2 - f(x, j)^2 + \sum_i p(x, j, i)\, w_2(x, i) - w_2(x, j) - \nu(x)^2 = 0, \quad j \in I$$

have solutions satisfying the following assumptions:

$$\sup_{x,i} (|w_1(x, i)| + |w_2(x, i)| + |\mu(x)| + |\nu(x)^2|) \leqq K_1,$$

$$\inf_x \nu(x)^2 \geqq K_2,$$

$$|w_1(x, j) - w_1(x', j)| + |w_2(x, j) - w_2(x', j)| + |\mu(x) - \mu(x')| + + |\nu(x) - \nu(x')| \leqq K|x - x'|, \quad j \in I, \quad x, x' \in R^1,$$

$$\nu(x) = \sqrt{\nu(x)^2}.$$

Proposition 3. *The probabilities $\{Q_n\}$ induced by $\{{}^nX_t\}$ converge weakly to the probability Q corresponding to the stochastic differential equation*

$$dX_t = \mu(X_t)\, dt + \nu(X_t)\, dW_t, \quad t \in \langle 0, 1 \rangle, \quad X_0 = x_0.$$

Proof. Assumptions (i), (iii), (iv) are satisfied because $a_n(x, j) = f(x, j)$, $b_n(x, j)^2 = \sum_k h(x, j, k)^2\, p(x, j, k) - f(x, j)^2$, $a(x) = \mu(x)$, $b(x) = \nu(x)$. Let us choose the function $\varphi(n) = o(n)$. Similarly as in System 1

$$\sup_{x,j} |a_n'(x, j) - \mu(x)| \xrightarrow[n\to\infty]{} 0.$$

Further,

$$b_n'(x, j)^2 = (n/\varphi(n))\, E_{xj}((1/n) \sum_{m=0}^{\varphi(n)-1} f(X_m, Y_m))^2 +$$

$$+ (n/\varphi(n))\, 2E_{xj}((1/n) \sum_{m=0}^{\varphi(n)-1} f(X_m, Y_m))\, ((1/\sqrt{n}) \sum_{k=0}^{\varphi(n)-1} (h(X_k, Y_k, Y_{k+1}) - f(X_k, Y_k))) +$$

$$+ (1/\varphi(n))\, E_{xj} \sum_{m=0}^{\varphi(n)-1} (h(X_m, Y_m, Y_{m+1}) - f(X_m, Y_m))^2 - (\varphi(n)/n)\, a_n'(x, j)^2 .$$

We have used the fact that $h(X_m, Y_m, Y_{m+1}) - f(X_m, Y_m)$ is a martingale difference. For arbitrary $x \in R^1$, $j \in I$

$$|b_n'(x, j)^2 - v(x)^2| \leqq K((\varphi(n)/n) + (\sqrt{\varphi(n)}/\sqrt{n})) +$$

$$+ (1/\varphi(n))\, |E_{xj} \sum_{m=0}^{\varphi(n)-1} (h(X_m, Y_m, Y_{m+1}) - f(X_m, Y_m))^2 - v(X_0)^2)| =$$

$$= K((\varphi(n)/n) + (\sqrt{\varphi(n)}/\sqrt{n}) + (1/\varphi(n))|E_{xj} \sum_{m=0}^{\varphi(n)-1} E\{w_2(X_m, Y_m) -$$

$$- w_2(X_m, Y_{m+1}) + v(X_m)^2 - v(X_0)^2/\mathscr{F}_m\}| .$$

If we use the same method of estimation as in System 1 we establish the condition (ii). To verify condition (v) it suffices to prove the validity of (5) for $\delta = 2$. If we use the property of martingale differences, we obtain

$$\sum_{k=0}^{n/\varphi(n)-1} E_{xj}({}^nX_{(k+1)\varphi(n)} - {}^nX_{k\varphi(n)})^4 \xrightarrow[n\to\infty]{} 0$$

which implies (v) in virtue of the existence of K_1, K_2 such that $|a_n'(x, j)| \leqq K_1$, $|b_n'(x, j)| \geqq K_2$ for sufficiently large values of n.

Note. It may be supposed that the functions $w(x, j)$ are uniformly equicontinuous instead of being Lipschitz continuous.

