# APPROXIMATION OF DYNAMIC AND QUASI-STATIC EVOLUTION PROBLEMS IN ELASTO-PLASTICITY BY CAP MODELS

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Abstract. This work is devoted to the analysis of elasto-plasticity models arising in soil mechanics. Contrary to the typical models mainly used for metals, it is required here to take into account plastic dilatancy due to the sensitivity of granular materials to hydrostatic pressure. The yield criterion thus depends on the mean stress and the elasticity domain is unbounded and not invariant in the direction of hydrostatic matrices. In the mechanical literature, so-called *cap models* have been introduced, where the elasticity domain is cut in the direction of hydrostatic stresses by means of a strain-hardening yield surface, called a cap. The purpose of this article is to study the well-posedness of plasticity models with unbounded elasticity sets in dynamical and quasi-static regimes. An asymptotic analysis as the cap is moved to infinity is also performed, which enables one to recover solutions to the uncapped model of perfect elasto-plasticity.

## 1. Introduction

Models of elasto-plasticity have the capacity to predict the appearance of permanent deformations in a material when a critical stress is reached. From a microscopic point of view, these so-called plastic deformations are the result of atomic defects due to intercrystalline slips inside a lattice, called dislocations. It is experimentally observed that plastic flows occur on very thin zones called slip bands, on which there is strain localization: these zones are macroscopically interpreted as discontinuity surfaces of the displacement. For this reason, it has turned out to be convenient to approximate these models by regularized ones, *e.g.*, of viscosity or strain-hardening type. We refer to the monographs [21, 24] for an exhaustive presentation of elasto-plasticity models.

The mathematical models of plasticity are highly nonlinear and this makes difficult the search of solutions. However, the variational principles of Hodge-Prager for the stress rate, and of Greenberg for the velocity, enable one to formulate the problem in a more tractable way. In particular, the development of convex analysis and the interpretation of the model of perfect elasto-plasticity as the sweeping process of a moving convex set in [29] have permitted to prove existence and uniqueness of the stress history. Other mathematical results have been obtained in [17] by means of constructive theory of partial differential equations including Galerkin approximations, regularization, and penalization. The existence and uniqueness problem for the stress has been solved in a quite satisfactory way, while the evolution problem for the velocity (or the displacement) encountered additional difficulties connected to the regularity of the strain tensor. This problem was avoided in [22] by means of a weak formulation, which was however too weak to obtain full information on the strain. In the footsteps of these works, the quasi-static case was studied in [36, 37] and the dynamical case in [3, 25] by means of different types of visco-plastic regularizations (see also [2, 40]). The difficulty was connected to the definition of the correct functional setting for kinematically admissible displacement fields which can exhibit discontinuities. It has been overcome by the introduction in [27, 38] of the space $BD$ of functions of bounded deformation (see [39] for

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a comprehensive treatment on that subject). More recently, the quasi-static case was revisited in [10] within the general framework of variational evolutions of rate independent processes (see [26]).

To formulate more precisely the problem, let us consider a bounded open set $\Omega \subset \mathbb{R}^n$ (in dimension $n = 2$ or 3) which stands for the reference configuration of an elasto-plastic body. We will work in the framework of small strain elasto-plasticity where the natural kinematic variable is the displacement field $u : \Omega \times [0, T] \to \mathbb{R}^n$ (or the velocity field $v := \dot{u}$). We denote by $Eu := (Du + Du^T)/2$ the linearized strain tensor which takes its values in the set $\mathbb{M}^{n\times n}_{sym}$ of symmetric $n \times n$ matrices. In small strain elasto-plasticity, $Eu$ decomposes additively in the following form:

$$Eu = e + p,$$

where $e : \Omega \times [0, T] \to \mathbb{M}^{n\times n}_{sym}$ is the elastic strain and $p : \Omega \times [0, T] \to \mathbb{M}^{n\times n}_{sym}$ the plastic strain. The elastic strain is related to the stress tensor $\sigma : \Omega \times [0, T] \to \mathbb{M}^{n\times n}_{sym}$ by means of Hooke's law $\sigma := \mathbb{C}e$, where $\mathbb{C}$ is the symmetric fourth order elasticity tensor. In a dynamical framework and in the presence of an external body load $f : \Omega \times [0, T] \to \mathbb{R}^n$, the equation of motion writes

$$\ddot{u} - \mathrm{div}\sigma = f \quad \text{in } \Omega \times [0, T].$$

Plasticity is characterized by the existence of a yield zone beyond which permanent strains appear. The stress tensor is indeed constrained to belong to a fixed closed and convex subset $K$ of $\mathbb{M}^{n\times n}_{sym}$:

$$\sigma \in K.$$

If $\sigma$ lies inside the interior of $K$, the material behaves elastically, so that unloading will bring back the body into its initial configuration ($p = 0$). On the other hand, if $\sigma$ reaches the boundary of $K$ (called the yield surface), a plastic flow may develop, so that, after unloading, there will remain a non-trivial permanent plastic strain $p$. Its evolution is described by means of the flow rule and is expressed with the Prandtl-Reuss law

$$\dot{p} \in N_K(\sigma),$$

where $N_K(\sigma)$ is the normal cone to $K$ at $\sigma$. From the theory of convex analysis, $N_K(\sigma) = \partial I_K(\sigma)$, *i.e.*, the subdifferential of the indicator function $I_K$ of the set $K$ ($I_K(\sigma) = 0$ if $\sigma \in K$, while $I_K(\sigma) = +\infty$ otherwise). Hence, from convex duality, the flow rule can be equivalently written as

$$\sigma : \dot{p} = \max_{\tau \in K} \tau : \dot{p} =: H(\dot{p}), \tag{1.1}$$

where $H : \mathbb{M}^{n\times n}_{sym} \to \mathbb{R}$ is the support function of $K$. This last formulation (1.1) is nothing but Hill's principle of maximum plastic work, and $H(\dot{p})$ denotes the plastic dissipation.

In general, the elasticity domain $K$ is expressed by means of a yield function $F : \mathbb{M}^{n\times n}_{sym} \to \mathbb{R}$ as

$$K := \{\sigma \in \mathbb{M}^{n\times n}_{sym} : F(\sigma) \le 0\}.$$

In this paper we are interested in yield functions $F$ of the form

$$F(\sigma) := \alpha\sigma_m + \kappa(\sigma_D) - k, \tag{1.2}$$

where $\kappa : \mathbb{M}^{n\times n}_D \to [0, +\infty)$ is typically a convex and positively 1-homogeneous function with $\kappa(0) = 0$, and $\alpha > 0$ and $k > 0$ are positive constants related to the cohesion and the coefficient of internal friction of the material, respectively. Here, $\mathbb{M}^{n\times n}_D = \{\sigma \in \mathbb{M}^{n\times n}_{sym} : \mathrm{tr}\,\sigma = 0\}$ is the space of all deviatoric and symmetric $n \times n$ matrices,

$$\sigma_D := \sigma - \frac{\mathrm{tr}\sigma}{n}\,\mathrm{Id} \in \mathbb{M}^{n\times n}_D \quad \text{and} \quad \sigma_m := \frac{\mathrm{tr}\sigma}{n} \in \mathbb{R}$$

are respectively the deviatoric and spherical part of $\sigma$, so that $\sigma = \sigma_D + \sigma_m\,\mathrm{Id}$. Thus, the set $K$ is actually a closed and convex cone with vertex on the axis of hydrostatic stresses. Several well known models are recovered here. For instance, the *Drucker-Prager model* corresponds to

$$\kappa(\sigma_D) = |\sigma_D|,$$

while the *Mohr-Coulomb model* corresponds to

$$\kappa(\sigma_D) = \max_{i,j}\{(\sigma_D)_i - (\sigma_D)_j\},$$

where $(\sigma_D)_i$, $i = 1, \dots, n$, are the eigenvalues of $\sigma_D$.

Note that if $\alpha = 0$, the yield function $F$ does not depend on the mean stress $\sigma_m$ and the set $K$ is invariant in the direction of hydrostatic stresses. This is usually the case for most of metals and alloys for which the influence of mean stress on yielding is generally negligible. In particular, the Drucker-Prager criterion reduces to the Von Mises criterion, while the Mohr-Coulomb criterion reduces to that of Tresca. This kind of models, called of Prandtl-Reuss type, have been studied in [10] in the quasi-static setting, and in [3, 25] in the dynamic one. A typical feature of them is that, since the plastic strain is a deviatoric measure (recall that the displacement has bounded deformation), materials obeying this kind of laws do not develop plastic (or permanent) volumetric changes, and the displacement field admits only tangential discontinuities.

Here the function $F$ does depend on the mean stress $\sigma_m$: in particular, the set $K$ is not invariant and actually unbounded in the direction of hydrostatic matrices. It turns out that such yield criterions are necessary when it is desired to apply plasticity theory to soils, rocks, and concrete (see [16, 34]). Indeed, the essential property of such materials is that they are composed of many small particles. Consequently, permanent deformations and plastic slips occur when these particles slide over one another, and thus, as in ductile metals, failure occur primarily in shear. However, a strong difference with metals is that the shear strength is strongly influenced by the compressive normal stress acting on the shear plane, and therefore by the hydrostatic pressure. The physical reason for this phenomenon is connected to the fact that the void between the particles is composed of water and air. When the material is loaded in compression, the void ratio decreases in an irreversible way, leading to a permanent volume change. Therefore, the intergranular interaction is governed by a Coulomb type law of friction, where the shear and normal stresses achieve a critical combination on the shearing plane, depending on the angle of internal friction and the cohesion of the grains. In conclusion, the sensitivity on hydrostatic pressure as well as plastic dilatancy are typical features of this kind of granular materials.

All these models are called perfectly plastic, referring to the fact that the yield surface is fixed and does not move during the evolution. For more sophisticated models with a work-hardening material, the yield function may depend on an additional internal variable describing the position of the yield surface. Typical hardening rules are the isotropic hardening, representing a global uniform expansion of the elastic domain in all directions with no change in shape, and the kinematic hardening, representing a translation of the yield surface in stress space by shifting its reference point. We mention the papers [5, 7, 8], where the relation of isotropic and/or kinematic hardening models with the Prandtl-Reuss model of perfect plasticity is studied, both in a quasi-static and dynamic setting.

We are interested in studying the following model of dynamical evolution in perfect elasto-plasticity. Let $f : \Omega \times [0, T] \to \mathbb{R}^n$ be a given body force and $w : \partial\Omega \times [0, T] \to \mathbb{R}^n$ be a boundary displacement. We consider an initial datum $(u_0, e_0, p_0) : \Omega \to \mathbb{R}^n \times \mathbb{M}^{n\times n}_{sym} \times \mathbb{M}^{n\times n}_{sym}$ satisfying $Eu_0 = e_0 + p_0$ in $\Omega$, $u_0 = w(0)$ on $\partial\Omega$, and $\sigma_0 := \mathbb{C}e_0 \in K$, and an initial velocity $v_0 : \Omega \to \mathbb{R}^n$ satisfying $v_0 = \dot{w}(0)$ on $\partial\Omega$. We look for a triplet $(u, e, p) : \Omega \times [0, T] \to \mathbb{R}^n \times \mathbb{M}^{n\times n}_{sym} \times \mathbb{M}^{n\times n}_{sym}$ with the properties

$$\begin{cases} Eu = e + p \text{ in } \Omega \times [0, T], \quad u = w \text{ on } \partial\Omega \times [0, T], \\ \sigma = \mathbb{C}e, \quad \sigma \in K \text{ in } \Omega \times [0, T], \\ \ddot{u} - \operatorname{div}\sigma = f \text{ in } \Omega \times [0, T], \\ \dot{p} \in N_K(\sigma) \text{ in } \Omega \times [0, T], \\ (u(0), e(0), p(0)) = (u_0, e_0, p_0), \quad \dot{u}(0) = v_0 \text{ in } \Omega. \end{cases} \tag{1.3}$$

For some geomaterials, it turns out that this kind of pressure-dependent models overestimate the yield stress and inadequately predict plastic dilatancy, which exceeds what is observed experimentally. In order to remedy these defects, a modified model has been introduced in [15], where the cone $K$ is cut in the direction of hydrostatic stresses through the use of a strain-hardening yield surface or cap. In [31, 32] a Drucker-Prager cap model with a hardening law on the cap surface has been studied: if the stress reaches the cap, it is pushed forward in such a way that, if the stress gets the same position at some subsequent time, it will then be an interior point of the new elasticity domain. In our framework this corresponds to introduce an auxiliary variable $\xi$, related to the position of the cap, and to consider the yield functions $F_i : \mathbb{M}^{n\times n}_{sym} \times \mathbb{R} \to \mathbb{R}$, for $i = 1, 2, 3$, defined by

$$\begin{cases} F_1(\sigma,\xi) := F(\sigma), \\ F_2(\sigma,\xi) := \lambda\xi - \sigma_m, \\ F_3(\sigma,\xi) := \xi, \end{cases}$$

where $\lambda \geq 1$. These functions are clearly convex, and we define the closed convex set $K_\lambda \subset \mathbb{M}^{n\times n}_{sym} \times \mathbb{R}$ by

$$K_\lambda := \{(\sigma,\xi) \in \mathbb{M}^{n\times n}_{sym} \times \mathbb{R} : F_i(\sigma,\xi) \leq 0 \text{ for } i = 1,2,3\}.$$

The hardening cap model reads as follows: find a quadruplet $(u, e, p, \xi) : \Omega \times [0, T] \to \mathbb{R}^n \times \mathbb{M}^{n\times n}_{sym} \times \mathbb{M}^{n\times n}_{sym} \times \mathbb{R}$ satisfying

$$\begin{cases} Eu = e + p \text{ in } \Omega\times[0,T], \quad u = w \text{ on } \partial\Omega\times[0,T], \\ \sigma = \mathbb{C}e,\ \xi = -z, \quad (\sigma,\xi) \in K_\lambda \text{ in } \Omega\times[0,T], \\ \ddot{u} - \mathrm{div}\sigma = f \text{ in } \Omega\times[0,T], \\ (\dot{p},\dot{z}) \in N_{K_\lambda}(\sigma,\xi) \text{ in } \Omega\times[0,T], \\ (u(0),e(0),p(0),\xi(0)) = (u_0,e_0,p_0,\xi_0), \quad \dot{u}(0) = v_0 \text{ in } \Omega. \end{cases} \tag{1.4}$$

Following [25, 3], the resolution of this dynamical problem can be performed by means of a vanishing viscosity method (see Theorems 3.1 and 4.1). In Theorem 5.1 we then prove existence and uniqueness of solutions for the uncapped dynamical problem (1.3), by showing convergence of the solution of (1.4) to a solution of (1.3), as $\lambda \to \infty$. In these results the particular structure (1.2) of the yield function $F$ is not relevant and we can assume $K$ to be any closed convex set containing the origin as an interior point. In particular, contrary to [3, 10], $K$ does not need to be bounded along any given direction. Concerning our choice of the cap, it is clearly motivated by the structure (1.2). We actually prove that the convergence result continues to hold if $(K_\lambda)$ is replaced by any increasing family $(\tilde{K}_\lambda)$ of closed convex sets satisfying the inclusion $K_\lambda \subset \tilde{K}_\lambda$ for every $\lambda$. Extension to other kinds of cap is not straightforward.

When the evolution is assumed to be slow, inertia terms (as thus the acceleration $\ddot{u}$) can be neglected, and the equations of motion in (1.3) and (1.4) become an equation of quasi-static equilibrium

$$-\mathrm{div}\sigma = f \quad \text{in } \Omega\times[0,T].$$

From a mathematical point of view, it may seem easier to deal with the quasi-static model instead of the dynamical one. Surprisingly, this observation turns out to be wrong. This is related to regularity issues of the stress and the displacement. Indeed it is known that the displacement can be discontinuous, and that the right functional setting to treat this problem is the space $BD(\Omega)$ of functions of bounded deformation. On the other hand, since the set $K$ is bounded in no direction, the best integrability one can hope for the stress is $\sigma \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$. Therefore, the flow rule (or equivalently Hills's principle of maximum plastic work) has to be written in a measure theoretic sense. Indeed, in (1.1) it is possible to define $H(\dot{p})$ using the theory of convex functions of a measure [20, 12, 13], while, according to [23, 19], it is also possible to give a sense to the stress/plastic strain duality product $\sigma:\dot p$ as a distribution (and even as a measure) by means of an integration by parts formula (see Definition 2.4 and formula (2.9) below). However, this definition makes sense provided $\sigma$ and $\dot u$ have enough space integrability. Indeed, taking smooth enough data, the natural regularities are either

$$\sigma \in L^n(\Omega;\mathbb{M}^{n\times n}_{sym}), \quad \dot u \in L^{n/(n-1)}(\Omega;\mathbb{R}^n),$$

or

$$\sigma \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym}), \quad \dot u \in L^2(\Omega;\mathbb{R}^n). \tag{1.5}$$

In the dynamic problem the control of the kinetic energy gives us a natural $L^2(\Omega;\mathbb{R}^n)$ bound for the velocity $\dot u$, so that the conditions in (1.5) are always fulfilled and the stress/strain duality is well defined in any dimension. The quasi-static case is unfortunately less manageable. Indeed, except for planar elasto-plasticity ($n = 2$), where the previous alternatives are clearly equivalent, in higher dimension we only have $\sigma \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ and $\dot u \in L^{n/(n-1)}(\Omega;\mathbb{R}^n)$ by Sobolev embedding. Consequently, higher regularity results would be needed, either for the stress or the velocity. To the best of our knowledge, such results are not available in the literature. However, we can still give a meaning to the flow rule in the quasi-static setting by writing it in terms of an energy equality, which reduces to the usual flow rule when the solutions are smooth enough. Existence of solutions in this sense is proved in Theorems 6.3 and 6.5.

This obstruction on the dimension was already present in related works on the subject. In [30] the authors study a Hencky plasticity problem with a Mohr-Coulomb yield function[1] and consider a formulation within the framework of a minimax problem. As usual in elasto-plasticity, the associated Lagrangian is defined on a non-reflexive Banach space, so that existence of saddle points is not ensured. Consequently, the Lagrangian needs to be relaxed and this turns out to be possible only if there exists a statically and plastically admissible stress $\sigma$ in $L^n(\Omega;\mathbb{M}^{n\times n}_{sym})$ (see [30, Lemma 3.2]). In the two-dimensional case, this condition is clearly satisfied provided the intersection between statically and plastically admissible stresses is not empty (since the stress is always at least squared integrable), while in the three-dimensional case the condition is in general not fulfilled. In the footsteps of this work, in [35] the author continues the study of the previous Mohr-Coulomb model by deriving a $H^1_{\rm loc}(\Omega;\mathbb{M}^{n\times n}_{sym})$ regularity property for the stress. Note that such estimates are standard for yield functions independent of the mean stress (see [4, 14]). In order to get such a regularity property in the Mohr-Coulomb case, the author employs a visco-plastic regularization of the constitutive law, similar to that we used in Section 3. He shows that the $H^1_{\rm loc}(\Omega;\mathbb{M}^{n\times n}_{sym})$ norm of the visco-plastic stress can be bounded in terms of the $L^2(\Omega;\mathbb{R}^n)$ norm of the visco-plastic displacement (see formula (3.33) in [35]). Unfortunately, this estimate is uniform (with respect to the viscosity parameter) only in dimension $n = 2$.

Let us now comment on our choice of boundary conditions: we only impose Dirichlet boundary conditions, which correspond to a hard device applied to the whole boundary. The case of a Neumann condition (even on a portion of the boundary) seems to be difficult to carry out, and the reason is again connected to regularity issues. Indeed, here $\sigma \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ with $\mathrm{div}\sigma \in L^2(\Omega;\mathbb{R}^n)$ (respectively, $\mathrm{div}\sigma \in L^n(\Omega;\mathbb{R}^n)$ in the quasi-static case) so that the normal stress $\sigma\nu$ is *a priori* defined as an element of $H^{-1/2}(\partial\Omega;\mathbb{R}^n)$. On the other hand, a trace theorem in $BD(\Omega)$ asserts that kinematically admissible displacements $u$ have a trace (still denoted by $u$) in $L^1(\partial\Omega;\mathbb{R}^n)$. Consequently, the displacement and the stress are not in duality on the boundary. In [10] and [3] this problem was avoided owing to a result of [23]: since the set $K$ is bounded in the direction of deviatoric matrices, only the tangential part of the normal stress is relevant on the boundary, which turns out to belong to $L^\infty(\partial\Omega;\mathbb{R}^n)$. Consequently, in that case, (the tangential part of) $\sigma\nu$

[1]In our terminology, the model studied in [30] actually corresponds to a Drucker-Prager model.

and $u$ are in duality on the boundary and this makes possible to take into account traction loads on a portion of $\partial\Omega$.

As a consequence, the only forces applied to our system are body loads, and, as usual in plasticity, we must ensure that the space of statically and plastically admissible stresses is not empty. In the quasi-static case, as observed in [29, 22], a stronger hypothesis is needed, namely a safe-load condition: the forces must derive from a potential $\chi$ well contained in the set $K$ (see (6.4)). This safety condition, which ensures that the body is not in a free flow, is necessary, from a mathematical point of view, in order to obtain an *a priori* estimate on the plastic strain rate. In the dynamical case we observe that this safe-load condition is no longer necessary thanks to the presence of the kinetic energy which is of higher order than the work of external forces.

The paper is organized as follows. In Section 2 we collect the main notation and introduce the different energies involved in our model; in particular, we define the dissipation functionals as convex functionals of measures and in terms of the stress/strain duality. In Section 3 we prove existence and uniqueness of solutions to a visco-plastic regularization of the dynamical cap model, similar to that of [25]. Our approach uses an implicit finite difference approximation of the underlying hyperbolic system. In Section 4 we perform a vanishing viscosity analysis in order to get existence and uniqueness of solutions of the dynamical elasto-plastic cap model. Section 5 is devoted to the asymptotic analysis of the previous cap model as the position of the cap is sent to infinity. In particular, we show existence and uniqueness of solutions to the dynamical elasto-plastic uncapped model. Eventually, in Section 6 we review the quasi-static case: in the framework of planar elasto-plasticity, we show existence of solutions to the quasi-static cap model, as well as its convergence when the cap is sent to infinity. In higher dimension existence is proved for a weaker formulation of the problem, where the flow rule is replaced by an energy equality.

## 2. The mathematical setting

Throughout the paper, $\Omega$ is a bounded connected open set in $\mathbb{R}^n$ with Lipschitz boundary. The Lebesgue measure in $\mathbb{R}^n$ and the $(n-1)$-dimensional Hausdorff measure are denoted by $\mathcal{L}^n$ and $\mathcal{H}^{n-1}$, respectively.

We use standard notation for Lebesgue and Sobolev spaces. In particular, for $1 \leq p \leq \infty$, the $L^p(\Omega)$-norms of the various quantities are denoted by $\|\cdot\|_p$.

The notation $\odot$ stands for the symmetrized tensor product between vectors in $\mathbb{R}^n$.

Given a locally compact set $E \subset \mathbb{R}^n$ and a Euclidean space $X$, we denote by $\mathcal{M}(E;X)$ (or simply $\mathcal{M}(E)$ if $X = \mathbb{R}$) the space of bounded Radon measures on $E$ with values in $X$, endowed with the norm $\|\mu\|_1 := |\mu|(E)$, where $|\mu| \in \mathcal{M}(E)$ is the total variation of the measure $\mu$. The Riesz Representation Theorem ensures that $\mathcal{M}(E;X)$ can be identified with the dual of $\mathcal{C}_0(E;X)$, the space of continuous functions $\varphi : E \to X$ vanishing on the boundary of $E$, *i.e.*, such that $\{|\varphi| \geq \varepsilon\}$ is compact for any $\varepsilon > 0$. For $\mu \in \mathcal{M}(E;X)$ we consider the Lebesgue decomposition $\mu = \mu^a + \mu^s$, where $\mu^a$ is absolutely continuous and $\mu^s$ is singular with respect to the Lebesgue measure $\mathcal{L}^n$. Moreover, if $\nu$ is a non-negative Radon measure over $E$, we denote by $\frac{d\mu}{d\nu}$ the Radon-Nikodym derivative of $\mu$ with respect to $\nu$.

Finally, $BD(\Omega)$ stands for the space of functions of bounded deformation in $\Omega$, *i.e.*, $u \in BD(\Omega)$ if $u \in L^1(\Omega;\mathbb{R}^n)$ and $Eu \in \mathcal{M}(\Omega;\mathbb{M}^{n\times n}_{sym})$, where $Eu := (Du + Du^T)/2$ and $Du$ is the distributional derivative of $u$. We refer to [39] for general properties of this space.

### 2.1. **The elastic energy.**

Let $\mathbb{C}$ be a fourth order tensor satisfying the usual symmetry conditions

$$\mathbb{C}_{ijkl} = \mathbb{C}_{klij} = \mathbb{C}_{jikl} \quad \text{for all } i,j,k,l \in \{1,\dots,n\}.$$

We assume that there exist constants $0 < \alpha_{\mathbb{C}} \leq \beta_{\mathbb{C}} < \infty$ such that

$$\alpha_{\mathbb{C}}|e|^2 \leq \mathbb{C}e : e \leq \beta_{\mathbb{C}}|e|^2 \quad \text{for all } e \in \mathbb{M}^{n\times n}_{sym}. \tag{2.1}$$

We define the elastic energy, for all $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, by

$$\mathcal{Q}(e) := \frac{1}{2}\int_\Omega \mathbb{C}e(x) : e(x)\,dx.$$

2.2. **The dissipation energies.** Let $K$ be a closed convex set in $\mathbb{M}^{n\times n}_{sym}$ satisfying

$$\{\sigma \in \mathbb{M}^{n\times n}_{sym} : |\sigma| \le \alpha_H\} \subset K \tag{2.2}$$

for some $\alpha_H > 0$.

Let $(K_\lambda)_{\lambda\ge 1}$ be a family of closed convex sets in $\mathbb{M}^{n\times n}_{sym} \times (-\infty, 0]$ such that

$$K_{\lambda_1} \subset K_{\lambda_2} \quad \text{for every } 1 \le \lambda_1 < \lambda_2 \tag{2.3}$$

and

$$\{(\sigma,\xi) \in K \times (-\infty,0] : \lambda\xi - \sigma_m \le 0\} \subset K_\lambda \quad \text{for every } \lambda \ge 1. \tag{2.4}$$

**Remark 2.1.** In the applications we have in mind, *e.g.* in the Drucker-Prager and the Mohr-Coulomb model, the set $K$ is typically unbounded in the direction of negative hydrostatic matrices: it is usually of the form

$$K := \{\sigma \in \mathbb{M}^{n\times n}_{sym} : \alpha\sigma_m + \kappa(\sigma_D) - k \le 0\},$$

where $\kappa : \mathbb{M}^{n\times n}_D \to [0,+\infty)$ is a convex and positively 1-homogeneous function with $\kappa(0) = 0$, while $\alpha > 0$ and $k > 0$ are given constants. The condition $\kappa(0) = 0$ guarantees that (2.2) is satisfied. In this case a natural choice for the sets $K_\lambda$ is simply

$$K_\lambda := \{(\sigma,\xi) \in K \times (-\infty,0] : \lambda\xi - \sigma_m \le 0\},$$

that clearly satisfy (2.3) and (2.4).

