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APPROXIMATION OF DISCONTINUOUS CURVES AND SURFACES BY DISCRETE SPLINES WITH TANGENT CONDITIONS

ABDELOUAHED KOUIBIA, MIGUEL PASADAS

Abstract. This paper concerns the construction of a discontinuous parametric curve or surface from a finite set of points and tangent conditions. The method is adapted from the theory of the discrete smoothing variational splines to introduce a discontinuity set and some tangent conditions. Such method is justified by a convergence result.

1. Introduction

The problem of the construction of discontinuous parametric curves and surfaces from some Lagrangean points and a set of tangent spaces is frequently encountered in CAGD, Geology, and other Earth Sciences.

The authors in [10] present a smoothing method for fitting parametric surfaces from sets of data points and tangent planes. In those papers and in [6, 7, 8, 10], the corresponding original curves or/and surfaces that are approximated do not present any discontinuities. So for its practical interest, we introduce some discontinuity in order to study an approximation problem of discontinuous curves and surfaces.

In this paper we present a discrete version of the previous problem in a finite element space by minimizing a quadratic functional from a set of Lagrangean data and another one of tangent conditions. The approximation of discontinuous functions from a set of scattered data points is usually a two steps: first, a detection algorithm is applied to localize the discontinuity sets, then the functions are reconstructed using a fitting method. In this work we propose a method for the second stage, based on the computation of discrete smoothing variational splines [7].

Specially, we study the following problem: given a differentiable function f in a subset Ω' of an open set $\Omega \subset \mathbb{R}^p$ with values in $\mathbb{R}^n$, $1 \le p < n \le 3$, whose first partial derivatives or she can present discontinuity in a subset F of $\overline{\Omega}$, such that $\Omega' = \Omega \backslash \overline{F}$, construct a function σ that approximates f in the given Lagrangean points of Ω' and whose tangent spaces at the points of an other given set of Ω' are close to the tangent spaces of f at the same points. To do this, firstly using the

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work of Arcangéli, Manzanilla and Torrens [1], we determine certain hypotheses about the set Ω' that allows to model the contingent discontinuities of f. Secondly, we study a method of smoothness which results from adapting to this context the theory of the discrete smoothing variational splines (c.f. [7]).

In Section 2, we briefly recall preliminary notation and some results. We study the problem of discrete smoothing variational splines with tangent conditions in Section 3. Section 4 is devoted to compute such spline and to prove a convergence result.

2. Notation

We denote by $\langle\cdot\rangle_{\mathbb{R}^n}$ and $\langle\cdot\,,\,\cdot\rangle_{\mathbb{R}^n}$ respectively, the Euclidean norm and the inner product in $\mathbb{R}^n$, with n, m and p belonging to $\mathbb{N}^*$. We denote by $\overline{E}$, δE and card$E$ respectively, the adherence, the bounded and the cardinal of E for each subset E of $\mathbb{R}^p$. Let us consider $\mathbb{R}^{N,n}$ the space of real matrices with N rows and n columns equipped with the inner product

$$\langle A, B\rangle_{N,n} = \sum_{i=1}^{N}\sum_{j=1}^{n} a_{ij}b_{ij}$$

and the corresponding norm

$$\langle A\rangle_{N,n} = \langle A, A\rangle_{N,n}^{1/2}.$$

For all $\alpha = (\alpha_1, \dots, \alpha_p) \in \mathbb{N}^p$, we write $|\alpha| = \sum_{i=1}^{p}\alpha_i$ and we indicate by ∂^α the operator of partial derivative

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\dots\partial x_p^{\alpha_p}}$$

Let ω be a nonempty open bounded set of $\mathbb{R}^p$ and we denote by $H^m(\omega;\mathbb{R}^n)$ the usual Sobolev space of (classes of) functions u belonging to $L^2(\omega;\mathbb{R}^n)$, together with all their partial derivative $\partial^\alpha u$ –in the distribution sense– of order $|\alpha| \le m$. This space is equipped with the inner product of order ℓ

$$(\,u\,,\,v\,)_{\ell,\omega,\mathbb{R}^n} = \Big(\sum_{|\alpha|=\ell}\int_\omega \langle\partial^\alpha u(x), \partial^\alpha v(x)\rangle_{\mathbb{R}^n}\Big)^{1/2}, \quad \ell = 0,\dots,m,$$

the corresponding semi–norms of order ℓ

$$|\,u\,|_{\ell,\omega,\mathbb{R}^n} = (\,u\,,\,u\,)_{\ell,\omega,\mathbb{R}^n}^{1/2},\ \ell = 0,\dots,m,$$

the norm

$$\|\,u\,\|_{m,\omega,\mathbb{R}^n} = \Big(\sum_{\ell=0}^{m}|\,u\,|_{\ell,\omega,\mathbb{R}^n}^2\Big)^{1/2}$$

and the corresponding inner product

$$((\,u\,,\,v\,))_{m,\omega,\mathbb{R}^n} = \sum_{\ell=0}^{m}(\,u\,,\,v\,)_{\ell,\omega,\mathbb{R}^n}.$$

