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APPROXIMATION OF OPERATOR SEMIGROUPS IN A BANACH SPACE

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In this paper we study the approximation of an abstract Cauchy problem and its convergence through the notion of semigroups of operators.

Let X be a Banach space, A a closed linear operator in X, and let us consider a differential equation

$$(d/dt)u(t) = Au(t) , \qquad t > 0 .$$

By an (abstract) Cauchy problem we mean the problem composed of this equation and the initial condition

$$u(0) = x_0 , \qquad x_0 \in X .$$

Our object is to find the solution to this problem. By a family of solution operators to the Cauchy problem we mean a one-parameter family $\{U(t); t > 0\}$ of linear operators defined on a linear manifold $D(\subset X)$ such that for each $x \in D$ the function $u(t) = U(t)x$ is a solution of the problem. To find the solution operators of our problem, we proceed as follows.

Let $\{A_n\}$ be a sequence of closed linear operators in X which are "regular" in comparison with A and approximate A in an appropriate sense, and let us consider a sequence of approximating equations

$$(d/dt)u_n(t) = A_n u_n(t) , \qquad t > 0 ,$$

under the initial conditions

$$u_n(0) = x_0 , \qquad n = 1, 2, \cdots .$$

Then the problem of our interest in this paper is to find sufficient conditions under which the solution operator of the original Cauchy problem is obtained as the limit of the solution operators of these approximating Cauchy problems. This kind of problem is sometimes reduced to the problem of convergence of semigroups of operators and many interesting results on the relation between the convergence of semigroups of operators and that of the corresponding generators have been obtained. For instance, see [3], [5], [9], [12] and [15].

Among others, we shall derive some sufficient conditions under which

(1) the solution operators $U_n(t)$ of the approximating Cauchy problems exist;

(2) the sequence $\{U_n(t)\}$ converges to the solution operator $U(t)$ of the original Cauchy problem in an appropriate sense as $n \to \infty$ (Section 2). It should be noted that the domain D of the solution operator $U(t)$ is not necessarily dense in X and $U(t)$ need not be bounded on D.

The solution operators constructed there are not necessarily extensible to semigroups of bounded linear operators. In fact, if A_n satisfies our assumptions and is densely defined in X, then it is proved only that A_n is the infinitesimal generator of a distribution semigroup. Hence, the limit operator A will not be more than the infinitesimal generator of a distribution semigroup. To obtain a semigroup of bounded linear operators which are bounded extensions of solution operators, we need some additional restrictions. By imposing a stability condition together with the above restrictions, we shall obtain some convergence theorems of semigroups of bounded linear operators (Section 3). The convergence (C_0)-, $(1, A)$-, $(0, A)$- and (A)-semigroups can be treated in view of these results.

Another aspect of our arguments is an abstract setting of semi-discrete finite difference approximation of a Cauchy problem. We shall introduce the notion of well-posedness of a Cauchy problem in terms of (distribution) semigroup, the semigroup being determined by the corresponding solution operator (Section 4), and show the relationship between the well-posedness and the convergence of semi-discrete difference approximation (Section 5). In this setting, our results are directly applicable to the convergence problem of the finite difference approximation of Cauchy problems in concrete function spaces.

1. **Preliminaries.** In this section we introduce some notations and basic notions which will be used in this paper.

Let X be a Banach space and let A be a linear operator from X into itself. We say simply that A is an *operator in* X. We denote the *domain* of A by $D(A)$, the *resolvent set* of A by $\rho(A)$ and the *resolvent* of A at $\xi \in \rho(A)$ by $R(\xi; A)$. For any closable operator B such that $\bar{B} = A$ ($\bar{B}$ denotes the *closure* of B), its domain $D(B)$ is called a *core* of A; in other words, a linear manifold $D(\subset D(A))$ is a core of A, if D is dense in $D(A)$ with respect to the graph norm of A. Let A be a closed operator in X. Then for any nonnegative integer k, $D(A^k)$ can be regarded as a Banach space with respect to the norm $\|\cdot\|_k$ defined by $\|x\|_k = \sum_{i=0}^{k} \|A^i x\|$, $x \in D(A^k)$, where $A^0 = I$ (the identity operator on X). We write $[D(A^k)]$ for the Banach space.

In this paper, we treat the resolvents of a countable number of closed operators. For the brevity in notation, we use the following notations: Let $\{A_n\}$ be a sequence of closed operators in a Banach space X. We then denote by $R_n(\xi)$ the resolvent of A_n at $\xi \in \rho(A_n)$. Also, we write $J_n(\lambda)$ for the operator $(I - \lambda A_n)^{-1} = \lambda^{-1}R_n(\lambda^{-1})$.

We denote by $\mathscr{B}(X, Y)$ the class of all bounded operators on a Banach space X into a Banach space Y. We write $\mathscr{B}(X)$ for $\mathscr{B}(X, X)$. Let $T \in \mathscr{B}(X)$. We mean by $T[S]$ the image of $S \subset X$ under T. If a sequence $\{T_n\}(\subset \mathscr{B}(X))$ converges to some $T(\in \mathscr{B}(X))$ in the sense of the strong operator topology, then we write $\text{s-lim}_{n\to\infty} T_n = T$.

Let A be an operator in X, $0 < \tau \leqq +\infty$, and let us consider a differential equation

$$(d/dt)u(t) = Au(t) , \qquad 0 < t < \tau , \tag{1.1}$$

where (d/dt) means the differentiation in the sense of the strong topology of X. We then formulate the following problem:

ACP. Given an element $x \in X$, find an X-valued function $u(t) = u(t; x)$, defined on $[0, \tau)$, such that

(i) $u(t)$ is continuously differentiable in $[0, \tau)$ (or in $(0, \tau)$),

(ii) for each $t \in (0, \tau)$, $u(t) \in D(A)$ and $u(t)$ satisfies (1.1),

(iii) $\lim_{t\to+0} u(t) = u(0) = x$.

This problem is called the *(abstract) Cauchy problem*, ACP, *on* $(0, \tau)$ *for* A *and* x and the function $u(t)$ satisfying (i), (ii) and (iii) is called the *solution* of the *ACP*. There are two alternatives for condition (i); the corresponding problems will be denoted by ACP_1 and ACP_2, respectively. We shall omit the term "on $(0, \tau)$" unless we need to specify the τ.