We define the support functions $H : \mathbb{M}^{n\times n}_{sym} \to [0,+\infty]$ and $H_\lambda : \mathbb{M}^{n\times n}_{sym} \times \mathbb{R} \to [0,+\infty]$ of $K$ and $K_\lambda$ by

$$H(p) := \sup_{\sigma\in K} \sigma : p \quad \text{and} \quad H_\lambda(p,z) := \sup_{(\sigma,\xi)\in K_\lambda} \{\sigma : p + \xi z\}.$$

Since $K$ and $K_\lambda$ are closed and convex, $H$ and $H_\lambda$ are convex, lower semicontinuous, and positively 1-homogeneous. We observe that if $1 \le \lambda_1 \le \lambda_2$, then $H_{\lambda_1} \le H_{\lambda_2}$ by (2.3). The following result shows that $H_\lambda$ monotonically increases to $H$.

**Lemma 2.2.** *For any* $(p,z) \in \mathbb{M}^{n\times n}_{sym} \times \mathbb{R}$ *we have*

$$\sup_{\lambda\ge 1} H_\lambda(p,z) = H(p) + I_{[0,+\infty)}(z),$$

*where* $I_{[0,+\infty)}$ *is the indicator function of* $[0,+\infty)$*, i.e.,* $I_{[0,+\infty)}(z) = 0$ *if* $z \ge 0$*, while* $I_{[0,+\infty)}(z) = +\infty$ *otherwise.*

*Proof.* Defining $K_\infty := \cup_{\lambda\ge 1} K_\lambda$, we have that for every $(p,z) \in \mathbb{M}^{n\times n}_{sym} \times \mathbb{R}$ and $\lambda \ge 1$,

$$H_\lambda(p,z) \le H_\infty(p,z) := \sup_{(\sigma,\xi)\in K_\infty} \{\sigma : p + \xi z\} = H(p) + I_{[0,+\infty)}(z),$$

where the last equality follows from the fact that $\overline{K}_\infty = K \times (-\infty,0]$ by (2.4). Hence we deduce that for any $(p,z) \in \mathbb{M}^{n\times n}_{sym} \times \mathbb{R}$,

$$\sup_{\lambda\ge 1} H_\lambda(p,z) \le H(p) + I_{[0,+\infty)}(z).$$

To prove the converse inequality, assume that $\sup_\lambda H_\lambda(p,z) < \infty$. If $(\sigma,\xi) \in K_\infty$, then $(\sigma,\xi) \in K_\lambda$ for some $\lambda$, and thus

$$\sup_{\lambda\ge 1} H_\lambda(p,z) \ge \sigma : p + \xi z.$$

Maximizing with respect to $(\sigma,\xi)\in K_\infty$ in the right-handside of the previous inequality yields

$$\sup_{\lambda\geq 1} H_\lambda(p,z)\geq H(p)+I_{[0,+\infty)}(z),$$

and the proof is complete. □

The functions $H$ and $H_\lambda$ enjoy a nice coercivity property. Indeed, by (2.2) we deduce that

$$\alpha_H|p|\leq H(p)\quad \text{ for all } p\in\mathbb{M}^{n\times n}_{sym}.$$

Moreover, since $B(0,\alpha_H)\times(-\infty,-\frac{\alpha_H}{\sqrt{n}}]\subset K_1\subset K_\lambda$ for all $\lambda\geq 1$ by (2.3) and (2.4), we have

$$\alpha_H|p|-\frac{\alpha_H}{\sqrt{n}}z\leq H_\lambda(p,z)\quad \text{ for all } (p,z)\in\mathbb{M}^{n\times n}_{sym}\times[0,+\infty), \tag{2.5}$$

and

$$z<0 \quad\text{implies that}\quad H_\lambda(p,z)=+\infty. \tag{2.6}$$

The dissipated energy is then defined, for all $(p,z)\in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\times L^2(\Omega)$, by

$$\mathcal{H}_\lambda(p,z):=\int_\Omega H_\lambda(p(x),z(x))\,dx.$$

As a consequence of the previous properties of $H_\lambda$, we infer that $\mathcal{H}_\lambda$ is sequentially weakly lower semicontinuous in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\times L^2(\Omega)$. It will also be useful to extend the definition of $\mathcal{H}_\lambda$ when $p\in\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym})$. According to [20], we define the non-negative Borel measures

$$H(p)(B):=\int_B H\left(\frac{dp}{d|p|}(x)\right)\,d|p|(x)$$

and

$$H_\lambda(p,z)(B):=\int_B H_\lambda\left(\frac{dp}{d\mathcal{L}^n}(x),z(x)\right)\,dx+\int_B H_\lambda\left(\frac{dp}{d|p^s|}(x),0\right)\,d|p^s|(x)$$

for any Borel set $B\subset\overline{\Omega}$. In general, by [20] the measures $H(p)$ and $H_\lambda(p,z)$ are not even locally finite. However, if further $H(p)$ and $H_\lambda(p,z)$ have finite mass, *i.e.*, if $H(p)$ and $H_\lambda(p,z)$ are bounded Radon measures, we can define the functionals

$$\mathcal{H}(p):=H(p)(\overline{\Omega})\quad\text{ and }\quad \mathcal{H}_\lambda(p,z):=H_\lambda(p,z)(\overline{\Omega}).$$

In that case, the results of [12, 13] apply and these convex functions of measures can be expressed by means of duality formulas. To this aim, let us extend $p$ and $z$ by zero on $\Omega'\setminus\overline{\Omega}$, where $\Omega'\subset\mathbb{R}^n$ is a bounded smooth open set containing $\overline{\Omega}$. As a consequence, $H(p)$ and $H_\lambda(p,z)$ are in $\mathcal{M}(\Omega')$ and thanks to [13, Theorem 2.1–(ii)], we get that

$$\begin{gathered}\int_{\Omega'}\varphi\,d[H(p)]=\sup\left\{\int_{\Omega'}\varphi\sigma:dp\ :\sigma\in\mathcal{C}^\infty_c(\Omega';K)\right\},\\ \int_{\Omega'}\varphi\,d[H_\lambda(p,z)]=\sup\left\{\int_{\Omega'}\varphi\sigma:dp+\int_{\Omega'}\xi z\varphi\,dx\ :(\sigma,\xi)\in\mathcal{C}^\infty_c(\Omega';K_\lambda)\right\},\end{gathered} \tag{2.7}$$

for any $\varphi\in\mathcal{C}_c(\Omega')$ with $\varphi\geq 0$, and in particular

$$\begin{gathered}\mathcal{H}(p)=\sup\left\{\int_{\overline{\Omega}}\sigma:dp\ :\sigma\in\mathcal{C}^\infty(\overline{\Omega};K)\right\},\\ \mathcal{H}_\lambda(p,z)=\sup\left\{\int_{\overline{\Omega}}\sigma:dp+\int_\Omega\xi z\,dx\ :(\sigma,\xi)\in\mathcal{C}^\infty(\overline{\Omega};K_\lambda)\right\}.\end{gathered} \tag{2.8}$$

Note also that Reshetnyak Theorem (see [1, Theorem 2.38]) still holds here, so that $\mathcal{H}$ and $\mathcal{H}_\lambda$ are sequentially weakly* lower semicontinuous in $\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym})$ and $\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym})\times L^2(\Omega)$, respectively.

2.3. **The total dissipation.** Let $(p,z) : [0,T] \to \mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym}) \times L^2(\Omega)$. The $\mathcal{H}_\lambda$-variation of $(p,z)$ on a time interval $[a,b]$, which will play the role of the total dissipation, is defined as

$$\mathcal{D}_\lambda(p,z;[a,b]) := \sup\Big\{ \sum_{j=1}^{N} \mathcal{H}_\lambda(p(t_j)-p(t_{j-1}), z(t_j)-z(t_{j-1})) : \\ a=t_0\le t_1\le\cdots\le t_N=b,\ N\in\mathbb{N}\Big\}.$$

The lower semicontinuity of $\mathcal{H}_\lambda$ ensures that $\mathcal{D}_\lambda(\cdot,\cdot;[a,b])$ is sequentially weakly* lower semicontinuous in $\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym}) \times L^2(\Omega)$ pointwise in time.

The following result enables one to write the total dissipation $\mathcal{D}_\lambda$ as a time integral when $p$ and $z$ are regular enough with respect to time. This property follows from an adaptation of [10, Theorem 7.1] (see also [6, Appendix]). Indeed, a careful inspection of the proof of that result shows that it is enough to have the functional $\mathcal{H}_\lambda$ lower semicontinuous, which is ensured in our case by Reshetnyak Theorem. In particular, the fact that the function $H_\lambda$ can take the value $+\infty$ does not affect the validity of the result.

**Proposition 2.3.** *Assume that $p\in AC([0,T];\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym}))$, $z\in AC([0,T];L^2(\Omega))$, and*

$$\mathcal{D}_\lambda(p,z;[0,T])<+\infty.$$

*Then, for a.e. $t\in[0,T]$, there exist $(\dot p(t),\dot z(t))\in\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym})\times L^2(\Omega)$ such that*

$$\begin{cases} \dfrac{p(s)-p(t)}{s-t} \rightharpoonup \dot p(t) \text{ weakly* in } \mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym}), \\ \dfrac{z(s)-z(t)}{s-t} \to \dot z(t) \text{ strongly in } L^2(\Omega), \end{cases} \qquad \text{as } s\to t.$$

*Moreover, the function $t\mapsto\mathcal{H}_\lambda(\dot p(t),\dot z(t))$ is measurable and for all $0\le a\le b\le T$,*

$$\mathcal{D}_\lambda(p,z;[a,b]) = \int_a^b \mathcal{H}_\lambda(\dot p(t),\dot z(t))\,dt.$$

2.4. **Duality between the stress and the plastic strain.** The duality pairing between stresses and plastic strains is *a priori* not well defined, since the former are only squared Lebesgue integrable, while the latter are measures. This is clearly an obstacle if one wishes to express in a pointwise sense Hill's principle of maximum plastic work (1.1). Using an integration by parts formula as in [23, 19], it is actually possible to give a sense to this duality pairing as a distribution, and even as a measure for solutions of the elasto-plasticity model.

**Definition 2.4.** Let $\sigma\in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ with $\mathrm{div}\sigma\in L^2(\Omega;\mathbb{R}^n)$, and let $p\in\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym})$ be such that there exists a triple $(u,e,w)\in(BD(\Omega)\cap L^2(\Omega;\mathbb{R}^n))\times L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\times H^1(\Omega;\mathbb{R}^n)$ satisfying $Eu=e+p$ in $\Omega$ and $p=(w-u)\odot\nu\mathcal{H}^{n-1}$ on $\partial\Omega$, where $\nu$ is the outer unit normal to $\partial\Omega$. We define the distribution $[\sigma:p]$ on $\mathbb{R}^n$ as

$$\langle[\sigma:p],\varphi\rangle = \int_\Omega \varphi(w-u)\cdot\mathrm{div}\sigma\,dx + \int_\Omega \sigma:[(w-u)\odot\nabla\varphi]\,dx + \int_\Omega \sigma:(Ew-e)\varphi\,dx$$

for every $\varphi\in\mathcal{C}^\infty_c(\mathbb{R}^n)$.

It is easy to check that the definition of $[\sigma:p]$ is independent of the choice of $(u,e,w)$, so that the distribution $[\sigma:p]$ is well defined. Moreover, since $[\sigma:p]$ is a distribution supported in $\overline{\Omega}$, we can define the duality product $\langle\sigma,p\rangle$ as

$$\langle\sigma,p\rangle := \langle[\sigma:p],1\rangle = \int_\Omega (w-u)\cdot\mathrm{div}\sigma\,dx + \int_\Omega \sigma:(Ew-e)\,dx. \tag{2.9}$$

**Remark 2.5.** Note that, contrary to [23, 19], here the stress $\sigma$ may not be in $L^\infty(\Omega;\mathbb{M}^{n\times n}_{sym})$; therefore, it is not clear how to prove at this stage that the distribution $[\sigma:p]$ is actually a measure. However, this property will be obtained afterwards for solutions of the plasticity problems (see Theorems 4.1, 5.1, 6.4, and 6.6).

Using this notion of stress/strain duality, the duality formulas (2.7) and (2.8) can be now extended to less regular statically and plastically admissible stresses. This property rests on a density result, together with the fact that, when the stress is smooth enough, the stress/strain duality reduces to the usual duality between continuous functions and measures. Indeed, according to the integration by parts formula in $BD(\Omega)$, if $\sigma\in\mathcal{C}^1(\overline{\Omega};\mathbb{M}^{n\times n}_{sym})$, we have

$$\langle[\sigma:p],\varphi\rangle=\int_{\overline{\Omega}}\varphi\sigma:dp\quad\text{ for all }\varphi\in\mathcal{C}^\infty_c(\mathbb{R}^n),\tag{2.10}$$

and

$$\langle\sigma,p\rangle=\int_{\overline{\Omega}}\sigma:dp.\tag{2.11}$$

Moreover, the following approximation result is an immediate adaptation of [10, Lemma 2.3].

**Lemma 2.6.** *Let $1\le p<\infty$ and $(\sigma,\xi)\in L^p(\Omega;\mathbb{M}^{n\times n}_{sym})\times L^2(\Omega)$ be such that* $\mathrm{div}\sigma\in L^p(\Omega;\mathbb{R}^n)$ *and $(\sigma(x),\xi(x))\in K_\lambda$ for a.e. $x\in\Omega$. There exists a sequence $(\sigma_k,\xi_k)\subset\mathcal{C}^\infty(\overline{\Omega};\mathbb{M}^{n\times n}_{sym})\times\mathcal{C}^\infty(\overline{\Omega})$ such that $(\sigma_k,\xi_k)\to(\sigma,\xi)$ strongly in $L^p(\Omega;\mathbb{M}^{n\times n}_{sym})\times L^2(\Omega)$,* $\mathrm{div}\sigma_k\to\mathrm{div}\sigma$ *strongly in $L^p(\Omega;\mathbb{R}^n)$, and $(\sigma_k(x),\xi_k(x))\in K_\lambda$ for all $x\in\Omega$ and all $k\in\mathbb{N}$.*

If $p\in\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym})$ with $\mathcal{H}(p)<+\infty$, then, using the duality formulas (2.7) and (2.8), together with (2.10), (2.11), and the approximation result [10, Lemma 2.3], we infer that for all $\varphi\in\mathcal{C}^\infty_c(\Omega')$ with $\varphi\ge 0$,

$$\int_{\Omega'}\varphi\,d[H(p)]=\sup\Big\{\langle[\sigma:p],\varphi\rangle:\sigma\in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\text{ with }\mathrm{div}\sigma\in L^2(\Omega;\mathbb{R}^n)$$
$$\text{and }\sigma(x)\in K\text{ for a.e. }x\in\Omega\Big\},\tag{2.12}$$

and in particular,

$$\mathcal{H}(p)=\sup\Big\{\langle\sigma,p\rangle:\sigma\in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\text{ with }\mathrm{div}\sigma\in L^2(\Omega;\mathbb{R}^n)$$
$$\text{and }\sigma(x)\in K\text{ for a.e. }x\in\Omega\Big\}.\tag{2.13}$$

Analogously, if $p\in\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym})$, $z\in L^2(\Omega)$, and $\mathcal{H}_\lambda(p,z)<+\infty$, then, by (2.7), (2.8), (2.10), (2.11), and Lemma 2.6, we infer that for all $\varphi\in\mathcal{C}^\infty_c(\Omega')$ with $\varphi\ge0$,

$$\int_{\Omega'}\varphi\,d[H_\lambda(p,z)]=\sup\Big\{\langle[\sigma:p],\varphi\rangle+\int_\Omega\xi z\varphi\,dx:(\sigma,\xi)\in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\times L^2(\Omega)\text{ with}$$
$$\mathrm{div}\sigma\in L^2(\Omega;\mathbb{R}^n)\text{ and }(\sigma(x),\xi(x))\in K_\lambda\text{ for a.e. }x\in\Omega\Big\},\tag{2.14}$$

and in particular,

$$\mathcal{H}_\lambda(p,z)=\sup\Big\{\langle\sigma,p\rangle+\int_\Omega\xi z\,dx:(\sigma,\xi)\in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\times L^2(\Omega)\text{ with}$$
$$\mathrm{div}\sigma\in L^2(\Omega;\mathbb{R}^n)\text{ and }(\sigma(x),\xi(x))\in K_\lambda\text{ for a.e. }x\in\Omega\Big\}.\tag{2.15}$$

**Remark 2.7.** Note that, if $p$ is associated to a displacement $u \in BD(\Omega) \setminus L^2(\Omega;\mathbb{R}^n)$, we can still define the distribution $[\sigma : p]$ for $\sigma \in L^n(\Omega;\mathbb{M}^{n\times n}_{sym})$ with $\mathrm{div}\sigma \in L^n(\Omega;\mathbb{R}^n)$, because of the embedding of $BD(\Omega)$ into $L^{n/(n-1)}(\Omega;\mathbb{R}^n)$. In that case, formulas (2.12)–(2.15) hold with supremum taken over all $\sigma \in L^n(\Omega;\mathbb{M}^{n\times n}_{sym})$ satisfying $\mathrm{div}\sigma \in L^n(\Omega;\mathbb{R}^n)$. Nevertheless, the definition of the duality will be a source of difficulties when dealing with the quasi-static case in dimension higher than two, because the velocity and/or the stress may miss the required integrability (see Theorems 6.4 and 6.6).

## 3. The dynamical visco-plastic cap model

The main result of this section is an existence result for a visco-plastic dynamical cap model. The kind of viscosity we use is not related to a regularization of the flow rule of Perzyna or Norton-Hoff type as in [36, 37, 40], but rather connected to the constitutive law. We consider a viscosity of Kelvin-Voigt type where the (visco-plastic) stress $\tilde{\sigma} = \mathbb{C}e + \varepsilon E\dot{u}$ is the sum of two terms. The first part $\sigma := \mathbb{C}e$ is the stress that originates from the elastic reaction to the deformation, while the second part $\varepsilon E\dot{u}$ is a damping term due to viscosity ($\varepsilon > 0$ is a viscosity coefficient). The model described below (Theorem 3.1) is similar to that studied in [25] (see also [33] for a related model involving a dissipation functional depending further on the gradient of the internal variable). In that reference, existence and uniqueness were proved by means of a Galerkin method, while here, we employ a time discretization procedure of the underlying hyperbolic system. However, our proofs are very close to those of [25].

Let us define the set of visco-plastic kinematically admissible fields in the following way: given a boundary displacement $\hat{w} \in H^1(\Omega;\mathbb{R}^n)$,

$$\mathcal{A}_{\mathrm{vp}}(\hat{w}) := \big\{(v,\eta,q) \in H^1(\Omega;\mathbb{R}^n) \times L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \times L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) : \\ Ev = \eta + q \text{ a.e. in } \Omega,\ v = \hat{w}\ \mathcal{H}^{n-1}\text{-a.e. on } \partial\Omega\big\}.$$

Let us now describe the assumptions on the data. We consider a body load

$$f \in AC([0,T]; L^2(\Omega;\mathbb{R}^n)) \tag{3.1}$$

and a boundary displacement that is the trace on $\partial\Omega \times [0,T]$ of a function

$$w \in H^2([0,T]; H^1(\Omega;\mathbb{R}^n)) \cap H^3([0,T]; L^2(\Omega;\mathbb{R}^n)). \tag{3.2}$$

Moreover, let $\lambda \geq 1$ and let $(u_0, e_0, p_0, z_0) \in \mathcal{A}_{\mathrm{vp}}(w(0)) \times L^2(\Omega)$ and $v_0 \in H^2(\Omega;\mathbb{R}^n)$ be initial data satisfying

$$\begin{cases} v_0 = \dot{w}(0)\ \mathcal{H}^{n-1}\text{-a.e. on } \partial\Omega, \\ (\sigma_0, \xi_0) := (\mathbb{C}e_0, -z_0) \in K_\lambda, \quad -\mathrm{div}\sigma_0 = f(0) \quad \text{a.e. in } \Omega. \end{cases} \tag{3.3}$$

**Theorem 3.1.** *Let $\varepsilon > 0$ and $\lambda \geq 1$. Assume (2.1)–(2.4) and (3.1)–(3.3). Then there exist unique*

$$\begin{cases} u_\varepsilon \in W^{2,\infty}([0,T]; L^2(\Omega;\mathbb{R}^n)) \cap H^2([0,T]; H^1(\Omega;\mathbb{R}^n)), \\ \sigma_\varepsilon,\ e_\varepsilon \in W^{1,\infty}([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_{sym})), \\ p_\varepsilon \in H^1([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_{sym})), \\ \xi_\varepsilon,\ z_\varepsilon \in W^{1,\infty}([0,T]; L^2(\Omega)), \end{cases}$$

*with the following properties: for all $t \in [0,T]$,*

$$\begin{cases} Eu_\varepsilon(t) = e_\varepsilon(t) + p_\varepsilon(t) \text{ a.e. in } \Omega, \quad u_\varepsilon(t) = w(t)\ \mathcal{H}^{n-1}\text{-a.e. on } \partial\Omega, \\ \sigma_\varepsilon(t) = \mathbb{C}e_\varepsilon(t), \quad \xi_\varepsilon(t) = -z_\varepsilon(t), \\ (\sigma_\varepsilon(t), \xi_\varepsilon(t)) \in K_\lambda \text{ a.e. in } \Omega, \end{cases} \tag{3.4}$$

*and*

$$\begin{cases} \ddot u_\varepsilon - \operatorname{div}(\sigma_\varepsilon + \varepsilon E\dot u_\varepsilon) = f & \text{a.e. in } \Omega\times(0,T),\\ (u_\varepsilon(0), e_\varepsilon(0), p_\varepsilon(0), z_\varepsilon(0)) = (u_0,e_0,p_0,z_0), & \dot u_\varepsilon(0) = v_0. \end{cases} \tag{3.5}$$

*Moreover, for a.e.* $t\in[0,T]$,

$$\dot z_\varepsilon(t)\geq 0 \quad \text{and} \quad (\dot p_\varepsilon(t), \dot z_\varepsilon(t)) \in N_{K_\lambda}(\sigma_\varepsilon(t),\xi_\varepsilon(t)) \quad \text{a.e. in } \Omega. \tag{3.6}$$

**Remark 3.2.** Note that (3.4) and (3.6) ensure that the map $t\mapsto \xi_\varepsilon(t)$ is non-increasing. Since $\xi_\varepsilon$ is an internal variable describing the position of the cap (see (2.4)), this property means that the cap is moving outwards as time proceeds. This is exactly the strain-hardening phenomenon we wanted to highlight on the evolution of the cap surface.

The remaining of this section is devoted to the proof of Theorem 3.1. The proof relies on a time discretization procedure, that is described in Subsection 3.1. Compactness of discrete-time evolutions is proved via *a priori* estimates in Subsection 3.2. The equation of motion and the flow rule are deduced in Subsections 3.3 and 3.4. In Subsection 3.5 we show uniqueness of the solution, while in Subsection 3.6 we prove higher regularity by establishing some *a posteriori* estimates.

3.1. **Time discretization.** Let us consider a partition of the time interval $[0,T]$ into $N_k$ subintervals of equal length $\delta_k$:

$$0=t_k^0<t_k^1<\cdots<t_k^{N_k}=T, \quad \text{with} \quad \delta_k:=\frac{T}{N_k}=t_k^i-t_k^{i-1}\to 0.$$

We define the discrete body loads by $f_k^i:=f(t_k^i)$ for all $i\in\{0,\dots,N_k\}$ and the discrete boundary values $\{w_k^i\}_{0\leq i\leq N_k}$ by

$$w_k^0:=w(0),\quad w_k^1=w(0)+\delta_k\dot w(0), \quad \text{and} \quad w_k^i=w(t_k^i) \text{ for all } i\in\{2,\dots,N_k\}.$$

Then we define inductively

$$(u_k^0,e_k^0,p_k^0,z_k^0):=(u_0,e_0,p_0,z_0),\quad (u_k^1,e_k^1,p_k^1,z_k^1):=(u_0,e_0,p_0,z_0)+\delta_k(v_0,0,Ev_0,0), \tag{3.7}$$

and, for all $i\in\{2,\dots,N_k\}$, we define $(u_k^i,e_k^i,p_k^i,z_k^i)$ as the solution of the following minimum problem

$$\min_{(v,\eta,q,\zeta)\in\mathcal{A}_{\rm vp}(w_k^i)\times L^2(\Omega)}\left\{\mathcal{Q}(\eta)+\frac{\varepsilon}{2\delta_k}\|Ev-Eu_k^{i-1}\|_2^2+\frac{1}{2\delta_k^2}\|v-2u_k^{i-1}+u_k^{i-2}\|_2^2 \right.$$
$$\left. +\frac12\|\zeta\|_2^2+\mathcal{H}_\lambda(q-p_k^{i-1},\zeta-z_k^{i-1})-\int_\Omega f_k^i\cdot v\,dx\right\}. \tag{3.8}$$

Korn's inequality together with the sequential weak lower semicontinuity in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\times L^2(\Omega)$ of $\mathcal{H}_\lambda$ imply that the previous minimum problem admits a solution denoted by $(u_k^i,e_k^i,p_k^i,z_k^i)\in\mathcal{A}_{\rm vp}(w_k^i)\times L^2(\Omega)$, which is unique by strict convexity of the functional. In particular, since $\mathcal{H}_\lambda(p_k^i-p_k^{i-1},z_k^i-z_k^{i-1})<\infty$, we deduce from (2.6) that

$$z_k^i\geq z_k^{i-1} \quad \text{a.e. in } \Omega. \tag{3.9}$$

We now derive the Euler-Lagrange equations as the first order optimality conditions of the previous minimum problems.