Given a function $f:\omega\to\mathbb{R}^n$, we denote by Im$Df(x)$ the image of the differential of f at the point $x\in\omega$, when this exists, i. e. the linear subspace generated by $\{\partial^\alpha f(x)\ :\ |\alpha| = 1\}$. Furthermore, if $1\le p<n\le 3$, we can consider f as the parameterization of a curve ($p=1$) or a surface ($p=2$) and, if f is differentiable at $x\in\omega$, the space Im$Df(x)$ is called the tangent space of f at x, sometimes when

$p = 2$ it is written by $T_x(f) = span\langle D_1 f(x), D_2 f(x)\rangle$, where $D_1 f$ and $D_2 f$ denote the first partial derivatives of f.

3. Discrete variational spline over Ω' with tangent conditions

The first step in order to develop this work is to have an adequate characterization over a set of discontinuity. Let us introduce the following definition due to R. Arcangéli, R. Manzanilla and J. J. Torrens [1].

Definition 3.1. Let Ω be a bounded open connected set of $\mathbb{R}^p$ with Lipschitz boundary and let F be a nonempty subset of $\overline{\Omega}$ such that, there exists a finite family $\{R_1, \dots, R_I\}$ of open connected subsets of Ω with Lipschitz boundary, verifying the following conditions:

(i) for all $i, j = 1, \dots, I,\ i \neq j,\ R_i \cap R_j = \emptyset$;
(ii) $\bigcup_{i=1}^{I} \overline{R}_i = \overline{\Omega}$;
(iii) $F \subset \delta R$, where $\cup_{i=1}^{I} R_i = R$;
(iv) F is contained in the interior of $\delta\Omega$ (equipped of the induced topology by $\mathbb{R}^p$) of $F \cap \delta\Omega$;
v) the interior in δR of $\overline{F} \cap \Omega$ is contained in F;
vi) $\overline{F} \cap \delta\Omega$ is contained in F.

It is said that the family $\{R_1, \dots, R_I\}$ represents F in Ω and we write $\Omega' = \Omega \backslash \overline{F}$.

We denote by $C_F^k(\Omega'; \mathbb{R}^n)$ the space of functions $\varphi \in C^k(\Omega'; \mathbb{R}^n)$ such that

$$\forall i = 1, \dots, I,\ \varphi\big|_{R_i} \in C^k(\overline{R}_i; \mathbb{R}^n).$$

Such space is equipped by the norm

$$\|\varphi\|_{C_F^k(\Omega';\mathbb{R}^n)} = \max_{1 \le i \le I} \|\varphi\big|_{R_i}\|_{C^k(\overline{R}_i;\mathbb{R}^n)}\,. \tag{3.1}$$

Now, we suppose that

$$m > \frac{p}{2} + 1. \tag{3.2}$$

Let Υ_0 be a curve or surface parameterized by a function $f \in H^m(\Omega'; \mathbb{R}^n)$, and A_1, A_2 be two ordered finite subsets of, respectively, N_1 and N_2 distinct points of $\overline{\Omega}$. For any $a \in A_1$, let us consider the linear form defined on $C_F^0(\Omega'; \mathbb{R}^n)$ by

$$\phi_a v = \begin{cases} v(a) & \text{if } a \in A_1 \backslash F, \\ v\big|_{R_i}(a) & \text{if } a \in A_1 \cap R_i \cap F,\ 1 \le i \le I, \end{cases} \tag{3.3}$$

and, for any $a \in A_2$ let Π_a be the operator defined on $C_F^1(\Omega'; \mathbb{R}^n)$ by

$$\Pi_a v = \begin{cases} \big(P_{S_a^\perp}(\frac{\partial v}{\partial x_j}(a))\big)_{1 \le j \le p} & \text{if } a \in A_2 \backslash F,\ 1 \le i \le I, \\ \big(P_{S_a^\perp}(\frac{\partial v|_{R_i}}{\partial x_j}(a))\big)_{1 \le j \le p} & \text{if } a \in A_2 \cap R_i \cap F,\ 1 \le i \le I, \end{cases} \tag{3.4}$$

where, for any $a \in A_2$, $P_{S_a^\perp}$ is the operator projection onto $S_a^\perp$, being $S_a^\perp$ the orthogonal complement of the linear space $S_a = \mathrm{Im}\mathrm{D}f(a)$. Finally, let

$$Lv = (\phi_a v)_{a \in A_1} \quad \text{and} \quad \Pi v = (\Pi_a v)_{a \in A_2}$$

and we suppose that

$$\mathrm{Ker} L \cap \tilde{P}_{m-1}(\Omega'; \mathbb{R}^n) = \{0\} \tag{3.5}$$

where $\tilde{P}_{m-1}(\Omega';\mathbb{R}^n)$ designs the space of functions over Ω' into $\mathbb{R}^n$ that are polynomials of total degree $\leq m-1$ respect to the set of variables over each connected component of Ω'.

