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APPROXIMATION OF INFINITE DIMENSIONAL TEICHMÜLLER SPACES[1]

BY

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ABSTRACT. By means of an exhaustion process it is shown that Teichmüller's metric and Kobayashi's metric are equal for infinite dimensional Teichmüller spaces. By the same approximation method important estimates coming from the Reich-Strebel inequality are extended to the infinite dimensional cases. These estimates are used to show that Teichmüller's metric is the integral of its infinitesimal form. They are also used to give a sufficient condition for a sequence to be an absolute maximal sequence for the Hamilton functional. Finally, they are used to give a new sufficient condition for a sequence of Beltrami coefficients to converge in the Teichmüller metric.

Introduction. The subject of this paper is Teichmüller spaces of infinitely generated Fuchsian groups. By approximation techniques involving theta series and finitely generated subgroups of a given group, we extend certain important results already known in the finite case to the infinitely generated case.

In §1 we set up the approximation technique and cite the necessary theorems involving Poincaré series and approximation by rational functions.

In §2 we consider Kobayashi's extremal problem for Teichmüller spaces with complex structure and prove the following new result. The theorem of Royden on the equality of the Kobayashi and Teichmüller metrics remains true in the infinite cases. These cases include Teichmüller spaces of groups of the first and second kind. In particular, the case of universal Teichmüller space is included.

In §3 we prove the important main inequality of Reich and Strebel [**14**]. Its most significant consequences are upper and lower estimates for the extremal value of the dilatation in a given Teichmüller class. The chief result of this section is that these upper and lower estimates hold even in the infinite dimensional cases.

In §4 we derive the well-known [**16**] infinitesimal form of Teichmüller's metric. We use this general form together with the Hamilton condition as developed by Reich and Strebel [**14**] to show that Teichmüller's metric is equal to the integral of its infinitesimal form. O'Byrne already obtained this result in [**6**, **7**, **13**]. The method used here is more direct and the theorem is proved in greater generality.

In §5, we give a sufficient condition for a sequence φ_n to be an absolute maximal sequence for the Hamilton functional $H[\mu]$. We do not know if this condition is also

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necessary. But if it were, then one would know that if $\nu \sim \mu$ and ν and μ are both extremal then $t\nu \sim t\mu$ for $0 \leqslant t \leqslant 1$.

§6 gives a condition under which the solution to Kobayashi's extremal problem is unique. It turns out to be the same as the classical condition under which it is known that Teichmüller's extremal problem has a unique solution. However, there is no known relationship between the two forms of uniqueness. Moreover, it is not known whether a geodesic joining two points in an infinite dimensional Teichmüller space is unique.

§7 gives a sufficient condition for a sequence of Beltrami coefficients to converge to zero in the Teichmüller metric. This sufficient condition is a consequence of the estimates in §3.

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1. The approximation method. Let Γ be a Fuchsian group with limit set $\Lambda \subseteq \hat{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$. Let C be a closed subset of $\hat{\mathbf{R}}$ which contains Λ and is invariant under Γ. When Γ is elementary or consists of the identity alone, we require that $(C - \Lambda)/\Gamma$ contain three or more points. In particular we will assume C contains $0, 1$ and ∞. Using conjugation by a Möbius transformation which fixes $\hat{\mathbf{R}}$, it is clear that this assumption involves no loss of generality. $M(\Gamma)$ is the set of all elements μ of $L_\infty(\mathbf{C})$ with support in the upper half plane U such that

$$(1.1)\qquad \begin{array}{ll}\text{(i)} & \|\mu\|_\infty < 1 \quad \text{and} \\ \text{(ii)} & \mu(Ax)\overline{A'(z)} = \mu(z)A'(z) \quad \text{for all } A \text{ in } \Gamma.\end{array}$$

We call such a μ a Beltrami coefficient. For each μ in $M(\Gamma)$, let $\tilde{\mu}$ be the extension of μ to the lower half plane given by the rule $\tilde{\mu}(\bar{z}) = \overline{\mu(z)}$ and let w_μ be the unique quasiconformal homeomorphism of the extended complex plane $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ which satisfies

$$(1.2)\qquad \frac{\partial}{\partial \bar{z}} w_\mu = \tilde{\mu} \frac{\partial}{\partial z} w_\mu \quad \text{and} \quad w_\mu(0) = 0, \quad w_\mu(1) = 1, \quad w_\mu(\infty) = \infty.$$

On $M(\Gamma)$ we put an equivalence relation. Two Beltrami coefficients μ and ν are equivalent ($\mu \sim \nu$) if $w_\mu(x) = w_\nu(x)$ for all x in C. The Teichmüller space $T(\Gamma)$ is $M(\Gamma)$ factored by this equivalence relation.

When Γ is infinitely generated or when $(C - \Lambda)/\Gamma$ is infinite, the Teichmüller space is an infinite dimensional manifold. On the other hand, if Γ is finitely generated and if $(C - \Lambda)/\Gamma$ is a finite set, then $T(\Gamma)$ is finite dimensional. In order to approximate the infinite dimensional case by finite dimensional ones, let Γ_n be a sequence of finitely generated subgroups of Γ with limit sets Λ_n and C_n a sequence of subsets of C satisfying:

$$(1.3)\qquad \begin{array}{ll}\text{(i)} & \Gamma_n \subseteq \Gamma_{n+1} \text{ and } \bigcup \Gamma_n = \Gamma, \\ \text{(ii)} & C_n \subseteq C_{n+1} \text{ and } \bigcup C_n = C, \\ \text{(iii)} & C_n \text{ is invariant under } \Gamma_n,\ C_n \supseteq \Lambda_n \text{ and } (C_n - \Lambda_n)/\Gamma_n \\ & \text{is a finite set,} \\ \text{(iv)} & 0, 1 \text{ and } \infty \text{ are elements of } C_1.\end{array}$$

It is clear that such a sequence of subgroups Γ_n of Γ and subsets C_n of C will exist for a given group Γ and closed set C. Notice that $T(\Gamma_n)$ is a set of equivalence classes of elements of $M(\Gamma_n)$. Two elements μ and ν are equivalent ($\mu \sim_n \nu$) if $w_\mu(x) = w_\nu(x)$ for all x in C_n.

Now let $\Omega = \hat{\mathbf{C}} - C$ and $\Omega_n = \hat{\mathbf{C}} - C_n$. Also, let $M(\Gamma_n, \Omega_n)$ be the set of complex-valued measurable functions μ with support in Ω_n satisfying (1.1) with the exception (1.1)(ii) is to hold for all A in Γ_n. Let $w = w^\mu$ be the unique holomorphic self-mapping of $\hat{\mathbf{C}}$ satisfying $\partial w/\partial \bar{z} = \mu \partial w/\partial z$ and normalized to fix 0, 1 and ∞. Define μ to be strongly equivalent to ν and write $\mu \equiv_n \nu$ if $w^\mu(x) = w^\nu(x)$ for all x in C_n and if w^μ is homotopic to w^ν to Ω_n. Using the notation of Kra in [**11**], define the Teichmüller space $\tilde{T}(\Gamma_n, \Omega_n)$ to be $M(\Gamma_n, \Omega_n)/\equiv_n$.

Let π: $M(\Gamma) \to M(\Gamma_n, \Omega_n)$ be defined by $\pi(\mu)(z) = \mu(z)$ for z in U, the upper half plane, and $\pi(\mu)(z) = 0$ for z in L, the lower half plane.