**Proposition 3.3.** *Let* $(u_k^i,e_k^i,p_k^i,z_k^i)\in\mathcal{A}_{\rm vp}(w_k^i)\times L^2(\Omega)$ *be defined by* (3.7) *and* (3.8)*. Then, for all* $i\in\{0,\dots,N_k\}$,

$$\begin{cases} Eu_k^i=e_k^i+p_k^i \text{ a.e. in } \Omega, \quad u_k^i=w_k^i\ \mathcal{H}^{n-1}\text{-a.e. on } \partial\Omega,\\ \sigma_k^i:=\mathbb{C}e_k^i,\ \xi_k^i:=-z_k^i,\quad (\sigma_k^i,\xi_k^i)\in K_\lambda \text{ a.e. in } \Omega, \end{cases}$$

*and for all* $i \in \{2, \dots, N_k\}$,

$$\begin{cases} \dfrac{u_k^i - 2u_k^{i-1} + u_k^{i-2}}{\delta_k^2} - \operatorname{div}\left(\sigma_k^i + \varepsilon \dfrac{Eu_k^i - Eu_k^{i-1}}{\delta_k}\right) = f_k^i \ \textit{a.e. in } \Omega, \\ (p_k^i - p_k^{i-1}, z_k^i - z_k^{i-1}) \in N_{K_\lambda}(\sigma_k^i, \xi_k^i) \ \textit{a.e. in } \Omega. \end{cases} \tag{3.10}$$

*Proof.* The first condition is a consequence of the fact that $(u_k^i, e_k^i, p_k^i, z_k^i) \in \mathcal{A}_{\rm vp}(w_k^i) \times L^2(\Omega)$ for all $i \geq 0$. Let us define $\sigma_k^i := \mathbb{C}e_k^i$ and $\xi_k^i := -z_k^i$. Clearly from (3.3) and (3.7) we have $(\sigma_k^0, \xi_k^0) \in K_\lambda$ and $(\sigma_k^1, \xi_k^1) \in K_\lambda$.

We now consider $i \geq 2$. For any $(v, \eta, q, \zeta) \in \mathcal{A}_{\rm vp}(0) \times L^2(\Omega)$ and $t > 0$, the quadruplet $(u_k^i, e_k^i, p_k^i, z_k^i) + t(v, \eta, q, \zeta) \in \mathcal{A}_{\rm vp}(w_k^i) \times L^2(\Omega)$ is admissible for the minimum problem (3.8). Hence choosing it as a competitor, dividing the resulting inequality by $t > 0$, and letting $t \to 0^+$ yield

$$\begin{aligned} 0 \leq \int_\Omega \sigma_k^i : \eta \, dx + \varepsilon \int_\Omega \frac{Eu_k^i - Eu_k^{i-1}}{\delta_k} : Ev \, dx + \int_\Omega \frac{u_k^i - 2u_k^{i-1} + u_k^{i-2}}{\delta_k^2} \cdot v \, dx + \int_\Omega z_k^i \zeta \, dx \\ + \mathcal{H}_\lambda(p_k^i - p_k^{i-1} + q, z_k^i - z_k^{i-1} + \zeta) - \mathcal{H}_\lambda(p_k^i - p_k^{i-1}, z_k^i - z_k^{i-1}) - \int_\Omega f_k^i \cdot v \, dx, \end{aligned} \tag{3.11}$$

where we used the convexity of $H_\lambda$. Taking in particular $(v, \eta, q, \zeta) = \pm(\varphi, E\varphi, 0, 0)$ where $\varphi \in \mathcal{C}_c^\infty(\Omega; \mathbb{R}^n)$, we infer that

$$\int_\Omega \left(\sigma_k^i + \varepsilon \frac{Eu_k^i - Eu_k^{i-1}}{\delta_k}\right) : E\varphi \, dx + \int_\Omega \frac{u_k^i - 2u_k^{i-1} + u_k^{i-2}}{\delta_k^2} \cdot \varphi \, dx = \int_\Omega f_k^i \cdot \varphi \, dx,$$

leading to the first equation in (3.10).

Let now $\hat{q} \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ and $\hat{\zeta} \in L^2(\Omega)$. Choosing $(v, \eta, q, \zeta) = (0, -\hat{q} + p_k^i - p_k^{i-1}, \hat{q} - p_k^i + p_k^{i-1}, \hat{\zeta} - z_k^i + z_k^{i-1})$ in (3.11) implies

$$\mathcal{H}_\lambda(\hat{q}, \hat{\zeta}) \geq \mathcal{H}_\lambda(p_k^i - p_k^{i-1}, z_k^i - z_k^{i-1}) + \int_\Omega \sigma_k^i : \big(\hat{q} - (p_k^i - p_k^{i-1})\big) \, dx + \int_\Omega \xi_k^i \big(\hat{\zeta} - (z_k^i - z_k^{i-1})\big) \, dx.$$

Localizing this inequality yields

$$\begin{aligned} H_\lambda(\hat{q}, \hat{\zeta}) \geq H_\lambda(p_k^i(x) - p_k^{i-1}(x), z_k^i(x) - z_k^{i-1}(x)) \\ + \sigma_k^i(x) : \big(\hat{q} - (p_k^i(x) - p_k^{i-1}(x))\big) + \xi_k^i(x)\big(\hat{\zeta} - (z_k^i(x) - z_k^{i-1}(x))\big), \end{aligned}$$

for all $(\hat{q}, \hat{\zeta}) \in \mathbb{M}^{n\times n}_{sym} \times \mathbb{R}$ and for a.e. $x \in \Omega$. This implies that $(\sigma_k^i, \xi_k^i) \in K_\lambda$ and $(p_k^i - p_k^{i-1}, z_k^i - z_k^{i-1}) \in N_{K_\lambda}(\sigma_k^i, \xi_k^i)$ a.e. in $\Omega$. ☐

3.2. *A priori* **estimates and compactness.** Owing to the Euler-Lagrange equations, we can derive some *a priori* estimates. We define three types of interpolations. We start with piecewise constant left-continuous interpolations defined by

$$\begin{gathered} u_k(0) := u_0, \quad w_k(0) := w(0), \quad f_k(0) := f(0), \quad e_k(0) := e_0, \\ \sigma_k(0) := \sigma_0, \quad p_k(0) := p_0, \quad \xi_k(0) := \xi_0, \quad z_k(0) := z_0, \end{gathered}$$

and, for all $t \in (t_k^{i-1}, t_k^i]$ and $i \in \{1, \dots, N_k\}$,

$$\begin{gathered} u_k(t) := u_k^i, \quad w_k(t) := w_k^i, \quad f_k(t) := f_k^i, \quad e_k(t) := e_k^i, \\ \sigma_k(t) := \sigma_k^i, \quad p_k(t) := p_k^i, \quad \xi_k(t) := \xi_k^i, \quad z_k(t) := z_k^i. \end{gathered}$$

We will also consider the piecewise affine interpolations given by

$$\begin{gathered} \hat{u}_k(0) := u_0, \quad \hat{w}_k(0) := w(0), \quad \hat{e}_k(0) := e_0, \quad \hat{\sigma}_k(0) := \sigma_0, \\ \hat{p}_k(0) := p_0, \quad \hat{\xi}_k(0) := \xi_0, \quad \hat{z}_k(0) := z_0, \end{gathered}$$

and, for every $t \in (t_k^{i-1}, t_k^i]$ and $i \in \{1, \dots, N_k\}$,

$$\hat u_k(t) := u_k^{i-1} + \frac{t - t_k^{i-1}}{\delta_k}(u_k^i - u_k^{i-1}), \quad \hat w_k(t) := w_k^{i-1} + \frac{t - t_k^{i-1}}{\delta_k}(w_k^i - w_k^{i-1}),$$
$$\hat e_k(t) := e_k^{i-1} + \frac{t - t_k^{i-1}}{\delta_k}(e_k^i - e_k^{i-1}), \quad \hat \sigma_k(t) := \sigma_k^{i-1} + \frac{t - t_k^{i-1}}{\delta_k}(\sigma_k^i - \sigma_k^{i-1}),$$
$$\hat \xi_k(t) := \xi_k^{i-1} + \frac{t - t_k^{i-1}}{\delta_k}(\xi_k^i - \xi_k^{i-1}), \quad \hat z_k(t) := z_k^{i-1} + \frac{t - t_k^{i-1}}{\delta_k}(z_k^i - z_k^{i-1}),$$
$$\hat p_k(t) := p_k^{i-1} + \frac{t - t_k^{i-1}}{\delta_k}(p_k^i - p_k^{i-1}).$$

Finally, let $\tilde u_k$, $\tilde w_k$, and $\breve w_k$ be quadratic interpolations of $\{u_k^i\}_{0 \le i \le N_k}$ and $\{w_k^i\}_{0 \le i \le N_k}$ satisfying $\tilde u_k(t_k^i) = u_k^i$, $\tilde w_k(t_k^i) = \breve w_k(t_k^i) = w_k^i$ for all $i \in \{0, \dots, N_k\}$, and

$$\ddot{\tilde u}_k(t) = \frac{u_k^i - 2u_k^{i-1} + u_k^{i-2}}{\delta_k^2}, \quad \ddot{\tilde w}_k(t) = \frac{w_k^i - 2w_k^{i-1} + w_k^{i-2}}{\delta_k^2}, \quad \ddot{\breve w}_k(t) = \frac{w_k^{i+1} - 2w_k^i + w_k^{i-1}}{\delta_k^2},$$

for all $t \in (t_k^{i-1}, t_k^i)$, $i \in \{1, \dots, N_k\}$, where we set $u_k^{-1} := u_0$, $w_k^{-1} := w(0)$, and $w_k^{N_k+1} := w(T)$.

Observe that for all $t \in [0, T]$,

$$\begin{cases} Eu_k(t) = e_k(t) + p_k(t) \text{ a.e. in } \Omega, \quad u_k(t) = w_k(t) \ \mathcal{H}^{n-1}\text{-a.e. on } \partial\Omega, \\ E\hat u_k(t) = \hat e_k(t) + \hat p_k(t) \text{ a.e. in } \Omega, \quad \hat u_k(t) = \hat w_k(t) \ \mathcal{H}^{n-1}\text{-a.e. on } \partial\Omega, \\ \sigma_k(t) = \mathbb{C}e_k(t),\ \xi_k(t) = -z_k(t), \quad (\sigma_k(t), \xi_k(t)) \in K_\lambda \text{ a.e. in } \Omega, \\ \hat\sigma_k(t) = \mathbb{C}\hat e_k(t),\ \hat\xi_k(t) = -\hat z_k(t), \quad (\hat\sigma_k(t), \hat\xi_k(t)) \in K_\lambda \text{ a.e. in } \Omega, \end{cases} \tag{3.12}$$

and for a.e. $t \in [\delta_k, T]$,

$$\begin{cases} \ddot{\tilde u}_k(t) - \operatorname{div}\big(\sigma_k(t) + \varepsilon E\dot{\hat u}_k(t)\big) = f_k(t) \text{ a.e. in } \Omega, \\ (\dot{\hat p}_k(t), \dot{\hat z}_k(t)) \in N_{K_\lambda}(\sigma_k(t), \xi_k(t)) \text{ a.e. in } \Omega. \end{cases} \tag{3.13}$$

Moreover (3.9) ensures that for all $0 \le s \le t \le T$,

$$z_k(s) \le z_k(t), \quad \hat z_k(s) \le \hat z_k(t), \quad \text{and} \quad \dot{\hat z}_k(t) \ge 0 \quad \text{a.e. in } \Omega. \tag{3.14}$$

We are now in a position to derive some *a priori* estimates.

**Proposition 3.4.** *There exists a constant $C > 0$ (independent of $k$, $\varepsilon$, and $\lambda$) such that*

$$\|z_k\|_{L^\infty(0,T;L^2(\Omega))} + \|e_k\|_{L^\infty(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))} + \|\dot{\hat u}_k\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^n))} + \sqrt{\varepsilon}\|E\dot{\hat u}_k\|_{L^2(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))}$$
$$\le C\left(\|v_0\|_2 + \|Ev_0\|_2 + \|e_0\|_2 + \|z_0\|_2 + \|\ddot w\|_{L^1(0,T;H^1(\Omega;\mathbb{R}^n))} + \|f\|_{L^1(0,T;L^2(\Omega;\mathbb{R}^n))}\right). \tag{3.15}$$

*Moreover, there exists a constant $C_\varepsilon > 0$ (independent of $k$) such that*

$$\|\hat u_k\|_{L^\infty(0,T;H^1(\Omega;\mathbb{R}^n))} + \|\dot{\hat z}_k\|_{L^2(0,T;L^2(\Omega))}$$
$$+ \|\dot{\hat e}_k\|_{L^2(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))} + \|\dot{\hat p}_k\|_{L^2(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))} \le C_\varepsilon. \tag{3.16}$$

Formally, estimate (3.15) is obtained by taking the velocity as test function in the equation of motion, and by using the monotone character of the variational inclusion in (3.13). Estimate (3.16) rests, in turn, on the bound on the viscous dissipation energy which enables one to control the strain rate. Note that this last estimate is $\varepsilon$-dependent, but this will not affect the convergence from discrete to continuous times.

*Proof of Proposition 3.4.* Let $i \geq 2$. We multiply the first line of (3.10) by $u_k^i - u_k^{i-1}$ and integrate by parts over $\Omega$. Using the kinematic compatibility $Eu_k^i - Eu_k^{i-1} = (e_k^i - e_k^{i-1}) + (p_k^i - p_k^{i-1})$, we obtain

$$\frac{1}{2}\left\|\frac{u_k^i - u_k^{i-1}}{\delta_k}\right\|_2^2 - \frac{1}{2}\left\|\frac{u_k^{i-1} - u_k^{i-2}}{\delta_k}\right\|_2^2 + \frac{1}{2}\left\|\frac{u_k^i - 2u_k^{i-1} + u_k^{i-2}}{\delta_k}\right\|_2^2 + \int_\Omega \sigma_k^i : (p_k^i - p_k^{i-1})\,dx$$
$$+ \mathcal{Q}(e_k^i) - \mathcal{Q}(e_k^{i-1}) + \mathcal{Q}(e_k^i - e_k^{i-1}) + \varepsilon\delta_k \left\|\frac{Eu_k^i - Eu_k^{i-1}}{\delta_k}\right\|_2^2$$
$$= \int_\Omega \left(\sigma_k^i + \varepsilon\frac{Eu_k^i - Eu_k^{i-1}}{\delta_k}\right) : (Ew_k^i - Ew_k^{i-1})\,dx + \int_\Omega \frac{u_k^i - 2u_k^{i-1} + u_k^{i-2}}{\delta_k^2} \cdot (w_k^i - w_k^{i-1})\,dx$$
$$+ \int_\Omega f_k^i \cdot (u_k^i - u_k^{i-1})\,dx - \int_\Omega f_k^i \cdot (w_k^i - w_k^{i-1})\,dx.$$

Combining the discrete flow rule in (3.10) with the fact that $(0,0) \in K_\lambda$, we have

$$\int_\Omega \sigma_k^i : (p_k^i - p_k^{i-1})\,dx \geq -\int_\Omega \xi_k^i (z_k^i - z_k^{i-1})\,dx = \frac{1}{2}\|z_k^i\|_2^2 - \frac{1}{2}\|z_k^{i-1}\|_2^2 + \frac{1}{2}\|z_k^i - z_k^{i-1}\|_2^2,$$

where the last equality follows from the identity $\xi_k^i = -z_k^i$. Hence,

$$\frac{1}{2}\left\|\frac{u_k^i - u_k^{i-1}}{\delta_k}\right\|_2^2 - \frac{1}{2}\left\|\frac{u_k^{i-1} - u_k^{i-2}}{\delta_k}\right\|_2^2 + \frac{1}{2}\|z_k^i\|_2^2 - \frac{1}{2}\|z_k^{i-1}\|_2^2$$
$$+ \mathcal{Q}(e_k^i) - \mathcal{Q}(e_k^{i-1}) + \varepsilon\delta_k \left\|\frac{Eu_k^i - Eu_k^{i-1}}{\delta_k}\right\|_2^2$$
$$\leq \int_\Omega \left(\sigma_k^i + \varepsilon\frac{Eu_k^i - Eu_k^{i-1}}{\delta_k}\right) : (Ew_k^i - Ew_k^{i-1})\,dx + \int_\Omega \frac{u_k^i - 2u_k^{i-1} + u_k^{i-2}}{\delta_k^2} \cdot (w_k^i - w_k^{i-1})\,dx$$
$$+ \int_\Omega f_k^i \cdot (u_k^i - u_k^{i-1})\,dx - \int_\Omega f_k^i \cdot (w_k^i - w_k^{i-1})\,dx.$$

Summing up for $i = 2$ to $j$, and using a discrete integration by parts, we deduce

$$\frac{1}{2}\left\|\frac{u_k^j - u_k^{j-1}}{\delta_k}\right\|_2^2 - \frac{1}{2}\|v_0\|_2^2 + \frac{1}{2}\|z_k^j\|_2^2 - \frac{1}{2}\|z_0\|_2^2 + \mathcal{Q}(e_k^j) - \mathcal{Q}(e_0) + \varepsilon\sum_{i=2}^{j}\delta_k \left\|\frac{Eu_k^i - Eu_k^{i-1}}{\delta_k}\right\|_2^2$$
$$\leq \sum_{i=2}^{j}\int_\Omega \left(\sigma_k^i + \varepsilon\frac{Eu_k^i - Eu_k^{i-1}}{\delta_k}\right) : (Ew_k^i - Ew_k^{i-1})\,dx$$
$$-\sum_{i=2}^{j-1}\int_\Omega (u_k^i - u_k^{i-1}) \cdot \frac{w_k^{i+1} - 2w_k^i + w_k^{i-1}}{\delta_k^2}\,dx - \int_\Omega v_0 \cdot \frac{w_k^2 - w_k^1}{\delta_k}\,dx$$
$$+ \int_\Omega \frac{u_k^j - u_k^{j-1}}{\delta_k} \cdot \frac{w_k^j - w_k^{j-1}}{\delta_k}\,dx + \sum_{i=2}^{j}\int_\Omega f_k^i \cdot (u_k^i - u_k^{i-1})\,dx - \sum_{i=2}^{j}\int_\Omega f_k^i \cdot (w_k^i - w_k^{i-1})\,dx.$$

Consequently, if $t \in (t_k^{j-1}, t_k^j]$, we have from Hölder's inequality

$$\frac{1}{2}\|\dot{\hat{u}}_k(t)\|_2^2 - \frac{1}{2}\|v_0\|_2^2 + \frac{1}{2}\|z_k(t)\|_2^2 - \frac{1}{2}\|z_0\|_2^2 + \mathcal{Q}(e_k(t)) - \mathcal{Q}(e_0) + \varepsilon \int_{\delta_k}^{t_k^j} \|E\dot{\hat{u}}_k(s)\|_2^2\, ds$$
$$\leq \Big(\sqrt{T}\|\sigma_k\|_{L^\infty(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))} + \varepsilon\|E\dot{\hat{u}}_k\|_{L^2(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))}\Big)\|E\dot{w}_k\|_{L^2(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))}$$
$$+ \Big(\|\ddot{w}_k\|_{L^1(0,T;L^2(\Omega;\mathbb{R}^n))} + \|\dot{w}\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^n))}\Big)\|\dot{\hat{u}}_k\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^n))} + \|v_0\|_2\|\dot{w}\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^n))}$$
$$+ \Big(\|\dot{\hat{u}}_k\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^n))} + \|\dot{w}_k\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^n))}\Big)\|f_k\|_{L^1(0,T;L^2(\Omega;\mathbb{R}^n))},$$

from which (3.15) follows.

To prove (3.16), we first observe that

$$\hat{u}_k(t) = u_0 + \int_0^t \dot{\hat{u}}_k(s)\, ds \quad \text{ and } \quad E\hat{u}_k(t) = Eu_0 + \int_0^t E\dot{\hat{u}}_k(s)\, ds.$$

Therefore, by (3.15) there exists some constant $C_\varepsilon > 0$, depending on $\varepsilon$, such that

$$\|\hat{u}_k\|_{L^\infty(0,T;H^1(\Omega;\mathbb{R}^n))} \leq C_\varepsilon.$$

Moreover, according to the kinematic compatibility (3.12), we have $E\dot{\hat{u}}_k(t) = \dot{\hat{e}}_k(t) + \dot{\hat{p}}_k(t)$ for a.e. $t \in [0,T]$. Multiplying this equality by $\dot{\hat{\sigma}}_k(t)$, integrating over $\Omega$, and using the coercivity condition (2.1) yield

$$\alpha_{\mathbb{C}}\|\dot{\hat{e}}_k(t)\|_2^2 \leq \int_\Omega \dot{\hat{\sigma}}_k(t) : \dot{\hat{e}}_k(t)\, dx = \int_\Omega \dot{\hat{\sigma}}_k(t) : E\dot{\hat{u}}_k(t)\, dx - \int_\Omega \dot{\hat{\sigma}}_k(t) : \dot{\hat{p}}_k(t)\, dx.$$

Since $(\sigma_k^{i-1}, \xi_k^{i-1}) \in K_\lambda$, by the discrete flow rule in (3.10) we deduce

$$\int_\Omega \sigma_k^i : (p_k^i - p_k^{i-1})\, dx + \int_\Omega \xi_k^i (z_k^i - z_k^{i-1})\, dx \geq \int_\Omega \sigma_k^{i-1} : (p_k^i - p_k^{i-1})\, dx + \int_\Omega \xi_k^{i-1}(z_k^i - z_k^{i-1})\, dx,$$

for all $i \geq 2$. Hence, if $t \in (t_k^{i-1}, t_k^i]$, we have that

$$\int_\Omega \dot{\hat{\sigma}}_k(t) : \dot{\hat{p}}_k(t)\, dx = \frac{1}{\delta_k^2}\int_\Omega (\sigma_k^i - \sigma_k^{i-1}) : (p_k^i - p_k^{i-1})\, dx$$
$$\geq -\frac{1}{\delta_k^2}\int_\Omega (\xi_k^i - \xi_k^{i-1})(z_k^i - z_k^{i-1})\, dx = \left\|\frac{z_k^i - z_k^{i-1}}{\delta_k}\right\|_2^2 = \|\dot{\hat{z}}_k(t)\|_2^2.$$

Consequently, we derive that

$$\|\dot{\hat{z}}_k(t)\|_2^2 + \alpha_{\mathbb{C}}\|\dot{\hat{e}}_k(t)\|_2^2 \leq \beta_{\mathbb{C}}\|\dot{\hat{e}}_k(t)\|_2\|E\dot{\hat{u}}_k(t)\|_2$$

for all $t \in [\delta_k, T]$, and by (3.15) we deduce that

$$\|\dot{\hat{z}}_k\|_{L^2(\delta_k,T;L^2(\Omega))} + \|\dot{\hat{e}}_k\|_{L^2(\delta_k,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))} \leq C_\varepsilon$$

for some constant $C_\varepsilon > 0$ independent of $k$. Moreover, by the relation $\dot{\hat{p}}_k = E\dot{\hat{u}}_k - \dot{\hat{e}}_k$, we also have that

$$\|\dot{\hat{p}}_k\|_{L^2(\delta_k,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))} \leq C_\varepsilon.$$

Since $\dot{\hat{e}}_k(t) = 0$, $\dot{\hat{p}}_k(t) = Ev_0$, and $\dot{\hat{z}}_k(t) = 0$ for all $t \in [0, \delta_k]$ by (3.7), this concludes the proof of (3.16). □

From the previous *a priori* estimates, we now deduce compactness results at fixed $\varepsilon > 0$. Indeed, as a consequence of (3.15), (3.16), and Korn's inequality, we can extract a subsequence (not relabeled), and find

$$u \in H^1([0,T]; H^1(\Omega;\mathbb{R}^n)) \cap W^{1,\infty}([0,T]; L^2(\Omega;\mathbb{R}^n))$$

such that

$$\begin{cases} \hat{u}_k \rightharpoonup u \text{ weakly in } H^1([0,T]; H^1(\Omega;\mathbb{R}^n)), \\ \dot{\hat{u}}_k \rightharpoonup \dot{u} \text{ weakly* in } L^\infty([0,T]; L^2(\Omega;\mathbb{R}^n)). \end{cases}$$

Moreover, since for a.e. $t \in [0,T]$,

$$\|\hat{u}_k(t) - u_k(t)\|_{H^1(\Omega;\mathbb{R}^n)} \leq 2\delta_k \|\dot{\hat{u}}_k(t)\|_{H^1(\Omega;\mathbb{R}^n)},$$

we deduce that

$$u_k \rightharpoonup u \text{ weakly in } L^2(0,T; H^1(\Omega;\mathbb{R}^n)).$$

Note that the previous weak convergences of the sequence $(\hat{u}_k)$ implies, by Ascoli-Arzelà Theorem, that $\hat{u}_k(t) \rightharpoonup u(t)$ weakly in $H^1(\Omega;\mathbb{R}^n)$ for all $t \in [0,T]$. But since $\hat{u}_k(t) = \hat{w}_k(t)$ $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$, and $\hat{w}_k(t) \to w(t)$ strongly in $H^1(\Omega;\mathbb{R}^n)$ (by the absolute continuity of $t \mapsto w(t)$ in $H^1(\Omega;\mathbb{R}^n)$), we infer by the continuity of the trace that $u(t) = w(t)$ $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$. Moreover, since $\hat{u}_k(0) = u_0$, we deduce that $u(0) = u_0$.