Now, suppose we are given:

- a subset $\mathcal{H}$ of $(0,+\infty)$ with 0 is an accumulation point;
- for all $h \in \mathcal{H}$, a partition $\mathcal{T}_h$ of $\overline{\Omega}$ made with rectangles or triangles K of disjoint interiors and diameter $h_K \leq h$ such that

$$\forall\, K \in \mathcal{T}_h,\ \overset{\circ}{K} \cap F = \emptyset; \tag{3.6}$$

each side of K is a side of another K' or a part of $\delta\Omega$ or a part of F; (3.7)

- for any $h \in \mathcal{H}$, a finite element space X_h constructed on $\mathcal{T}_h$ such that

$$X_h \subset H^m(\Omega') \cap C_F^k(\Omega'),\ k+1 \geq m; \tag{3.8}$$

- for any $h \in \mathcal{H}$, a parametric finite element space V_h constructed from X_h by $V_h = (X_h)^n$, and from (3.8) satisfies

$$V_h \subset H^m(\Omega';\mathbb{R}^n) \cap C_F^k(\overline{\Omega'};\mathbb{R}^n). \tag{3.9}$$

Now, given $\tau \geq 0$ and $\varepsilon > 0$, let $J_{\varepsilon\tau}$ be the functional defined on V_h by

$$J_{\varepsilon\tau}(v) = \langle Lv - Lf\rangle^2_{N_1,n} + \tau\langle \Pi v\rangle^2_{N_2,pn} + \varepsilon\,|v|^2_{m,\Omega',\mathbb{R}^n}\,. \tag{3.10}$$

Remark 3.2. We observe that the functional $J_{\varepsilon\tau}(v)$ contains different terms which can be interpreted as follows:

- The first term, $\langle Lv - Lf\rangle^2_{N_1,n}$, indicates how well v approaches f in a discrete least squares sense.
- The second term, $\langle \Pi v\rangle^2_{N_2,pn}$, indicates how well, for any point $a \in A_2$, the tangent spaces $\mathrm{Im}Df(a)$ and $\mathrm{Im}Dv(a)$ are really close.
- The last term, $|v|^2_{m,\Omega',\mathbb{R}^n}$, measures the degree of smoothness of v in order to reduce, as much as possible, any unwanted oscillations.

We note that the parameters τ and ε control the relative weights corresponding, respectively, to the last two terms.

Now, we consider the following minimization problem:
Find an approximating curve or surface Υ of Υ_0 parameterized by a function $\sigma^h_{\varepsilon\tau}$ belonging to V_h from the data $\{f(a) : a \in A_1\}$ and $\{S_a : a \in A_2\}$, such that $\sigma^h_{\varepsilon\tau}$ minimizes the functional $J_{\varepsilon\tau}$ on V_h, i.e. find $\sigma^h_{\varepsilon\tau}$ such that

$$\sigma^h_{\varepsilon\tau} \in V_h, \quad \text{and for all } v \in V_h, \quad J_{\varepsilon\tau}(\sigma^h_{\varepsilon\tau}) \leq J_{\varepsilon\tau}(v). \tag{3.11}$$

Theorem 3.3. *The problem (3.11) has a unique solution, called discrete smoothing variational spline with tangent conditions in Ω' relative to A_1, A_2, f, τ and ε, which is also the unique solution of the following variational problem:*
Find $\sigma^h_{\varepsilon\tau}$ such that $\sigma_{\varepsilon\tau} \in V_h$ and for all $v \in V_h$,

$$\langle L\sigma^h_{\varepsilon\tau}, Lv\rangle_{N_1,n} + \tau\langle \Pi\sigma^h_{\varepsilon\tau}, \Pi v\rangle_{N_2,pn} + \varepsilon(\sigma^h_{\varepsilon\tau}, v)_{m,\Omega',\mathbb{R}^n} = \langle Lf, Lv\rangle_{N_1,n}.$$

Proof. Taking into account (3.2), (3.5) and that the norm

$$[[v]] = \left(\langle Lv\rangle^2_{N_1,n} + \tau\langle \Pi v\rangle^2_{N_2,pn} + \varepsilon|v|^2_{m,\Omega',\mathbb{R}^n}\right)^{1/2}$$

is equivalent in V_h to the norm $\|\cdot\|_{m,\Omega',\mathbb{R}^n}$ (cf. [1, Proposition 4.1]), one easily checks that the symmetric bilinear form $\widetilde{a} : V_h \times V_h \longrightarrow \mathbb{R}$ given by

$$\widetilde{a}(u,v) = \langle Lu, Lv\rangle_{N_1,n} + \tau\langle \Pi u, \Pi v\rangle_{N_2,pn} + \varepsilon(u,v)_{m,\Omega',\mathbb{R}^n}$$

is continuous and V_h–elliptic. Likewise, the linear form

$$\varphi : v \in V_h \longmapsto \varphi(v) = \langle Lf, Lv \rangle_{N_1,n}$$

is continuous. The result is then a consequence of the Lax–Milgram Lemma (see [3]). □