LEMMA 1.1. *Let $C = \hat{\mathbf{R}}$. For μ and ν in $M(\Gamma)$ and $\mu \sim \nu$ one has $\pi(\mu) \equiv_n \pi(v)$.*

PROOF. It is well known [**6**] that $w_\mu(x) = w_\nu(x)$ for all in $\hat{\mathbf{R}}$ implies $w^{\pi(\mu)}(x) = w^{\pi(\nu)}(x)$ for all x in $\hat{\mathbf{R}}$ and, hence, for all x in C_n. Furthermore, by Ahlfors [**1**, p. 119] there is a homotopy h_t: $U \to w^{\pi(\mu)}(U)$ for which $h_0(z) = w^{\pi(\mu)}(z)$ for z in U and $h_1(z) = w^{\pi(\nu)}(z)$ for z in U and $h_t(x) = w^{\pi(\mu)}(x) = w^{\pi(\nu)}(x)$ for x in $\hat{\mathbf{R}}$ and $0 \leq t \leq 1$. This homotopy extends to a homotopy h_t from Ω_n to $w^{\pi(\mu)}(\Omega_n)$ by setting $h_t(z) = w^{\pi(\mu)}(z) = w^{\pi(\nu)}(z)$ for z in $L \cup \mathbf{R}$. It follows that $\pi(\mu)$ and $\pi(\nu)$ are strongly equivalent and this completes the proof.

Lemma 1.1 implies that the mapping π induces a mapping from $T(\Gamma)$ to $\tilde{T}(\Gamma_n, \Omega_n)$. We denote the induced mapping by the same letter π. Since the complex structures on $T(\Gamma)$ and on $\tilde{T}(\Gamma_n, \Omega_n)$ are inherited from $M(\Gamma)$ and $M(\Gamma_n, \Omega_n)$, this induced mapping is holomorphic.

Now suppose μ is extremal in its class in $M(\Gamma)$. By this we mean that $k = \|\mu\|_\infty \leq \|\nu\|_\infty$ for all ν in $M(\Gamma)$ for which $\nu \sim \mu$. Let

$$k_n|\eta_n|/\eta_n \tag{1.4}$$

be extremal in the class of $\pi(\mu)$ in $M(\Gamma_n, \Omega_n)$ under the relation $\equiv_n$. Here $0 \leq k_n < 1$ and η_n is an integrable, holomorphic, quadratic differential for Γ_n on Ω_n. That is, η_n is holomorphic on Ω_n, $\eta_n(Az)A'(z)^2 = \eta_n(z)$ for z in Ω_n and

$$\iint_{\Omega_n/\Gamma} |\eta_n| dx\, dy < \infty.$$

The class of $\pi(\mu)$ in $M(\Gamma_n, \Omega_n)$ possesses an extremal element of the form (1.4) and it is unique if $\pi(\mu)$ is not equivalent to zero under the relation $\equiv_n$. This fact is Teichmüller's theorem applied to the space $\tilde{T}(\Gamma_\mu, \Omega_n)$, which is isomorphic to the Teichmüller space of the Fuchsian group of the first kind obtained by lifting Γ_n under the universal covering mapping of U onto Ω_n.

Obviously $k_n \leq k_{n+1} \leq k$ for all n since the equivalence relations induced by $\sim$, $\equiv_{n+1}$ and $\equiv_n$ are progressively finer. The fact that $\sim$ is finer than $\equiv_{n+1}$ is Lemma 1.1 and depends on the assumption that $C = \hat{\mathbf{R}}$. The fact that $\equiv_{n+1}$ is finer than $\equiv_n$ follows from the inclusion $C_n \subset C_{n+1}$.

LEMMA 1.2. *Suppose $C = \hat{\mathbf{R}}$. Let μ be extremal in its class in $M(\Gamma)$ and $k = \|\mu\|_\infty$. Assume $k_n|\eta_n|/\eta_n$ in (1.4) is extremal in the class of $\pi(\mu)$ in $M(\Gamma_n, \Omega_n)$ under the equivalence relation $\equiv_n$. Then $\lim_{n\to\infty} k_n = k$.*

PROOF. Consider the mappings w^{ν_n} where $\nu_n = k_n|\eta_n|/\eta_n$. By hypothesis $w^{\nu_n}(x) = w^{\pi(\mu)}(x)$ for all x in C_n. Let w^ν be a normalized limit of some subsequence of w^{ν_n}. Such a limit exists because $\|\nu_n\|_\infty = k_n \leq k < 1$ for all n. Also $\nu(A\bar{z})\overline{A'(z)} = \nu(z)A'(z)$ for all A in Γ. It is clear from (1.3)(iv) and from the hypothesis $C = \mathbf{R}$ that $w^\mu(x) = w^\nu(x)$ for all x in $\mathbf{R}$ and that therefore ν restricted to the upper half plane is equivalent to μ. Moreover, ν restricted to L is trivial in $M(\Gamma, L)$, (but ν might not be identically zero in L). By the fact that μ is extremal in its class, it follows that $\|\nu \mid U\|_\infty \geq \|\mu\|_\infty = k$. But if $k_n \leq k - \varepsilon$ for all n and for some positive ε, one would have $\|\nu\|_\infty \leq k - \varepsilon$, a contradiction. Hence the lemma follows.

Let $M_{\text{symm}}(\Gamma_n, \Omega_n)$ be those μ in $M(\Gamma_n, \Omega_n)$ for which $\mu(\bar{z}) = \overline{\mu(z)}$. From Teichmüller's theorem, for a given μ in $M_{\text{symm}}(\Gamma_n, \Omega_n)$, there is an extremal element equivalent under $\equiv_n$ to μ of the form

$$k'_n|\varphi_n|/\varphi_n \tag{1.5}$$

where $0 \leq k'_n < 1$ and φ_n is a symmetric, integrable, holomorphic, quadratic differential for Γ_n on Ω_n. Let μ in $M_{\text{symm}}(\Gamma, \Omega)$ be extremal under the equivalence relation induced by the set C. Since $C_n \subset C_{n+1} \subset C$, one has $k'_n \leq k'_{n+1} \leq k$ for all n. An argument similar to but easier than the proof of Lemma 1.2 gives the following lemma and so we omit its proof.

LEMMA 1.3. $\lim_{n\to\infty} k'_n = k$.

REMARK. In contrast to Lemma 1.2, in this lemma we do not assume $C = \hat{\mathbf{R}}$. This is because it is obvious that when μ and ν are symmetric and $\mu \sim \nu$, then $\mu \equiv_n \nu$.

Now let ω be a fundamental domain for Γ in U and $\tilde{\omega}$ a fundamental domain for Γ in Ω. Similarly, let ω_n and $\tilde{\omega}_n$ be fundamental domains for Γ_n acting on U and Ω_n respectively. It is clear that we may pick these fundamental domains so that $\omega \subset \omega_{n+1} \subset \omega_n$ and $\tilde{\omega}_n$ is the union of ω_n and its complex conjugate and an appropriate part of the real axis.

The Poincaré theta series operator is defined by $\Theta F = \sum F(B(z))B'(z)^2$ where the summation is taken over all B in the group Γ and Θ_n is given by the same summation except it is taken over all B in the group Γ_n. Let D be a domain in $\hat{\mathbf{C}}$ and $A(D)$ be the set of all holomorphic functions F defined on D such that $\|F\| = \iint_D |F(z)|\,dx\,dy$ is finite and $|F(z)| = O(|z|^{-4})$ as $z \to \infty$ if ∞ is an interior point of D. In our applications D will be either U or Ω.