An application of System 3:

We consider for $n = 1, 2. \ldots$ a Markov chain $\{{}^nY_m\}$ with a finite state space I, controlled by the chain $\{{}^nX_m\}$ where $\{{}^nX_m\}$ is the reward up to the time m. Let the control be a continuous mapping $u : x \in R^1 \to u_x \in \prod_{i\in I} Z(i) \equiv \mathscr{U}$ where $Z(i)$ is a bounded closed subset of R^s. For fixed $\mathbf{z} \in \mathscr{U}$ the chain $\{{}^nY_m\}$ has the transition probability matrix $\|p(i, j, \mathbf{z}(i))\|_{i,j\in I}$, which involves only one closed class of recurrent states. The reward accumulated up to the time m $\{{}^nX_m\}$ is the sum of the rewards at the times $0, 1, \ldots, m - 1$:

$$(9) \qquad {}^nX_m = {}^nX_0 + \sum_{k=1}^{m-1} \sigma(u_{{}^nX_k}({}^nY_k), {}^nY_k, {}^nX_{k+1}), \quad {}^nX_0 = x_0 .$$

Let us suppose that the functions $p(i, j, z)$, $\sigma(z, i, j)$ are continuous functions on $Z(i)$ for each $i, j \in I$. We set

$$^{n}\varrho_1(z, i) = \sum_{j \in I} p(i, j, z)\, {}^{n}\sigma(z, i, j)\,,$$
$$^{n}\varrho_2(z, i) = \sum_{j \in I} p(i, j, z)\, {}^{n}\sigma(z, i, j)^2\,.$$

We assume $|{}^{n}\sigma(z, i, j)| \leqq K/\sqrt{n}$, $i, j \in I$, $z \in Z(i)$, and

$$\lim_{n \to \infty} n\, {}^{n}\varrho_1(z, i) = r_1(z, i)\,,$$
$$\lim_{n \to \infty} n\, {}^{n}\varrho_2(z, i) = r_2(z, i)$$

uniformly in $z \in Z(i)$, $i \in I$.

For $\mathbf{z} \in \mathcal{U}$, let $a(\mathbf{z})$, $\{v_{1i}(\mathbf{z}), i \in I\}$ and $b^2(\mathbf{z})$, $\{v_{2i}(\mathbf{z}), i \in I\}$ denote the solutions of the equations

$$r_1(\mathbf{z}(i), i) + \sum_j p(i, j, \mathbf{z}(i))\, v_{1j}(\mathbf{z}) - v_{1i}(\mathbf{z}) - a(\mathbf{z}) = 0\,, \quad i \in I\,,$$
$$r_2(\mathbf{z}(i), i) + \sum_j p(i, j, \mathbf{z}(i))\, v_{2j}(\mathbf{z}) - v_{2i}(\mathbf{z}) - b(\mathbf{z})^2 = 0\,, \quad i \in I\,, \quad b(\mathbf{z}) = \sqrt{b(\mathbf{z})^2}\,.$$

Let $a(u_x)$ and $b(u_x)$ be Lipschitz continuous and $b(u_x) \geqq K > 0$. Let us note that the functions $\{v_{1j}(\mathbf{z}), v_{2j}(\mathbf{z}), j \in I\}$, $a(\mathbf{z})$, $b(\mathbf{z})$ are continuous on a compact set and therefore bounded. *The probabilities* $\{Q_n\}$ *induced by the processes* $\{{}^{n}X_t\}$ *converge to the probability* Q *corresponding to the stochastic differential equation*

$$\mathrm{d}X_t = a(u_{X_t})\, \mathrm{d}t + b(u_{X_t})\, \mathrm{d}W_t\,, \quad t \in \langle 0, 1 \rangle\,, \quad X_0 = x_0\,.$$

Proof. We can express (9) in the form

$$^{n}X_{m+1} = {}^{n}X_m + (1/n)\, {}^{n}f({}^{n}X_m, {}^{n}Y_m) + (1/\sqrt{n})\, ({}^{n}h({}^{n}X_m, {}^{n}Y_m, {}^{n}Y_{m+1}) - {}^{n}f({}^{n}X_m, {}^{n}Y_m))$$

with ${}^{n}f(x, i) = n\, {}^{n}\varrho_1(u_x(i), i)$,

$$^{n}h(x, i, j) = \sqrt{n}({}^{n}\sigma(u_x(i), i, j) - {}^{n}\varrho_1(u_x(i), i)) + n\, {}^{n}\varrho_1(u_x(i), i)\,.$$