A one-parameter family $\{T(t); t > 0\} \subset \mathscr{B}(X)$ is called a *semigroup (of bounded operators) on* X, if it has the following properties;

$$T(t + s) = T(t)T(s) , \qquad t, s > 0 , \tag{1.2}$$

$$\operatorname*{s-lim}_{t\to t_0} T(t) = T(t_0) , \qquad t_0 > 0 . \tag{1.3}$$

For a semigroup $\{T(t); t > 0\}$ on X, we define the *infinitesimal generator* A_0 by $A_0 x = \lim_{h\to+0} A_h x$, $A_h = h^{-1}[T(h) - I]$, whenever the limit exists. If A_0 is closable, then $A = \bar{A}_0$ is called the *complete infinitesimal generator*. Also, we define the *type* ω_0 by $\omega_0 = \lim_{t\to+\infty} t^{-1} \log \| T(t) \| (< +\infty)$. The set $\Sigma = \{x \in X; \lim_{t\to+0} T(t)x = x\}$ is called the *continuity set*. We denote by $R_0(\lambda)$, $\text{Re}(\lambda) > \omega_0$, the operator which is defined by the Laplace transform of $T(t)x$, $x \in \Sigma$:

$$R_0(\lambda)x = \int_0^\infty e^{-\lambda t} T(t)x dt , \qquad x \in \Sigma ,$$

where the integral is taken in the sense of Bochner.

Let $\{T(t); t > 0\}$ be a semigroup on X and let us consider the following conditions;

(S_1) $X_0 = \cup_{t>0} T(t)[X]$ is dense in X,

(S_2) there is a real number $\omega_1(>\omega_0)$ and for each λ with $\mathrm{Re}(\lambda) > \omega_1$, there exists an invertible operator $R(\lambda) \in \mathscr{B}(X)$ such that $R(\lambda)x = R_0(\lambda)x$ for $x \in X_0$,

$(S_3; k)$ there are a nonnegative integer k and a complex number λ_0 with $\mathrm{Re}(\lambda_0) > \omega_1$ such that $R(\lambda_0)^k[X] \subset \Sigma$.

If a semigroup $\{T(t); t > 0\}$ on X satisfies conditions (S_1) and (S_2), then its generator A_0 is closable and densely defined. Also, the closure $A(=\bar{A}_0)$ has the following properties: $\{\lambda; \mathrm{Re}(\lambda) > \omega_1\} \subset \rho(A)$ and $R(\lambda; A) = R(\lambda)$ for $\mathrm{Re}(\lambda) > \omega_1$. Therefore, condition $(S_3; k)$ can be restated as follows;

$(S_3; k)$ there is a nonnegative integer k such that $D(A^k) \subset \Sigma$.

There are a countable number of alternatives for condition $(S_3; k)$; a semigroup satisfying (S_1), (S_2) and $(S_3; k)$ will be called the *semigroup of class* $(C_{(k)})$. In view of well-known facts for classes (C_0), $(1, A)$, $(0, A)$ and (A) (for the definitions and the detailed properties of these classes, see [2]), it is easily seen that $(C_0) = (C_{(0)})$, $(1, A) \subset (0, A) \subset (C_{(1)})$ and $(A) \subset (C_{(2)})$ in the set theoretical sense. The basic properties of classes $(C_{(k)})$ are mentioned in [7].

2. Convergence of Solution Operators. Let A be a closed operator in a Banach space X, ω a real number and let τ be a fixed positive number. Let us consider the following conditions:

(I; ω) $\{\xi; \xi > \omega\} \subset \rho(A)$,

(II; k) there are a nonnegative integer k and a positive number M such that

$$\| \xi^m R(\xi; A)^m x \| \leqq M \| x \|_k$$

for $x \in D(A^k)$, $\xi > \omega$ and $m \geqq 1$ with $m/\xi \in (0, \tau)$.

We denote by $G_1(\omega, k, \tau)$ the set of all closed operators in X satisfying (I; ω) and (II; k). Let $A \in G_1(\omega, k, \tau)$ and $b = 2k + 1$. Then it can be proved (see [7; Th. 2.3]) that *there is a uniquely determined one-parameter family* $\{U(t); t \in (0, \tau)\} (\subset \mathscr{B}([D(A^b)], X))$ *such that*

(a) *for each* $x \in D(A^b)$ *and each* $t \in (0, \tau)$, $U(t)x = \lim_{m\to\infty} (I - (t/m)A)^{-m}x$ *exists uniformly in* $(0, \tau)$, $U(t)x$ *is strongly continuous in* $(0, \tau)$ *and* $U(t)x \to x$ *as* $t \to +0$,

(b) *for each* $x \in D(A^b)$ *and each* $t \in (0, \tau)$, $\| U(t)x \| \leqq M \| x \|_k$,

(c) *for each positive integer* p *and each* $t \in (0, \tau)$, $U(t)$ *maps* $D(A^{b+p})$ *into* $D(A^p)$ *and* $A^p U(t) = U(t) A^p$ *on* $D(A^{b+p})$,

(d) *for each* $x \in D(A^{2b})$ *and* $t, s > 0$ *with* $t + s \in (0, \tau)$, $U(t + s)x = U(t) U(s)x$,

(e) *for each nonnegative integer* p *and each* $x \in D(A^{b+p+1})$, *there is a positive number* $\beta = \beta(p, x)$ *such that*

$$\| U(t)x - U(s)x \|_p \leqq \beta |t - s| , \qquad t, s \in (0, \tau) ,$$

(f) *for each* $x \in D(A^{b+1})$ *and each* $t \in (0, \tau)$, $U(t)x - x = \int_0^t U(s)Ax\, ds = \int_0^t A U(s)x\, ds$. *If in addition,* $D(A)$ *is dense in* X, *then all assertions mentioned above hold for* $b = k$.

The above-mentioned states that for each $x \in D(A^{b+1})$, $u(t) = U(t)x$ is a unique solution of ACP_1 for A with the initial value x. In view of this, we call $U(t)$ the *solution operator of* ACP_1 *for* A.