Now suppose D is contained in the set of discontinuity of Γ. Let $A(D, \Gamma)$ be the set of all holomorphic functions φ in D such that $\varphi(Az)A'(z)^2 = \varphi(z)$ for all A in Γ and z in D and $\|\varphi\| = \iint_{D/\Gamma} |\varphi(z)|\,dx\,dy$ is finite. Finally, if D is symmetric with respect to complex conjugation, let $A_s(D)$ and $A_s(D, \Gamma)$ be the subsets of $A(D)$ and $A(D, \Gamma)$ whose elements satisfy $F(\bar{z}) = \overline{F(z)}$.

The following well-known result [**3**, **11**] is stated for special domains D.

LEMMA 1.4. *Let D be the upper half plane U or the set $\Omega = \hat{\mathbf{C}} - C$ defined above. Then $\Theta: A(D) \to A(D, \Gamma)$ is a continuous surjective linear operator between Banach spaces. Θ has norm less than or equal to one and the image of the unit ball contains the ball of radius one-third. If D is symmetric under complex conjugation, the same statements are true for the restriction of Θ to a map from $A_s(D)$ onto $A_s(D, \Gamma)$.*

Let $R(\Omega)$ be the subset of $A(\Omega)$ which consists of rational functions with at most simple poles lying in $C = \hat{\mathbf{C}} - \Omega$. Similarly, $R_s(\Omega)$ is the subset of $A_s(\Omega)$ consisting of rational functions with at most simple poles in C.

The following lemma is well known.

LEMMA 1.5 [**2**, **11**]. *$R(\Omega)$ is dense in $A(\Omega)$ and $R_s(\Omega)$ is dense in $A_s(\Omega)$.*

2. Extremal holomorphic mappings from a disk into $T(\Gamma)$. In this section we assume $C = \hat{\mathbf{R}}$, and hence, $T(\Gamma)$ is a complex manifold. The cases when Γ is finitely or infinitely generated and either of the first or second kind are included. Let $\mathfrak{F}$ be the family of holomorphic functions from the unit disk Δ into $T(\Gamma)$. Let P and Q be elements of $T(\Gamma)$. Among all f in $\mathfrak{F}$ which map 0 into P and a positive number r into Q, we consider the problem of finding the minimum value of r.

THEOREM 2.1. *The minimum value of r for this problem is the number $k = \|\mu\|_\infty$ where μ is the Beltrami coefficient of an extremal quasiconformal mapping joining P to Q. A function for which this minimum value is achieved is any holomorphic function f of the form $f(z) = [z\mu/\|\mu\|_\infty]$ where μ is the Beltrami coefficient of an extremal mapping joining P to Q.*

PROOF. The crux of the matter is to show that for every f in $\mathfrak{F}$ which takes 0 into P and a positive number r into $Q = [\mu]$ one has the inequality $r \geqslant k$, since the mapping described in the second part of the theorem will map Δ into $T(\Gamma)$ taking 0 into P and k into Q.

To see that $r \geqslant k$ we rely on the fact that Royden has proved this in the finite dimensional case [**16**]. His proof requires no alteration in the case where $T(\Gamma)$ has elliptic or parabolic elements. In our situation we have holomorphic mappings

$$\Delta \xrightarrow{f} T(\Gamma) \xrightarrow{\pi} \tilde{T}(\Gamma_n, \Omega_n).$$

($\tilde{T}(\Gamma_n, \Omega_n)$ is defined in §1 just before Lemma 1.1.) We assume without loss of generality that $T(\Gamma)$ is based at the point P so that P is represented by the zero Beltrami coefficient. We know that $\pi \circ f(0) = 0$ and $\pi \circ f(r)$ is representable in the form $k_n|\eta_n|/\eta_n$. By Royden's theorem $r \geqslant k_n$ for all n. By Lemma 1.2, $r \geqslant k$.

COROLLARY 2.1. *The Kobayashi and Teichmüller metrics on $T(\Gamma)$ coincide.*

PROOF. Let d_K be Kobayashi's (pseudo) metric and d_T be Teichmüller's metric. Let $d_1(P, Q) = \frac{1}{2}\log(1 + r)/(1 - r)$ where r is the minimum value achieved in the extremal problem of Theorem 2.1. Let

$$d_n(P, Q) = \inf \sum_{i=1}^{n} d_1(P_i, P_{i-1})$$

where the infimum is taken over all points $P_0, \ldots, P_n$ in $T(\Gamma)$ such that $P_0 = P$ and $P_n = Q$. Obviously, $d_{n+1} \leqslant d_n$ for all n. By definition, $d_K(P, Q) = \lim_{n\to\infty} d_n(P, Q)$. It is clear that if d_1 satisfies the triangle inequality, then $d_1 = d_n$ for all n and, hence, $d_1 = d_K$. By the preceding theorem, d_1 does satisfy the triangle inequality since it is identical to Teichmüller's metric. Therefore, $d_1 = d_T = d_K$.

3. The fundamental inequality of Reich and Strebel for $T(\Gamma)$. In this section we do not require C to be all of $\hat{\mathbf{R}}$ and hence $T(\Gamma)$ may not have complex structure. As usual, Ω is the complement in $\hat{\mathbf{C}}$ of C. In its most elementary form the main inequality of Reich and Strebel [**14**, **15**, **17**] contains two variables μ and φ. Here μ is a trivial element of $M(\Gamma)$, that is $w_\mu(x) = x$ for all x in C and φ is in $A_s(\Omega, \Gamma)$. The inequality says

$$\left|\operatorname{Re} \iint_\omega \frac{\mu\varphi}{1-|\mu|^2}\, dx\, dy\right| \leqslant \iint_\omega \frac{|\mu|^2|\varphi|}{1-|\mu|^2}\, dx\, dy \tag{3.1}$$

where ω is a fundamental domain for Γ in U. In the case where $C = \hat{\mathbf{R}}$, we may drop the real part symbol in the left side of (3.1). Also, in this case $A_s(\Omega, \Gamma)$ is replaced by $A(U, \Gamma)$. The inequality (3.1) follows easily from Teichmüller's inequality [**5**] in finite dimensional cases. Thus we assume it is known when $A_s(\Omega, \Gamma)$ is finite dimensional and show how to prove it for infinite dimensional cases. The hypothesis that μ is trivial in $M(\Gamma)$ implies its symmetric extension is trivial in $M(\Gamma_n, \Omega_n)$ since $C_n \subset C$. Thus we assume

$$\operatorname{Re} \iint_{\omega_n} \frac{\mu\varphi_n}{1-|\mu|^2} \leqslant \iint_{\omega_n} \frac{|\mu|^2|\varphi_n|}{1-|\mu|^2} \tag{3.2}$$

for φ_n in $A_s(\Omega_n, \Gamma_n)$.

In order to view (3.1) as a limit of (3.2), we need the following lemma.

LEMMA 3.1. *Let Θ_n be the Poincaré theta series operators for the groups Γ_n in* (3.1). *Let ω_n and ω be fundamental domains for Γ_n and Γ in the upper half plane U with $\omega \subset \omega_n$. Suppose G is a measurable function defined on U and $\iint_U |G|\, dx\, dy < \infty$. Then*

$$\lim_{n\to\infty} \iint_{\omega_n} |\Theta_n G| = \iint_\omega |\Theta G| \quad \textit{and} \quad \lim_{n\to\infty} \iint_\omega |\Theta_n G - \Theta G| = 0.$$

PROOF. Start with the inequality

$$\left|\iint_{\omega_n} |\Theta_n G| - \iint_\omega |\Theta G|\right| \leqslant \iint_{\omega_n - \omega} |\Theta_n G| + \iint_\omega \big|\, |\Theta_n G| - |\Theta G|\, \big|. \tag{3.3}$$

The second term on the right-hand side of (3.3) is less than or equal to

$$\iint_\omega \sum \left|G(Az)A'(z)^2\right| \tag{3.4}$$

where the sum is over all A in $\Gamma - \Gamma_n$. We can select n_0 so that for $n \geqslant n_0$ this sum is less than $\varepsilon/2$ since the theta series of an integrable function converges absolutely.