Using again (3.16), we get that

$$\begin{cases} \hat{e}_k \rightharpoonup e \text{ weakly in } H^1([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_{sym})), \\ \hat{p}_k \rightharpoonup p \text{ weakly in } H^1([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_{sym})), \\ \hat{z}_k \rightharpoonup z \text{ weakly in } H^1([0,T]; L^2(\Omega)), \end{cases} \tag{3.17}$$

for some $e$, $p \in H^1([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$, and some $z \in H^1([0,T]; L^2(\Omega))$. Applying again Ascoli-Arzelà Theorem, we obtain that for all $t \in [0,T]$,

$$\begin{cases} \hat{e}_k(t) \rightharpoonup e(t) \text{ weakly in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym}), \\ \hat{p}_k(t) \rightharpoonup p(t) \text{ weakly in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym}), \\ \hat{z}_k(t) \rightharpoonup z(t) \text{ weakly in } L^2(\Omega). \end{cases}$$

In particular, since $(\hat{e}_k(0), \hat{p}_k(0), \hat{z}_k(0)) = (e_0, p_0, z_0)$, we deduce that

$$(e(0), p(0), z(0)) = (e_0, p_0, z_0).$$

Moreover, for all $t \in [0,T]$, we have

$$(u(t), e(t), p(t)) \in \mathcal{A}_{\rm vp}(w(t)),$$

and by (3.14), we get that for all $0 \leq s \leq t \leq T$,

$$z(s) \leq z(t) \quad \text{a.e. in } \Omega.$$

Then, for a.e. $t \in [0,T]$, we have

$$(\dot{u}(t), \dot{e}(t), \dot{p}(t)) \in \mathcal{A}_{\rm vp}(\dot{w}(t)), \quad \dot{z}(t) \geq 0. \tag{3.18}$$

On the other hand, since for all $t \in [0,T]$,

$$\begin{cases} \|\hat{e}_k(t) - e_k(t)\|_{L^2(\Omega;\mathbb{M}^{n\times n}_{sym})} \leq 2\delta_k \|\dot{\hat{e}}_k(t)\|_{L^2(\Omega;\mathbb{M}^{n\times n}_{sym})}, \\ \|\hat{p}_k(t) - p_k(t)\|_{L^2(\Omega;\mathbb{M}^{n\times n}_{sym})} \leq 2\delta_k \|\dot{\hat{p}}_k(t)\|_{L^2(\Omega;\mathbb{M}^{n\times n}_{sym})}, \\ \|\hat{z}_k(t) - z_k(t)\|_{L^2(\Omega)} \leq 2\delta_k \|\dot{\hat{z}}_k(t)\|_{L^2(\Omega)}, \end{cases} \tag{3.19}$$

the following convergences hold true:

$$\begin{cases} e_k \rightharpoonup e \text{ weakly* in } L^\infty(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym})),\\ p_k \rightharpoonup p \text{ weakly in } L^2(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym})),\\ z_k \rightharpoonup z \text{ weakly* in } L^\infty(0,T;L^2(\Omega)). \end{cases}$$

Finally, for all $t\in[0,T]$ we have $(\hat\sigma_k(t),\hat\xi_k(t)) = (\mathbb{C}\hat e_k(t), -\hat z_k(t)) \rightharpoonup (\mathbb{C}e(t),-z(t)) =: (\sigma(t),\xi(t))$ weakly in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\times L^2(\Omega)$. Since $(\hat\sigma_k(t),\hat\xi_k(t))\in K_\lambda$ a.e. in $\Omega$ by (3.12) and $K_\lambda$ is a closed and convex set, we infer that for all $t\in[0,T]$

$$(\sigma(t),\xi(t))\in K_\lambda \quad \text{a.e. in } \Omega. \tag{3.20}$$

3.3. **Weak formulation of the equation of motion.** At this stage we do not have enough time regularity on the velocity $\dot u$ to write the initial condition $\dot u(0)=v_0$. As usual in hyperbolic equations, this condition will be expressed by giving a weak formulation of the equation of motion.

**Proposition 3.5.** *For every $\varphi\in\mathcal{C}^\infty_c(\Omega\times[0,T);\mathbb{R}^n)$,*

$$-\int_0^T\int_\Omega \dot u\cdot\dot\varphi\,dx\,dt + \int_0^T\int_\Omega \big(\sigma+\varepsilon E\dot u\big):E\varphi\,dx\,dt = \int_0^T\int_\Omega f\cdot\varphi\,dx\,dt + \int_\Omega v_0\varphi(0)\,dx.$$

*Proof.* Let $\varphi\in\mathcal{C}^\infty_c(\Omega\times[0,T);\mathbb{R}^n)$ and define the right-continuous piecewise constant and piecewise affine interpolations by

$$\begin{cases} \varphi_k(t) := \varphi(t_k^{i-1}),\\ \hat\varphi_k(t) := \varphi(t_k^{i-1}) + \dfrac{t-t_k^i}{\delta_k}(\varphi(t_k^i)-\varphi(t_k^{i-1})), \end{cases} \qquad \text{for all } t\in[t_k^{i-1},t_k^i),\ i\in\{1,\dots,N_k\}.$$

We multiply the first equation in (3.10) by $\varphi(t_k^{i-1})$ and integrate by parts over $\Omega$. Since, for $k$ large enough, we have

$$\sum_{i=2}^{N_k}\delta_k\int_\Omega \frac{u_k^i-2u_k^{i-1}+u_k^{i-2}}{\delta_k^2}\cdot\varphi(t_k^{i-1})\,dx = -\sum_{i=1}^{N_k}\delta_k\int_\Omega\frac{u_k^i-u_k^{i-1}}{\delta_k}\cdot\frac{\varphi(t_k^i)-\varphi(t_k^{i-1})}{\delta_k}\,dx - \int_\Omega v_0\varphi(0)\,dx,$$

we deduce that

$$-\sum_{i=1}^{N_k}\delta_k\int_\Omega\frac{u_k^i-u_k^{i-1}}{\delta_k}\cdot\frac{\varphi(t_k^i)-\varphi(t_k^{i-1})}{\delta_k}\,dx + \sum_{i=2}^{N_k}\delta_k\int_\Omega\left(\sigma_k^i+\varepsilon\left(\frac{Eu_k^i-Eu_k^{i-1}}{\delta_k}\right)\right):E\varphi(t_k^{i-1})\,dx = \sum_{i=2}^{N_k}\delta_k\int_\Omega f_k^i\cdot\varphi(t_k^{i-1})\,dx + \int_\Omega v_0\varphi(0)\,dx,$$

hence,

$$-\int_0^T\int_\Omega \dot{\hat u}_k\cdot\dot{\hat\varphi}_k\,dx\,dt + \int_{\delta_k}^T\int_\Omega\big(\sigma_k+\varepsilon E\dot{\hat u}_k\big):E\varphi_k\,dx\,dt = \int_{\delta_k}^T\int_\Omega f_k\cdot\varphi_k\,dx\,dt + \int_\Omega v_0\varphi(0)\,dx.$$

Note that $\varphi_k\to\varphi$ strongly in $L^2(0,T;H^1(\Omega;\mathbb{R}^n))$, and $\dot{\hat\varphi}_k\to\dot\varphi$ strongly in $L^2(0,T;L^2(\Omega;\mathbb{R}^n))$. Thus since, by the absolute continuity of $t\mapsto f(t)$ in $L^2(\Omega;\mathbb{R}^n)$ we have $f_k\to f$ strongly in $L^2(0,T;L^2(\Omega;\mathbb{R}^n))$, we get the desired result by passing to the limit as $k\to\infty$ in the previous expression, and by using the weak convergences established for the sequences $(\hat u_k)$ and $(\sigma_k)$. □

**Remark 3.6.** As a consequence of Proposition 3.5, we have $\ddot{u} \in L^2(0,T;H^{-1}(\Omega;\mathbb{R}^n))$. Hence since $\dot{u}-\dot{w} \in L^2(0,T;H^1_0(\Omega;\mathbb{R}^n))$ and $\ddot{u}-\ddot{w} \in L^2(0,T;H^{-1}(\Omega;\mathbb{R}^n))$, we deduce from [18, Section 5.9, Theorem 3] that $\dot{u} \in \mathcal{C}([0,T];L^2(\Omega;\mathbb{R}^n))$. Therefore,

$$\ddot{u} - \operatorname{div}(\sigma + \varepsilon E\dot{u}) = f \quad \text{ in } L^2(0,T;H^{-1}(\Omega;\mathbb{R}^n)),$$

and the initial condition is expressed in the standard way

$$\dot{u}(0) = v_0. \tag{3.21}$$

3.4. **Flow rule.** We now derive the flow rule. To this end we need to improve the weak convergences established so far for the elastic strain and the hardening variable, into strong convergences. The strategy rests on an idea of [25]: it consists formally in multiplying the equations of motion by the velocity, both at the discrete and at the continuous time level, and in applying the discrete flow rule by taking $(\sigma(t),\xi(t))$ as competitor in the underlying variational inequality.

**Lemma 3.7.** *The following strong convergences hold:*

$$\begin{cases} e_k,\ \hat{e}_k \to e \text{ strongly in } L^2(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym})),\\ z_k,\ \hat{z}_k \to z \text{ strongly in } L^2(0,T;L^2(\Omega)).\end{cases}$$

*Proof.* For every $t \in (0,T]$ and for every $k$, we set $[t]_k := t^i_k$ where $i \in \{1,\dots,N_k\}$ is such that $t \in (t^{i-1}_k, t^i_k]$. According to Remark 3.6, we have for all $\varphi \in L^2(0,T;H^1_0(\Omega;\mathbb{R}^n))$,

$$\int_0^T \langle \ddot{u}(s), \varphi(s)\rangle\, ds + \int_0^T\int_\Omega (\sigma+\varepsilon E\dot{u}) : E\varphi\, dx\, ds = \int_0^T\int_\Omega f\cdot\varphi\, dx\, ds,$$

where the brackets $\langle\cdot,\cdot\rangle$ denote the duality pairing between $H^1_0(\Omega;\mathbb{R}^n)$ and $H^{-1}(\Omega;\mathbb{R}^n)$. Hence, taking $\varphi = (\dot{\hat{u}}_k - \dot{\hat{w}}_k)\chi_{[\delta_k,[t]_k]}$ as test function in the above relation and in the first equation of (3.13), and subtracting the resulting expressions yield

$$\int_{\delta_k}^{[t]_k} \langle \ddot{\hat{u}}_k(s) - \ddot{u}(s), \dot{\hat{u}}_k(s) - \dot{\hat{w}}_k(s)\rangle\, ds + \int_{\delta_k}^{[t]_k}\int_\Omega \big((\sigma_k + \varepsilon E\dot{\hat{u}}_k) - (\sigma+\varepsilon E\dot{u})\big) : (E\dot{\hat{u}}_k - E\dot{\hat{w}}_k)\, dx\, ds$$
$$= \int_{\delta_k}^{[t]_k}\int_\Omega (f_k - f)\cdot(\dot{\hat{u}}_k - \dot{\hat{w}}_k)\, dx\, ds.$$

By (3.15) the sequence $(\dot{\hat{u}}_k)$ is uniformly bounded in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^n))$. Since $f_k \to f$ strongly in $L^1(0,T;L^2(\Omega;\mathbb{R}^n))$, this implies that the integral in the right-handside of the previous equality tends to zero as $k\to\infty$. Moreover, the initial condition (3.7) and Proposition 3.4 ensure that

$$\lim_{k\to\infty}\Bigg(\int_{\delta_k}^{[t]_k} \langle \ddot{\hat{u}}_k(s) - \ddot{u}(s), \dot{\hat{u}}_k(s) - \dot{\hat{w}}_k(s)\rangle\, ds$$
$$+ \int_0^{[t]_k}\int_\Omega \big((\sigma_k + \varepsilon E\dot{\hat{u}}_k) - (\sigma+\varepsilon E\dot{u})\big) : (E\dot{\hat{u}}_k - E\dot{\hat{w}}_k)\, dx\, ds\Bigg) = 0.$$

According to (3.13) and (3.15), the sequence $\big(\|\ddot{\hat{u}}_k\|_{L^2(\delta_k,T;H^{-1}(\Omega;\mathbb{R}^n))}\big)_{k\in\mathbb{N}}$ is uniformly bounded with respect to $k$. Thus since $\sigma_k \rightharpoonup \sigma$ and $E\dot{\hat{u}}_k \rightharpoonup E\dot{u}$ weakly in $L^2(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$, and $E\dot{\hat{w}}_k \to E\dot{w}$ strongly in $L^2(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$, we infer by dominated convergence that

$$\lim_{k\to\infty}\int_0^T\Bigg(\int_{\delta_k}^{[t]_k} \langle \ddot{\hat{u}}_k(s) - \ddot{u}(s), \dot{\hat{u}}_k(s) - \dot{\hat{w}}_k(s)\rangle\, ds + \int_0^{[t]_k}\int_\Omega (\sigma_k - \sigma) : E\dot{\hat{u}}_k\, dx\, ds$$
$$+ \varepsilon\int_0^{[t]_k}\int_\Omega |E\dot{\hat{u}}_k - E\dot{u}|^2\, dx\, ds\Bigg)\, dt = 0. \tag{3.22}$$

We now estimate the first two integrals of (3.22). Let us start with

$$I_k^1 := \int_0^T \int_0^{[t]_k} \int_\Omega (\sigma_k - \sigma) : E\dot{\hat{u}}_k \, dx\, ds\, dt.$$

By the kinematic compatibility $E\dot{\hat{u}}_k = \dot{\hat{e}}_k + \dot{\hat{p}}_k$ a.e. in $\Omega \times [0,T]$ we have

$$\int_0^{[t]_k} \int_\Omega (\sigma_k - \sigma) : E\dot{\hat{u}}_k \, dx\, dt = \int_0^{[t]_k} \int_\Omega (\sigma_k - \sigma) : (\dot{\hat{e}}_k + \dot{\hat{p}}_k)\, dx\, dt,$$

but since $(\dot{\hat{p}}_k, \dot{\hat{z}}_k) \in N_{K_\lambda}(\sigma_k, \xi_k)$ by (3.13) and $(\sigma, \xi) \in K_\lambda$ by (3.20), we deduce that

$$\int_0^{[t]_k} \int_\Omega (\sigma_k - \sigma) : E\dot{\hat{u}}_k \, dx\, dt \geq \int_0^{[t]_k} \int_\Omega (\sigma_k - \sigma) : \dot{\hat{e}}_k \, dx\, dt - \int_0^{[t]_k} \int_\Omega (\xi_k - \xi)\dot{\hat{z}}_k \, dx\, dt.$$

Hence, recalling that $\xi = -z$ and $\xi_k = -z_k$, we have

$$\limsup_{k\to\infty} I_k^1 \geq \limsup_{k\to\infty} \left( \int_0^T \int_0^{[t]_k} \int_\Omega (\sigma_k - \sigma) : \dot{\hat{e}}_k \, dx\, ds\, dt + \int_0^T \int_0^{[t]_k} \int_\Omega (z_k - z) : \dot{\hat{z}}_k \, dx\, ds\, dt \right).$$

Using the fact that $\sigma_k \rightharpoonup \sigma$ weakly in $L^2(0,T; L^2(\Omega; \mathbb{M}^{n\times n}_{sym}))$, $z_k \rightharpoonup z$ weakly in $L^2(0,T; L^2(\Omega))$, and that $\sigma_k - \hat{\sigma}_k \to 0$ strongly in $L^2(0,T; L^2(\Omega; \mathbb{M}^{n\times n}_{sym}))$ and $z_k - \hat{z}_k \to 0$ strongly in $L^2(0,T; L^2(\Omega))$, we obtain thanks to the bounds (3.16)

$$\begin{aligned}
&\limsup_{k\to\infty} I_k^1 \\
&\geq \limsup_{k\to\infty} \left( \int_0^T \int_0^{[t]_k} \int_\Omega (\hat{\sigma}_k - \sigma) : (\dot{\hat{e}}_k - \dot{e})\, dx\, ds\, dt + \int_0^T \int_0^{[t]_k} \int_\Omega (\hat{z}_k - z) : (\dot{\hat{z}}_k - \dot{z})\, dx\, ds\, dt \right) \\
&\qquad = \limsup_{k\to\infty} \left( \int_0^T \mathcal{Q}(\hat{e}_k([t]_k) - e([t]_k))\, dt + \frac{1}{2}\int_0^T \|\hat{z}_k([t]_k) - z([t]_k)\|_2^2\, dt \right),
\end{aligned}$$

because $\hat{e}_k(0) - e(0) = 0$ and $\hat{z}_k(0) - z(0) = 0$. Since $\hat{e}_k([t]_k) = e_k(t)$, $\hat{z}_k([t]_k) = z_k(t)$ and $e \in H^1([0,T]; L^2(\Omega; \mathbb{M}^{n\times n}_{sym}))$, $z \in H^1([0,T]; L^2(\Omega))$, we conclude that

$$\limsup_{k\to\infty} I_k^1 \geq \limsup_{k\to\infty} \left( \int_0^T \mathcal{Q}(e_k(t) - e(t))\, dt + \frac{1}{2}\int_0^T \|z_k(t) - z(t)\|_2^2\, dt \right). \tag{3.23}$$

We now estimate

$$I_k^2 := \int_0^T \int_{\delta_k}^{[t]_k} \langle \ddot{\hat{u}}_k(s) - \ddot{u}(s), \dot{\hat{u}}_k(s) - \dot{\hat{w}}_k(s) \rangle\, ds\, dt.$$

Let us further split the previous integral as

$$I_k^2 = \int_0^T \int_{\delta_k}^{[t]_k} \langle \ddot{\hat{w}}_k(s) - \ddot{u}(s), \dot{\hat{u}}_k(s) - \dot{\hat{w}}_k(s) \rangle\, ds\, dt + \int_0^T \int_{\delta_k}^{[t]_k} \langle \ddot{\hat{u}}_k(s) - \ddot{\hat{w}}_k(t), \dot{\hat{u}}_k(s) - \dot{\hat{w}}_k(s) \rangle\, ds\, dt.$$

Since $\ddot{\hat{w}}_k \chi_{[\delta_k, [t]_k]} \to \ddot{w}\chi_{[0,t]}$ strongly in $L^2(0,T; L^2(\Omega; \mathbb{R}^n))$, while $\dot{\hat{u}}_k - \dot{\hat{w}}_k \rightharpoonup \dot{u} - \dot{w}$ weakly in $L^2(0,T; H^1_0(\Omega; \mathbb{R}^n))$, we get that

$$\begin{aligned}
\lim_{k\to\infty} \int_0^T \int_{\delta_k}^{[t]_k} \langle \ddot{\hat{w}}_k(s) - \ddot{u}(s), \dot{\hat{u}}_k(s) - \dot{\hat{w}}_k(s) \rangle\, ds\, dt &= \int_0^T \int_0^t \langle \ddot{w}(s) - \ddot{u}(s), \dot{u}(s) - \dot{w}(s) \rangle\, ds\, dt \\
&= \int_0^T \left( \frac{\|\dot{u}(0) - \dot{w}(0)\|_2^2}{2} - \frac{\|\dot{u}(t) - \dot{w}(t)\|_2^2}{2} \right) dt, \quad (3.24)
\end{aligned}$$

by [18, Section 5.9, Theorem 3] and Remark 3.6. On the other hand, if $s \in (t_k^{i-1}, t_k^i]$ (for some $i \in \{2, \dots, N_k\}$), then

$$\begin{aligned}
&\langle \ddot{\tilde{u}}_k(s) - \ddot{\tilde{w}}_k(s), \dot{\hat{u}}_k(s) - \dot{\hat{w}}_k(s) \rangle \\
&\quad = \int_\Omega \Big( \frac{u_k^i - 2u_k^{i-1} + u_k^{i-2}}{\delta_k^2} - \frac{w_k^i - 2w_k^{i-1} + w_k^{i-2}}{\delta_k^2} \Big) \cdot \Big( \frac{u_k^i - u_k^{i-1}}{\delta_k} - \frac{w_k^i - w_k^{i-1}}{\delta_k} \Big)\, dx \\
&\qquad \geq \frac{1}{2\delta_k} \Big\| \frac{u_k^i - u_k^{i-1}}{\delta_k} - \frac{w_k^i - w_k^{i-1}}{\delta_k} \Big\|_2^2 - \frac{1}{2\delta_k} \Big\| \frac{u_k^{i-1} - u_k^{i-2}}{\delta_k} - \frac{w_k^{i-1} - w_k^{i-2}}{\delta_k} \Big\|_2^2 .
\end{aligned}$$

Hence, summing up for $i = 2$ to $j$, where $j \in \{2, \dots, N_k\}$ is such that $t_k^j = [t]_k$, leads to

$$\int_{\delta_k}^{[t]_k} \langle \ddot{\tilde{u}}_k(s) - \ddot{\tilde{w}}_k(s), \dot{\hat{u}}_k(s) - \dot{\hat{w}}_k(s) \rangle \, ds \geq \frac{\|\dot{\hat{u}}_k(t) - \dot{\hat{w}}_k(t)\|_2^2}{2} - \frac{\|v_0 - \dot{w}(0)\|_2^2}{2},$$

where we used (3.7). Integrating this last inequality between 0 and $T$, and taking the liminf as $k \to \infty$ yield

$$\liminf_{k\to\infty} \int_0^T \int_{\delta_k}^{[t]_k} \langle \ddot{\tilde{u}}_k(s) - \ddot{\tilde{w}}_k(s), \dot{\hat{u}}_k(s) - \dot{\hat{w}}_k(s) \rangle \, ds \, dt \geq \int_0^T \left( \frac{\|\dot{u}(t) - \dot{w}(t)\|_2^2}{2} - \frac{\|v_0 - \dot{w}(0)\|_2^2}{2} \right) dt. \tag{3.25}$$

Gathering (3.24) and (3.25) together with the velocity initial condition (3.21) leads to

$$\liminf_{k\to\infty} I_k^2 \geq 0. \tag{3.26}$$

Finally, in view of (3.22), (3.23), and (3.26), we obtain that

$$\lim_{k\to\infty} \int_0^T \left( \|e_k(t) - e(t)\|_2^2 + \|z_k(t) - z(t)\|_2^2 \right) dt = 0.$$

Eventually the strong convergences of the sequences $(\hat{e}_k)$ and $(\hat{z}_k)$ follow from (3.19) and (3.16). □

We are now in position to derive the flow rule.

**Corollary 3.8.** *For a.e.* $t \in [0, T]$,

$$(\dot{p}(t), \dot{z}(t)) \in N_{K_\lambda}(\sigma(t), \xi(t)) \qquad a.e.\ in\ \Omega.$$

*Proof.* According to (3.13) we have that for all $t \in [\delta_k, T]$ and a.e. $x \in \Omega$,

$$(\dot{\hat{p}}_k(t), \dot{\hat{z}}_k(t)) \in N_{K_\lambda}(\sigma_k(t), \xi_k(t)).$$

Thus for all $(\hat{\sigma}, \hat{\xi}) \in L^2(\Omega \times (0, T); K_\lambda)$ we have that

$$\int_{\delta_k}^T \int_\Omega \left( (\sigma_k - \hat{\sigma}) : \dot{\hat{p}}_k + (\xi_k - \hat{\xi}) \dot{\hat{z}}_k \right) dx \, dt \geq 0.$$

By Lemma 3.7 we have that $\sigma_k \to \sigma$ strongly in $L^2(0, T; L^2(\Omega; \mathbb{M}^{n\times n}_{sym}))$ and $\xi_k \to \xi$ strongly in $L^2(0, T; L^2(\Omega))$, while (3.17) ensures that $\dot{\hat{p}}_k \rightharpoonup \dot{p}$ weakly in $L^2(0, T; L^2(\Omega; \mathbb{M}^{n\times n}_{sym}))$ and $\dot{\hat{z}}_k \rightharpoonup \dot{z}$ weakly in $L^2(0, T; L^2(\Omega))$. Therefore, passing to the limit in the previous inequality yields

$$\int_0^T \int_\Omega \left( (\sigma - \hat{\sigma}) : \dot{p} + (\xi - \hat{\xi}) \dot{z} \right) dx \, dt \geq 0,$$

and the result follows by a standard localization argument. □

**Remark 3.9.** Since $N_{K_\lambda} = \partial I_{K_\lambda}$ and $(\sigma(t), \xi(t)) \in K_\lambda$ a.e. in $\Omega$ for every $t \in [0, T]$, we deduce by convex duality that the flow rule is equivalent to each one of the following formulations:

(i) for a.e. $x \in \Omega$ and a.e. $t \in [0,T]$,

$$(\dot p(t), \dot z(t)) \in \partial I_{K_\lambda}(\sigma(t), \xi(t));$$

(ii) for a.e. $x \in \Omega$ and a.e. $t \in [0,T]$,

$$(\sigma(t), \xi(t)) \in \partial H_\lambda(\dot p(t), \dot z(t));$$

(iii) for a.e. $x \in \Omega$ and a.e. $t \in [0,T]$,

$$H_\lambda(\dot p(t), \dot z(t)) = \sigma(t) : \dot p(t) + \xi(t)\dot z(t);$$

(iv) for a.e. $x \in \Omega$ and a.e. $t \in [0,T]$,

$$\sigma(t) : \dot p(t) + \xi(t)\dot z(t) \geq \tau : \dot p(t) + \eta : \dot z(t) \quad \text{for every } (\tau, \eta) \in K_\lambda.$$

Note that condition (iv) is precisely Hill's principle of maximum plastic work.

3.5. **Uniqueness of the solution.** So far we have established the existence of weak solutions to the dynamical visco-plastic cap model described in Theorem 3.1. We now show the uniqueness.

Let us consider two solutions $(u_1, e_1, p_1, z_1, \sigma_1, \xi_1)$ and $(u_2, e_2, p_2, z_2, \sigma_2, \xi_2)$. Subtracting the equations of motions leads to

$$\ddot u_1 - \ddot u_2 - \operatorname{div}\big((\sigma_1 + \varepsilon E\dot u_1) - (\sigma_2 + \varepsilon E\dot u_2)\big) = 0 \quad \text{ in } L^2(0,T; H^{-1}(\Omega;\mathbb{R}^n)),$$

and since $\dot u_1 - \dot u_2 \in L^2(0,T; H^1_0(\Omega;\mathbb{R}^n))$, we infer that

$$\int_0^t \langle \ddot u_1(s) - \ddot u_2(s), \dot u_1(s) - \dot u_2(s)\rangle\, ds + \int_0^t \int_\Omega (\sigma_1 - \sigma_2) : (E\dot u_1 - E\dot u_2)\, dx\, ds + \varepsilon \int_0^t \int_\Omega |E\dot u_1 - E\dot u_2|^2\, dx\, ds = 0. \tag{3.27}$$

Since $\ddot u_1 - \ddot u_2 \in L^2(0,T; H^{-1}(\Omega;\mathbb{R}^n))$, we get from [18, Section 5.9, Theorem 3] that

$$\int_0^t \langle \ddot u_1(s) - \ddot u_2(s), \dot u_1(s) - \dot u_2(s)\rangle\, ds = \frac{\|\dot u_1(t) - \dot u_2(t)\|_2^2}{2} \tag{3.28}$$

since $\dot u_1(0) = \dot u_2(0) = v_0$. On the other hand, since $E\dot u_1 - E\dot u_2 = (\dot e_1 - \dot e_2) + (\dot p_1 - \dot p_2)$ by kinematic compatibility, we obtain

$$\begin{aligned}
\int_0^t \int_\Omega (\sigma_1 - \sigma_2) : (E\dot u_1 - E\dot u_2)\, dx\, ds
&= \int_0^t \int_\Omega (\sigma_1 - \sigma_2) : (\dot e_1 - \dot e_2)\, dx\, ds + \int_0^t \int_\Omega (\sigma_1 - \sigma_2) : (\dot p_1 - \dot p_2)\, dx\, ds \\
&= \mathcal{Q}(e_1(t) - e_2(t)) + \int_0^t \int_\Omega (\sigma_1 - \sigma_2) : (\dot p_1 - \dot p_2)\, dx\, ds,
\end{aligned} \tag{3.29}$$

since $e_1(0) = e_2(0) = e_0$. In order to estimate the last integral we use the flow rule as well as the fact that $(\sigma_1, \xi_1)$, $(\sigma_2, \xi_2) \in K_\lambda$ a.e. in $\Omega \times (0,T)$. Indeed,

$$\begin{aligned}
\int_0^t \int_\Omega (\sigma_1 - \sigma_2) : \dot p_1\, dx\, ds &\geq -\int_0^t \int_\Omega (\xi_1 - \xi_2)\dot z_1\, dx\, ds = \int_0^t \int_\Omega (z_1 - z_2)\dot z_1\, dx\, ds, \\
\int_0^t \int_\Omega (\sigma_2 - \sigma_1) : \dot p_2\, dx\, ds &\geq -\int_0^t \int_\Omega (\xi_2 - \xi_1)\dot z_2\, dx\, ds = \int_0^t \int_\Omega (z_2 - z_1)\dot z_2\, dx\, ds,
\end{aligned}$$

and summing up, we deduce that

$$\int_0^t \int_\Omega (\sigma_1 - \sigma_2) : (\dot p_1 - \dot p_2)\, dx\, ds \geq \int_0^t \int_\Omega (z_1 - z_2)(\dot z_1 - \dot z_2)\, dx\, ds = \frac{\|z_1(t) - z_2(t)\|_2^2}{2}, \tag{3.30}$$

since $z_1(0) = z_2(0) = z_0$. Gathering (3.27)–(3.30) yields $e_1 = e_2$ (hence $\sigma_1 = \sigma_2$), $z_1 = z_2$ (hence $\xi_1 = \xi_2$) and $\dot u_1 = \dot u_2$. But since $u_1(0) = u_2(0) = u_0$, we deduce that $u_1 = u_2$ and finally that $p_1 = p_2$.