Let $\{A_n\}$ be a sequence of closed operators in X which belong to the same class $G_1(\omega, k, \tau)$. Then each A_n has the resolvent $R_n(\xi)$ at $\xi > \omega$. In this section we consider ACP_1's for the A_n, $n = 1, 2, \cdots$, on a fixed interval $(0, \tau)$ and discuss the relation between the convergence of the solution operators $U_n(t)$ and the convergence of $R_n(\xi)$.

Our main results of this section are the following:

THEOREM 2.1. *Let* $\{A_n\} \subset G_1(\omega, k, \tau)$. *Assume the following conditions;*

(I) *orf each* $\xi > \omega$, $\sup_n \| R_n(\xi) \| < + \infty$,

(II) *there is a positive number* M, *independent of* n, *such that*

$$\| \xi^m R_n(\xi)^m x \| \leqq M \| x \|_{k,n}$$

for $x \in D(A_n^k)$, $\xi > \omega$ *and* $m \geqq 1$ *with* $m/\xi \in (0, \tau)$, *where* $\| x \|_{k,n} = \sum_{i=0}^k \| A_n^i x \|$,

(III) *for some* $\xi_0 > \omega$, *there is an invertible operator* $R(\xi_0) \in \mathscr{B}(X)$ *such that* $R(\xi_0) = \text{s-lim}_{n\to\infty} R_n(\xi_0)$.

Then there exists an operator $A \in G_1(\omega, k, \tau)$ *satisfying*

(a) $R(\xi) = \text{s-lim}_{n\to\infty} R_n(\xi)$ *for* $\xi > \omega$, *where* $R(\xi)$ *is the resolvent of* A *at* ξ,

(b) *for each* $x \in X$ *and* $\xi > \omega$, *let* $y_n = R_n(\xi)^{2k+1} x$ *and* $y = R(\xi)^{2k+1}x$, *then* $\lim_{n\to\infty} y_n = y$ *and* $\lim_{n\to\infty} U_n(t)y_n = U(t)y$ *hold uniformly in* $(0, \tau)$, *where* $U_n(t)$ *and* $U(t)$ *are the corresponding solution operators of* ACP_1's *for* A_n *and* A, *respectively.*

THEOREM 2.2. *In addition to conditions* (I)–(III) *of Theorem* 2.1, *assume that each* A_n *is densely defined and also that* $R(\xi_0)[X]$ *is dense in* X. *Then* A *is densely defined and the assertion* (b) *holds for* k, *instead of* $2k + 1$.

We shall prove these theorems by several lemmas. In the following, we assume conditions (I)–(III).

LEMMA 2.3. *Let* $\phi(\xi) = \sup_n \| R_n(\xi) \|$ *for* $\xi > \omega$. *Then* $\phi(\xi)$ *is uniformly* *bounded on each compact subinterval of* (ω, ∞).

PROOF. It suffices to show that for each $\xi_0 > \omega$, $\phi(\xi)$ is uniformly bounded in some neighbourhood of ξ_0. From the resolvent equations

$$R_n(\xi) = R_n(\xi_0)[I + (\xi_0 - \xi)R_n(\xi)] , \qquad \xi > \omega, n \geqq 1 ,$$

it follows that $\phi(\xi) \leqq \phi(\xi_0) + |\xi - \xi_0|\phi(\xi)\phi(\xi_0)$; this means that $\phi(\xi)$ is uniformly bounded in a small neighbourhood of ξ_0. *q.e.d.*

The following is obtained by the same method as in T. Kato [3; Th. IX. 2.17]. (His argument is done in a half-plane $\{\xi; \mathrm{Re}(\xi) > \omega\}$.)

LEMMA 2.4. *For every* $\xi > \omega$, *there is an operator* $R(\xi) \in \mathscr{B}(X)$ *such that* $R(\xi) = \text{s-lim}_{n\to\infty} R_n(\xi)$ *holds uniformly for* ξ *on each compact subinterval of* (ω, ∞). *Moreover,* $\{R(\xi); \xi > \omega\}$ *satisfies the resolvent equation:*

$$R(\xi) - R(\eta) = (\eta - \xi)R(\xi)R(\eta) , \qquad \xi, \eta > \omega .$$

LEMMA 2.5. *Put* $A = \xi_0 - [R(\xi_0)]^{-1}$. *Then* A *is a closed operator with* $D(A) = R(\xi_0)[X]$, $\{\xi; \xi > \omega\} \subset \rho(A)$, *and* $R(\xi; A) = R(\xi)$ *for* $\xi > \omega$. *Furthermore,* A *satisfies condition* (II; k) *for the constant* M *which is given in* (II). *Therefore,* $A \in G_1(\omega, k, \tau)$.

PROOF. The first half is evident from condition (III) and Lemma 2.4. In order to prove the last half, it suffices to show that

$$\lim_{n\to\infty} A_n^i R_n(\xi_0)^k x = A^i R(\xi_0)^k x$$

for $x \in X$ and $i = 0, 1, \cdots, k$. But this follows from condition (I), Lemma 2.4 and the relations:

$$A_n^i R_n(\xi_0)^k x = [\xi_0 R_n(\xi_0) - I]^i R_n(\xi_0)^{k-i} x , \qquad i = 0, 1, \cdots, k . \qquad q.e.d.$$

By this lemma, Theorem 2.1 (a) has been proved. Also, by the result mentioned at the begining of this section, we can obtain the family $\{U(t); t \in (0, \tau)\}$ of solution operators of ACP_1 for A such that $U(t)x = \lim_{m\to\infty} J(t/m)^m x$ for $x \in D(A^{2k+1})$, where $J(t/m) = (I - (t/m)A)^{-1}$. We then prove Theorem 2.1 (b) after preparing the following three lemmas; the first is given in [1; Lem. 1.4], the second is a linear version of [1; Lem. 1.3] and the third is crucial for the proof of (b).