The first term on the right side of (3.3) is less than or equal to

$$\iint_{U-\Gamma_n\omega} |G|. \tag{3.5}$$

Since $\Gamma_n\omega$ is an increasing sequence of sets whose union is U and since G is integrable, (3.5) approaches zero. The proof of the second part of the lemma is similar so we omit it.

THEOREM 3.1 [**5**, **17**]. *The inequality* (3.1) *holds whenever $w_\mu(x) = x$ for all x in C and for all φ in $A_s(\Omega, \Gamma)$. The space $A_s(\Omega, \Gamma)$ may be finite or infinite dimensional.*

PROOF. Let F be in $R_s(\Omega)$ and have simple poles in the closed set C_{n_0}. Since we are assuming that (3.2) is true for finite dimensional spaces and since $w_\mu(x) = x$ for all x in C_{n_0}, we know that (3.2) is true with φ_n replaced by $\Theta_n F$ for all $n \geqslant n_0$. Since μ is a Beltrami differential for Γ and $\Gamma_{n_0} \subset \Gamma$, one has

$$\iint_{\omega_n} \frac{(\Theta_n F)\mu}{1-|\mu|^2}\,dx\,dy = \iint_{\omega} \frac{(\Theta F)\mu}{1-|\mu|^2}\,dx\,dy.$$

Since $|\mu|$ is automorphic for Γ, on letting the function G of Lemma 3.1 be $|\mu|^2 F/(1-|\mu|^2)$, it follows that the right side of (3.2) approaches

$$\iint_{\omega} \frac{|\mu|^2|\Theta F|}{1-|\mu|^2}\,dx\,dy \quad \text{as } n \to \infty.$$

Now Lemma 1.5 tells us that $R_s(\Omega)$ is dense in $A_s(\Omega)$. This fact together with the fact that Θ is surjective proves (3.1) in the general case.

By elementary calculations (3.1) extends to an inequality involving three variables φ, μ and ν_1. This inequality is called the main inequality of Reich and Strebel [**17**]. Here φ is any element of $A_s(\Gamma, \Omega)$ with $\iint_\omega |\varphi| = 1$, μ is any element of $M(\Gamma)$ and ν_1 is any element of $M(w_\mu \Gamma (w_\mu)^{-1})$ for which $w_{\nu_1} \circ w_\mu$ has trivial Beltrami coefficient in $M(\Gamma)$. Let $p(z) = \partial w_\mu(z)/\partial z$.

THEOREM 3.2 [**17**]. *For all φ, μ and ν_1 as described above*

$$1 \leqslant \iint_{\omega} |\varphi| \frac{|1-\mu\varphi/|\varphi||^2}{1-|\mu|^2} \cdot \frac{|1-\nu_1(\bar{p}\varphi/p|\varphi|)\theta|^2}{1-|\nu_1|^2}\,dx\,dy \tag{3.6}$$

where $\theta(z) = (1-\bar{\mu}\bar{\varphi}/|\varphi|)/(1-\mu\varphi/|\varphi|)$.

PROOF. One applies (3.1) to the trivial Beltrami coefficient of the mapping $w_{\nu_1} \circ w_\mu$. Then the verification is a routine calculation. The proof given in [**14**] is different.

LEMMA 3.2 [**17**]. *Let k_0 be the minimum value of $\|\nu\|_\infty$ where $\nu \sim \mu$ and $K_0 = (1+k_0)/(1-k_0)$. Then for all φ in $A_s(\Gamma, \Omega)$ with $\|\varphi\| = \iint_\omega |\varphi|\,dx\,dy = 1$, one has*

$$K_0^{-1} \leqslant \iint_{\omega} |\varphi| \frac{|1-\mu\varphi/|\varphi||^2}{1-|\mu|^2}\,dx\,dy. \tag{3.7}$$

PROOF. Let $\nu \sim \mu$ and ν_1 satisfy $w_{\nu_1} \circ w_\nu(z) = z$ for all z and suppose ν is extremal in the class of μ. Then $w_{\nu_1} \circ w_\mu$ is trivial and $k_0 = \|\nu_1\|_\infty$. The second fraction in the integrand in (3.6) is bounded by $(1 + k_0)^2/(1 - k_0^2)$. Thus (3.7) follows.

For the finite dimensional cases the inequality (3.7) also follows from the Teichmüller inequality [**5**].

To state the next theorem, we introduce the functional

$$I[\mu] = \sup \operatorname{Re} \iint_\omega \frac{\mu\varphi}{1 - |\mu|^2}\, dx\, dy \tag{3.8}$$

where the supremum is taken over all φ in $A_s(\Gamma, \Omega)$ with norm one. The following theorem is proved in [**14**]. We include it for the sake of completeness.

THEOREM 3.3 [**14**]. *Let k_0 be the smallest value of $\|\nu\|_\infty$ where $\nu \sim \mu$ in $M(\Gamma)$ and let $k = \|\mu\|_\infty$. Then*

$$I[\mu] \leq \frac{k_0}{1 + k_0} + \frac{k^2}{1 - k^2}. \tag{3.9}$$

PROOF. Subtracting 1 from both sides of (3.7) and expanding out the numerator of the integrand on the right-hand side, you get

$$\frac{1 - k_0}{1 + k_0} - 1 \leq -2 \operatorname{Re} \iint_\omega \frac{\mu\varphi}{1 - |\mu|^2} + 2 \iint_\omega \frac{|\mu|^2 |\varphi|}{1 - |\mu|^2}\, dx\, dy.$$

This simplifies to

$$\operatorname{Re} \iint_\omega \frac{\mu\varphi}{1 - |\mu|^2} \leq \frac{k_0}{1 + k_0} + \iint_\omega \frac{|\mu|^2 |\varphi|}{1 - |\mu|^2}.$$

On taking suprema over all φ with norm one and using the fact that $\|\mu\|_\infty = k$, this leads to (3.9).

The next theorem gives a lower bound for $I[\mu]$. It appears in [**14**], although the fact that it remains valid when Γ is infinitely generated is a new result.

THEOREM 3.4. *Let k_0 be the minimum value of $\|\nu\|_\infty$ where $\nu \sim \mu$ in $M(\Gamma)$ and let $k = \|\mu\|_\infty$. Then*

$$I[\mu] \geq \frac{k_0}{1 - k_0} - \frac{k^2}{1 - k^2}. \tag{3.10}$$

PROOF. Let k_n' be the minimum value of $\|\nu\|_\infty$ where $\nu \equiv_n \mu$. From Lemma 1.3 we know that $k_n' \leq k_0$ and k_n' converges to k_0. Let $K_n' = (1 + k_n')/(1 - k_n')$. By applying inequality (3.6) in the finite dimensional case and letting ν_1 be determined by the conditions $w_{\nu_1} \circ w_\nu(z) = z$ and $\nu = k_n' |\varphi_n|/\varphi_n$, the Teichmüller extremal Beltrami coefficient in the class of μ under the relation $\equiv_n$, we get

$$K_n' \leq \iint_{\omega_n} |\varphi_n| \frac{|1 + \mu\varphi_n/|\varphi_n||^2}{1 - |\mu|^2} \tag{3.11}$$

where φ_n is an element of $A_s(\Omega_n, \Gamma_n)$ and $\iint_{\omega_n} |\varphi_n| = 1$. Subtracting 1 from both sides of (3.11) and expanding out the numerator of the integrand, (3.11) leads to

$$\frac{k_n'}{1-k_n'} \leqslant \operatorname{Re} \iint_{\omega_n} \frac{\mu\varphi_n}{1-|\mu|^2} + \iint_{\omega_n} \frac{|\mu|^2|\varphi_n|}{1-|\mu|^2}. \tag{3.12}$$