**Remark 3.10.** Owing to the uniqueness of the solution, there is actually no need to extract subsequences in all weak and strong convergences obtained so far.

3.6. *A posteriori* **estimates.** The object of this subsection is to prove some higher time regularity of the velocity $\dot u$, the stress $\sigma$, and the hardening cap variable $\xi$ with uniform estimates with respect to $\varepsilon$ and $\lambda$. This is essential to give a pointwise meaning to the equation of motion. The argument of proof is based on the difference quotient method and exploits the monotone character of the system in a similar way to [25]. Note that the assumption on the initial velocity $v_0 \in H^2(\Omega;\mathbb{R}^n)$ is crucial in the proof of this result.

**Proposition 3.11.** *There exists a constant $C > 0$ (independent of $\varepsilon$ and $\lambda$) such that*

$$\begin{aligned}\sup_{t\in[0,T]} \|\ddot u(t)\|_2^2 + \varepsilon \int_0^T \int_\Omega |E\ddot u|^2\,dx\,ds + \sup_{t\in[0,T]} \|\dot\sigma(t)\|_2^2 + \sup_{t\in[0,T]} \|\dot z(t)\|_2^2 \\ \le C\Big(\varepsilon^2\|\Delta v_0\|_2^2 + \|Ev_0\|_2^2 + \|\dot f\|^2_{L^1(0,T;L^2(\Omega;\mathbb{R}^n))} \\ + \|\dddot w\|^2_{L^1(0,T;L^2(\Omega;\mathbb{R}^n))} + \|E\ddot w\|^2_{L^2(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))}\Big).\end{aligned} \tag{3.31}$$

*Proof.* Let us extend continuously the fields for $s < 0$ by setting

$$\begin{cases} u(s) = u_0 + sv_0,\ w(s) = w(0) + s\dot w(0),\ f(s) = f(0), \\ e(s) = e_0,\ \sigma(s) = \sigma_0, \\ p(s) = p_0, \\ z(s) = z_0,\ \xi(s) = \xi_0. \end{cases} \tag{3.32}$$

In the following we will use the notation

$$\partial_t^h g(s) := \frac{g(s) - g(s-h)}{h}.$$

Let $t \in [0,T]$ and $h < t$. Using the equation of motion, we have for all $\varphi \in L^2(0,T+h;H^1_0(\Omega;\mathbb{R}^n))$,

$$\begin{aligned}&\int_0^T \langle \ddot u(s), \varphi(s)\rangle\,ds + \int_0^T \int_\Omega \big(\sigma(s) + \varepsilon E\dot u(s)\big) : E\varphi(s)\,dx\,ds = \int_0^T\int_\Omega f(s)\cdot\varphi(s)\,dx\,ds, \\ &\int_h^{T+h} \langle \ddot u(s-h), \varphi(s)\rangle\,ds + \int_h^{T+h} \int_\Omega \big(\sigma(s-h) + \varepsilon E\dot u(s-h)\big) : E\varphi(s)\,dx\,ds \\ &\qquad = \int_h^{T+h}\int_\Omega f(s-h)\cdot\varphi(s)\,dx\,ds.\end{aligned}$$

Taking the difference of the two previous equalities with the test function $\varphi = \chi_{[0,t]}\partial_t^h(\dot u - \dot w)$ yields

$$\int_0^t \langle \partial_t^h \ddot u(s), \partial_t^h(\dot u - \dot w)(s)\rangle\, ds + \int_0^t\int_\Omega \partial_t^h(\sigma + \varepsilon E\dot u)(s) : E\big(\partial_t^h(\dot u - \dot w)\big)(s)\, dx\, ds$$
$$- \int_0^t\int_\Omega \partial_t^h f(s)\cdot \partial_t^h(\dot u - \dot w)\, dx\, ds$$
$$= \frac{1}{h}\int_0^h\int_\Omega \big[f(0)\cdot \partial_t^h(\dot u - \dot w)(s) - (\sigma_0 + \varepsilon E v_0) : E\big(\partial_t^h(\dot u - \dot w)\big)(s)\big]\, dx\, ds$$
$$= \frac{1}{h}\int_0^h\int_\Omega \varepsilon \Delta v_0 \cdot \partial_t^h(\dot u - \dot w)(s)\, dx\, ds,$$

where we used the initial condition $-\mathrm{div}\sigma_0 = f(0)$ a.e. in $\Omega$. Hence thanks to the Cauchy-Schwarz inequality,

$$\int_0^t \langle \partial_t^h \ddot u(s), \partial_t^h(\dot u - \dot w)(s)\rangle\, ds + \int_0^t\int_\Omega \partial_t^h(\sigma + \varepsilon E\dot u)(s) : E\big(\partial_t^h(\dot u - \dot w)\big)(s)\, dx\, ds$$
$$\leq \big(\varepsilon\|\Delta v_0\|_2 + \|\partial_t^h f\|_{L^1(0,T;L^2(\Omega;\mathbb{R}^n))}\big) \sup_{s\in[0,T]} \|\partial_t^h(\dot u - \dot w)(s)\|_2. \quad (3.33)$$

Next using the kinematic compatibility for the rates $E\dot u = \dot e + \dot p$ a.e. on $\Omega\times[0,T]$, we get for all $\tau \in L^2(0,T+h;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$,

$$\int_0^T\int_\Omega E\dot u(s) : \tau(s)\, dx\, ds = \int_0^T\int_\Omega \dot e(s) : \tau(s)\, dx\, ds + \int_0^T\int_\Omega \dot p(s) : \tau(s)\, dx\, ds,$$
$$\int_h^{T+h}\int_\Omega E\dot u(s-h) : \tau(s)\, dx\, ds = \int_h^{T+h}\int_\Omega \dot e(s-h) : \tau(s)\, dx\, ds + \int_h^{T+h}\int_\Omega \dot p(s-h) : \tau(s)\, dx\, ds.$$

Taking the difference of the two previous relations with the test function $\tau = \chi_{[0,t]}(\partial_t^h \sigma)$ yields

$$\int_0^t\int_\Omega E\partial_t^h \dot u(s) : \partial_t^h\sigma(s)\, dx\, ds - \int_0^t\int_\Omega \partial_t^h \dot e(s) : \partial_t^h\sigma(s)\, dx\, ds - \int_0^t\int_\Omega \partial_t^h \dot p(s) : \partial_t^h\sigma(s)\, dx\, ds$$
$$= -\frac{1}{h}\int_0^h\int_\Omega E v_0 : \partial_t^h\sigma(s)\, dx\, ds \leq \|Ev_0\|_2 \sup_{s\in[0,h]}\|\partial_t^h\sigma(s)\|_2. \quad (3.34)$$

According to the flow rule, since for a.e. $s\in(0,T)$,

$$(\dot p(s), \dot z(s)) \in N_{K_\lambda}(\sigma(s), \xi(s)), \quad (\dot p(s-h), \dot z(s-h)) \in N_{K_\lambda}(\sigma(s-h), \xi(s-h)),$$

we have

$$\int_0^t\int_\Omega \partial_t^h \dot p(s) : \partial_t^h \sigma(s)\, dx\, ds$$
$$= \frac{1}{h^2}\int_0^t\int_\Omega \dot p(s) : \big(\sigma(s) - \sigma(s-h)\big)\, dx\, ds + \frac{1}{h^2}\int_0^t\int_\Omega \dot p(s-h) : \big(\sigma(s-h) - \sigma(s)\big)\, dx\, ds$$
$$\geq -\frac{1}{h^2}\int_0^t\int_\Omega \dot z(s)\big(\xi(s) - \xi(s-h)\big)\, dx\, ds - \frac{1}{h^2}\int_0^t\int_\Omega \dot z(s-h)\big(\xi(s-h) - \xi(s)\big)\, dx\, ds$$
$$= \frac{1}{h^2}\int_0^t\int_\Omega \dot z(s)\big(z(s) - z(s-h)\big)\, dx\, ds + \frac{1}{h^2}\int_0^t\int_\Omega \dot z(s-h)\big(z(s-h) - z(s)\big)\, dx\, ds$$
$$= \int_0^t\int_\Omega \partial_t^h \dot z(s)\partial_t^h z(s)\, dx\, ds. \quad (3.35)$$

Gathering (3.33)–(3.35), we obtain that

$$\begin{aligned}
&\int_0^t \langle \partial_t^h(\ddot u-\ddot w)(s), \partial_t^h(\dot u-\dot w)(s)\rangle\, ds + \varepsilon \int_0^t\int_\Omega |\partial_t^h(E\dot u)|^2\, dx\, ds\\
&\qquad + \int_0^t\int_\Omega \partial_t^h \dot e(s) : \partial_t^h\sigma(s)\, dx\, ds + \int_0^t\int_\Omega \partial_t^h\dot z(s)\partial_t^h z(s)\, dx\, ds\\
&\quad\le \big(\varepsilon\|\Delta v_0\|_2 + \|\partial_t^h f\|_{L^1(0,T;L^2(\Omega;\mathbb{R}^n))} + \|\partial_t^h \ddot w\|_{L^1(0,T;L^2(\Omega;\mathbb{R}^n))}\big) \sup_{s\in[0,T]} \|\partial_t^h(\dot u-\dot w)(s)\|_2\\
&\qquad + \Big(\|Ev_0\|_2 + \|E(\partial_t^h\dot w)\|_{L^1(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))}\Big) \sup_{s\in[0,T]} \|\partial_t^h\sigma(s)\|_2\\
&\qquad\qquad + \varepsilon\|E(\partial_t^h\dot w)\|_{L^2(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))}\|E(\partial_t^h\dot u)\|_{L^2(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))},
\end{aligned}$$

and thus, since $\partial_t^h(\dot u-\dot w)\in L^2(0,T;H^1_0(\Omega;\mathbb{R}^n))$ and $\partial_t^h(\ddot u-\ddot w)\in L^2(0,T;H^{-1}(\Omega;\mathbb{R}^n))$, we get from [18, Section 5.9, Theorem 3] that

$$\begin{aligned}
&\frac{\|\partial_t^h(\dot u-\dot w)(t)\|_2^2}{2} + \varepsilon\int_0^t\int_\Omega |\partial_t^h(E\dot u)|^2\, dx\, ds + \mathcal{Q}(\partial_t^h e(t)) + \frac{\|\partial_t^h z(t)\|_2^2}{2}\\
&\quad\le \big(\varepsilon\|\Delta v_0\|_2 + \|\partial_t^h f\|_{L^1(0,T;L^2(\Omega;\mathbb{R}^n))} + \|\partial_t^h \ddot w\|_{L^1(0,T;L^2(\Omega;\mathbb{R}^n))}\big) \sup_{s\in[0,T]} \|\partial_t^h(\dot u-\dot w)(s)\|_2\\
&\qquad + \Big(\|Ev_0\|_2 + \|E(\partial_t^h\dot w)\|_{L^1(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))}\Big) \sup_{s\in[0,T]} \|\partial_t^h\sigma(s)\|_2\\
&\qquad + \varepsilon\|E(\partial_t^h\dot w)\|_{L^2(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))}\|E(\partial_t^h\dot u)\|_{L^2(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))}\\
&\qquad\qquad + \frac{\|\partial_t^h(\dot u-\dot w)(0)\|_2^2}{2} + \mathcal{Q}(\partial_t^h e(0)) + \frac{\|\partial_t^h z(0)\|_2^2}{2}.
\end{aligned}$$

Hence, applying Young's inequality, and according to the choice of the extensions (3.32), we obtain

$$\begin{aligned}
&\sup_{t\in[0,T]}\|\partial_t^h(\dot u-\dot w)(t)\|_2^2 + \varepsilon\int_0^T\int_\Omega |\partial_t^h(E\dot u)|^2\, dx\, ds + \sup_{t\in[0,T]}\|\partial_t^h\sigma(t)\|_2^2 + \sup_{t\in[0,T]}\|\partial_t^h z(t)\|_2^2\\
&\quad\le C\Big(\varepsilon^2\|\Delta v_0\|_2^2 + \|\partial_t^h f\|^2_{L^1(0,T;L^2(\Omega;\mathbb{R}^n))} + \|\partial_t^h\ddot w\|^2_{L^1(0,T;L^2(\Omega;\mathbb{R}^n))} + \|Ev_0\|_2^2\\
&\qquad\qquad + \|E(\partial_t^h\dot w)\|^2_{L^2(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))}\Big),
\end{aligned}$$

for some constant $C>0$ independent of $\varepsilon$, $\lambda$ and $h$. Letting $h\to 0$ leads to

$$\begin{aligned}
&\sup_{t\in[0,T]}\|\ddot u(t)\|_2^2 + \varepsilon\int_0^T\int_\Omega |E\ddot u|^2\, dx\, ds + \sup_{t\in[0,T]}\|\dot\sigma(t)\|_2^2 + \sup_{t\in[0,T]}\|\dot z(t)\|_2^2\\
&\le C\Big(\varepsilon^2\|\Delta v_0\|_2^2 + \|\dot f\|^2_{L^1(0,T;L^2(\Omega;\mathbb{R}^n))} + \|Ev_0\|_2^2 + \|\dddot w\|^2_{L^1(0,T;L^2(\Omega;\mathbb{R}^n))} + \|E\ddot w\|^2_{L^2(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))}\Big),
\end{aligned}$$

which is (3.31). □

## 4. The dynamical elasto-plastic cap model

In this section, we pass to the limit as the viscosity parameter $\varepsilon$ tends to zero in order to recover a solution for the dynamical elasto-plastic cap model (1.4) from the visco-plastic evolutions obtained in Theorem 3.1. In this case, due to a lack of coercivity in reflexive Sobolev spaces, the space of kinematically admissible fields needs to be relaxed in the following way: given a boundary displacement $\hat{w} \in H^1(\Omega;\mathbb{R}^n)$, we set

$$\mathcal{A}_{\rm dyn}(\hat{w}) := \Big\{(v,\eta,q) \in (BD(\Omega)\cap L^2(\Omega;\mathbb{R}^n)) \times L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \times \mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym}) : \\ Ev = \eta + q \text{ in } \Omega, \quad q = (\hat{w}-v)\odot \nu \mathcal{H}^{n-1} \text{ on } \partial\Omega\Big\}.$$

Indeed, it may happen that a kinematically admissible displacement $v$ does not match the prescribed boundary value $\hat{w}$ on some portion of the boundary (of positive $\mathcal{H}^{n-1}$ measure). In that case, on this portion of the boundary a plastic strain $q$ must develop, compatible with the fact that $q$ is the jump part of the measure $Ev$.

The main result of this section is the following existence and uniqueness result for a dynamical elasto-plastic cap model, obtained as a vanishing viscosity limit of the visco-plastic cap evolutions constructed in Theorem 3.1.

**Theorem 4.1.** *Let $\lambda \geq 1$. Assume (2.1)–(2.4) and (3.1)–(3.3). Then there exist unique*

$$\begin{cases} u \in AC([0,T];BD(\Omega)) \cap W^{2,\infty}([0,T];L^2(\Omega;\mathbb{R}^n)), \\ \sigma,\ e \in W^{1,\infty}([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_{sym})), \\ p \in AC([0,T];\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym})), \\ \xi,\ z \in W^{1,\infty}([0,T];L^2(\Omega)), \end{cases}$$

*with the following properties: for all $t \in [0,T]$,*

$$\begin{cases} Eu(t) = e(t) + p(t) \text{ in } \Omega, \quad p(t) = (w(t)-u(t))\odot\nu\mathcal{H}^{n-1} \text{ on } \partial\Omega, \\ \sigma(t) = \mathbb{C}e(t), \quad \xi(t) = -z(t), \\ (\sigma(t),\xi(t)) \in K_\lambda \text{ a.e. in } \Omega, \end{cases} \tag{4.1}$$

*and*

$$\begin{cases} \ddot{u} - \mathrm{div}\sigma = f \quad \text{a.e. in } \Omega\times(0,T), \\ (u(0),e(0),p(0),z(0)) = (u_0,e_0,p_0,z_0), \quad \dot{u}(0) = v_0. \end{cases} \tag{4.2}$$

*Moreover, for a.e. $t \in [0,T]$,*

$$\dot{z}(t) \geq 0 \quad \text{a.e. in } \Omega \tag{4.3}$$

*and the distribution $[\sigma(t):\dot{p}(t)]$ is a measure in $\mathcal{M}(\overline{\Omega})$ satisfying*

$$H_\lambda(\dot{p}(t),\dot{z}(t)) = [\sigma(t):\dot{p}(t)] + \xi(t)\dot{z}(t) \quad \text{in } \mathcal{M}(\overline{\Omega}). \tag{4.4}$$

**Remark 4.2.** Within the proof of Theorem 4.1, we will also prove the following bound: there exists a constant $C>0$ (independent of $\lambda$) such that

$$\|\ddot{u}\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^n))} + \|\dot{e}\|_{L^\infty(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))} + \|\dot{z}\|_{L^\infty(0,T;L^2(\Omega))} \\ \leq C\Big(\|Ev_0\|_2 + \|\dot{f}\|_{L^1(0,T;L^2(\Omega;\mathbb{R}^n))} + \|\dddot{w}\|_{L^1(0,T;L^2(\Omega;\mathbb{R}^n))} + \|E\ddot{w}\|_{L^2(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))}\Big). \tag{4.5}$$

The remaining of the section is devoted to the proof of Theorem 4.1. In Subsection 4.1 we prove compactness of the dynamical visco-plastic evolutions. The flow rule is derived in Subsection 4.2, while uniqueness is shown in Subsection 4.3

4.1. *A priori* **estimates and compactness.** In order to apply the result of Theorem 3.1, we first need to regularize the initial data. According to [11, Lemma 5.1], there exists a sequence $(u_{0,\varepsilon}) \subset H^1(\Omega;\mathbb{R}^n)$ such that $u_{0,\varepsilon} = w(0)$ $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$, $u_{0,\varepsilon} \to u_0$ strongly in $L^1(\Omega;\mathbb{R}^n)$, and $Eu_{0,\varepsilon} \rightharpoonup Eu_0$ weakly* in $\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym})$. Setting $p_{0,\varepsilon} = Eu_{0,\varepsilon} - e_0$, we get that $(u_{0,\varepsilon},e_0,p_{0,\varepsilon}) \in \mathcal{A}_{\rm vp}(w(0))$. Moreover, using a standard approximation argument we can find a sequence $(v_{0,\varepsilon}) \subset H^2(\Omega;\mathbb{R}^n)$ such that $v_{0,\varepsilon} \to v_0$ strongly in $H^1(\Omega;\mathbb{R}^n)$ and $\varepsilon\Delta v_{0,\varepsilon} \to 0$ strongly in $L^2(\Omega;\mathbb{R}^n)$. For every $\varepsilon > 0$

let $u_\varepsilon$, $e_\varepsilon$, $\sigma_\varepsilon$, $p_\varepsilon$, $z_\varepsilon$, $\xi_\varepsilon$ be the unique solution of (3.4)–(3.6) with initial data $(u_{0,\varepsilon}, e_0, p_{0,\varepsilon}, z_0, v_{0,\varepsilon})$, whose existence is guaranteed by Theorem 3.1.

We first deduce an energy equality, which will enable us to get additional *a priori* estimates uniformly with respect to $\varepsilon$. It will also be a crucial ingredient in the derivation of the flow rule (4.4).

**Proposition 4.3.** *For every $t \in [0, T]$,*

$$\begin{aligned}\mathcal{Q}(e_\varepsilon(t)) + \frac{1}{2}\|z_\varepsilon(t)\|_2^2 + \int_0^t \mathcal{H}_\lambda(\dot{p}_\varepsilon(s), \dot{z}_\varepsilon(s))\, ds + \frac{1}{2}\|\dot{u}_\varepsilon(t)\|_2^2 + \varepsilon \int_0^t \int_\Omega |E\dot{u}_\varepsilon|^2\, dx\, ds \\ = \mathcal{Q}(e_0) + \frac{1}{2}\|z_0\|_2^2 + \frac{1}{2}\|v_{0,\varepsilon}\|_2^2 + \int_0^t \int_\Omega (\sigma_\varepsilon + \varepsilon E\dot{u}_\varepsilon) : E\dot{w}\, dx\, ds \\ + \int_0^t \int_\Omega \ddot{u}_\varepsilon \cdot \dot{w}\, dx\, ds + \int_0^t \int_\Omega f \cdot (\dot{u}_\varepsilon - \dot{w})\, dx\, ds. \qquad (4.6)\end{aligned}$$

*Proof.* Multiplying the equation of motion (3.5) by $(\dot{u}_\varepsilon - \dot{w})\chi_{[0,t]} \in L^2(0, T; H^1_0(\Omega; \mathbb{R}^n))$, and integrating by parts yield

$$\int_0^t \int_\Omega \ddot{u}_\varepsilon \cdot (\dot{u}_\varepsilon - \dot{w})\, dx\, ds + \int_0^t \int_\Omega (\sigma_\varepsilon + \varepsilon E\dot{u}_\varepsilon) : (E\dot{u}_\varepsilon - E\dot{w})\, dx\, ds = \int_0^t \int_\Omega f \cdot (\dot{u}_\varepsilon - \dot{w})\, dx\, ds.$$

On the other hand, by the kinematic compatibility for the rates (3.18) $E\dot{u}_\varepsilon = \dot{e}_\varepsilon + \dot{p}_\varepsilon$ a.e. in $\Omega \times [0, T]$ and the flow rule in (3.6) we have

$$\begin{aligned}\int_0^t \int_\Omega \sigma_\varepsilon : E\dot{u}_\varepsilon\, dx\, ds &= \int_0^t \int_\Omega \sigma_\varepsilon : \dot{e}_\varepsilon\, dx\, ds + \int_0^t \mathcal{H}_\lambda(\dot{p}_\varepsilon(s), \dot{z}_\varepsilon(s))\, ds - \int_0^t \int_\Omega \xi_\varepsilon \dot{z}_\varepsilon\, dx\, ds \\ &= \mathcal{Q}(e_\varepsilon(t)) - \mathcal{Q}(e_0) + \int_0^t \mathcal{H}_\lambda(\dot{p}_\varepsilon(s), \dot{z}_\varepsilon(s))\, ds + \frac{1}{2}\|z_\varepsilon(t)\|_2^2 - \frac{1}{2}\|z_0\|_2^2,\end{aligned}$$

where we used that $\xi_\varepsilon = -z_\varepsilon$. Combining the two previous equalities and applying [18, Section 5.9, Theorem 3], we obtain (4.6). ☐

**Remark 4.4.** Since $\operatorname{div}(\sigma_\varepsilon + \varepsilon E\dot{u}_\varepsilon) = \ddot{u}_\varepsilon - f \in L^2(0, T; L^2(\Omega; \mathbb{R}^n))$, we deduce that the normal trace $(\sigma_\varepsilon + \varepsilon E\dot{u}_\varepsilon)\nu \in L^2(0, T; H^{-1/2}(\partial\Omega; \mathbb{R}^n))$. Using an integration by parts, the term

$$\int_0^t \int_\Omega (\sigma_\varepsilon + \varepsilon E\dot{u}_\varepsilon) : E\dot{w}\, dx\, ds + \int_0^t \int_\Omega \ddot{u}_\varepsilon \cdot \dot{w}\, dx\, ds + \int_0^t \int_\Omega f \cdot (\dot{u}_\varepsilon - \dot{w})\, dx\, ds$$

can be equivalently rewritten as

$$\int_0^t \langle (\sigma_\varepsilon(s) + \varepsilon E\dot{u}_\varepsilon(s))\nu, \dot{w}(s)\rangle\, ds + \int_0^t \int_\Omega f \cdot \dot{u}_\varepsilon\, dx\, ds,$$

where the bracket $\langle \cdot, \cdot \rangle$ stands for the duality pairing between $H^{-1/2}(\partial\Omega; \mathbb{R}^n)$ and $H^{1/2}(\partial\Omega; \mathbb{R}^n)$, which is precisely a weak formulation of the power of internal and external forces.