LEMMA 2.6. *Let* $p \geqq q$ *be positive integers and let* α, β *be positive numbers satisfying* $\alpha + \beta = 1$. *Then*

$$\text{(i)} \qquad \sum_{j=0}^{q} {}_pC_j \alpha^j \beta^{p-j}(q - j) \leqq [(p\alpha - q)^2 + p\alpha\beta]^{1/2} ,$$

$$\text{(ii)} \qquad \sum_{j=q}^{p} {}_{j-1}C_{q-1} \alpha^q \beta^{j-q}(p - j) \leqq [q\beta/\alpha^2 + (q\beta/\alpha + q - p)^2]^{1/2} .$$

LEMMA 2.7. *Let $p \geqq q$ be positive integers and let λ, μ be positive numbers satisfying $\lambda \geqq \mu$ and $\lambda^{-1} > \omega$. Then*

$$J_n(\mu)^p - J_n(\lambda)^q = \sum_{j=0}^{q} {}_pC_j\alpha^j\beta^{p-j}J_n(\mu)^p[I - J_n(\lambda)^{q-j}] + \sum_{j=q}^{p} {}_{j-1}C_{q-1}\alpha^q\beta^{j-q}J_n(\mu)^j[J_n(\mu)^{p-j} - I]\,,$$

where $\alpha = \mu/\lambda$ and $\beta = 1 - \mu/\lambda$.

LEMMA 2.8. *Let $x \in X$ and $\xi > \omega$ be fixed. Then there is a positive number C, independent of $n = 0, 1, 2, \cdots$, such that*

$$\left\| U_n(t)R_n(\xi)^{2k+1}x - J_n\left(\frac{t}{m}\right)^m R_n(\xi)^{2k+1}x \right\| \leqq tC\|x\|/\sqrt{m} \tag{2.1}$$

for $t \in (0, \tau)$ and m sufficiently large, and

$$\| U_n(t)R_n(\xi)^{2k+1}x - R_n(\xi)^{2k+1}x \| \leqq tC\|x\| \tag{2.2}$$

for $t \in (0, \tau)$, where A_0 means the limit operator A and $U_0(t)$ means the solution operator $U(t)$ corresponding to A.

PROOF. We first prove (2.1). By conditions (I) and (II), a positive number C, independent of n, can be found such that

$$\begin{aligned} &\|J_n(\mu)^jJ_n(\nu)^i[J_n(\nu) - I]R_n(\xi)^{2k+1}x\| \\ &\leqq \nu\|J_n(\mu)^jR_n(\xi)^k\|\,\|J_n(\nu)^{i+1}R_n(\xi)^k\|\,\|A_nR_n(\xi)\|\,\|x\| \\ &\leqq \frac{\nu}{2}C\|x\| \end{aligned} \tag{2.3}$$

for $n = 0, 1, 2, \cdots, \nu \geqq \mu$ and i, j with $1 \leqq i + 1, j \leqq p < \tau/\nu$. Combining Lemma 2.7 with (2.3) and then applying Lemma 2.6 with $\lambda = t/q$ and $\mu = t/p$, we obtain

$$\begin{aligned} &\left\| J_n\left(\frac{t}{p}\right)^p R_n(\xi)^{2k+1}x - J_n\left(\frac{t}{q}\right)^q R_n(\xi)^{2k+1}x \right\| \\ &\leqq \frac{1}{2}C\left\{\frac{t}{q}\sum_{j=0}^{q} {}_pC_j\alpha^j\beta^{p-j}(q-j) + \frac{t}{p}\sum_{j=q}^{p} {}_{j-1}C_{q-1}\alpha^q\beta^{j-q}(p-j)\right\}\|x\| \\ &\leqq tC\left(\frac{1}{q} - \frac{1}{p}\right)^{1/2}\|x\| \end{aligned} \tag{2.4}$$

for $t \in (0, \tau)$ and p, q sufficiently large. Letting $p \to \infty$ in (2.4), we obtain (2.1).

Next, let us prove (2.2). By (2.3), we obtain

$$\left\| J_n\left(\frac{t}{p}\right)^p R_n(\xi)^{2k+1}x - R_n(\xi)^{2k+1}x \right\|$$

$$\leqq \sum_{j=0}^{p-1} \left\| J_n\left(\frac{t}{p}\right)^j \left[J_n\left(\frac{t}{p}\right) - I \right] R_n(\xi)^{2k+1} x \right\|$$
$$\leqq \frac{t}{p} \sum_{j=1}^{p} \left\| J_n\left(\frac{t}{p}\right)^j R_n(\xi)^k \right\| \, \| A_n R_n(\xi) \| \, \| R_n(\xi)^k \| \, \| x \|$$
$$\leqq tC \| x \|$$

for $t \in (0, \tau)$ and p sufficiently large. Letting $p \to \infty$, we obtain (2.2). *q.e.d.*

Remark 2.9. The inequality (2.4) shows that if $A \in G_1(\omega, k, \tau)$, then for each $x \in D(A^{2k+1})$, the sequence $\{(I - (t/p)A)^{-p}x\}$ forms a Cauchy sequence which converges uniformly in $(0, \tau)$. This proves the existence of the solution operator $U(t)$. We note that the estimate obtained here is more precise than that given in [7].

Theorem 2.1 (b) is proved by the following:

LEMMA 2.10. *Let $\xi > \omega$ be fixed. Then*

$$\text{s-lim}_{n \to \infty} U_n(t) R_n(\xi)^{2k+1} = U(t) R(\xi)^{2k+1}$$

holds uniformly in $(0, \tau)$.

PROOF. Let $x \in X$. It follows from (2.2) that for any $\varepsilon > 0$, there is a positive number $\delta = \delta(\varepsilon)$ such that

$$\begin{aligned} (2.5) \quad & \| U_n(t)R_n(\xi)^{2k+1}x - U(t)R(\xi)^{2k+1}x \| \\ & \leqq \| U_n(t)R_n(\xi)^{2k+1}x - R_n(\xi)^{2k+1}x \| + \| R_n(\xi)^{2k+1}x - R(\xi)^{2k+1}x \| \\ & \quad + \| U(t)R(\xi)^{2k+1}x - R(\xi)^{2k+1}x \| \\ & \leqq \frac{\varepsilon}{2} + \| R_n(\xi)^{2k+1}x - R(\xi)^{2k+1}x \| , \end{aligned}$$