Since $\iint_{\omega_n} |\varphi_n| = 1$, the second term on the right in (3.12) is bounded by $k^2/(1-k^2)$. Pick F_n in $A_s(\Omega)$ such that $\Theta_n F_n = \varphi_n$. Clearly,

$$\iint_{\omega_n} \frac{\mu\varphi_n}{1-|\mu|^2} = \iint_{U} \frac{\mu F_n}{1-|\mu|^2} = \iint_{\omega} \frac{\mu\Theta F_n}{1-|\mu|^2}$$

and $\iint_\omega |\Theta F_n| \leqslant \iint_{\omega_n} |\varphi_n| = 1$. Hence

$$\frac{k_n'}{1-k_n'} \leqslant \operatorname{Re} \iint_{\omega} \frac{\mu\Theta F_n}{1-|\mu|^2} + \frac{k^2}{1-k^2}. \tag{3.13}$$

The facts that (3.13) holds for every n, that $k_n'/(1-k_n')$ increases to $k_0/(1-k_0)$ and that ΘF_n is an element of $A_{\mathrm{symm}}(\Gamma, \Omega)$ for which $\iint_\omega |\Theta F_n| \leqslant 1$ all taken together imply (3.10).

An important application of Theorems 3.3 and 3.4 is the calculation of the infinitesimal form of Teichmüller's metric. We arrive at the infinitesimal form for both the finite and infinite dimensional cases at the same time. For the case of universal Teichmüller space ($\Gamma = \{\text{identity}\}$) this result appears in [**14**].

THEOREM 3.5. *Let $k_0(t)$ be the minimum value of $\|\nu\|_\infty$ where $\nu \sim t\mu$ and let $\dot{k}_0 = \lim_{t\to 0+} k_0(t)/t$. Then*

$$\dot{k}_0 = \sup \operatorname{Re} \iint_\omega \mu\varphi \tag{3.14}$$

where the supremum is over all φ in $A_s(\Omega, \Gamma)$ where $\iint_\omega |\varphi| = 1$.

PROOF. From (3.10)

$$\frac{k_0(t)}{1-k_0(t)} \leqslant I[t\mu] + \frac{t^2\|\mu\|_\infty^2}{1-t^2\|\mu\|_\infty^2}.$$

Dividing both sides by $t > 0$ and letting $t \to 0$ we get

$$\overline{\lim_{t\to 0+}}\,(k_0(t)/t) \leqslant \sup_{\|\varphi\|=1} \operatorname{Re} \iint_\omega \mu\varphi.$$

A similar manipulation of inequality (3.9) yields

$$\underline{\lim_{t\to 0+}}\, k_0(t)/t \geqslant \sup_{\|\varphi\|=1} \operatorname{Re} \iint_\omega \mu\varphi.$$

Putting these last two inequalities together, one gets (3.14).

A second consequence of Theorems 3.3 and 3.4 is Reich's and Strebel's necessary and sufficient condition for extremality, which follows.

THEOREM 3.6 [**14**]. *Let $\mu \in M(\Gamma)$ and $k = \|\mu\|_\infty$. Let k_0 be the minimum value of $\|\nu\|_\infty$ such that $w_\nu(x) = w_\mu(x)$ for all x in C. Then $k = k_0$ if, and only if, $I[\mu] = k/(1-k^2)$.*

PROOF. Suppose $k = k_0$. Then from Theorem 3.4

$$I[\mu] \geqslant (k/(1-k)) - (k^2/(1-k^2)) = k/(1-k^2).$$

Since the opposite inequality is obvious, this proves the first half of the theorem.

Conversely, suppose $I[\mu] = k/(1-k^2)$. Then from Theorem 3.3, $k/(1-k^2) \leqslant (k_0/(1+k_0)) + k^2/(1-k^2)$. This implies $k/(1+k) \leqslant k_0/(1+k_0)$, and hence, $k \leqslant k_0$. Since obviously $k \geqslant k_0$, this proves the second half of the theorem.

The condition that $I[\mu] = k/(1-k^2)$ is very close to what is called Hamilton's condition.

Let $H[\mu]$ be the functional defined by

$$H[\mu] = \sup \operatorname{Re} \iint_\omega \mu\varphi \tag{3.15}$$

where the supremum is over all φ in $A_s(\Omega, \Gamma)$ with $\iint_\omega |\varphi| = 1$. By elementary methods [**14**] one can show that $H[\mu] = k$ if, and only if, $I[\mu] = k/(1-k^2)$. In fact a sequence φ_n in $A_s(\Omega, \Gamma)$ with $\iint_\omega |\varphi_n|\, dx\, dy = 1$ will realize the maximum k for $H[\mu]$ if, and only if, the same sequence φ_n realizes the maximum $k/(1-k^2)$ for $I[\mu]$. Hence, Theorem 3.6 has the following

COROLLARY [**10**, **14**]. *A necessary and sufficient condition for μ to be extremal is that $H[\mu] = k$.*

The necessity in this corollary was proved by Hamilton [**10**] and the sufficiency by Reich and Strebel [**14**].

4. The infinitesimal form of Teichmüller's metric. If μ is an extremal element in its class in $M(\Gamma)$ and if $k_0 = \|\mu\|_\infty$, then the Teichmüller distance d from the class $[0]$ to the class $[\mu]$ is by definition

$$d([0],[\mu]) = \tfrac{1}{2}\log(1+k_0)/(1-k_0). \tag{4.1}$$

For economy in notation we will write $d(0, \mu)$ in place of $d([0],[\mu])$. By Theorem 3.5 it is immediate that for $t \geqslant 0$

$$d(0, t\mu) = tH[\mu] + o(t) \tag{4.2}$$

where $H[\mu]$ is defined in (3.15).

Using the translation mapping $w_\sigma \mapsto w_\tau = w_\sigma \circ (w_\mu)^{-1}$, the Teichmüller space $T(\Gamma)$ formed from $M(\Gamma)$ is isometrically transformed into $T(w_\mu \Gamma w_\mu^{-1})$ and the point $[\mu]$ is transformed into $[0]$. By translating the formula (4.2) in this way, (4.2) yields the infinitesimal form of Teichmüller's metric at every point $[\mu]$ in $T(\Gamma)$. The infinitesimal metric $F([\mu], \nu)$ will be a nonnegative function, homogeneous in the variable ν at every point $[\mu]$ in $T(\Gamma)$. By definition, $F([\mu], \nu)$ is the derivative with respect to t at $t = 0$ of the function $d(\mu, \mu + t\nu)$. If we let R_μ be right translation by w_μ, from (4.2) we find that

$$d(\mu, \mu + t\nu) = d\big(0, R_\mu^{-1}(\mu + t\nu)\big) = t \sup \operatorname{Re} \iint \varphi S(\nu) + o(t),$$

where S is the derivative of R_μ^{-1} at μ. Here, the integral is over a fundamental domain ω_μ for $\Gamma_\mu = w_\mu \Gamma (w_\mu)^{-1}$ and the supremum is over all φ in $A_s(\Omega_\mu, \Gamma_\mu)$ with $\iint_{\omega_\mu} |\varphi| = 1$. We are using the obvious notation $\Omega_\mu = w_\mu(\Omega)$. Calculation of the Beltrami coefficient of $w_\sigma \circ (w_\mu)^{-1}$ yields

$$R_\mu^{-1}(\sigma) = \left[\frac{\sigma - \mu}{1 - \bar{\mu}\sigma} \cdot \frac{1}{\theta}\right] \circ w_\mu^{-1} \tag{4.3}$$

where $\theta = \bar{p}/p$ and $p = \partial w_\mu / \partial z$. Letting $\sigma = \mu + t\nu$, we find $S(\nu) = \nu/(1 - |\mu|^2)\theta$ and

$$F([\mu], \nu) = \sup \operatorname{Re} \iint_{\omega_\mu} \varphi(w) \left[\frac{\nu}{1 - |\mu|^2} \cdot \frac{1}{\theta}\right] du\, dv \tag{4.4}$$

where the supremum is over all φ in $A_s(\Omega_\mu, \Gamma_\mu)$ for which $\iint_{\omega_\mu} |\varphi| = 1$ and $w = u + iv$.