Difference Equation with a Renewal Process

We consider for $n = 1, 2, \ldots$ the sequence of random variables $\{{}^{n}X_m\}$, where $\{{}^{n}X_m\}$ fulfil the stochastic difference equations

$$^{n}X_{m+1} = {}^{n}X_m - (\beta/n)\, {}^{n}X_m + (1/\sqrt{n})\, (f(Y_m) - \lambda)\, \gamma + (1/\sqrt{n})\, f(Y_m)\, \varepsilon_m \sigma\,,$$
$$^{n}X_0 = x_0\,, \quad \beta, \lambda, \gamma, \sigma, > 0$$

and $\{Y_m\}$ is an alternating renewal process with the *on-time* T_1 and *off-time* T_2, where $ET_1^2 + ET_2^2 < \infty$. The function f assumes the value 1, if the system is on, and the value O, if the system is off. The random variables $\{\varepsilon_m\}$ are the same as in System 1.

The next result is stated without a proof.

Proposition 4. *Let* $\lambda = ET_1/(ET_1 + ET_2)$,

$$\Theta = E((ET_2)\,T_1 - (ET_1)\,T_2)^2/(ET_1 + ET_2)^3 .$$

Then the probabilities $\{Q_n\}$ *converge weakly to the probability* Q *corresponding to the stochastic differential equation*

$$\mathrm{d}X_t = -\beta X_t\,\mathrm{d}t + \sqrt{(\gamma^2\theta + \sigma^2\lambda)}\,\mathrm{d}W_t\,, \quad t \in \langle 0, 1\rangle\,, \quad X_0 = x_0\,.$$

References

[1] *I. I. Gichman, A. V. Skorochod:* Theory of Random Processes III (in Russian). Nauka, Moscow 1975.

[2] *P. Mandl:* A connection between Controlled Markov Chains and Martingales. Kybernetika 9 (1973), 237–241.

[3] *P. Mandl:* On aggregating controlled Markov chains, in Jaroslav Hájek Memorial Volume (to appear).

[4] *P. Morton:* On the Optimal Control of Stationary Diffusion Processes with Inaccessible Boundaries and no Discounting. J. Appl. Probability 8 (1971), 551–560.

Souhrn

APROXIMACE DVOUROZMĚRNÝCH MARKOVOVÝCH ŘETĚZCŮ JEDNOROZMĚRNÝMI DIFUSNÍMI PROCESY

Daniela Kuklíková

Článek pojednává o některých případech difusní aproximace. Navazuje na práci [3], kde je navrhováno, jak řídit velké množství rychle se měnících jednotek. Tam se předpokládá, že každá jednotka tvoří Markovův řetězec, který se mění v časech $0, \Delta, 2\Delta, \ldots$, přičemž změna stavu jednotky přináší zisk. Celkový zisk je pak součtem zisků získaných jednotkami. Uvažuje se stacionární řízení, které je funkcí celkového zisku. Chování celkového zisku při $\Delta \to 0$ lze popsat difusním procesem, splňujícím stochastickou diferenciální rovnici, nezávisející na stavech jednotek. Tohoto faktu lze využít při nalezení optimálního řízení vzhledem ke kriteriu, jež není aditivní. Úkolem tohoto článku je podat obecnou metodu řešení podobných případů a ukázat více systémů, kde lze snížit rozměr úlohy navrhovanou metodou. To jest speciálně aproximovat dvourozměrné náhodné veličiny jednorozměrným difusním procesem tak, že se jedna ze souřadnic nahradí určitou její charakteristikou, např. stacionární střední hodnotou. Na čtyřech typech posloupností $\{({}^nX_m, {}^nY_m),\ m = 0, 1, \ldots\}$, $n = 1, 2, \ldots$ je ukázáno, že je to možno provést například tehdy, když tendence ke stacionárnímu stavu je u $\{{}^nY_m\}$ větší než u $\{{}^nX_m\}$. V prvé části uvádíme větu, kde vyvozujeme podmínky pro aproximaci řetězce $\{{}^nX_m\}$ stochastickou diferenciální rovnicí, která závisí na řetězci $\{{}^nY_m\}$ pouze prostřednictvím jeho stacionárního rozdělení. V druhé části uvádíme jednotlivé systémy a zkoumáme, zda splňují předpoklady konvergenční věty.

Author's address: RNDr. *Daniela Kuklíková*, Matematicko-fyzikální fakulta UK, Sokolovská 83, 186 00 Praha 8.