We are now in a position to derive some *a priori* estimates on the sequence of dynamical visco-plastic evolutions. According to Proposition 3.11, there exists a constant $C_1 > 0$ (independent of $\varepsilon$ and $\lambda$) such that

$$\begin{aligned}\sup_{t\in[0,T]} \|\ddot{u}_\varepsilon(t)\|_2 + \sqrt{\varepsilon}\|E\ddot{u}_\varepsilon\|_{L^2(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))} + \sup_{t\in[0,T]} \|\dot{e}_\varepsilon(t)\|_2 + \sup_{t\in[0,T]} \|\dot{z}_\varepsilon(t)\|_2 \\ \leq C_1\Big(\varepsilon\|\Delta v_{0,\varepsilon}\|_2 + \|Ev_{0,\varepsilon}\|_2 + \|\dot{f}\|_{L^1(0,T;L^2(\Omega;\mathbb{R}^n))} \\ + \|\dddot{w}\|_{L^1(0,T;L^2(\Omega;\mathbb{R}^n))} + \|E\ddot{w}\|_{L^2(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))}\Big). \qquad (4.7)\end{aligned}$$

Moreover, as a consequence of Proposition 4.3, and by applying the coercivity properties (2.1), (2.5), and Hölder inequality, we deduce that there is a constant $C > 0$ (independent of $\varepsilon$ and $\lambda$) such that

$$\frac{\alpha_{\mathbb{C}}}{2}\|e_\varepsilon(t)\|_2^2 + \frac{1}{2}\|z_\varepsilon(t)\|_2^2 + \alpha_H\int_0^t \|\dot p_\varepsilon(s)\|_1\,ds + \frac{1}{2}\|\dot u_\varepsilon(t)\|_2^2 + \varepsilon\int_0^t\int_\Omega |E\dot u_\varepsilon|^2\,dx\,ds$$
$$\leq C\Big(1 + \sup_{s\in[0,T]}\|e_\varepsilon(s)\|_2 + \sup_{s\in[0,T]}\|\dot u_\varepsilon(s)\|_2 + \varepsilon\|E\dot u_\varepsilon\|_{L^2(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))}\Big)$$
$$+\frac{\alpha_H}{\sqrt{n}}\int_0^T \|\dot z_\varepsilon(s)\|_1\,ds.$$

Here we also used the fact that $(\ddot u_\varepsilon)$ is uniformly bounded in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^n))$ by (4.7), and that $(v_{0,\varepsilon})$ is uniformly bounded in $L^2(\Omega;\mathbb{R}^n)$. Since $\dot z_\varepsilon$ is uniformly bounded in $L^\infty(0,T;L^2(\Omega))$ by (4.7), we conclude by Young inequality that

$$\sup_{t\in[0,T]}\|\dot u_\varepsilon(t)\|_2 + \sqrt{\varepsilon}\|E\dot u_\varepsilon\|_{L^2(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))} + \sup_{t\in[0,T]}\|e_\varepsilon(t)\|_2 + \sup_{t\in[0,T]}\|z_\varepsilon(t)\|_2 \leq C_2, \quad (4.8)$$

$$\int_0^T \|\dot p_\varepsilon(s)\|_1\,ds \leq C_3, \quad (4.9)$$

for some constants $C_2 > 0$ and $C_3 > 0$, both independent of $\varepsilon$ and $\lambda$.

Let now $\Omega'$ be a smooth and bounded open set such that $\overline{\Omega} \subset \Omega'$. For every $t \in [0,T]$ we extend $u_\varepsilon(t)$, $e_\varepsilon(t)$, $p_\varepsilon(t)$, and $z_\varepsilon(t)$ to $\Omega'$ by setting $u_\varepsilon(t) = w(t)$, $e_\varepsilon(t) = Ew(t)$, $p_\varepsilon(t) = 0$, $z_\varepsilon(t) = 0$ in $\Omega' \setminus \Omega$. Clearly the bounds (4.7), (4.8), and (4.9) remain satisfied if we replace $\Omega$ by $\Omega'$.

By (4.7) and (4.8) we deduce the existence of three functions $v \in W^{1,\infty}([0,T];L^2(\Omega';\mathbb{R}^n))$, $e \in W^{1,\infty}([0,T];L^2(\Omega';\mathbb{M}^{n\times n}_{sym}))$, and $z \in W^{1,\infty}([0,T];L^2(\Omega'))$ such that, up to a subsequence,

$$\dot u_\varepsilon \rightharpoonup v \text{ weakly* in } W^{1,\infty}([0,T];L^2(\Omega';\mathbb{R}^n)), \quad (4.10)$$
$$e_\varepsilon \rightharpoonup e \text{ weakly* in } W^{1,\infty}([0,T];L^2(\Omega';\mathbb{M}^{n\times n}_{sym})), \quad (4.11)$$
$$z_\varepsilon \rightharpoonup z \text{ weakly* in } W^{1,\infty}([0,T];L^2(\Omega')). \quad (4.12)$$

By Ascoli-Arzelà Theorem and the bounds (4.7) and (4.8), we also have that for every $t \in [0,T]$,

$$\dot u_\varepsilon(t) \rightharpoonup v(t) \text{ weakly in } L^2(\Omega';\mathbb{R}^n), \quad (4.13)$$
$$e_\varepsilon(t) \rightharpoonup e(t) \text{ weakly in } L^2(\Omega';\mathbb{M}^{n\times n}_{sym}), \quad (4.14)$$
$$z_\varepsilon(t) \rightharpoonup z(t) \text{ weakly in } L^2(\Omega'). \quad (4.15)$$

Moreover, by (3.6) we have

$$\dot z(t) \geq 0 \quad \text{a.e. in } \Omega, \quad \text{for a.e. } t \in [0,T]. \quad (4.16)$$

We next establish some compactness on the sequence $(p_\varepsilon)$ of plastic strains. By (4.9) and Helly Theorem (see [26, Theorem 3.2] or [10, Lemma 7.2]) we deduce the existence of a subsequence (not relabeled) and of a function $p \in BV([0,T];\mathcal{M}(\Omega';\mathbb{M}^{n\times n}_{sym}))$ such that for every $t \in [0,T]$,

$$p_\varepsilon(t) \rightharpoonup p(t) \text{ weakly* in } \mathcal{M}(\Omega';\mathbb{M}^{n\times n}_{sym}). \quad (4.17)$$

Finally we state some compactness properties for the sequence $(u_\varepsilon)$ of displacements. By (4.14) and (4.17) we have that $Eu_\varepsilon(t) \rightharpoonup e(t) + p(t)$ weakly* in $\mathcal{M}(\Omega';\mathbb{M}^{n\times n}_{sym})$. Since $u_\varepsilon(t) = w(t)$ a.e. on $\Omega' \setminus \Omega$, we deduce by the Poincaré-Korn inequality (see [39, Chapter II, Remark 2.5 (ii)]) that the sequence $(u_\varepsilon(t))$ is uniformly bounded in $BD(\Omega')$. Therefore, for every $t \in [0,T]$ there exist a subsequence $(\varepsilon_j)$, possibly depending on $t$, and a function $u(t) \in BD(\Omega')$ such that $u_{\varepsilon_j}(t) \rightharpoonup u(t)$ weakly* in $BD(\Omega')$. Since $u(t) = w(t)$ a.e. in $\Omega' \setminus \Omega$ and $Eu(t) = e(t) + p(t)$ in $\Omega'$, we conclude again by the Poincaré-Korn inequality that the limit $u(t)$ is uniquely determined. Therefore, the convergence result holds for the whole sequence, that is,

$$u_\varepsilon(t) \rightharpoonup u(t) \text{ weakly}^* \text{ in } BD(\Omega') \quad \text{for every } t \in [0,T]. \tag{4.18}$$

In particular, we have shown that for every $t \in [0,T]$,

$$Eu(t) = e(t) + p(t) \text{ in } \Omega, \quad p(t) = (w(t) - u(t)) \odot \nu \mathcal{H}^{n-1} \text{ on } \partial\Omega. \tag{4.19}$$

Moreover, by (4.13) and (4.18) we infer that $v(t) = \dot{u}(t)$ for every $t \in [0,T]$, hence

$$u \in W^{2,\infty}([0,T]; L^2(\Omega;\mathbb{R}^n)).$$

Clearly, from the convergences of the initial data $(u_{0,\varepsilon})$ and $(v_{0,\varepsilon})$, the initial conditions $u(0) = u_0$, $e(0) = e_0$, $p(0) = p_0$, $z(0) = z_0$, and $\dot{u}(0) = v_0$ are satisfied. Inequality (4.5) is an immediate consequence of (4.7).

We define $\sigma(t) := \mathbb{C}e(t)$ and $\xi(t) := -z(t)$. Since $(\sigma_\varepsilon(t), \xi_\varepsilon(t)) \in K_\lambda$ a.e. in $\Omega$ and $K_\lambda$ is a closed and convex set, by (4.14) and (4.15) we immediately deduce that $(\sigma(t), \xi(t)) \in K_\lambda$ a.e. in $\Omega$.

Let $\varphi \in C_c^\infty(\Omega \times (0,T); \mathbb{R}^n)$. By the equation of motion in (3.5) we have

$$\int_0^T \int_\Omega \ddot{u}_\varepsilon \cdot \varphi \, dx \, dt + \int_0^T \int_\Omega (\sigma_\varepsilon + \varepsilon E\dot{u}_\varepsilon) : E\varphi \, dx \, dt = \int_0^T \int_\Omega f \cdot \varphi \, dx \, dt.$$

Since $\varepsilon E\dot{u}_\varepsilon \to 0$ strongly in $L^2(0,T; L^2(\Omega; \mathbb{M}^{n\times n}_{sym}))$ by (4.8), we can pass to the limit in the above equality and obtain

$$\int_0^T \int_\Omega \ddot{u} \cdot \varphi \, dx \, dt + \int_0^T \int_\Omega \sigma : E\varphi \, dx \, dt = \int_0^T \int_\Omega f \cdot \varphi \, dx \, dt,$$

which implies

$$\ddot{u} - \operatorname{div}\sigma = f \quad \text{ a.e. in } \Omega \times (0,T). \tag{4.20}$$

4.2. **Strong convergences and flow rule.** In order to derive the flow rule, we first show that the weak convergences established in the previous section can be improved into strong convergences. The proof strategy is similar to that of Lemma 3.7.

**Lemma 4.5.** *The following strong convergences hold:*

$$\dot{u}_\varepsilon \to \dot{u} \textit{ strongly in } L^\infty(0,T; L^2(\Omega;\mathbb{R}^n)), \tag{4.21}$$

$$e_\varepsilon \to e \textit{ strongly in } L^\infty(0,T; L^2(\Omega;\mathbb{M}^{n\times n}_{sym})), \tag{4.22}$$

$$z_\varepsilon \to z \textit{ strongly in } L^\infty(0,T; L^2(\Omega)), \tag{4.23}$$

$$\sqrt{\varepsilon} E\dot{u}_\varepsilon \to 0 \textit{ strongly in } L^2(0,T; L^2(\Omega;\mathbb{M}^{n\times n}_{sym})). \tag{4.24}$$

*Proof.* Let $t \in [0,T]$. We multiply the equations of motions (3.5) and (4.20) by $\dot{u}_\varepsilon(t) - \dot{w}(t)$, integrate by parts over $\Omega$ and subtract the resulting expressions. In this way we obtain

$$\begin{aligned} &\int_\Omega (\ddot{u}_\varepsilon(t) - \ddot{u}(t)) \cdot \dot{u}_\varepsilon(t) \, dx + \int_\Omega (\sigma_\varepsilon(t) - \sigma(t)) : E\dot{u}_\varepsilon(t) \, dx + \varepsilon \int_\Omega |E\dot{u}_\varepsilon(t)|^2 \, dx \\ &\qquad = \int_\Omega (\ddot{u}_\varepsilon(t) - \ddot{u}(t)) \cdot \dot{w}(t) \, dx + \int_\Omega (\sigma_\varepsilon(t) + \varepsilon E\dot{u}_\varepsilon(t) - \sigma(t)) : E\dot{w}(t) \, dx. \end{aligned}$$

By the kinematic compatibility (3.18) for the rates $E\dot{u}_\varepsilon(t) = \dot{e}_\varepsilon(t) + \dot{p}_\varepsilon(t)$ a.e. in $\Omega$ we have

$$\int_\Omega (\sigma_\varepsilon(t) - \sigma(t)) : E\dot{u}_\varepsilon(t) \, dx = \int_\Omega (\sigma_\varepsilon(t) - \sigma(t)) : \dot{e}_\varepsilon(t) \, dx + \int_\Omega (\sigma_\varepsilon(t) - \sigma(t)) : \dot{p}_\varepsilon(t) \, dx.$$

Moreover, combining the flow rule in (3.6) with the fact that $(\sigma(t),\xi(t))\in K_\lambda$ a.e. in $\Omega$, we deduce that

$$\int_\Omega(\sigma_\varepsilon(t)-\sigma(t)):\dot p_\varepsilon(t)\,dx\ge-\int_\Omega(\xi_\varepsilon(t)-\xi(t)):\dot z_\varepsilon(t)\,dx=\int_\Omega(z_\varepsilon(t)-z(t)):\dot z_\varepsilon(t)\,dx.$$

Therefore,

$$\begin{aligned}\int_\Omega(\ddot u_\varepsilon(t)-\ddot u(t))\cdot(\dot u_\varepsilon(t)-\dot u(t))\,dx&+\int_\Omega(\sigma_\varepsilon(t)-\sigma(t)):(\dot e_\varepsilon(t)-\dot e(t))\,dx\\&+\int_\Omega(z_\varepsilon(t)-z(t))(\dot z_\varepsilon(t)-\dot z(t))\,dx+\varepsilon\int_\Omega|E\dot u_\varepsilon(t)|^2\,dx\\\le\int_\Omega(\ddot u_\varepsilon(t)-\ddot u(t))\cdot(\dot w(t)-\dot u(t))\,dx&-\int_\Omega(\sigma_\varepsilon(t)-\sigma(t)):\dot e(t)\,dx-\int_\Omega(z_\varepsilon(t)-z(t))\dot z(t)\,dx\\&+\int_\Omega(\sigma_\varepsilon(t)+\varepsilon E\dot u_\varepsilon(t)-\sigma(t)):E\dot w(t)\,dx.\end{aligned}$$

By integrating with respect to time on $[0,t]$ we obtain

$$\begin{aligned}&\frac12\|\dot u_\varepsilon(t)-\dot u(t)\|_2^2+\mathcal{Q}(e_\varepsilon(t)-e(t))+\frac12\|z_\varepsilon(t)-z(t)\|_2^2+\varepsilon\int_0^t\int_\Omega|E\dot u_\varepsilon(s)|^2\,dx\,ds\\&\le\frac12\|v_{0,\varepsilon}-v_0\|_2^2-\int_0^t\int_\Omega(\ddot u_\varepsilon(s)-\ddot u(s))\cdot(\dot u(s)-\dot w(s))\,dx\,ds-\int_0^t\int_\Omega(\sigma_\varepsilon(s)-\sigma(s)):\dot e(s)\,dx\,ds\\&\qquad-\int_0^t\int_\Omega(z_\varepsilon(s)-z(s))\dot z(s)\,dx\,ds+\int_0^t\int_\Omega(\sigma_\varepsilon(s)+\varepsilon E\dot u_\varepsilon(s)-\sigma(s)):E\dot w(s)\,dx\,ds,\end{aligned}$$

where we used the fact that $\dot u_\varepsilon(0)-\dot u(0)=v_{0,\varepsilon}-v_0$, $e_\varepsilon(0)-e(0)=0$, and $z_\varepsilon(0)-z(0)=0$. Since the right-handside converges to 0 by (4.8) and (4.10)–(4.12), by (2.1) we deduce (4.21)–(4.24). □

The strong convergence properties proved in Lemma 4.5 allow us to pass to the limit in the energy balance (4.6) and to deduce an energy inequality for the limit evolution. Indeed, the energy balance (4.6) can be rewritten between two times $0\le t_1\le t_2\le T$ as

$$\begin{aligned}\mathcal{Q}(e_\varepsilon(t_2))+\frac12\|z_\varepsilon(t_2)\|_2^2+\int_{t_1}^{t_2}\mathcal{H}_\lambda(\dot p_\varepsilon(s),\dot z_\varepsilon(s))\,ds+\frac12\|\dot u_\varepsilon(t_2)\|_2^2+\varepsilon\int_{t_1}^{t_2}\int_\Omega|E\dot u_\varepsilon|^2\,dx\,ds\\=\mathcal{Q}(e_\varepsilon(t_1))+\frac12\|z_\varepsilon(t_1)\|_2^2+\frac12\|\dot u_\varepsilon(t_1)\|_2^2+\int_{t_1}^{t_2}\int_\Omega(\sigma_\varepsilon+\varepsilon E\dot u_\varepsilon):E\dot w\,dx\,ds\\+\int_{t_1}^{t_2}\int_\Omega\ddot u_\varepsilon\cdot\dot w\,dx\,ds+\int_{t_1}^{t_2}\int_\Omega f\cdot(\dot u_\varepsilon-\dot w)\,dx\,ds.\end{aligned}$$

Owing to Lemma 4.5, Proposition 2.3, and the lower semicontinuity of the dissipation $\mathcal{D}_\lambda$, we deduce the following energy inequality:

$$\begin{aligned}&\mathcal{Q}(e(t_2))+\frac12\|z(t_2)\|_2^2+\mathcal{D}_\lambda(p,z;[t_1,t_2])+\frac12\|\dot u(t_2)\|_2^2\\&\le\mathcal{Q}(e(t_1))+\frac12\|z(t_1)\|_2^2+\frac12\|\dot u(t_1)\|_2^2+\int_{t_1}^{t_2}\int_\Omega(\sigma:E\dot w+\ddot u\cdot\dot w)\,dx\,ds+\int_{t_1}^{t_2}\int_\Omega f\cdot(\dot u-\dot w)\,dx\,ds.\end{aligned}\tag{4.25}$$

A first consequence of the above energy inequality is a control on the time increments of the mapping $t\mapsto p(t)$, that guarantees absolute continuity with respect to time (recall that at this stage we only know that $t\mapsto p(t)$ has bounded variation from $[0,T]$ into $\mathcal{M}(\overline\Omega;\mathbb{M}^{n\times n}_{sym})$).

**Lemma 4.6.** *We have $p \in AC([0,T];\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym}))$ and $u \in AC([0,T];BD(\Omega))$. Moreover, for a.e. $t \in [0,T]$*

$$\|\dot u(t)\|_{BD(\Omega)} + \|\dot p(t)\|_1 \le C(\|\dot e(t)\|_2 + \|\ddot u(t)\|_2 + \|\dot z(t)\|_2 + \|\dot w(t)\|_{H^1(\Omega;\mathbb{R}^n)} + \|\dot f(t)\|_2), \tag{4.26}$$

*where $C$ is a positive constants depending on $\alpha_H$, $\beta_{\mathbb{C}}$, $n$, $\Omega$, $\sup_t \|e(t)\|_2$, $\sup_t \|z(t)\|_2$, $\sup_t \|\dot u(t)\|_2$, $\sup_t \|\ddot u(t)\|_2$, and $\sup_t \|\dot w(t)\|_2$. Finally, we have*

$$(\dot u(t), \dot e(t), \dot p(t)) \in \mathcal{A}_{\rm dyn}(\dot w(t)) \tag{4.27}$$

*for a.e. $t \in [0,T]$.*

*Proof.* We recall that by (2.5) and (4.16)

$$\mathcal{D}_\lambda(p,z;[t_1,t_2]) \ge \mathcal{H}_\lambda(p(t_2)-p(t_1), z(t_2)-z(t_1)) \ge \alpha_H\|p(t_2)-p(t_1)\|_1 - \frac{\alpha_H}{\sqrt{n}}\|z(t_2)-z(t_1)\|_1.$$

Combining this inequality with (4.25) and the Cauchy-Schwarz inequality yields

$$\begin{aligned}
\alpha_H\|p(t_2)-p(t_1)\|_1 \le\ & \sup_{t\in[0,T]}\|\sigma(t)\|_2\,\|e(t_2)-e(t_1)\|_2 + \sup_{t\in[0,T]}\|\dot u(t)\|_2\,\|\dot u(t_2)-\dot u(t_1)\|_2 \\
& + \sup_{t\in[0,T]}\|\sigma(t)\|_2 \int_{t_1}^{t_2}\|E\dot w(s)\|_2\,ds + \sup_{t\in[0,T]}\|\ddot u(t)\|_2\int_{t_1}^{t_2}\|\dot w(s)\|_2\,ds \\
& + \sup_{t\in[0,T]}(\|\dot u(t)\|_2 + \|\dot w(t)\|_2)\int_{t_1}^{t_2}\|\dot f(s)\|_2\,ds \\
& + \sup_{t\in[0,T]}\|z(t)\|_2\,\|z(t_2)-z(t_1)\|_2 + \frac{\alpha_H}{\sqrt{n}}\|z(t_2)-z(t_1)\|_1.
\end{aligned}$$

This implies that $p \in AC([0,T];\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym}))$. By the kinematic compatibility and the Poincaré-Korn inequality we deduce that $u \in AC([0,T];BD(\Omega))$ and (4.26) holds. Finally, inclusion (4.27) is a consequence of (4.19) and [10, Lemma 5.5]. □

We are now in position to derive the flow rule.

**Proposition 4.7.** *For a.e. $t \in [0,T]$, the distribution $[\sigma(t):\dot p(t)]$ is a measure in $\mathcal{M}(\overline{\Omega})$ and satisfies*

$$H_\lambda(\dot p(t), \dot z(t)) = [\sigma(t):\dot p(t)] + \xi(t)\dot z(t) \quad \text{in } \mathcal{M}(\overline{\Omega}).$$

*Proof.* According to Lemma 4.6 and Proposition 2.3, we have

$$\mathcal{D}_\lambda(p,z;[t_1,t_2]) = \int_{t_1}^{t_2}\mathcal{H}_\lambda(\dot p(s),\dot z(s))\,ds.$$

Dividing the energy inequality (4.25) by $t_2-t_1$ (with $t_2>t_1$) and sending $t_1$ to $t_2 = t$, we obtain that

$$\begin{aligned}
\int_\Omega \sigma(t):\dot e(t)\,dx + \int_\Omega z(t)\dot z(t)\,dx + \mathcal{H}_\lambda(\dot p(t),\dot z(t)) + \int_\Omega \dot u(t)\cdot\ddot u(t)\,dx \\
\le \int_\Omega(\sigma(t):E\dot w(t) + \ddot u(t)\cdot\dot w(t))\,dx + \int_\Omega f(t)\cdot(\dot u(t)-\dot w(t))\,dx
\end{aligned}$$

for a.e. $t \in [0,T]$. Using the equation of motion, the previous inequality can be equivalently written as

$$\mathcal{H}_\lambda(\dot p(t),\dot z(t)) \le \int_\Omega \sigma(t):(E\dot w(t)-\dot e(t))\,dx + \int_\Omega {\rm div}\sigma(t)\cdot(\dot w(t)-\dot u(t))\,dx + \int_\Omega \xi(t)\dot z(t)\,dx.$$

We notice that by (4.27), for a.e. $t\in[0,T]$ we have $\dot u(t) \in BD(\Omega)\cap L^2(\Omega;\mathbb{R}^n)$, $\dot e(t) \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, $\dot p(t) \in \mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym})$, $E\dot u(t) = \dot e(t) + \dot p(t)$ in $\Omega$, $\dot p(t) = (\dot w(t)-\dot u(t))\odot\nu\mathcal{H}^{n-1}$ on $\partial\Omega$, while $\sigma(t) \in$

$L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ and $\mathrm{div}\sigma(t) \in L^2(\Omega;\mathbb{R}^n)$. Thus, we can use the duality introduced in Definition 2.4 and (2.9) to get that

$$\mathcal{H}_\lambda(\dot p(t), \dot z(t)) \le \langle \sigma(t), \dot p(t)\rangle + \int_\Omega \xi(t)\dot z(t)\, dx \tag{4.28}$$

for a.e. $t \in [0,T]$.

Inequality (4.28) implies that $H_\lambda(\dot p(t), \dot z(t)) \in \mathcal{M}(\Omega')$ for a.e. $t \in [0,T]$. Therefore, by (2.14), we have that for every $\varphi \in \mathcal{C}^\infty_c(\Omega')$, $\varphi \ge 0$ and a.e. $t \in [0,T]$,

$$\int_{\Omega'} \varphi\, d[H_\lambda(\dot p(t), \dot z(t))] \ge \langle [\sigma(t) : \dot p(t)], \varphi\rangle + \int_\Omega \xi(t)\dot z(t)\varphi\, dx$$

which implies that

$$H_\lambda(\dot p(t), \dot z(t)) \ge [\sigma(t) : \dot p(t)] + \xi(t)\dot z(t),$$

where the inequality above is intended in the sense of distributions on $\Omega'$. In other words, $H_\lambda(\dot p(t), \dot z(t)) - [\sigma(t) : \dot p(t)] - \xi(t)\dot z(t)$ is a non negative distribution, hence a Radon measure with zero total variation by (4.28). Therefore, for a.e. $t \in [0,T]$

$$[\sigma(t) : \dot p(t)] \in \mathcal{M}(\Omega')$$

and

$$H_\lambda(\dot p(t), \dot z(t)) = [\sigma(t) : \dot p(t)] + \xi(t)\dot z(t) \quad \text{ in } \mathcal{M}(\Omega').$$

Thanks to our choice of the extensions this last relation implies (4.4) and completes the proof of the proposition. □

4.3. **Uniqueness of the solution.** The proof of Theorem 4.1 will be complete once uniqueness is proved. This property is established in a similar way to the case $\varepsilon > 0$. However, because of the lack of regularity of the plastic strain rate, one cannot simply repeat the proof of Subsection 3.5. In particular, some extra care is needed due to the presence of the duality pairing between the plastic strain rate and the stress.