LEMMA 4.1 [**6**]. *The function F from the tangent bundle of $T(\Gamma)$ to $\mathbf{R}$ is continuous.*

PROOF. Since right translation R_μ is an isometry, it suffices to show that F is continuous at $\mu = 0$. In other words, we must show that $|F(0, \nu) - F(\mu, \nu_1)| < \varepsilon$ if $\|\mu\|_\infty < \delta$ and $\|\nu - \nu_1\| < \delta$. Since F is homogeneous and sublinear in the second variable, it suffices to prove the same inequality with ν_1 replaced by ν and for $\|\nu\|_\infty < 1$. We claim it suffices to prove the inequality

$$F(0, \nu) < F(\mu, \nu) + \varepsilon \quad \text{for } \|\mu\|_\infty < \delta \text{ and } \|\nu\|_\infty < 1. \tag{4.5}$$

For, if we show this, then by applying the translation mapping R_μ^{-1}, we find that

$$F\left(\mu_1, \left(R_\mu^{-1}\right)_0'(\nu)\right) < F\left(0, \left(R_\mu^{-1}\right)_\mu'(\nu)\right) + \varepsilon, \tag{4.6}$$

where μ_1 is the Beltrami coefficient of the inverse mapping of w_μ. Of course, by $(R_\mu^{-1})_\sigma'(\nu)$ we mean

$$\lim_{t \to 0} (1/t)\left[R_\mu^{-1}(\sigma + t\nu) - R_\mu^{-1}(\sigma)\right].$$

Moreover, by using formula (4.3) one sees that

$$\left(R_\mu^{-1}\right)_\mu'(\nu) = \left[\frac{\nu}{1 - |\mu|^2} \cdot \frac{1}{\theta}\right] \circ w_\mu^{-1},$$

$$\left(R_\mu^{-1}\right)_0'(\nu) = \left[\nu\left(1 - |\mu|^2\right)\left(\frac{1}{\theta}\right)\right] \circ w_\mu^{-1}.$$

It is clear that the L_∞-norm of the difference between the two quantities in (4.6) is bounded by a constant times $\|\mu\|_\infty^2 \|\nu\|_\infty$. Thus, for sufficiently small $\|\mu\|_\infty$, from (4.5) we obtain the inequality $F(\mu_1, \nu) < F(0, \nu) + 2\varepsilon$.

Hence, we have reduced the lemma to proving (4.5). Now, we observe that it is sufficient to consider the case when $C = \hat{\mathbf{R}}$. For, if C is a proper subset of $\hat{\mathbf{R}}$, we may consider the universal covering of the domain $\Omega = \hat{\mathbf{C}} - C$. Under this covering the symmetric Beltrami coefficients lift to Beltrami coefficients defined on the universal covering surface U which are compatible with an anticonformal involution. Although the group Γ may lift to a group of either the first or second kind acting on U, the equivalence relation on $M_{\text{symm}}(\Omega, \Gamma)$ lifts to the relation determined by saying $\mu \sim \nu$ if $w_\mu(x) = w_\nu(x)$ for all x in $\hat{\mathbf{R}}$.

Let Θ_μ be the theta series operator for the group Γ_μ and so Θ_0 is Θ. By (4.4), there is an element φ in $A(\Gamma, U)$ such that $\|\varphi\| = 1$ and $F(0, \nu) < \iint_\omega \varphi\nu \, dx\, dy + \varepsilon/2$. We then can find G in $A(U)$ such that $\Theta G = \varphi$. Clearly,

$$\iint_\omega \varphi\nu \, dx\, dy = \iint_U G\nu \, dx\, dy.$$

We must find an element ψ in $A(\Gamma_\mu, U)$ such that $\|\psi\| \leq 1$ and

$$\iint_U G\nu \, dx\, dy \leq \operatorname{Re} \iint_{\omega_\mu} \psi \left[\frac{\mu}{1 - |\mu|^2} \cdot \frac{1}{\theta} \right] \circ w_\mu^{-1} + \frac{\varepsilon}{2}$$

if $\|\mu\|_\infty$ is sufficiently small. As our candidate for ψ we pick $\Theta_\mu G / \|\Theta_\mu G\|$. Clearly, to complete the proof of Lemma 4.1, it suffices to prove the following

LEMMA 4.2. *Let G be in $A(U)$. Then $\iint_{\omega_\mu} |\Theta_\mu G|$ approaches $\iint_\omega |\Theta G|$ as $\|\mu\|_\infty$ approaches zero.*

PROOF. Let A_μ^n be an enumeration of Γ_μ and Θ_{μ^n} be the truncation of Θ_μ to the first n elements of Γ_μ and Θ_n the corresponding summation for Γ. Clearly, $\iint_{\omega_\mu} |\Theta_{\mu^n} G|$ converges to $\iint_\omega |\Theta_n G|$ as $\|\mu\|_\infty \to 0$. In order to pass to the sum over all elements of Γ_μ and Γ, we will show that for any $\varepsilon > 0$, there exists n_0 and $\delta > 0$, such that for $n \geq n_0$ and $\|\mu\|_\infty < \delta$

$$\iint_{\omega_\mu} \sum_{n > n_0} \left| G(A_\mu^n z) A_\mu^{n\prime}(z)^2 \right| < \varepsilon. \tag{4.7}$$

To simplify notation, let U be replaced by the unit disk Δ and assume the groups Γ and Γ_μ act on Δ. Pick $r < 1$ so that $\iint_{r < |z| < 1} |G| < \varepsilon$ and pick n_0 so that $D = \bigcup_{n=1}^{n_0} A^n(\omega)$ contains the disk or radius $(r+1)/2$. Now $\omega_\mu = w_\mu(\omega)$ and hence $w_\mu(D) = \bigcup_{n=1}^{n_0} A_\mu^n(\omega_\mu)$. Since w_μ satisfies a Hölder condition of order $\alpha = 1/K$ where $K = (1 + \|\mu\|_\infty)/(1 - \|\mu\|_\infty)$, we see that for sufficiently small $\|\mu\|_\infty$, $w_\mu(D) \supseteq \{z : |z| < r\}$. This implies (4.7) is bounded by $\iint_{r < |z| < 1} |G|$ which is less than ε. This completes the proofs of Lemmas 4.1 and 4.2.

Now let $\bar{d}$ be the integrated form of (4.4). This means $\bar{d}(p, q) = \inf L(\gamma)$ where γ is a piecewise smooth path joining p to q, that is, $\gamma(0) = p$, $\gamma(t_0) = q$ and $L(\gamma) = \int_0^{t_0} F(\gamma(t), \gamma'(t))\, dt$. It is a general and elementary fact that if F is a continuous function on the tangent space, then $d \leq \bar{d}$.