4. Computation and convergence result

Let us see how to compute the *discrete smoothing variational spline with tangent conditions*. To do this, for any $h \in \mathcal{H}$, let I and $\{w_1, \dots, w_I\}$ be the dimension and a basis of X_h, respectively, and let us denote by $\{e_1, e_2, \dots, e_n\}$ the canonical basis of $\mathbb{R}^n$. Then, the family $\{v_1, \dots, v_Z\}$, with $Z = nI$, is a basis of V_h, where for $i = 1, \dots, I$ and $\ell = 1, 2, \dots, n$,

$$j = n(i-1) + \ell, \quad v_j = w_i e_\ell.$$

Thus, for any $h \in \mathcal{H}$, the function $\sigma^h_{\varepsilon\tau}$ can be expressed as

$$\sigma^h_{\varepsilon\tau} = \sum_{i=1}^{Z} \beta_i v_i,$$

with the unknown $\beta_i \in \mathbb{R}$, for $i = 1, \dots, Z$.

Applying Theorem 3.3, we obtain that the vector $\beta = (\beta_i)_{1 \le i \le Z} \in \mathbb{R}^Z$ is the solution of the following linear system of order Z:

$$\big(\mathcal{A}^T\mathcal{A} + \tau\, \mathcal{P}^T\mathcal{P} + \varepsilon\mathcal{R}\big)\beta = \mathcal{A}^T b, \tag{4.1}$$

where

$$\begin{gathered}
\mathcal{A} = \big(v_j(a_i)\big)_{1 \le i \le N_1 \ \ 1 \le j \le Z};\\
\mathcal{P} = \big(\Pi_{a_i} v_j\big)_{1 \le i \le N_2 \ \ 1 \le j \le Z};\\
\mathcal{R} = \big((v_i, v_j)_{m,\Omega',\mathbb{R}^n}\big)_{1 \le i,j \le Z};\\
b = \big(f(a_i)\big)_{1 \le i \le N_1}.
\end{gathered}$$

We point out that $\mathcal{A}^T\mathcal{A} + \tau\mathcal{P}^T\mathcal{P} + \varepsilon\mathcal{R}$ of the linear system given in (4.1) is a band matrix which is symmetric positive definite.

Now, under adequate hypotheses, we shall show that the *discrete smoothing variational spline with tangent conditions* converges to f. Suppose that we are given:

- a subset $\mathcal{D}$ of $(0, +\infty)$ with 0 as an accumulation point;
- for all $d \in \mathcal{D}$, two subsets A_1^d and A_2^d of respectively $N_1 = N_1(d)$ and $N_2 = N_2(d)$ distinct points of $\overline{\Omega}$;
- for all $d \in \mathcal{D}$ and any $a \in A_1^d$, let us consider the linear form defined on $C^0_F(\Omega'; \mathbb{R}^n)$ by
$$\phi^d_a v = \begin{cases} v(a) & \text{if } a \in A_1^d \backslash F,\\ v|_{R_i}(a) & \text{if } a \in A_1^d \cap R_i \cap F,\ 1 \le i \le I; \end{cases}$$
- for all $d \in \mathcal{D}$ and any $a \in A_2^d$, let Π^d_a be the operator defined in $C^1_F(\Omega'; \mathbb{R}^n)$ by
$$\Pi^d_a v = \begin{cases} \big(P_{S_a^\perp}\big(\frac{\partial v}{\partial x_j}(a)\big)\big)_{1 \le j \le p} & \text{if } a \in A_2^d \backslash F,\ 1 \le i \le I,\\ \big(P_{S_a^\perp}\big(\frac{\partial v|_{R_i}}{\partial x_j}(a)\big)\big)_{1 \le j \le p} & \text{if } a \in A_2^d \cap R_i \cap F,\ 1 \le i \le I, \end{cases}$$

where for any $a \in A_2^d$, $P_{S_a^\perp}$ is the operator projection onto $S_a^\perp$, being $S_a^\perp$ the orthogonal complement of the linear space $S_a = \mathrm{Im}\mathrm{D}f(a)$.