for $t \in (0, \delta)$. On the other hand, by virtue of (2.1),

$$\begin{aligned} (2.6) \quad & \| U_n(t)R_n(\xi)^{2k+1}x - U(t)R(\xi)^{2k+1}x \| \\ & \leqq \left\| U_n(t)R_n(\xi)^{2k+1}x - J_n\left(\frac{t}{m}\right)^m R_n(\xi)^{2k+1}x \right\| \\ & \quad + \left\| J_n\left(\frac{t}{m}\right)^m R_n(\xi)^{2k+1}x - J_n\left(\frac{t}{m}\right)^m R(\xi)^{2k+1}x \right\| \\ & \quad + \left\| J_n\left(\frac{t}{m}\right)^m R(\xi)^{2k+1}x - J\left(\frac{t}{m}\right)^m R(\xi)^{2k+1}x \right\| \\ & \quad + \left\| J\left(\frac{t}{m}\right)^m R(\xi)^{2k+1}x - U(t)R(\xi)^{2k+1}x \right\| \end{aligned}$$

$$\leqq 2tC\|x\|/\sqrt{m} + \left[\frac{m}{t}\phi\left(\frac{m}{t}\right)\right]^m \|R_n(\xi)^{2k+1}x - R(\xi)^{2k+1}x\|$$
$$+ \left\|J_n\left(\frac{t}{m}\right)^m R(\xi)^{2k+1}x - J\left(\frac{t}{m}\right)^m R(\xi)^{2k+1}x\right\|$$

for $t \in [\delta, \tau)$ and m sufficiently large. Choosing m_0 sufficiently large so that $2\tau C\|x\|/\sqrt{m_0} < \varepsilon/3$ and then applying Lemmas 2.4 and 2.5 to the remaining terms of (2.6), we see that for each $\varepsilon > 0$, there is an integer $N = N(\varepsilon)$ such that

$$\|U_n(t)R_n(\xi)^{2k+1}x - U(t)R(\xi)^{2k+1}x\| < \varepsilon \text{ for } n \geqq N \text{ and } t \in [\delta, \tau)\ .$$

This estimate and (2.5) imply that $\lim_{n\to\infty} U_n(t)R_n(\xi)^{2k+1}x = U(t)R(\xi)^{2k+1}x$ holds uniformly in $(0, \tau)$. *q.e.d.*

The following lemma proves Theorem 2.2.

LEMMA 2.11. *Under the assumption of Theorem* 2.2,

$$\text{s-lim}_{n\to\infty} U_n(t)R_n(\xi)^k = U(t)R(\xi)^k$$

holds uniformly in $(0, \tau)$ *for each* $\xi > \omega$.

PROOF. First, in this case, we recall that $U_n(t)$ and $U(t)$ are well-defined on $D(A_n^k)$ and on $D(A^k)$, respectively. Now, let $x \in D(A^{k+1})$ and $\lambda > 0$ be sufficiently small. Then

$$\begin{aligned}
&\|U_n(t)R_n(\xi)^k x - U(t)R(\xi)^k x\| \\
&\quad \leqq \|U_n(t)R_n(\xi)^k x - U_n(t)R_n(\xi)^k J_n(\lambda)^{k+1}x\| \\
&\qquad + \|U_n(t)R_n(\xi)^k J_n(\lambda)^{k+1}x - U(t)R(\xi)^k J(\lambda)^{k+1}x\| \\
&\qquad + \|U(t)R(\xi)^k J(\lambda)^{k+1}x - U(t)R(\xi)^k x\| \\
&\quad \leqq C(\|x - J_n(\lambda)^{k+1}x\| + \|x - J(\lambda)^{k+1}x\|) \\
&\qquad + \|U_n(t)R_n(\xi)^k J_n(\lambda)^{k+1}x - U(t)R(\xi)^k J(\lambda)^{k+1}x\|
\end{aligned}$$

for $t \in (0, \tau)$, where C is a positive number, independent of n and λ. From the proof of Lemma 2.8, it is easily seen that the last term of the above inequality converges to 0 uniformly in $(0, \tau)$ as $n \to \infty$. Therefore,

$$\begin{aligned}
&\limsup_{n\to\infty} \|U_n(t)R_n(\xi)^k x - U(t)R(\xi)^k x\| \\
&\quad \leqq 2C\|x - J(\lambda)^{k+1}x\| \leqq 2\lambda C \sum_{i=1}^{k+1} \|J(\lambda)^i Ax\| \leqq 2\lambda(k+1)CM\|Ax\|_k\ .
\end{aligned}$$

Since $\lambda > 0$ is arbitrary, $\lim_{n\to\infty} U_n(t)R_n(\xi)^k x = U(t)R(\xi)^k x$ holds uniformly in $(0, \tau)$ for $x \in D(A^{k+1})$. It is evident that the convergence holds uniformly in $(0, \tau)$ for all $x \in X$, because $D(A^{k+1})$ is dense in X and $\|U_n(t)R_n(\xi)^k\|$ is uniformly bounded for $t \in (0, \tau)$ and n. *q.e.d.*

Finally we give a variation of Theorem 2.2 which is convenient in applications.

Let $\{A_n\}$ be a sequence of densely defined closed operators in X which belong to the same class $G_1(\omega, k, \tau)$ and let us consider the following conditions;

(III′) there exist a densely defined closed operator A and a set D contained in $D(A)$ and $\bigcap_{n\geqq 1} D(A_n)$ such that

(i) $\rho(A) \cap \{\xi; \xi > \omega\}$ is non-empty,

(ii) D is a core of A,

(iii) $\lim_{n\to\infty} A_n x = Ax$ for $x \in D$.

The condition (III′) will be proposed as the consistency condition in the finite difference method (see Section 5). Employing this condition, instead of (III), we obtain the following:

COROLLARY 2.12. *Let* $\{A_n\} \subset G_1(\omega, k, \tau)$ *be a sequence of densely defined closed operators in* X *satisfying conditions* (I), (II) *and* (III′). *Then the conclusion of Theorem* 2.2 *holds. If in addition, we assume that* D *is contained in* $D(A^k)$ *and* $\bigcap_{n\geqq 1} D(A_n^k)$ *and* $\lim_{n\to\infty} A_n^i x = A^i x$ *for* $x \in D$ *and* $i = 2, 3, \cdots, k$, *then for each* $x \in D$, $\lim_{n\to\infty} U_n(t)x = U(t)x$ *holds uniformly in* $(0, \tau)$.