Let us consider two solutions $(u_1, e_1, p_1, z_1, \sigma_1, \xi_1)$ and $(u_2, e_2, p_2, z_2, \sigma_2, \xi_2)$. Subtracting the equations of motions leads to

$$\ddot u_1 - \ddot u_2 - \mathrm{div}(\sigma_1 - \sigma_2) = 0 \quad \text{ in } L^2(0,T;L^2(\Omega;\mathbb{R}^n)),$$

and since $\dot u_1 - \dot u_2 \in L^2(0,T;L^2(\Omega;\mathbb{R}^n))$, we infer that

$$\int_0^t \int_\Omega (\ddot u_1 - \ddot u_2)\cdot(\dot u_1 - \dot u_2)\, dx\, ds - \int_0^t \int_\Omega \mathrm{div}(\sigma_1 - \sigma_2)\cdot(\dot u_1 - \dot u_2)\, dx\, ds = 0. \tag{4.29}$$

Since $\ddot u_1 - \ddot u_2 \in L^2(0,T;L^2(\Omega;\mathbb{R}^n))$, we get that

$$\int_0^t \int_\Omega (\ddot u_1(s) - \ddot u_2(s))\cdot(\dot u_1(s) - \dot u_2(s))\, dx\, ds = \frac{\|\dot u_1(t) - \dot u_2(t)\|_2^2}{2} \tag{4.30}$$

where we used that $\dot u_1(0) = \dot u_2(0) = v_0$. On the other hand, by the stress-strain duality (see Definition 2.4 and (2.9)), we get that

$$\begin{aligned} -\int_0^t &\int_\Omega \mathrm{div}(\sigma_1(s) - \sigma_2(s))\cdot(\dot u_1(s) - \dot u_2(s))\, dx\, ds \\ &= \int_0^t \int_\Omega (\sigma_1(s) - \sigma_2(s)) : (\dot e_1(s) - \dot e_2(s))\, dx\, dt + \int_0^t \langle \sigma_1(s) - \sigma_2(s), \dot p_1(s) - \dot p_2(s)\rangle\, ds \\ &= \mathcal{Q}(e_1(t) - e_2(t)) + \int_0^t \langle \sigma_1(s) - \sigma_2(s), \dot p_1(s) - \dot p_2(s)\rangle\, ds, \end{aligned} \tag{4.31}$$

since $e_1(0) = e_2(0) = e_0$. We now estimate the last integral. We first note that, for $i = 1,2$, $(\sigma_i(s), \xi_i(s)) \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \times L^2(\Omega)$, $\mathrm{div}\sigma_i(s) \in L^2(\Omega;\mathbb{R}^n)$, and $(\sigma_i(s), \xi_i(s)) \in K_\lambda$ a.e. in $\Omega$ for all $s \in [0,T]$. Therefore, by the flow rule (4.4), the duality formula (2.15) and the kinematic compatibility for the rates (4.27), we get that for a.e. $s \in [0,T]$,

$$\langle \sigma_1(s), \dot p_1(s)\rangle + \int_\Omega \xi_1(s)\dot z_1(s)\,dx = \mathcal{H}_\lambda(\dot p_1(s), \dot z_1(s)) \geq \langle \sigma_2(s), \dot p_1(s)\rangle + \int_\Omega \xi_2(s)\dot z_1(s)\,dx,$$
$$\langle \sigma_2(s), \dot p_2(s)\rangle + \int_\Omega \xi_2(s)\dot z_2(s)\,dx = \mathcal{H}_\lambda(\dot p_2(s), \dot z_2(s)) \geq \langle \sigma_1(s), \dot p_2(s)\rangle + \int_\Omega \xi_1(s)\dot z_2(s)\,dx.$$

Summing up both previous inequalities and integrating in time yields

$$\int_0^t \langle \sigma_1(s) - \sigma_2(s), \dot p_1(s) - \dot p_2(s)\,ds \geq \int_0^t \int_\Omega (z_1 - z_2)(\dot z_1 - \dot z_2)\,dx\,ds = \frac{\|z_1(t) - z_2(t)\|_2^2}{2}, \tag{4.32}$$

since $z_1(0) = z_2(0) = z_0$. Gathering (4.29)–(4.32) yields $e_1 = e_2$ (hence $\sigma_1 = \sigma_2$), $z_1 = z_2$ (hence $\xi_1 = \xi_2$), and $\dot u_1 = \dot u_2$. But since $u_1(0) = u_2(0) = u_0$, we deduce that $u_1 = u_2$ and finally that $p_1 = p_2$.

**Remark 4.8.** Thanks to the uniqueness of the solution, there is actually no need to extract subsequences in all weak and strong convergences obtained so far.

## 5. Convergence of the dynamic cap model

In this section we characterize the asymptotic behaviour of the elasto-plastic dynamic evolutions studied in the previous section when the cap is sent to infinity. In that way we recover a solution of (1.3), namely of a model of perfect elasto-plasticity with a pressure-sensitive yield criterion.

To this aim, we consider an initial data $(u_0, e_0, p_0) \in \mathcal{A}_{\rm dyn}(w(0))$ and $v_0 \in H^1(\Omega;\mathbb{R}^n)$ satisfying

$$\begin{cases} v_0 = \dot w(0) \quad \mathcal{H}^{n-1}\text{-a.e. on } \partial\Omega, \\ \sigma_0 := \mathbb{C}e_0 \in K, \quad -\mathrm{div}\sigma_0 = f(0) \quad \text{a.e. in } \Omega. \end{cases} \tag{5.1}$$

We have the following convergence result.

**Theorem 5.1.** *Assume* (2.1)–(2.4), (3.1), (3.2), *and* (5.1). *For every $\lambda \geq 1$ let $u_\lambda$, $e_\lambda$, $\sigma_\lambda$, $p_\lambda$, $z_\lambda$, $\xi_\lambda$ be the solution to* (4.1)–(4.4) *constructed in Theorem 4.1 with initial data $(u_0, e_0, p_0, z_0)$ and $v_0$, where we set*

$$\xi_0 := (\sigma_0)_m - |(\sigma_0)_m| \quad \textit{and} \quad z_0 := -\xi_0. \tag{5.2}$$

*Then there exist unique*

$$\begin{cases} u \in AC([0,T]; BD(\Omega)) \cap W^{2,\infty}([0,T]; L^2(\Omega;\mathbb{R}^n)), \\ e,\ \sigma \in W^{1,\infty}([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_{sym})), \\ p \in AC([0,T]; \mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym})), \end{cases}$$

*such that for all $t \in [0,T]$,*

$$u_\lambda(t) \rightharpoonup u(t) \ \textit{weakly}^* \ \textit{in } BD(\Omega), \tag{5.3}$$
$$e_\lambda(t) \rightharpoonup e(t) \ \textit{weakly in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym}), \tag{5.4}$$
$$p_\lambda(t) \rightharpoonup p(t) \ \textit{weakly}^* \ \textit{in } \mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym}), \tag{5.5}$$

*and*

$$u_\lambda \to u \ \textit{strongly in } W^{1,\infty}([0,T]; L^2(\Omega;\mathbb{R}^n)), \tag{5.6}$$
$$e_\lambda \to e \ \textit{strongly in } L^\infty(0,T; L^2(\Omega;\mathbb{M}^{n\times n}_{sym})), \tag{5.7}$$

*as $\lambda \to +\infty$. For every $t \in [0,T]$ we have*

$$\begin{cases} Eu(t) = e(t) + p(t) \text{ in } \Omega, \quad p(t) = (w(t) - u(t)) \odot \nu \mathcal{H}^{n-1} \text{ on } \partial\Omega, \\ \sigma(t) = \mathbb{C}e(t), \\ \sigma(t) \in K \text{ a.e. in } \Omega, \end{cases}$$

*Moreover,*

$$\begin{cases} \ddot{u} - \operatorname{div}\sigma = f \quad \text{a.e. in } \Omega \times (0,T), \\ (u(0), e(0), p(0)) = (u_0, e_0, p_0), \quad \dot{u}(0) = v_0, \end{cases}$$

*and for a.e. $t \in [0,T]$ the distribution $[\sigma(t) : \dot{p}(t)]$ is a measure in $\mathcal{M}(\overline{\Omega})$ satisfying*

$$H(\dot{p}(t)) = [\sigma(t) : \dot{p}(t)] \text{ in } \mathcal{M}(\overline{\Omega}). \tag{5.8}$$

The proof of the theorem relies on the structural properties of the sets $K_\lambda$. Indeed, conditions (2.3) and (2.4) will play an important role in establishing the strong convergences (5.6) and (5.7), and the flow rule (5.8).

*Proof of Theorem 5.1.* From condition (2.4) and the choice (5.2) of $\xi_0$ it is immediate to see that $\xi_0$, $z_0 \in L^2(\Omega)$ and $(\sigma_0, \xi_0) \in K_1 \subset K_\lambda$ a.e. in $\Omega$, for all $\lambda \geq 1$. Consequently, $(u_0, e_0, p_0, z_0)$ and $v_0$ are admissible initial data for Theorem 4.1, and $u_\lambda$, $e_\lambda$, $\sigma_\lambda$, $p_\lambda$, $z_\lambda$, $\xi_\lambda$ are well defined.

The proof is split into several steps.

**Step 1:** ***A priori*** **estimates and compactness.** From (4.5) it follows that

$$\|\ddot{u}_\lambda\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^n))} + \|\dot{e}_\lambda\|_{L^\infty(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))} + \|\dot{z}_\lambda\|_{L^\infty(0,T;L^2(\Omega))} \leq C_1, \tag{5.9}$$

for some positive constant $C_1 > 0$ independent of $\lambda$. On the other hand, by the energy inequality (4.25) at times $t_1 = 0$ and $t_2 = t$ we have

$$\begin{aligned} &\frac{\alpha_{\mathbb{C}}}{2}\|e_\lambda(t)\|_2^2 + \frac{1}{2}\|z_\lambda(t)\|_2^2 + \frac{1}{2}\|\dot{u}_\lambda(t)\|_2^2 \\ &\quad \leq \frac{\beta_{\mathbb{C}}}{2}\|e_0\|_2^2 + \frac{1}{2}\|z_0\|_2^2 + \frac{1}{2}\|v_0\|_2^2 + \|\sigma_\lambda\|_{L^\infty(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))}\|E\dot{w}\|_{L^1(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))} \\ &\qquad + \|\ddot{u}_\lambda\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^n))}\|\dot{w}\|_{L^1(0,T;L^2(\Omega;\mathbb{R}^n))} \\ &\qquad + \|f\|_{L^1(0,T;L^2(\Omega;\mathbb{R}^n))}\big(\|\dot{u}_\lambda\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^n))} + \|\dot{w}\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^n))}\big) \end{aligned}$$

for every $t \in [0,T]$. This, together with (5.9), implies that

$$\|e_\lambda\|_{L^\infty(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))} + \|z_\lambda\|_{L^\infty(0,T;L^2(\Omega))} + \|\dot{u}_\lambda\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^n))} \leq C_2,$$

for some constant $C_2 > 0$ independent of $\lambda$. Moreover, combining (4.26) with the previous estimates yields

$$\|\dot{p}_\lambda(t)\|_1 \leq C_3(1 + \|\dot{f}(t)\|_2)$$

for a.e. $t \in [0,T]$ and some constant $C_3 > 0$ independent of $\lambda$.

From the previous bounds, we deduce the existence of functions $v \in W^{1,\infty}([0,T]; L^2(\Omega;\mathbb{R}^n))$, $e \in W^{1,\infty}([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$, and $z \in W^{1,\infty}([0,T]; L^2(\Omega))$ such that, up to subsequences,

$$\dot{u}_\lambda \rightharpoonup v \text{ weakly* in } W^{1,\infty}([0,T]; L^2(\Omega;\mathbb{R}^n)), \tag{5.10}$$

$$e_\lambda \rightharpoonup e \text{ weakly* in } W^{1,\infty}([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_{sym})), \tag{5.11}$$

$$z_\lambda \rightharpoonup z \text{ weakly* in } W^{1,\infty}([0,T]; L^2(\Omega)). \tag{5.12}$$

By Ascoli-Arzelà Theorem we also have that

$$\begin{aligned} \dot{u}_\lambda(t) &\rightharpoonup v(t) \text{ weakly in } L^2(\Omega;\mathbb{R}^n),\\ e_\lambda(t) &\rightharpoonup e(t) \text{ weakly in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym}),\\ z_\lambda(t) &\rightharpoonup z(t) \text{ weakly in } L^2(\Omega) \end{aligned} \tag{5.13}$$

for every $t \in [0,T]$. Moreover, (4.3) implies that

$$\dot{z}(t) \geq 0 \text{ a.e. in } \Omega \tag{5.14}$$

for a.e. $t \in [0,T]$. Finally, again by Ascoli-Arzelà Theorem we deduce the existence of a function $p \in AC([0,T];\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym}))$ such that (5.5) is satisfied for every $t \in [0,T]$.

**Step 2: Kinematic compatibility, equation of motion, and initial condition.** Arguing as in the proof of Theorem 4.1, one can show the existence of $u \in AC([0,T];BD(\Omega)) \cap W^{2,\infty}([0,T];L^2(\Omega;\mathbb{R}^n))$ such that (5.3) is satisfied, $v(t) = \dot{u}(t)$ and

$$Eu(t) = e(t) + p(t) \text{ in } \Omega, \quad p(t) = (w(t) - u(t)) \odot \nu\mathcal{H}^{n-1} \text{ on } \partial\Omega$$

for every $t \in [0,T]$, so that the kinematic compatibility holds. Moveover, according to [10, Lemma 5.5], we get, for a.e. $t \in [0,T]$,

$$E\dot{u}(t) = \dot{e}(t) + \dot{p}(t) \text{ in } \Omega, \quad \dot{p}(t) = (\dot{w}(t) - \dot{u}(t)) \odot \nu\mathcal{H}^{n-1} \text{ on } \partial\Omega. \tag{5.15}$$

Clearly the equation of motion and the initial conditions $u(0) = u_0$, $e(0) = e_0$, $p(0) = p_0$, $z(0) = z_0$, and $\dot{u}(0) = v_0$ are satisfied.

**Step 3: Stress constraint.** We set $\sigma(t) := \mathbb{C}e(t)$ and $\xi(t) := -z(t)$. From the inclusion $(\sigma_\lambda(t), \xi_\lambda(t)) \in K_\lambda$ a.e. in $\Omega$ for every $\lambda \geq 1$, it follows that $(\sigma_\lambda(t), \xi_\lambda(t)) \in K \times (-\infty, 0]$ a.e. in $\Omega$ for every $\lambda \geq 1$. By (5.4) and (5.13) we deduce that $(\sigma(t), \xi(t)) \in K \times (-\infty, 0]$ a.e. in $\Omega$, which implies that $\sigma(t) \in K$ a.e. in $\Omega$ for every $t \in [0,T]$.

**Step 4: Strong convergences.** We now prove the strong convergences (5.6), (5.7), together with

$$z_\lambda \to z \text{ strongly in } L^\infty(0,T;L^2(\Omega)). \tag{5.16}$$

For every $\lambda \geq 1$ we define

$$\zeta_\lambda := z - \frac{1}{\lambda}(\sigma_m - |\sigma_m|).$$

Using the fact that $z \in W^{1,\infty}([0,T];L^2(\Omega))$ and $\sigma \in W^{1,\infty}([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$, we get that $\zeta_\lambda \in W^{1,\infty}([0,T];L^2(\Omega))$ and

$$\zeta_\lambda \to z \text{ strongly in } W^{1,\infty}([0,T];L^2(\Omega)), \tag{5.17}$$

as $\lambda \to \infty$. Moreover, according to (2.4), $(\sigma(t), -\zeta_\lambda(t)) \in K_\lambda$ a.e. in $\Omega$. Since $\mathrm{div}\sigma(t) \in L^2(\Omega;\mathbb{R}^n)$, by integration of (4.4) and the duality formula (2.15), we have

$$\langle \sigma(t), \dot{p}_\lambda(t)\rangle - \int_\Omega \zeta_\lambda(t)\dot{z}_\lambda(t)\,dx \leq \langle \sigma_\lambda(t), \dot{p}_\lambda(t)\rangle + \int_\Omega \xi_\lambda(t)\dot{z}_\lambda(t)\,dx$$

for a.e. $t \in [0,T]$. Using the definition of the stress-strain duality (2.9), the kinematic compatibility for the rate (4.27), and the equation of motion this can be rewritten as

$$\begin{aligned} \int_\Omega \dot{e}_\lambda(t) : (\sigma_\lambda(t) - \sigma(t))\,dx + \int_\Omega \dot{z}_\lambda(t)(z_\lambda(t) - \zeta_\lambda(t))\,dx + \int_\Omega \dot{u}_\lambda(t) \cdot (\ddot{u}_\lambda(t) - \ddot{u}(t))\,dx \\ \leq \int_\Omega \dot{w}(t) \cdot (\ddot{u}_\lambda(t) - \ddot{u}(t))\,dx + \int_\Omega E\dot{w}(t) : (\sigma_\lambda(t) - \sigma(t))\,dx. \end{aligned}$$

By integrating with respect to time we obtain

$$
\begin{aligned}
&\mathcal{Q}(e_\lambda(t)-e(t))+\frac{1}{2}\|z_\lambda(t)-\zeta_\lambda(t)\|_2^2+\frac{1}{2}\|\dot u_\lambda(t)-\dot u(t)\|_2^2\\
&\quad\le \frac{1}{2}\|z(0)-\zeta_\lambda(0)\|_2^2-\int_0^t\int_\Omega \dot e(s):(\sigma_\lambda(s)-\sigma(s))\,dx\,ds-\int_0^t\int_\Omega \dot\zeta_\lambda(s)(z_\lambda(s)-\zeta_\lambda(s))\,dx\,ds\\
&\qquad+\int_0^t\int_\Omega(\dot w(s)-\dot u(s))\cdot(\ddot u_\lambda(s)-\ddot u(s))\,dx\,ds+\int_0^t\int_\Omega E\dot w(s):(\sigma_\lambda(s)-\sigma(s))\,dx\,ds,
\end{aligned}
$$

where we used that $e_\lambda(0)=e(0)$, $z_\lambda(0)=z(0)$, and $\dot u_\lambda(0)=\dot u(0)$. As $\lambda\to\infty$, the right-handside converges to 0 by (5.10)–(5.12) and (5.17). Owing to (2.1), we deduce (5.6), (5.7), and

$$z_\lambda-\zeta_\lambda\to 0 \text{ strongly in } L^\infty(0,T;L^2(\Omega)),$$

which, together with (5.17), implies (5.16).

**Step 5: Flow rule.** Owing to the strong convergences proved in the previous step, we are now in a position to pass to the limit into the energy inequality (4.25). We first observe that for every $0\le t_1\le t_2\le T$ and every $\lambda_1\ge 1$ we have by lower semicontinuity of the total dissipation and Proposition 2.3

$$
\begin{aligned}
&\int_{t_1}^{t_2}\mathcal{H}_{\lambda_1}(\dot p(s),\dot z(s))\,ds=\mathcal{D}_{\lambda_1}(p,z,[t_1,t_2])\\
&\qquad\le\liminf_{\lambda\to\infty}\mathcal{D}_{\lambda_1}(p_\lambda,z_\lambda,[t_1,t_2])=\liminf_{\lambda\to\infty}\int_{t_1}^{t_2}\mathcal{H}_{\lambda_1}(\dot p_\lambda(s),\dot z_\lambda(s))\,ds\\
&\qquad\le\liminf_{\lambda\to\infty}\int_{t_1}^{t_2}\mathcal{H}_{\lambda}(\dot p_\lambda(s),\dot z_\lambda(s))\,ds.
\end{aligned}
$$

Note that the last inequality follows from the monotonicity of $H_\lambda$ with respect to $\lambda$. By monotone convergence, letting $\lambda_1\to\infty$ and applying Lemma 2.2 yield

$$\int_{t_1}^{t_2}\mathcal{H}(\dot p(s))\,ds\le\liminf_{\lambda\to\infty}\int_{t_1}^{t_2}\mathcal{H}_\lambda(\dot p_\lambda(s),\dot z_\lambda(s))\,ds,$$

where we used (5.14). Thus by passing to the limit in (4.25) we obtain

$$
\begin{aligned}
&\mathcal{Q}(e(t_2))+\frac{1}{2}\|z(t_2)\|_2^2+\int_{t_1}^{t_2}\mathcal{H}(\dot p(s))\,ds+\frac{1}{2}\|\dot u(t_2)\|_2^2\\
&\qquad\le\mathcal{Q}(e(t_1))+\frac{1}{2}\|z(t_1)\|_2^2+\frac{1}{2}\|\dot u(t_1)\|_2^2\\
&\qquad\quad+\int_{t_1}^{t_2}\int_\Omega(\sigma:E\dot w+\ddot u\cdot\dot w)\,dx\,ds+\int_{t_1}^{t_2}f\cdot(\dot u-\dot w)\,dx\,ds. \qquad (5.18)
\end{aligned}
$$

Since $\dot z(t)\ge 0$ a.e. in $\Omega$, we have $z(t_2)\ge z(t_1)\ge z_0\ge 0$ a.e. in $\Omega$; thus,

$$
\begin{aligned}
&\mathcal{Q}(e(t_2))+\int_{t_1}^{t_2}\mathcal{H}(\dot p(s))\,ds+\frac{1}{2}\|\dot u(t_2)\|_2^2\\
&\qquad\le\mathcal{Q}(e(t_1))+\frac{1}{2}\|\dot u(t_1)\|_2^2+\int_{t_1}^{t_2}\int_\Omega(\sigma:E\dot w+\ddot u\cdot\dot w)\,dx\,ds+\int_{t_1}^{t_2}f\cdot(\dot u-\dot w)\,dx\,ds.
\end{aligned}
$$

By differentiation with respect to time, the duality formula (2.9) and the kinematic compatibility for the rates (5.15), this is equivalent to

$$\mathcal{H}(\dot p(t))\le\langle\sigma(t),\dot p(t)\rangle$$

for a.e. $t \in [0,T]$. Using next (2.12), and arguing as in the proof of Theorem 4.1, one can prove that this last inequality yields, in turn, (5.8).

**Step 6: Uniqueness.** The proof of the uniqueness of the solution is identical to that of Theorem 4.1, and rests on the stress-strain duality (2.9) as well as on the duality formula (2.13). In particular, the uniqueness ensures that there is no need to extract subsequences to get the previous convergences. □

**Remark 5.2.** Integrating the flow rule (5.8), and using the duality formula (2.9), together with the kinematic compatibility for the rates (5.15), one can show that the energy inequality is actually an equality:

$$\mathcal{Q}(e(t)) + \int_0^t \mathcal{H}(\dot p(s))\,ds + \frac{1}{2}\|\dot u(t)\|_2^2 = \mathcal{Q}(e(0)) + \frac{1}{2}\|v_0\|_2^2 + \int_0^t \int_\Omega (\sigma : E\dot w + \ddot u \cdot \dot w)\,dx\,ds + \int_0^t \int_\Omega f \cdot (\dot u - \dot w)\,dx\,ds.$$

As a consequence, since $t \mapsto z(t)$ is non-decreasing, comparing with (5.18), we deduce that $z(t) = z_0$ (and thus $\xi(t) = \xi_0$) for all $t \in [0,T]$.

## 6. The quasi-static case

In the dynamical elasto-plastic models studied in the previous sections the kinetic energy gives a natural $L^2(\Omega;\mathbb{R}^n)$ bound on the velocity field, which is crucial in order to define the duality between the stress and the plastic strain rate. Unfortunately, in the quasi-static case, the velocity only belongs, in general, to $BD(\Omega)$ and thus, it is at most in $L^{n/(n-1)}(\Omega;\mathbb{R}^n)$. Consequently, without any further regularity result at our disposal, the stress-strain duality might not be well defined, except of course in the two dimensional setting. This is clearly an obstacle in order to express the flow rule in a measure theoretic sense as we did in (4.4) for the cap model, and in (5.8) for the perfect elasto-plastic model. However, in higher dimension it is possible to give a weak sense to the flow rule by means of an energy equality.

6.1. **Quasi-static elasto-plastic cap model.** Using a variational evolution approach similar to that of [10], we can show an existence result for solutions of a quasi-static elasto-plastic cap model. In this context, since the kinetic energy is no longer controlled, given a boundary displacement $\hat w \in H^1(\Omega;\mathbb{R}^n)$, the space of kinematically admissible fields is defined by

$$\mathcal{A}_{\rm qs}(\hat w) := \Big\{(v,\eta,q) \in BD(\Omega) \times L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \times \mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym}) : Ev = \eta + q \text{ in } \Omega, \quad q = (\hat w - v) \odot \nu \mathcal{H}^{n-1} \text{ on } \partial\Omega\Big\}.$$

Let us fix $\lambda \geq 1$ and consider a body load $f$ satisfying

$$f \in AC([0,T]; L^n(\Omega;\mathbb{R}^n)), \tag{6.1}$$

and a boundary displacement which is the trace on $\partial\Omega \times [0,T]$ of a function

$$w \in AC([0,T]; H^1(\Omega;\mathbb{R}^n)). \tag{6.2}$$

We also consider an initial datum $(u_0, e_0, p_0, z_0) \in \mathcal{A}_{\rm qs}(w(0)) \times L^2(\Omega)$ satisfying the stability condition

$$\mathcal{Q}(e_0) + \frac{1}{2}\|z_0\|_2^2 - \int_\Omega f(0)\cdot u_0\,dx \leq \mathcal{Q}(\eta) + \mathcal{H}_\lambda(q - p_0, \zeta - z_0) + \frac{1}{2}\|\zeta\|_2^2 - \int_\Omega f(0)\cdot v\,dx, \tag{6.3}$$

for any $(v,\eta,q,\zeta) \in \mathcal{A}_{\rm qs}(w(0)) \times L^2(\Omega)$, and we define $(\sigma_0,\xi_0) := (\mathbb{C}e_0, -z_0)$.

Contrary to the dynamical case, we need also to assume the following safe-load condition: there exist $\chi \in AC([0,T]; L^n(\Omega;\mathbb{M}^{n\times n}_{sym}))$, $\vartheta \in AC([0,T]; L^2(\Omega))$, and a constant $\alpha_0 > 0$ such that for every $t \in [0,T]$

$$\begin{cases} -\mathrm{div}\chi(t) = f(t) & \text{a.e. in } \Omega, \\ (\chi(t)+\tau, \vartheta(t)) \in K_\lambda & \text{a.e. in } \Omega \end{cases} \tag{6.4}$$

for every $\tau \in \mathbb{M}^{n\times n}_{sym}$ with $|\tau| \le \alpha_0$.

As explained in the introduction, the validity of the safe-load condition ensures a control on the plastic strain (rate) from a control on the dissipation functional. Indeed, the following result establishes a coercivity property of the functional $p \mapsto \mathcal{H}_\lambda(p,z) - \langle \chi(t), p\rangle$.

**Proposition 6.1.** *Assume* (6.4). *Let $\hat w \in H^1(\Omega;\mathbb{R}^n)$ and let $(u,e,p,z) \in \mathcal{A}_{\rm qs}(\hat w) \times L^2(\Omega)$. Then for every $t \in [0,T]$ the following coercivity estimate holds:*

$$\mathcal{H}_\lambda(p,z) - \langle \chi(t), p\rangle \ge \alpha_0 \|p\|_1 - \alpha_1 \|z\|_2,$$

*where $\alpha_1 := \|\vartheta\|_{L^\infty(0,T;L^2(\Omega))}$.*

*Proof.* We notice that the duality $\langle \chi(t), p\rangle$ is well defined owing to Remark 2.7. Moreover, we can assume $\mathcal{H}_\lambda(p,z) < \infty$, otherwise the result is trivial. By the duality formula (2.15) and Remark 2.7 we have

$$\mathcal{H}_\lambda(p,z) - \langle \chi(t), p\rangle = \sup\Big\{ \langle \sigma - \chi(t), p\rangle + \int_\Omega \xi z\, dx : (\sigma,\xi) \in L^n(\Omega;\mathbb{M}^{n\times n}_{sym}) \times L^2(\Omega) \text{ with } \mathrm{div}\sigma \in L^n(\Omega;\mathbb{R}^n) \text{ and } (\sigma(x),\xi(x)) \in K_\lambda \text{ a.e. in } \Omega \Big\}.$$

Using (2.11) and (6.4), this implies that

$$\begin{aligned} \mathcal{H}_\lambda(p,z) - \langle \chi(t), p\rangle &\ge \sup\Big\{ \int_{\overline\Omega} \tau : dp + \int_\Omega \vartheta(t) z\, dx : \tau \in \mathcal{C}^\infty(\overline\Omega;\mathbb{R}^n) \text{ and } |\tau| \le \alpha_0 \text{ in } \overline\Omega \Big\} \\ &= \alpha_0 \|p\|_1 + \int_\Omega \vartheta(t) z\, dx, \end{aligned}$$

from which the thesis immediately follows. □

The following result concerns the optimality conditions of a suitable minimum problem arising in the definition of the incremental evolution.