THEOREM 4.1. *For any $T(\Gamma)$, $d = \bar{d}$, that is, Teichmüller's metric is the integral of its infinitesimal form.*

PROOF. This theorem is elementary for finite dimensional Teichmüller spaces and is mentioned in [**16**, p. 370]. The general case is treated by O'Byrne [**13**, p. 326] by means of a general theorem concerning quotients of Finsler structures. Here, we prove this result more directly and in slightly greater generality by using the explicit formula for the metric (4.4) and by using Theorem 3.6.

We must show $\bar{d} \leq d$. Assume $\|\mu\|_\infty = 1$ and $k\mu$ is extremal so $d(0, [k\mu]) = \frac{1}{2} \log(1 + k)/(1 - k)$.

Let $\gamma(t) = [t\mu]$, $0 \leqslant t \leqslant k$. We will show that $L(\gamma) = d(0, [k\mu])$ and it will follow that $\bar{d} \leqslant d$. Since $\gamma'(t) = \mu$; we must calculate $F([t\mu], \mu)$. We know that $w_{t\mu}$ and $(w_{t\mu})^{-1}$ are both extremal. The Beltrami coefficient of $(w_{t\mu})^{-1}$ is $-t\mu/\theta$ where $\theta = \bar{p}/p$ and $p = \partial w_{t\mu}/\partial z$. Since $-t\mu/\theta$ is extremal, Theorem 3.6 tells us that

$$\sup \operatorname{Re} \iint_{\omega_\mu} \varphi \frac{t\mu}{1 - t^2|\mu|^2} \cdot \frac{1}{\theta}\, du\, dv = \frac{t}{1 - t^2} \tag{4.8}$$

where the supremum is over all φ in $A_s(\Omega_\mu, \Gamma_\mu)$ with $\iint_{\omega_\mu}|\varphi| = 1$. From (4.4) and (4.8) one sees that $F([t\mu], \mu) = 1/(1 - t^2)$, and therefore,

$$\int_0^k F(\gamma(t), \gamma'(t))\, dt = \int_0^k \frac{dt}{1 - t^2} = \tfrac{1}{2} \log \frac{1 + k}{1 - k}.$$

5. Maximal sequences. A sequence φ_n in $A_s(\Omega, \Gamma)$ with $\iint_\omega |\varphi_n| = 1$ is called a maximal sequence for the functional $H[\mu]$ in (3.15) if $H[\mu] = \lim_{n\to\infty} \operatorname{Re} \iint_\omega \mu\varphi_n$. It is called an absolute maximal sequence if $\|\mu\|_\infty = \lim_{n\to\infty} \operatorname{Re} \iint_\omega \mu\varphi_n$.

THEOREM 5.1. *Suppose μ is extremal in $M(\Gamma)$ and $\|\mu\|_\infty = k$. Suppose φ_n is in $A_s(\Omega, \Gamma)$ and $\iint_\omega |\varphi_n| = 1$. If $[k_n|\varphi_n|/\varphi_n]$ converges in the Teichmüller metric to $[\mu]$, then φ_n is an absolute maximal sequence for μ.*

PROOF. Since μ is extremal and $\|\mu\|_\infty = k$, one has $d(0, \mu) = \frac{1}{2}\log(1 + k)/(1 - k)$. Since $k_n|\varphi_n|/\varphi_n$ is extremal, $d(0, k_n|\varphi_n|/\varphi_n) = \frac{1}{2}\log(1 + k_n)/(1 - k_n)$. But $|d(0, \mu) - d(0, k_n|\varphi_n|/\varphi_n)| \leqslant d(\mu, k_n|\varphi_n|/\varphi_n)$ and the latter quantity approaches zero, by hypothesis, so we know K_n approaches K. Let $\frac{1}{2}\log \tilde{K}_n(t)$ be the Teichmüller distance from $[t\nu]$ to $[k_n|\varphi_n|/\varphi_n]$. From Theorem 5.1 of **[9]** one knows that

$$\frac{\tilde{K}_n(t)}{K_n} \geqslant \frac{1}{1 + 2t \operatorname{Re} \iint_\omega \nu\varphi_n + O(t^2)} \tag{5.1}$$

where the constant in $O(t^2)$ depends only on $\|\nu\|_\infty$ and k. Since $\lim_{n\to\infty} \frac{1}{2}\log \tilde{K}_n(t) = d(t\nu, \mu)$ and $\lim_{n\to\infty} K_n = K$, taking the limit in (5.1) as $n \to \infty$ yields

$$d(t\nu, \mu) \geqslant \tfrac{1}{2}\log K - t \varliminf \operatorname{Re} \iint_\omega \nu\varphi_n + O(t^2). \tag{5.2}$$

From the fact that $[t\mu]$ is a geodesic (by the proof of Theorem 4.1), one has

$$d(0, \mu) = d(0, t\mu) + d(t\mu, \mu) \quad \text{for } 0 \leqslant t \leqslant 1. \tag{5.3}$$

The derivative with respect to t of the left side of (5.3) is zero and, since the first term on the right of (5.3) is differentiable from the right, the second term must also be differentiable and this second term will have derivative given by the first order term in (5.2). Combining these facts, we obtain

$$\lim_{n\to\infty} \operatorname{Re} \iint_\omega \mu\varphi_n = \sup_{\|\varphi\|=1} \operatorname{Re} \iint \mu\varphi = k \tag{5.4}$$

which shows that φ_n is an absolute maximal sequence for μ.

This theorem is of interest only for infinite dimensional Teichmüller spaces.

6. Uniqueness for Kobayashi's extremal problem. We can now show that the extremal function f for the extremal problem posed in §2 is uniquely determined when the point $P = [0]$ and the point $Q = [k|\varphi|/\varphi]$ where φ is integrable.

THEOREM 6.1. *Let* $[0]$ *and* $[k|\varphi|/\varphi]$ *be points in Teichmüller space* $T(\Gamma)$ *and assume* $C = \hat{\mathbf{R}}$ *and* $\int\int_\omega |\varphi| = 1$. *If* f *is any holomorphic function from* Δ *into* $T(\Gamma)$ *for which* $f(0) = [0]$ *and* $f(r) = [k|\varphi|/\varphi]$ *where* $r > 0$, *then* $r > k$ *unless* $f(z) = [z|\varphi|/\varphi]$, *in which case* $r = k$.

PROOF. By Theorem 2.1, $r \geq k$ so what remains to be shown is that if $r = k$ then $f(z) = [z|\varphi|/\varphi]$. Let d_Δ be the Poincaré metric on the unit disk Δ. Since the Poincaré metric is Kobayashi's metric for Δ and since holomorphic mappings are contracting under Kobayashi metrics, one has

$$d(f(s), f(t)) \leq d_\Delta(s, t). \tag{6.1}$$

By the triangle inequality

$$d(f(0), f(k)) \leq d(f(0), f(s)) + d(f(s), f(k)) \leq d_\Delta(0, s) + d_\Delta(s, k). \tag{6.2}$$