PROOF. In a similar way to [3; Th. VIII. 1.5], it is proved that under condition (I), (III′) implies that $\text{s-lim}_{n\to\infty} R_n(\xi) = R(\xi)$ for $\xi \in \rho(A) \cap \{\xi; \xi > \omega\}$. Therefore, it suffices to prove only the second assertion under the additional assumption. Let $x \in D$, then $x = R(\xi)^k z$ for some $z \in X$ and $\xi > \omega$. By assumption, $z_n = (\xi - A_n)^k x$ converges to z as $n \to \infty$. Now we have

$$\begin{aligned} &\| U_n(t) - U(t)x \| \\ &\quad \leqq \| U_n(t)R_n(\xi)^k z_n - U_n(t)R_n(\xi)^k z \| + \| U_n(t)R_n(\xi)^k z - U(t)R(\xi)^k z \| \\ &\quad \leqq C \| z_n - z \| + \| U_n(t)R_n(\xi)^k z - U(t)R(\xi)^k z \| \end{aligned}$$

for $t \in (0, \tau)$, where C is a positive number, independent of n. Applying Theorem 2.2, we see that $\lim_{n\to\infty} U_n(t)x = U(t)x$ holds uniformly in $(0, \tau)$ for each $x \in D$. *q.e.d.*

3. Convergence of Semigroups.

In this section we restrict ourselves to the semigroups of class $(C_{(k)})$ and discuss the convergence of the semigroups. The results obtained can be applied to the convergence problem of solution operators of ACP_2 on $(0, \infty)$.

We first mention the characterization ([7; Th. 6.12]) of $(C_{(k)})$-semigroup in terms of the complete infinitesimal generator:

An operator A *in a Banach space* X *is the complete infinitesimal* *generator of a* $(C_{(k)})$*-semigroup if and only if* A *is a densely defined closed operator satisfying*

$(\mathrm{I}; \omega)$ $\{\xi; \xi > \omega\} \subset \rho(A)$,

$(\mathrm{II}_{\exp}; k)$ *there are a real number* $\omega_1 \geqq \omega$ *and a positive number* M *such that*

$$\|R(\xi; A)^m x\| \leqq M(\xi - \omega_1)^{-m}\|x\|_k \quad \text{for} \quad x \in D(A^k),\ \xi > \omega_1 \quad \text{and} \quad m \geqq 1\,,$$

$(F; k)$ *for any* $\varepsilon > 0$ *and* $x \in D(A^k)$, *there are a positive number* M_ε *and a real number* $\eta_0 = \eta_0(\varepsilon, x)$ *such that*

$$\|\xi^m R(\xi; A)^m x\| \leqq M_\varepsilon \|x\| \text{ for } \xi > \eta_0 \text{ and } m \geqq 1 \text{ with } m/\xi \in [\varepsilon, 1/\varepsilon]\,.$$

Using the same notation as in [7], we denote by $G_2(\omega, k)$ the set of all closed operators in X satisfying $(\mathrm{I}; \omega)$ and $(\mathrm{II}_{\exp}; k)$. If $A \in G_2(\omega, k)$, then $A \in G_1(\gamma, k, \tau)$ for every $\tau > 0$, where $\gamma > \max\{0, \omega_1\}$. Hence, for an operator $A \in G_2(\omega, k)$ we have the same assertions as stated at the beginning of Section 2.

Let $\{T(t); t > 0\}$ be a semigroup of class $(C_{(k)})$ and A be its complete infinitesimal generator. Then the characterization mentioned above states that for each $x \in D(A^{k+1})$, $T(t)x$ is a unique solution of ACP_1 for A on $(0, \infty)$ with the initial value x. Also, it is proved in [7; Cor. 6.9] that for any $x \in D(A^k)$, $T(t)x$ is a unique solution of ACP_2 for A on $(0, \infty)$ with the initial value x. This means that for a $(C_{(k)})$-semigroup $\{T(t); t > 0\}$, the restriction $T(t)|D(A^k)$ of $T(t)$ on $D(A^k)$ is the solution operator of ACP_2 for its complete infinitesimal generator A. In this sense, $T(t)$ itself might be called the solution operator of ACP_2 for A.

REMARK 3.1. (1) It can be proved that an operator A in X is the infinitesimal generator of a regular distribution semigroup if and only if A is densely defined and for every $\tau > 0$, there exists a nonnegative integer $k(\tau)$ such that $A \in G_1(\omega, k(\tau), \tau)$. Also, it is proved that an operator A in X is the infinitesimal generator of an exponential distribution semigroup if and only if A is densely defined and $A \in G_2(\omega, k)$ for some nonnegative integer k. For the proofs of these propositions, see [7; Ths. 5.4, 5.5] and [8].

(2) In our case, the operators A have non-empty resolvent sets. The arguments on the operator belonging to $G_1(\omega, k, \tau)$ or $G_2(\omega, k)$ can be extended to the case of the operator whose resolvent set is empty. The detailed study on this matter will be seen in the forthcoming paper [6].

The main result of this section is the following:

THEOREM 3.2. *Let* $\{T_n(t); t > 0\}$ *be a sequence of* $(C_{(k)})$*-semigroups and* $\{A_n\}$ *be a sequence of the corresponding complete infinitesimal generators* *satisfying the following conditions;*

(I) *there is a real number* ω *such that for each* $\xi > \omega$, $\sup_n \|R_n(\xi)\| < +\infty$, *where* $R_n(\xi)$ *denotes the resolvent of* A_n *at* $\xi > \omega$,

($\mathrm{II}_{\exp}$) *there are a real number* $\omega_1 \geqq \omega$ *and a positive number* M, *independent of* n, *such that* $\|T_n(t)x\| \leqq Me^{\omega_1 t}\|x\|_{k,n}$ *for* $x \in D(A_n^k)$ *and* $t > 0$,

(III) *for some* $\xi_0 > \omega$, *there exists an invertible operator* $R(\xi_0) \in \mathscr{B}(X)$ *such that* $R(\xi_0) = \text{s-lim}_{n\to\infty} R_n(\xi_0)$ *and* $R(\xi_0)[X]$ *is dense in* X,

(IV) *for each* $x \in X$ *and* $t > 0$, $\sup_n \|T_n(t)x\| < +\infty$.