Finally, for any $d \in \mathcal{D}$ let

$$L^d v = \big(\phi_a^d v\big)_{a \in A_1^d} \quad \text{and} \quad \Pi^d v = \big(\Pi_a^d v\big)_{a \in A_2^d}.$$

We suppose that

$$\ker L^d \cap \tilde{P}_{m-1}(\Omega'; \mathbb{R}^n) = \{0\} \tag{4.2}$$

and that

$$\sup_{x \in \Omega'} \min_{a \in A_1^d} \langle x - a \rangle_{\mathbb{R}^p} = d. \tag{4.3}$$

Now, for each $d \in \mathcal{D}$, let $\tau = \tau(d) \geq 0$, $\varepsilon = \varepsilon(d) > 0$ and let $J_{\varepsilon\tau}^d$ be the functional defined in V_h as $J_{\varepsilon\tau}$ in (3.10) with L^d and Π^d instead of L and Π respectively. Finally, let $\sigma_{\varepsilon\tau}^{dh}$ be the *discrete smoothing variational spline with tangent conditions* in Ω' relative to A_1^d, A_2^d, f, τ and ε, which is the minimum of $J_{\varepsilon\tau}^d$ in V_h.

To prove the convergence of $\sigma_{\varepsilon\tau}^{dh}$ to f, under suitable hypotheses, we need the following results.

Proposition 4.1. *Let $B_0 = \{b_{01}, \dots, b_{0\triangle}\}$ be a $\tilde{P}_{m-1}(\Omega'; \mathbb{R}^n)$-unisolvent subset of points of R. Then, there exists $\eta > 0$ such that if $\mathcal{B}_\eta$ designs the set of $\triangle$-uplet $B = \{b_1, \dots, b_\triangle\}$ of points of Ω' satisfying the condition:*
for $j = 1, \dots, \triangle$ and $\langle b_j - b_{0j} \rangle_{\mathbb{R}^p} < \eta$, the application

$$[[v]]_{m,\Omega'}^B = \Big(\sum_{j=1}^{\triangle} \langle v(b_j) \rangle_{\mathbb{R}^n}^2 + |\ v\ |_{m,\Omega',\mathbb{R}^n}^2 \Big)^{1/2},$$

defined, for all $B \in \mathcal{B}_\eta$, is a norm on $H^m(\Omega'; \mathbb{R}^n)$, uniformly equivalent over $\mathcal{B}_\eta$ to the usual Sobolev norm $\| \cdot \|_{m,\Omega',\mathbb{R}^n}$.

The proof of this proposition is analogous to the proof of [1, proposition 6.2].

Now, we assume that the family $(X_h)_{h \in \mathcal{H}}$ is such that there exists a linear operator $\rho_h : L^2(\Omega'; \mathbb{R}^3) \to V_h$ satisfying

(i) For all $l = 0, \dots, m$, and all $y \in H^m(\Omega'; \mathbb{R}^3)$,

$$|y - \rho_h y|_{l,\Omega',\mathbb{R}^3} \leq C h^{m-l} |\vec{y}|_{m,\Omega',\mathbb{R}^3}; \tag{4.4}$$

(ii) For all $y \in H^m(\Omega'; \mathbb{R}^3)$,

$$\lim_{h \to 0} |y - \rho_h y|_{m,\Omega',\mathbb{R}^3} = 0\,.$$

Also assume that the cardinality of the subsets A_1^d and A_2^d satisfies

$$\max\{N_1(d), N_2(d)\} \leq C\, d^{-p}, \tag{4.5}$$

and that the family $(\mathcal{T}_h)_{h \in \mathcal{H}}$ satisfies the inverse assumptions (c.f. P. G. Ciarlet [3]):

$$\exists \nu > 0,\ \forall h \in \mathcal{H},\ \forall K \in \mathcal{T}_h,\ \frac{h}{h_K} \leq \nu. \tag{4.6}$$

Lemma 4.2. *Assume that (4.2), (4.3), (4.4), (4.5), (4.6) hold. Then, there exists a constant $C > 0$ such that for any $y \in H^m(\Omega'; \mathbb{R}^3)$, $d \in \mathcal{D}$ and $h \in \mathcal{H}$, one has*

$$\sum_{a\in A} \langle (\rho_h y - y)(a)\rangle^2_{\mathbb{R}^3} \le C h^{2m-2} |y|^2_{m,\Omega',\mathbb{R}^3}, \tag{4.7}$$

$$\sum_{b\in B^d} \langle D_i(\rho_h y - y)(b)\rangle^2_{\mathbb{R}^3} \le C \frac{h^{2m-4}}{d^2} |y|^2_{m,\Omega',\mathbb{R}^3},\ i = 1, 2\,. \tag{4.8}$$