**Lemma 6.2.** *Let $\hat w \in H^1(\Omega;\mathbb{R}^n)$, $f \in L^n(\Omega;\mathbb{R}^n)$, $(u,e,p,z) \in \mathcal{A}_{\rm qs}(\hat w) \times L^2(\Omega)$, and $(\sigma,\xi) := (\mathbb{C}e, -z)$. Then the following conditions are equivalent:*

(a) *$(u,e,p,z)$ is a solution of*

$$\min_{(v,\eta,q,\zeta) \in \mathcal{A}_{\rm qs}(\hat w) \times L^2(\Omega)} \Big\{ \mathcal{Q}(\eta) + \frac12 \|\zeta\|_2^2 + \mathcal{H}_\lambda(q-p, \zeta - z) - \int_\Omega f \cdot v\, dx \Big\};$$

(b) *$(\sigma,\xi)$ satisfies*

$$-\int_\Omega \sigma : \eta\, dx + \int_\Omega \xi\zeta\, dx + \int_\Omega f \cdot v\, dx \le \mathcal{H}_\lambda(q,\zeta)$$

*for every $(v,\eta,q,\zeta) \in \mathcal{A}_{\rm qs}(0) \times L^2(\Omega)$.*

*If (a) or (b) holds, then the following conditions are satisfied:*

(c) *$(\sigma,\xi) \in K_\lambda$ and $-\mathrm{div}\sigma = f$ a.e. in $\Omega$.*

*If, in addition, $\sigma \in L^n(\Omega;\mathbb{M}^{n\times n}_{sym})$ or $u \in L^2(\Omega;\mathbb{R}^n)$, then (c) is equivalent to (a) and (b).*

*Proof.* The proof is an immediate adaptation of [10, Theorem 3.6], once we notice that if $\sigma \in L^n(\Omega;\mathbb{M}^{n\times n}_{sym})$ or $u \in L^2(\Omega;\mathbb{R}^n)$ the stress-strain duality is well defined by Remark 2.7. □

We are now in a position to prove the first main result of this section.

**Theorem 6.3.** *Let $\lambda \geq 1$. Assume (2.1)–(2.4) and (6.1)–(6.4). Then there exist*

$$\begin{cases} u \in AC([0,T];BD(\Omega)),\\ \sigma,\ e \in AC([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_{sym})),\\ p \in AC([0,T];\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym})),\\ \xi,\ z \in AC([0,T];L^2(\Omega)),\end{cases}$$

*with $(u(0),e(0),p(0),z(0)) = (u_0,e_0,p_0,z_0)$, satisfying the following properties: for all $t \in [0,T]$*

$$\begin{cases} Eu(t) = e(t)+p(t) \text{ in } \Omega, \quad p(t) = (w(t)-u(t))\odot \nu\mathcal{H}^{n-1} \text{ on } \partial\Omega,\\ \sigma(t) = \mathbb{C}e(t), \quad \xi(t) = -z(t),\end{cases} \tag{6.5}$$

*and*

$$\begin{cases} -\operatorname{div}\sigma(t) = f(t) \text{ a.e. in } \Omega,\\ (\sigma(t),\xi(t)) \in K_\lambda.\end{cases}$$

*Moreover, for a.e. $t \in [0,T]$*

$$\dot z(t) \geq 0 \quad \text{a.e. in } \Omega, \tag{6.6}$$

*and the following energy equality holds for all $t \in [0,T]$,*

$$\mathcal{Q}(e(t)) + \frac{1}{2}\|z(t)\|_2^2 + \int_0^t \mathcal{H}_\lambda(\dot p(s),\dot z(s))\,ds = \mathcal{Q}(e_0) + \frac{1}{2}\|z_0\|_2^2 + \int_0^t\int_\Omega \sigma(s) : E\dot w(s)\,dx\,ds - \int_0^t\int_\Omega f(s)\cdot(\dot w(s)-\dot u(s))\,dx\,ds. \tag{6.7}$$

*Proof.* We give a sketch of the proof, which is based on the standard incremental variational approach as in [26, 10]. We introduce a time discretization

$$0 = t_k^0 < t_k^1 < \cdots < t_k^{N_k} = T, \quad \text{with} \quad \delta_k := \sup_{1\leq i\leq N_k}(t_k^i - t_k^{i-1}) \to 0$$

of the time interval $[0,T]$. For each $i = 0$, we set $(u_k^0,e_k^0,p_k^0,z_k^0) := (u_0,e_0,p_0,z_0)$ and for all $i \in \{1,\dots,N_k\}$ we define $(u_k^i,e_k^i,p_k^i,z_k^i) \in \mathcal{A}_{\rm qs}(w(t_k^i))\times L^2(\Omega)$ as a solution of the minimum problem

$$\min_{(v,\eta,q,\zeta)\in\mathcal{A}_{\rm qs}(w(t_k^i))\times L^2(\Omega)} \Big\{\mathcal{Q}(\eta) + \frac{1}{2}\|\zeta\|_2^2 + \mathcal{H}_\lambda(q-p_k^{i-1},\zeta-z_k^{i-1}) - \int_\Omega f(t_k^i)\cdot v\,dx\Big\}, \tag{6.8}$$

whose existence is ensured by the direct method, owing to Proposition 6.1, the Poincaré-Korn inequality and the safe load condition (6.4).

By minimality and (2.6), we have that $z_k^i \geq z_k^{i-1}$ a.e. in $\Omega$ for all $i \in \{1,\dots,N_k\}$. Moreover, since $H_\lambda$ satisfies the triangle inequality, the quadruplet $(u_k^i,e_k^i,p_k^i,z_k^i)$ is also a solution of

$$\min_{(v,\eta,q,\zeta)\in\mathcal{A}_{\rm qs}(w(t_k^i))\times L^2(\Omega)} \Big\{\mathcal{Q}(\eta) + \frac{1}{2}\|\zeta\|_2^2 + \mathcal{H}_\lambda(q-p_k^{i},\zeta-z_k^{i}) - \int_\Omega f(t_k^i)\cdot v\,dx\Big\}.$$

By Lemma 6.2, setting $(\sigma_k^i,\xi_k^i) := (\mathbb{C}e_k^i,-z_k^i)$, we have

$$-\int_\Omega \sigma_k^i : \eta\,dx - \int_\Omega z_k^i\zeta\,dx + \int_\Omega f(t_k^i)\cdot v\,dx \leq \mathcal{H}_\lambda(q,\zeta) \tag{6.9}$$

for every $(v,\eta,q,\zeta) \in \mathcal{A}_{\rm qs}(0) \times L^2(\Omega)$. Moreover, using the minimality property in (6.8), the following discrete energy inequality can be proved: for all $j \in \{1,\dots,N_k\}$,

$$\begin{aligned}\mathcal{Q}(e_k^j) - \int_\Omega \chi(t_k^j):(e_k^j - Ew(t_k^j))\,dx + \frac{1}{2}\|z_k^j\|_2^2 \\ + \sum_{i=1}^{j}\Big(\mathcal{H}_\lambda(p_k^i - p_k^{i-1}, z_k^i - z_k^{i-1}) - \langle \chi(t_k^i), p_k^i - p_k^{i-1}\rangle\Big) \\ \le \mathcal{Q}(e_0) - \int_\Omega \chi(0):(e_0 - Ew(0))\,dx + \frac{1}{2}\|z_0\|_2^2 + \sum_{i=1}^{j}\int_{t_k^{i-1}}^{t_k^i}\int_\Omega \sigma_k^{i-1}:E\dot w(s)\,dx\,ds \\ - \sum_{i=1}^{j}\int_{t_k^{i-1}}^{t_k^i}\int_\Omega \dot\chi(s):(e_k^{i-1} - Ew(t_k^{i-1}))\,dx\,ds + o(1) \quad \text{as } k\to\infty. \quad (6.10)\end{aligned}$$

Let $u_k(t)$, $e_k(t)$, $p_k(t)$, $z_k(t)$, $\sigma_k(t)$, and $\xi_k(t)$ be the piecewise constant right-continuous interpolations of $\{u_k^i\}_{0\le i\le N_k}$, $\{e_k^i\}_{0\le i\le N_k}$, $\{p_k^i\}_{0\le i\le N_k}$, $\{z_k^i\}_{0\le i\le N_k}$, $\{\sigma_k^i\}_{0\le i\le N_k}$, and $\{\xi_k^i\}_{0\le i\le N_k}$. By Proposition 6.1 and (6.10) we can deduce some *a priori* estimates on the interpolations. By Helly's Theorem they imply that, up to a subsequence, $p_k(t) \rightharpoonup p(t)$ weakly* in $\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym})$ and $z_k(t) \rightharpoonup z(t)$ weakly in $L^2(\Omega)$ for all $t\in[0,T]$, where $p \in BV([0,T];\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym}))$ and $z \in BV([0,T];L^2(\Omega))$ with $z(t)\ge z(s)$ for all $0\le s\le t\le T$. Owing to a uniqueness argument, we also deduce that, for the same subsequence, $e_k(t)\rightharpoonup e(t)$ weakly in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ and $u_k(t)\rightharpoonup u(t)$ weakly* in $BD(\Omega)$ for all $t\in[0,T]$, for some weakly measurable map $e:[0,T]\to L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ and some weakly* measurable map $u:[0,T]\to BD(\Omega)$, satisfying $(u(t),e(t),p(t))\in\mathcal{A}_{\rm qs}(w(t))$. Passing to the lower limit in (6.10) we obtain an energy inequality for the limit evolution.

For all $t\in[0,T]$, we define $(\sigma(t),\xi(t)) := (\mathbb{C}e(t), -z(t))$. Passing to the limit in the Euler-Lagrange equation (6.9) leads to

$$-\int_\Omega \sigma(t):\eta\,dx - \int_\Omega z(t)\zeta\,dx + \int_\Omega f(t)\cdot v\,dx \le \mathcal{H}_\lambda(q,\zeta) \quad (6.11)$$

for every $(v,\eta,q,\zeta)\in\mathcal{A}_{\rm qs}(0)\times L^2(\Omega)$. By Lemma 6.2 this implies that $-{\rm div}\sigma(t) = f(t)$ and $(\sigma(t),\xi(t))\in K_\lambda$ a.e. in $\Omega$. This is true also at time $t=0$ owing to the assumptions on the initial datum.

The converse energy inequality can be proved as in [10, Theorem 4.7]. The argument is based on a use of the minimality property together with an approximation of $t\mapsto p(t)$ and $t\mapsto e(t)$ by means of piecewise constant mappings. For $p$, this is possible since the map $t\mapsto p(t)$ has (at most) countably many discontinuity points for the strong $\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym})$-topology. For what concerns $t\mapsto e(t)$, this can be done by approximating its $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$-Bochner integral by suitable Riemann sums (see, *e.g.*, [9, Lemma 4.12]).

Arguing as in [10, Theorem 5.2], we deduce from the energy equality that $u$, $e$, $p$, and $z$ are absolutely continuous in time. Moreover, the following estimate holds true:

$$\|\dot e\|_{L^1(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))} + \|\dot z\|_{L^1(0,T;L^2(\Omega))} + \|\dot p\|_{L^1(0,T;\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym}))} + \|\dot u\|_{L^1(0,T;BD(\Omega))} \le C \quad (6.12)$$

where $C>0$ is a constant depending on the norms $\|\chi\|_{AC([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))}$, $\|w\|_{AC([0,T];H^1(\Omega;\mathbb{R}^n))}$, $\|f\|_{AC([0,T];L^n(\Omega;\mathbb{R}^n))}$, $\sup_t\|\vartheta(t)\|_2$, $\sup_t\|e(t)\|_2$, $\sup_t\|p(t)\|_1$ and $\sup_t\|z(t)\|_2$, but independent of $\lambda$. We note that in order to derive estimate (6.12), we use at some point the coercivity property of Proposition 6.1 and the following estimate for the stress/strain duality: for all $\tau\in L^n(\Omega;\mathbb{M}^{n\times n}_{sym})$ with ${\rm div}\tau\in L^n(\Omega;\mathbb{R}^n)$, and for all $s\in(0,T)$,

$$|\langle\tau,p(s)\rangle| \le C\big(\|{\rm div}\tau\|_n + \|\tau\|_2\big)\big(\|e(s)\|_2 + \|p(s)\|_1 + \|w(s)\|_{H^1(\Omega;\mathbb{R}^n)}\big),$$

for some constant $C > 0$ depending only on $\Omega$. In particular, $\dot{z}(t) \geq 0$ for a.e. $t \in [0, T]$ and a.e. $x \in \Omega$, and according to Proposition 2.3, we can write the energy equality as follows:

$$\mathcal{Q}(e(t)) - \int_\Omega \chi(t) : (e(t) - Ew(t))\, dx + \frac{1}{2}\|z(t)\|_2^2 + \int_0^t \mathcal{H}_\lambda(\dot{p}(s), \dot{z}(s))\, ds - \langle \chi(t), p(t) \rangle$$
$$= \mathcal{Q}(e_0) - \int_\Omega \chi(0) : (e_0 - Ew(0))\, dx + \frac{1}{2}\|z_0\|_2^2 - \langle \chi(0), p_0 \rangle$$
$$+ \int_0^t \int_\Omega \sigma(s) : E\dot{w}(s)\, dx\, ds - \int_0^t \int_\Omega \dot{\chi}(s) : (e(s) - Ew(s))\, dx\, ds - \int_0^t \langle \dot{\chi}(s), p(s) \rangle\, ds \quad (6.13)$$

for every $t \in [0, T]$. By applying again the duality formula (2.9), the safe-load condition (6.4), and by integrating by parts in time the force integral, the formula above reduces to (6.7). This completes the proof of the theorem. □

The following result states a more precise formulation of the energy equality as a flow rule, whenever additional integrability properties are satisfied for the stress and/or the velocity.

**Theorem 6.4.** *Under the same assumptions of Theorem 6.3, assume further that either $\sigma(t) \in L^n(\Omega; \mathbb{M}^{n\times n}_{sym})$ or $\dot{u}(t) \in L^2(\Omega; \mathbb{R}^n)$ for a.e. $t \in [0, T]$. Then the distribution $[\sigma(t) : \dot{p}(t)]$ is well defined for a.e. $t \in [0, T]$, and it is a measure in $\mathcal{M}(\overline{\Omega})$ satisfying*

$$H_\lambda(\dot{p}(t), \dot{z}(t)) = [\sigma(t) : \dot{p}(t)] + \xi(t) z(t) \quad \text{in } \mathcal{M}(\overline{\Omega}), \quad \text{for a.e. } t \in [0, T]. \quad (6.14)$$

*Moreover, the stress $\sigma$ and the cap variable $\xi$ are unique.*

*Proof.* By Definition 2.4, if $\dot{u}(t) \in L^2(\Omega; \mathbb{R}^n)$, then the distribution $[\tau : \dot{p}(t)]$ is well defined for every $\tau \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ with $\operatorname{div}\tau \in L^2(\Omega; \mathbb{R}^n)$. Equality (6.14) is obtained from (6.7) exactly as in the proof of Proposition 4.7. By Remark 2.7, since $\operatorname{div}\sigma(t) = -f(t) \in L^n(\Omega; \mathbb{R}^n)$, the same conclusion holds if $\sigma(t) \in L^n(\Omega; \mathbb{M}^{n\times n}_{sym})$.

Once the stress-strain duality pairing is defined, it is possible to argue as in [10, Theorem 5.9] to establish the uniqueness of $\sigma$ and $\xi$. □

Note that the assumptions of Theorem 6.4 are clearly satisfied for $n = 2$. However, it is not clear to us if such integrability properties for the stress and/or the velocity are true in higher dimension.

6.2. **Convergence of the quasi-static cap model.** In this section we characterize the asymptotic behaviour of the quasi-static evolutions studied in Theorems 6.3 and 6.4 when the cap is sent to infinity. We observe that the abstract theory of evolutionary Γ-convergence for rate-independent systems developed in [28] cannot be directly applied here. Indeed, this method prescribes to study separately the Γ-limit of the stored-energy functionals and that of the dissipation distances and then to couple them through the construction of a joint recovery sequence. This approach is not suited to our case, since coercivity in the full strain $Eu$ can be achieved only by a simultaneous control on the stored energy and the dissipation.

We consider a body load $f$ satisfying (6.1) together with the following safe-load condition: there exist $\chi \in AC([0, T]; L^n(\Omega; \mathbb{M}^{n\times n}_{sym}))$ and a constant $\alpha_0 > 0$ such that for every $t \in [0, T]$

$$\begin{cases} -\operatorname{div}\chi(t) = f(t) & \text{a.e. in } \Omega, \\ \chi(t) + \tau \in K & \text{a.e. in } \Omega \end{cases} \quad (6.15)$$

for every $\tau \in \mathbb{M}^{n\times n}_{sym}$ with $|\tau| \leq \alpha_0$. We also consider an initial datum $(u_0, e_0, p_0) \in \mathcal{A}_{\text{qs}}(w(0))$ satisfying the stability condition

$$\mathcal{Q}(e_0) - \int_\Omega f(0) \cdot u_0\, dx \leq \mathcal{Q}(\eta) + \mathcal{H}(q - p_0) - \int_\Omega f(0) \cdot v\, dx, \quad (6.16)$$

for any $(v, \eta, q) \in \mathcal{A}_{\text{qs}}(w(0))$.

**Theorem 6.5.** *Assume* (2.1)–(2.4), (6.1), (6.2), (6.15), *and* (6.16). *For every* $\lambda \geq 1$ *let* $u_\lambda$, $e_\lambda$, $\sigma_\lambda$, $p_\lambda$, $z_\lambda$, $\xi_\lambda$ *be the solution to* (6.5)–(6.7) *constructed in Theorem 6.3 with initial datum* $(u_0, e_0, p_0, z_0)$, *where* $z_0$ *is given by* (5.2). *Assume further that either* $u_0 \in L^2(\Omega; \mathbb{R}^n)$ *or* $\sigma_0 := \mathbb{C}e_0 \in L^n(\Omega; \mathbb{M}^{n\times n}_{sym})$. *Then there exist a subsequence* $(\lambda_k) \nearrow \infty$, *as* $k \to \infty$, *and*

$$\begin{cases} u \in AC([0,T]; BD(\Omega)), \\ \sigma,\ e \in AC([0,T]; L^2(\Omega; \mathbb{M}^{n\times n}_{sym})), \\ p \in AC([0,T]; \mathcal{M}(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})), \end{cases}$$

*with* $(u(0), e(0), p(0)) = (u_0, e_0, p_0)$, *such that for all* $t \in [0,T]$

$$\begin{aligned} &u_{\lambda_k}(t) \rightharpoonup u(t) \ \text{weakly}^* \text{ in } BD(\Omega), \\ &e_{\lambda_k}(t) \to e(t) \ \text{strongly in } L^2(\Omega; \mathbb{M}^{n\times n}_{sym}), \\ &p_{\lambda_k}(t) \rightharpoonup p(t) \ \text{weakly}^* \text{ in } \mathcal{M}(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym}), \end{aligned} \tag{6.17}$$

*as* $k \to \infty$. *For every* $t \in [0,T]$, *we have*

$$\begin{cases} Eu(t) = e(t) + p(t) \ \text{in } \Omega, \quad p(t) = (w(t) - u(t)) \odot \nu \mathcal{H}^{n-1} \ \text{on } \partial\Omega, \\ \sigma(t) = \mathbb{C}e(t), \end{cases}$$

*and*

$$\begin{cases} -\mathrm{div}\sigma(t) = f(t) \ \text{a.e. in } \Omega, \\ \sigma(t) \in K \ \text{a.e. in } \Omega. \end{cases}$$

*Moreover, the following energy equality holds*

$$\mathcal{Q}(e(t)) + \int_0^t \mathcal{H}(\dot{p}(s))\, ds = \mathcal{Q}(e_0) + \int_0^t \int_\Omega \sigma(s) : E\dot{w}(s)\, dx\, ds - \int_0^t \int_\Omega f(s) \cdot (\dot{w}(s) - \dot{u}(s))\, dx\, ds. \tag{6.18}$$

*Proof.* In order to apply Theorem 6.3, we set $\vartheta := \chi_m - |\chi_m| - \alpha_0/\sqrt{n} \in AC([0,T]; L^2(\Omega))$ and we observe that

$$(\chi(t) + \tau, \vartheta(t)) \in K_\lambda \quad \text{in } \Omega$$

for every $\tau \in \mathbb{M}^{n\times n}_{sym}$ with $|\tau| \leq \alpha_0$ and every $\lambda \geq 1$, so that the body load $f$ satisfies (6.4) as well. Moreover, since $\sigma_0 \in L^n(\Omega; \mathbb{M}^{n\times n}_{sym})$ or $u_0 \in L^2(\Omega; \mathbb{R}^n)$ by assumption, Lemma 6.2 ensures that $(u_0, e_0, p_0, z_0)$ is a solution of

$$\min_{(v,\eta,q,\zeta) \in \mathcal{A}_{\mathrm{qs}}(w(0)) \times L^2(\Omega)} \Big\{ \mathcal{Q}(\eta) + \frac{1}{2}\|\zeta\|_2^2 + \mathcal{H}_\lambda(q - p_0, \zeta - z_0) - \int_\Omega f(0) \cdot v\, dx \Big\}.$$

Consequently, $(u_0, e_0, p_0, z_0)$ is an admissible initial datum for Theorem 6.3, and $u_\lambda$, $e_\lambda$, $\sigma_\lambda$, $p_\lambda$, $z_\lambda$, $\xi_\lambda$ are well defined.

Some *a priori* estimates for the sequences $(u_\lambda)$, $(e_\lambda)$, $(p_\lambda)$, and $(z_\lambda)$ can be obtained from the uniform estimate (6.12) and the energy equality (6.13). They imply the existence of functions $u \in AC([0,T]; BD(\Omega))$, $e \in AC([0,T]; L^2(\Omega; \mathbb{M}^{n\times n}_{sym}))$, $p \in AC([0,T]; \mathcal{M}(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym}))$, and $z \in AC([0,T]; L^2(\Omega))$ such that, on a subsequence denoted $(\lambda_k)$, $u_{\lambda_k}(t) \rightharpoonup u(t)$ weakly* in $BD(\Omega)$, $e_{\lambda_k}(t) \rightharpoonup e(t)$ weakly in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, $p_{\lambda_k}(t) \rightharpoonup p(t)$ weakly* in $\mathcal{M}(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})$, and $z_{\lambda_k}(t) \rightharpoonup z(t)$ weakly in $L^2(\Omega)$. Moreover, we have that $(u(0), e(0), p(0), z(0)) = (u_0, e_0, p_0, z_0)$, $Eu(t) = e(t) + p(t)$ in $\Omega$, $p(t) = (w(t) - u(t)) \odot \nu \mathcal{H}^{n-1}$ on $\partial\Omega$ for every $t \in [0,T]$, and $\dot{z}(t) \geq 0$ a.e. in $\Omega$, for a.e. $t \in [0,T]$. In particular, we infer that $z(t) \geq z_0$ a.e. in $\Omega$ for every $t \in [0,T]$.

Setting $\sigma(t) := \mathbb{C}e(t)$, passing to the limit in (6.11) as $k \to \infty$, and applying Lemma 2.2, we obtain

$$-\int_\Omega \sigma(t) : \eta \, dx - \int_\Omega z(t)\zeta \, dx + \int_\Omega f(t) \cdot v \, dx \leq \mathcal{H}(q)$$

for every $(v, \eta, q, \zeta) \in \mathcal{A}_{\rm qs}(0) \times L^2(\Omega)$ with $\zeta \geq 0$ in $\Omega$. For $\zeta \equiv 0$ this implies that

$$-\int_\Omega \sigma(t) : \eta \, dx + \int_\Omega f(t) \cdot v \, dx \leq \mathcal{H}(q)$$

for every $(v, \eta, q) \in \mathcal{A}_{\rm qs}(0)$. By [10, Theorem 3.6] this condition is equivalent to saying that $(u(t), e(t), p(t))$ minimizes the functional

$$\mathcal{Q}(e) + \mathcal{H}(p - p(t)) - \int_\Omega f(t) \cdot u \, dx$$

over all $(u, e, p) \in \mathcal{A}_{\rm qs}(w(t))$. This, in turn, implies that $\sigma(t) \in K$ and $-\mathrm{div}\sigma(t) = f(t)$ a.e. in $\Omega$.

Furthermore, arguing as in the proof of Theorem 5.1, we deduce the following energy inequality:

$$\begin{aligned} \mathcal{Q}(e(t)) - \int_\Omega \chi(t) : (e(t) - Ew(t)) \, dx + \int_0^t \mathcal{H}(\dot{p}(s)) \, ds - \langle \chi(t), p(t) \rangle \\ \leq \mathcal{Q}(e_0) - \int_\Omega \chi(0) : (e_0 - Ew(0)) \, dx - \langle \chi(0), p_0 \rangle \\ + \int_0^t \int_\Omega \sigma(s) : E\dot{w}(s) \, dx \, ds - \int_0^t \int_\Omega \dot{\chi}(s) : (e(s) - Ew(s)) \, dx \, ds - \int_0^t \langle \dot{\chi}(s), p(s) \rangle \, ds. \end{aligned}$$

Finally, we argue as in the proof of Theorem 6.3 to show that this inequality is actually an equality leading to (6.18) by (2.9), (6.15), and integration by parts with respect to $t$. The strong convergence (6.17) can be proved as in [10, Theorem 4.8]. $\square$

Once again, provided the stress and/or the velocity have enough integrability in such a way that the stress-strain duality is well defined, one can write the flow rule in a measure theoretic sense.

**Theorem 6.6.** *Under the same assumptions of Theorem 6.5, assume further that either* $\sigma(t) \in L^n(\Omega; \mathbb{M}^{n \times n}_{sym})$ *or* $\dot{u}(t) \in L^2(\Omega; \mathbb{R}^n)$ *for a.e.* $t \in [0, T]$. *Then the distribution* $[\sigma(t) : \dot{p}(t)]$ *is well defined for a.e.* $t \in [0, T]$, *and it is a measure in* $\mathcal{M}(\overline{\Omega})$ *satisfying*

$$H(\dot{p}(t)) = [\sigma(t) : \dot{p}(t)] \quad \text{in } \mathcal{M}(\overline{\Omega}), \quad \text{for a.e. } t \in [0, T].$$

*Moreover, the stress* $\sigma$ *is unique.*

We again observe that the assumptions of Theorem 6.6 are clearly satisfied in the two-dimensional setting.

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