Then we have

(a) *there exists a closed operator* A *such that* A *generates a* $(C_{(k)})$-*semigroup* $\{T(t); t > 0\}$ *and* $R(\xi_0) = R(\xi_0; A)$,

(b) $\text{s-lim}_{n\to\infty} T_n(t) = T(t)$ *for* $t > 0$ *and the convergence is uniform for* t *on each compact subinterval of* $(0, \infty)$.

We prove this theorem by the following successive lemmas. We first mention a result which was obtained in [9; Lem. 2]:

LEMMA 3.3. *Assume condition* (IV). *Then there exist a positive number* γ *and a nonnegative nonincreasing function* $\psi(t)$ *of negative type such that*

$$\sup_n \|e^{-\gamma t}T_n(t)\| \leqq \psi(t) \quad \text{for} \quad t > 0 .$$

Here the function $\psi(t)$ *is said to be of negative type, if*

$$\limsup_{t\to+\infty} t^{-1}\log\psi(t) < 0 .$$

On the basis of this lemma, we can reduce Theorem 3.2 to the case of semigroups of negative type (i.e., the type $\omega_0 < 0$) by considering the equivalent semigroups $\{e^{-\gamma_0 t}T_n(t); t > 0\}$, where $\gamma_0 = \max\{\omega_1, \gamma\}$ (cf. [9]). Hence, we assume that $\omega = \omega_1 = \gamma = 0$ in the following.

LEMMA 3.4. *Assume condition* ($\mathrm{II}_{\exp}$). *Then*

(3.1) $\|\xi^m R_n(\xi)^m x\| \leqq M\|x\|_{k,n}$ *for* $x \in D(A_n^k)$, $\xi > 0$ *and* $m \geqq 1$.

Consequently, all A_n *belong to* $G_2(0, k)$.

PROOF. Differentiating

$$R_n(\xi)x = \int_0^\infty e^{-\xi t}T_n(t)x\,dt , \qquad x \in D(A_n^k),\ \xi > 0 ,$$

with respect to ξ, we have

$$R_n(\xi)^m x = \frac{1}{(m-1)!}\int_0^\infty t^{m-1}e^{-\xi t}T_n(t)x\,dt , \qquad m \geqq 1 . \tag{3.2}$$

We obtain (3.1) by applying condition ($\mathrm{II}_{\exp}$) to (3.2). *q.e.d.*

From this lemma, it is seen that for every fixed $\tau > 0$, the sequence $\{A_n\}(\subset G_2(0, k))$ satisfies the assumptions of Theorem 2.2. Hence, it is proved that there exists a densely defined closed operator $A \in G_2(0, k)$ such that $R(\xi_0) = R(\xi_0; A)$. Thus, if the following lemma is proved, then we see from the characterization theorem that A generates a $(C_{(k)})$-semigroup $\{T(t); t > 0\}$; hence Theorem 3.2 (a) is proved.

LEMMA 3.5. *A satisfies condition* $(F; k)$.

PROOF. Let $x \in D(A^k)$, $x \neq 0$, and let $x = R(\xi_0)^k y$ for some $y \in X$. Then (3.2) implies that

$$R_n(\xi)^m R_n(\xi_0)^k y = \frac{1}{(m-1)!}\int_0^\infty t^{m-1} e^{-\xi t} T_n(t) R_n(\xi_0)^k y dt\,, \qquad \xi > 0,\, m \geqq 1.$$

Employing Lemma 3.3, we have

$$\begin{aligned} (3.3) \qquad & \frac{1}{(m-1)!}\int_{\varepsilon/2}^\infty \xi^m t^{m-1} e^{-\xi t} \| T_n(t) R_n(\xi_0)^k y \| dt \\ & \leqq \psi\left(\frac{\varepsilon}{2}\right) \| R_n(\xi_0) y \| \frac{1}{(m-1)!}\int_{\varepsilon/2}^\infty \xi^m t^{m-1} e^{-\xi t} dt \\ & \leqq \psi\left(\frac{\varepsilon}{2}\right) \| R_n(\xi_0) y \|\,. \end{aligned}$$

On the other hand, by condition $(\mathrm{II}_{\exp})$, we have

$$\begin{aligned} (3.4) \qquad & \frac{1}{(m-1)!}\int_0^{\varepsilon/2} \xi^m t^{m-1} e^{-\xi t} \| T_n(t) R_n(\xi_0)^k y \| dt \\ & \leqq \frac{M}{(m-1)!}\int_0^{\varepsilon/2} \xi^m t^{m-1} e^{-\xi t} dt \| R_n(\xi_0)^k y \|_{k,n} \\ & \leqq \frac{M}{(m-1)!}\int_0^{\varepsilon\xi/2} s^{m-1} e^{-s} ds \| R_n(\xi_0)^k y \|_{k,n} \\ & \leqq 2\frac{M}{m} \| R_n(\xi_0)^k y \|_{k,n} \end{aligned}$$

for $m \geqq \varepsilon\xi$. The last step of (3.4) is derived from the inequality ([2; p. 374])

$$\frac{1}{(m-1)!}\int_0^{\delta'\xi} s^{m-1} e^{-s} ds \leqq q/m(1-q)^2\,, \qquad m \geqq \delta\xi\,,$$

with $\delta = \varepsilon$, $\delta' = \varepsilon/2$ and $q = \delta'/\delta = 1/2$. Combining (3.3) and (3.4), and then passing to the limit as $n \to \infty$, we obtain

$$(3.5) \qquad \| \xi^m R(\xi)^m R(\xi_0)^k y \| \leqq \psi\left(\frac{\varepsilon}{2}\right) \| x \| + 2\frac{M}{m} \| x \|_k \qquad \text{for } m \geqq \varepsilon\xi\,.$$

Set $\eta_0 = 2M\|x\|_k/\varepsilon\|x\|$. Then the second term of the right hand side of (3.5) is majorized by $\|x\|$ for $\xi > \eta_0$ and $m \geqq \varepsilon\xi$. Thus, we obtain

$$\|\xi^m R(\xi)^m x\| \leqq \left[\psi\left(\frac{\varepsilon}{2}\right) + 1\right]\|x\| \qquad \text{for } \xi > \eta_0 \text{ and } m \geqq \varepsilon\xi\,. \qquad q.e.d.$$

Theorem 3.2 (b) is proved by the following:

LEMMA 3.6. *s-$\lim_{n\to\infty} T_n(t) = T(t)$ holds uniformly for t on each compact subinterval of* $(0, \infty)$.