Proof. Reasoning as in [1, Lemma 6.1], we deduce that there exists a constant $C > 0$ such that for any $y \in H^m(\Omega; \mathbb{R}^3)$, $d \in \mathcal{D}$ and $h \in \mathcal{H}$, and for any $K \in \mathcal{T}_h$, one has

$$\max_{u\in K} \langle y(u)\rangle^2_{\mathbb{R}^3} \le C h^{-2} \sum_{\ell=0}^{m} h^{2\ell} |y|^2_{\ell,K,\mathbb{R}^3}, \tag{4.9}$$

$$\max_{u\in K} \langle D_i y(u)\rangle^2_{\mathbb{R}^3} \le C h^{-2} \sum_{\ell=0}^{m-1} h^{2\ell} |y|^2_{\ell+1,K,\mathbb{R}^3},\ i = 1, 2. \tag{4.10}$$

Thus, taking $\rho_h y - y$ instead of y in (4.9), we deduce that

$$\begin{aligned}\sum_{a\in A} \langle (\rho_h y - y)(a)\rangle^2_{\mathbb{R}^3} &\le \sum_{K\in\mathcal{T}_h} \sum_{a\in A\cap K} \langle (\rho_h y - y)(a)\rangle^2_{\mathbb{R}^3}\\ &\le C h^{-2} N_1 \sum_{K\in\mathcal{T}_h} \sum_{\ell=0}^{m} h^{2\ell} |\rho_h y - y|^2_{\ell,K,\mathbb{R}^3}\\ &\le C h^{-2} N_1 \sum_{\ell=0}^{m} h^{2\ell} |\rho_h y - y|^2_{\ell,\Omega',\mathbb{R}^3}\,.\end{aligned}$$

Hence, from (4.4), we have

$$\begin{aligned}\sum_{a\in A} \langle (\rho_h y - y)(a\rangle^2_{\mathbb{R}^3} &\le C h^{-2} N_1 \sum_{\ell=0}^{m} h^{2\ell} h^{2m-2\ell} |y|^2_{m,\Omega',\mathbb{R}^3}\\ &\le C h^{-2} N_1 (m+1) h^{2m} |y|^2_{m,\Omega',\mathbb{R}^3},\end{aligned}$$

and we conclude that (4.7) holds. Analogously, taking $\rho_h y - y$ instead of y in (4.10) we deduce, for $i = 1, 2$, that

$$\sum_{b\in B^d} \langle D_i(\rho_h y - y)(b)\rangle^2_{\mathbb{R}^3} \le C h^{-2} N_2 \sum_{\ell=0}^{m-1} h^{2\ell} |\rho_h y - y|^2_{\ell+1,\Omega',\mathbb{R}^3}.$$

Hence, from (4.4), we have

$$\sum_{b\in B^d} \langle D_i(\rho_h y - y\rangle^2_{\mathbb{R}^3} \le C h^{-2} N_2 m h^{2m-2} |y|^2_{m,\Omega',\mathbb{R}^3},$$

and, from (4.5), we conclude that (4.8) holds. □

Now we state the main result.

Theorem 4.3. *Assume that* (4.2)–(4.6) *hold, and that*

$$\varepsilon = o(d^{-p}),\ \ d \to 0, \tag{4.11}$$

$$\frac{\tau\, h^{2m-4}}{d^p\, \varepsilon} = o(1),\ \ d \to 0, \tag{4.12}$$

$$\frac{h^{2m}}{d^p\, \varepsilon} = o(1),\ \ d \to 0, \tag{4.13}$$

Then, one has

$$\lim_{d\to 0} \|\sigma_{\varepsilon\tau}^{dh} - f\|_{m,\Omega',\mathbb{R}^n} = 0. \tag{4.14}$$

Proof. Step 1. For all $d \in \mathcal{D}$, from Theorem 3.3 we have

$$J_{\varepsilon\tau}^d(\sigma_{\varepsilon\tau}^{dh}) \le J_{\varepsilon\tau}^d(\rho_h f)$$

where, for each $h \in \mathcal{H}$, ρ_h is the operator given in (4.4), which means that

$$|\sigma_{\varepsilon\tau}^{dh}|_{m,\Omega',\mathbb{R}^n}^2 \le \frac{1}{\varepsilon}\langle L^d(\rho_h f - f)\rangle_{N_1,n}^2 + \frac{\tau}{\varepsilon}\langle \Pi^d(\rho_h f)\rangle_{N_2,pn}^2 + |\rho_h f|_{m,\Omega',\mathbb{R}^n}^2.$$