For $0 \leq s \leq k$ the right-hand side of (6.2) is $d_\Delta(0, k)$. Since we are assuming $r = k$, the left- and right-hand sides of (6.2) are equal and so from (6.1) the corresponding terms are equal. We have

$$d(f(s), f(k)) = d_\Delta(s, k) \tag{6.3}$$

and

$$d(f(0), f(s)) = d_\Delta(0, s) \tag{6.4}$$

for each s with $0 \leq s \leq k$. Now let $f(s) = [\mu]$ where μ is extremal in its class. Since the Teichmüller distance from 0 to $[\mu]$ is $\frac{1}{2}\log(1 + s)/(1 - s)$, this implies that $\|\mu\|_\infty = s$. Choose σ so that the τ which satisfies $w_\tau = w_\sigma \circ w_\mu$ is in the same class as $k|\varphi|/\varphi$ and so that σ is extremal in its class. From (6.3) this implies $\|\sigma\|_\infty = (k - s)/1(1 - sk)$ so

$$\left\| \frac{\tau - \mu}{1 - \bar{\mu}\tau} \right\|_\infty = \frac{k - s}{1 - sk}. \tag{6.5}$$

Since $|\mu| \leq s$, one cannot have $|\tau| > k$ on any set of positive measure and have equality in (6.5). Thus $\|\tau\|_\infty \leq k$. It follows that $\tau = k|\varphi|/\varphi$ since $k|\varphi|/\varphi$ is unique extremal in its class. Hence we have equality in (6.5) with τ replaced by $k|\varphi|/\varphi$. We know that $|\mu| \leq s$ almost everywhere. If on any set of positive measure μ were not equal to $s|\varphi|/\varphi$, then again (6.5) could not be an equality. Hence $\mu = s|\varphi|/\varphi$ almost everywhere. This argument holds for every s with $0 \leq s \leq k$. Hence the two functions $f(z)$ and $g(z) = [z|\varphi|/\varphi]$ are identical for all z in Δ and this completes the proof.

Notice that in the step where we concluded that $\tau = k|\varphi|/\varphi$, we only used that $k|\varphi|/\varphi$ is unique extremal in its class. For the next step we also needed to know that $s|\varphi|/\varphi$ for $0 \leq s \leq k$ is unique extremal in its class. It is only for differentials of Teichmüller form $k|\varphi|/\varphi$ where φ has finite norm that we can make this conclusion.

7. A criterion for Teichmüller convergence. Let

$$\delta^2(\mu) = \sup \iint_\omega |\mu|^2 |\varphi| dx\, dy \tag{7.1}$$

where the supremum is over all φ in $A_s(\Omega, \Gamma)$ with $\iint_\omega |\varphi| = 1$. Recall that the Teichmüller distance $d(0, \mu)$ depends on the closed subset C of $\hat{\mathbf{R}}$. Specifically, $d(0, \mu) = \frac{1}{2} \log K_0$ where K_0 is the minimal dilatation of a mapping w_ν and $\nu \in M(\Gamma)$ and $w_\nu(x) = w_\mu(x)$ for all x in C. The expression (7.1) also depends on C since $C = \hat{\mathbf{C}} - \Omega$ and the supremum is over a set of differentials φ in $A_s(\Omega, \Gamma)$.

THEOREM 7.1. *Suppose Γ is a finitely generated Fuchsian group (possibly the identity) and μ is in $M(\Gamma)$ and $|\mu| \leqslant k < 1$. Let k_0 be the minimal value of $\|\nu\|_\infty$ where $\nu \sim \mu$. Then $k_0 \leqslant 2(1-k)^{-1}\delta(\mu)$ and, in particular, $d(0, \mu)$ converges to zero if $\|\mu\|_\infty \leqslant k$ and if $\delta(\mu)$ converges to zero.*

PROOF. When $A_s(\Omega, \Gamma)$ is finite dimensional one has the inequality

$$K_0 \leqslant \iint_\omega |\varphi| \frac{|1 + \mu\varphi/|\varphi||^2}{1 - |\mu|^2} \tag{7.2}$$

where φ is a holomorphic quadratic differential with $\iint_\omega |\varphi| = 1$ and φ is uniquely determined by the condition that $k|\varphi|/\varphi \sim \mu$. We can extend (7.2) to the case where C is an arbitrary closed subset of $\hat{\mathbf{R}}$, invariant under Γ and containing Λ, but the group Γ remains fixed. The inequality becomes

$$K_0 \leqslant \sup \iint_\omega |\varphi| \frac{|1 + \mu\varphi/|\varphi||^2}{1 - |\mu|^2} \tag{7.3}$$

where the supremum is over all φ in $A_s(\Omega, \Gamma)$ with $\iint_\omega |\varphi| = 1$. This is because as the subsets C_n increase to C, the spaces $A_s(\Omega_n, \Gamma)$ form an increasing sequence whose union is dense in $A_s(\Omega, \Gamma)$. Subtracting 1 from both sides of (7.3) and expanding out the numerator on the right-hand side and dividing by 2, we obtain

$$\frac{k_0}{1-k_0} \leqslant \sup \operatorname{Re} \iint \frac{\mu\varphi}{1 - |\mu|^2} + \sup \iint \frac{|\mu|^2 |\varphi|}{1 - |\mu|^2}. \tag{7.4}$$

The denominators in both integrals on the right-hand side of (7.4) are bounded below by $1 - k^2$ and, therefore,

$$\frac{k_0}{1-k_0}(1 - k^2) \leqslant \sup \iint_\omega |\mu\varphi| + \sup \iint_\omega |\mu|^2 |\varphi|, \tag{7.5}$$

where the suprema are again over all φ with $\iint_\omega |\varphi| = 1$. The first integral on the right side is less than or equal to

$$\left(\iint |\mu|^2 |\varphi| \right)^{1/2} \left(\iint |\varphi| \right)^{1/2}. \tag{7.6}$$

But the right-hand integral in (7.6) is normalized to equal one. Hence we have $k_0(1-k_0)^{-1}(1-k^2) \leqslant \delta + \delta^2$ which obviously implies $k_0 \leqslant 2(1-k)^{-1}\delta$. This proves the theorem.

REMARK. A common fallacy is to assume that $|\mu_n| \leqslant k < 1$ and $\mu_n \to 0$ in the pointwise sense implies $d(0, \mu_n) \to 0$. Although it is true that under these circumstances $w_{\mu_n}(z)$ will converge uniformly on compact subsets to z, the following simple example shows that $d(0, \mu_n)$ does not necessarily converge to zero.

EXAMPLE. Let $\mu_n = \frac{1}{10} c_n \bar{z}^n (1 - |z|^2)^2$ for $|z| < 1$ where

$$c_n = (1 + 4/n)^{n/2}((n+4)/4)$$

and let $\varphi_n = b_n z^n$ where $b_n = (n+2)/2\pi$. Then it is easy to see that $\|\mu_n\|_\infty = \frac{1}{10}$ and $\iint |\varphi_n| = 1$. Also μ_n converges in the bounded pointwise sense to zero and so $w_{\mu_n}(z)$ converges to z uniformly on Δ.

Let k_n be the minimal value of $\|\nu\|_\infty$ for which $\nu \sim \mu_n$. From inequality (3.9) one finds

$$\frac{k_n}{1 + k_n} \geqslant I[\mu_n] - \frac{1}{99}.$$

On the other hand,

$$I[\mu_n] = \int_0^{2\pi} \int_0^1 \frac{1}{10} b_n c_n r^{2n+1} (1 - r^2)^2 \, dr \, d\theta$$

which equals

$$\frac{1}{10}\left(1 + \frac{4}{n}\right)^{n/2}\left(\frac{n+4}{4}\right)\frac{1}{(n+1)(n+3)}$$

and this approaches .04618.

Thus we have a sequence of points $[\mu_n]$ in universal Teichmüller space such that

(i) $\|\mu_n\|_\infty = \frac{1}{10}$,

(ii) $d(0, \mu_n) \geqslant \frac{1}{30}$ for all n, and

(iii) $\mu_n(z)$ converges pointwise to 0.

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