From (4.4) and Lemma 4.2, we obtain

$$\langle L^d(\rho_h f - f)\rangle_{N_1,n}^2 \le C N_1 h^{2m} |f|_{m,\Omega',\mathbb{R}^n}^2$$

and taking into account

$$\langle \Pi^d(\rho_h f)\rangle_{N_2,pn}^2 \le C p N_2 h^{2m-4} |f|_{m,\Omega',\mathbb{R}^n}^2,$$

$$|\rho_h f|_{m,\Omega',\mathbb{R}^n}^2 = \mathrm{o}(1) + |f|_{m,\Omega',\mathbb{R}^n}^2,\ d \to 0\,,$$

we deduce from (4.5) and Lemma 4.2 that there exist $C_1 > 0$ and $C_2 > 0$ such that

$$|\sigma_{\varepsilon\tau}^{dh}|_{m,\Omega',\mathbb{R}^n}^2 \le \big(\frac{C_1 h^{2m}}{d^p \varepsilon} + \frac{C_2 h^{2m-4}\tau}{d^p \varepsilon} + 1\big)|f|_{m,\Omega',\mathbb{R}^n}^2 + \mathrm{o}(1),\ d \to 0, \tag{4.15}$$

$$\langle L^d(\sigma_{\varepsilon\tau}^{dh} - f)\rangle_{N_1,n}^2 = \mathrm{o}(\varepsilon), \quad \text{as } d \to 0. \tag{4.16}$$

Let $B_0 = \{b_{01}, \dots, b_{0\Delta}\}$ be a $\tilde{P}_{m-1}(\Omega')$-unisolvent subset of points of R and let η be the constant of the Proposition 4.1. Obviously, there exists $\eta' \in (0, \eta]$ such that

$$\overline{B}(b_{0j}, \eta') \subset \overline{R} \quad \text{for } j = 1, \dots, \Delta\,.$$

From (4.3), for all $d \in \mathcal{D}$, $d < \eta'$, $j = 1, \dots, \Delta$,

$$\overline{B}(b_{0j}, \eta' - d) \subset \bigcup_{a \in A_1^d \cap \overline{B}(b_{0j},\eta')} \overline{B}(a, d)\,.$$

If $\mathcal{N}_j = \mathrm{card}(A_1^d \cap \overline{B}(b_{0j}, \eta'))$, it follows: for all $d \in \mathcal{D}$, $d < \eta'$, $j = 1, \dots, \Delta$,

$$(\eta' - d)^p \le \mathcal{N}_j d^p\,.$$

Consequently, for any $d_0 \in (0, \eta')$, all $d \in \mathcal{D}$, $d \le d_0$, $j = 1, \dots, \Delta$,

$$\mathcal{N}_j \ge (\eta' - d_0)^p d^{-p}\,. \tag{4.17}$$

Now, from (4.11) and (4.16) it follows that for $j = 1, \dots, \Delta$,

$$\sum_{a \in A_1^d \cap \overline{B}(b_{0j},\eta')} \langle(\sigma_{\varepsilon\tau}^{dh} - f)(a)\rangle_{\mathbb{R}^n}^2 = o(d^{-p}), \quad \text{as } d \to 0\,. \tag{4.18}$$

If a_j^d is a point of $A_1^d \cap \overline{B}(b_{0j}, \eta')$ such that

$$\langle(\sigma_{\varepsilon\tau}^{dh} - f)(a_j^d)\rangle_{\mathbb{R}^n} = \min_{a \in A_1^d \cap \overline{B}(b_{0j},\eta')} \langle(\sigma_{\varepsilon\tau}^{dh} - f)(a)\rangle_{\mathbb{R}^n}\,,$$

it follows from (4.17) and (4.18) that for $j = 1, \dots, \Delta$,

$$\langle (\sigma^{dh}_{\varepsilon\tau} - f)(a^d_j)\rangle_{\mathbb{R}^n} = o(1), \quad \text{as } d \to 0\,. \tag{4.19}$$

We denote by B^d the set $\{a^d_1, \dots, a^d_\Delta\}$. Applying the Proposition 4.1 with $B = B^d$, for d sufficiently close to 0, it results from (4.12), (4.13), (4.15) and (4.19) that There exists $C > 0$, $\alpha > 0$, such that for all $d \in \mathcal{D}$, $d \le \alpha$ we have

$$\|\sigma^{dh}_{\varepsilon\tau}\|_{m,\Omega',\mathbb{R}^n} \le C\,.$$

This implies that the family $(\sigma^{dh}_{\varepsilon\tau})_{d\in\mathcal{D},\, d\le\alpha}$ is bounded in $H^m(\Omega';\mathbb{R}^n)$. Then, there exists a sequence $(\sigma^{d_l h_l}_{\varepsilon_l\tau_l})_{l\in\mathbb{N}}$, extracted from such family, with $\lim_{l\to+\infty} d_l = 0$, $\lim_{l\to+\infty} h_l = 0$, $\varepsilon_l = \varepsilon(d_l)$, $\tau_l = \tau(d_l)$, $\varepsilon_l = o(d_l^{-p})$, $\frac{\tau_l\, h_l^{2m-4}}{d_l^p\, \varepsilon_l} = \mathrm{o}(1)$, $\frac{h_l^{2m}}{d_l^p\, \varepsilon_l} = \mathrm{o}(1)$, as $l \to +\infty$, and an element $f^* \in H^m(\Omega';\mathbb{R}^n)$ such that

$$f^* \text{ converges weakly to } \sigma^{d_l h_l}_{\varepsilon_l\tau_l} \quad \text{in} H^m(\Omega';\mathbb{R}^n) \text{ as } l \to +\infty\,. \tag{4.20}$$

Step 2. Arguing by contradiction, it is easy to prove that $f^* = f$.
Step 3. From (4.20) and taking into account that $f^* = f$ and $H^m(\Omega';\mathbb{R}^n)$ is compactly injected in $H^{m-1}(\Omega';\mathbb{R}^n)$ we have:

$$f = \lim_{l\to+\infty} \sigma^{d_l h_l}_{\varepsilon_l\tau_l} \quad \text{in} \quad H^{m-1}(\Omega';\mathbb{R}^n)\,. \tag{4.21}$$

Consequently,

$$\lim_{l\to+\infty} ((\sigma^{d_l h_l}_{\varepsilon_l\tau_l}, f))_{m-1,\Omega',\mathbb{R}^n} = \|f\|^2_{m-1,\Omega',\mathbb{R}^n}\,.$$

Using again (4.20) and that $f = f^*$, we obtain

$$\begin{aligned}\lim_{l\to+\infty} (\sigma^{d_l h_l}_{\varepsilon_l\tau_l}, f)_{m,\Omega',\mathbb{R}^n} &= \lim_{l\to+\infty} \Big(((\sigma^{d_l h_l}_{\varepsilon_l\tau_l}, f))_{m,\Omega',\mathbb{R}^n} - ((\sigma^{d_l h_l}_{\varepsilon_l\tau_l}, f))_{m-1,\Omega',\mathbb{R}^n} \Big)\\ &= |f|^2_{m,\Omega',\mathbb{R}^n}\,.\end{aligned} \tag{4.22}$$

Since, for all $l \in \mathbb{N}$,

$$|\sigma^{d_l h_l}_{\varepsilon_l\tau_l} - f|^2_{m,\Omega',\mathbb{R}^n} = |\sigma^{d_l h_l}_{\varepsilon_l\tau_l}|^2_{m,\Omega',\mathbb{R}^n} + |f|^2_{m,\Omega',\mathbb{R}^n} - 2(\sigma^{d_l h_l}_{\varepsilon_l\tau_l}, f)_{m,\Omega',\mathbb{R}^n}\,.$$

From (4.15) and (4.22) we deduce

$$\lim_{l\to+\infty} |\sigma^{d_l h_l}_{\varepsilon_l\tau_l} - f|_{m,\Omega',\mathbb{R}^n} = 0\,,$$

which, together with (4.21), imply

$$\lim_{l\to+\infty} \|\sigma^{d_l h_l}_{\varepsilon_l\tau_l} - f\|_{m,\Omega',\mathbb{R}^n} = 0\,.$$

Step 4. To complete this proof we will argue by contradiction. Suppose that (4.14) does'nt hold. Then, there exist a real number $\mu > 0$ and three sequences $(d_{l'})_{l'\in\mathbb{N}}$, $(h_{l'})_{l'\in\mathbb{N}}$, $(\varepsilon_{l'})_{l'\in\mathbb{N}}$, and $(\tau_{l'})_{l'\in\mathbb{N}}$, with

$$\lim_{l'\to+\infty} d_{l'} = 0, \quad h_{l'} = h(d_{l'}), \quad \varepsilon_{l'} = \varepsilon(d_{l'}),$$

$$\tau_{l'} = \tau(d_{l'}), \quad \varepsilon_{l'} = o(d_{l'}^{-p}),$$

$$\frac{\tau_{l'} h_{l'}^{2m-4}}{d_{l'}^p \varepsilon_{l'}} = o(1), \quad \frac{h_{l'}^{2m}}{d_{l'}^p \varepsilon_{\ l'}} = o(1),$$

as $l' \to +\infty$, such that for all $l' \in \mathbb{N}$,

$$\|\sigma^{d_{l'} h_{l'}}_{\varepsilon_{l'}\tau_{l'}} - f\|_{m,\Omega',\mathbb{R}^n} \ge \mu\,. \tag{4.23}$$

Now, the sequence $(\sigma^{d_{l'} h_{l'}}_{\varepsilon_{l'} \tau_{l'}})_{l' \in \mathbb{N}}$ is bounded in $H^m(\Omega'; \mathbb{R}^n)$. Then, the reasoning of Steps 1–3 shows that there exists a subsequence convergent to f, which is a contradiction with (4.23). □

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Abdelouahed Kouibia
Centre d'Etudes Universitaire de Taza, Taza, Maroc
E-mail address: kouibia@ugr.es

Miguel Pasadas
Departamento de Matemática Aplicada, ETSI Caminos, C. y P., Universidad de Granada, Severo Ochoa s/n, E-18071 Granada, Spain
E-mail address: mpasadas@ugr.es Fax (34) 958 243131