PROOF. We note that $U_n(t) = T_n(t)\,|\,D(A_n^k)$ and $U(t) = T(t)\,|\,D(A^k)$ are the solution operators of ACP's for A_n and A, respectively, and that $U_n(t)$ converges to $U(t)$ uniformly in every finite interval $(0, \tau)$ in the sense of Lemma 2.11. Now, let $x \in X$. Since $D(A^k)$ is dense in X, for any $\varepsilon > 0$, there is an element $y = R(\xi_0)^k z \in D(A^k)$, $z \in X$, such that $\|x - y\| < \varepsilon$. Therefore,

$$\begin{aligned}
&\|T_n(t)x - T(t)x\| \\
&\quad \leqq \|T_n(t)x - T_n(t)y\| + \|T_n(t)R(\xi_0)^k z - T_n(t)R_n(\xi_0)^k z\| \\
&\qquad + \|T_n(t)R_n(\xi_0)^k z - T(t)R(\xi_0)^k z\| + \|T(t)y - T(t)x\| \\
&\quad \leqq \psi(t)(\|x - y\| + \|R(\xi_0)^k z - R_n(\xi_0)^k z\|) + \|T(t)\|\,\|y - x\| \\
&\qquad + \|T_n(t)R_n(\xi_0)^k z - T(t)R(\xi_0)^k z\|\,.
\end{aligned}$$

Thus, we obtain

$$\limsup_{n\to\infty} \|T_n(t)x - T(t)x\| \leqq (\psi(t) + \|T(t)\|)\varepsilon\,.$$

This means that $\lim_{n\to\infty} T_n(t)x = T(t)x$ for $t > 0$. Since both $\psi(t)$ and $\|T(t)\|$ are uniformly bounded on each compact subinterval of $(0, \infty)$, this convergence is uniform for t in such subintervals. *q.e.d.*

Finally, let us consider the following condition, instead of (I) and $(\mathrm{II}_{\exp})$:
(I′) there are a real number ω and a polinomial p of degree $l \geqq 0$ with nonnegative coefficients such that $\sup_n \|R_n(\lambda)\| \leqq p(|\lambda|)$ for λ with $\mathrm{Re}(\lambda) > \omega$.

It is proved in [7; Th. 4.7] that under condition (I′), all A_n belong to $G_2(\gamma, l + 2)$, where $\gamma > \max\{0, \omega\}$. By virtue of this fact, we can obtain a variation of Theorem 3.2. The conclusion obtained, however, is somewhat weaker than Theorem 3.2, as mentioned below.

THEOREM 3.7. *Let $\{T_n(t); t > 0\}$ be a sequence of semigroups satisfying conditions* (I′), (III) *and* (IV). *Then the conclusion of Theorem* 3.2 *holds with k replaced by $l + 2$.*

This theorem gives an extension of the theorem on convergence of (A)-semigroups which is obtained in [9; Th. (A)].

As an analogous result to Corollary 2.12, we give a result which is a variation of Theorem 3.2 and is convenient in applications.

COROLLARY 3.8. *Let* $\{T_n(t); t > 0\}$ *be a sequence of* $(C_{(k)})$*-semigroups satisfying conditions* (I), $(\mathrm{II}_{\exp})$, (III′) *and* (IV). *Then the conclusion of Theorem* 3.2 *holds.*

4. **Well-Posedness.** Let us consider an ACP

$$(d/dt)u(t) = Au(t)\ , \qquad u(0) = x\ , \tag{4.1}$$

for an operator A in a Banach space X. In this section we propose some kinds of well-posedness of ACP (4.1) and discuss the relationships between them and various notions of semigroups. Throughout this section we assume that A is a densely defined closed operator and soppose that there is a family $\{U(t); t > 0\}$ of operators, defined on a core D of A, such that for any $x \in D$, $u(t) = U(t)x$ is a unique solution of ACP_1 for A with the initial value x such that $u(t) \in D$ for $t > 0$.

When each $U(t)$ is bounded on D, $U(t)$ can be extended to $T(t) \in \mathscr{B}(X)$ because of the denseness of D. In our setting, $U(t)$ may be called a solution operator of ACP (4.1). But, we again call this extended operator $T(t)$ a solution operator of ACP (4.1). Obviously, $\{T(t); t > 0\}$ forms a semigroup of bounded operators on X. We call this semigroup a *semigroup of solution operators of* ACP (4.1).

Now we say that ACP (4.1) is *well-posed in the sense of semigroup of bounded operators* (simply, *S.G.-well-posed*) if there is a semigroup $\{T(t); t > 0\}$ of bounded operators such that $T(t)x = U(t)x$ for $x \in D$ and $t > 0$. If the semigroup is a $(C_{(k)})$-semigroup for some nonnegative integer k, then we say that ACP (4.1) is $(C_{(k)})$*-well-posed.*

For the $(C_{(k)})$-well-posedness, we have the following:

THEOREM 4.1. ACP (4.1) *is* $(C_{(k)})$*-well-posed if and only if* A *is the complete infinitesimal generator of a* $(C_{(k)})$*-semigroup. In this case,* $A \in G_2(\omega, k)$ *for some* ω.

PROOF. One direction is evident from the characterization theorem of $(C_{(k)})$-semigroup (see Section 3). To prove the converse, let $\{T(t); t > 0\}$ be the $(C_{(k)})$-semigroup of solution operators of ACP (4.1) and let $x \in D$. Since $T(t)x$ is a solution of ACP_1 (4.1),

$$\lim_{h \to +0} (T(h)x - x)/h = Ax\ ,$$

which implies that $Ax = A_0 x$ for $x \in D$, where A_0 is the infinitesimal generator of $\{T(t); t > 0\}$. This implies that $D(A) \subset D(\bar{A}_0)$ and $Ax = \bar{A}_0 x$ for $x \in D(A)$, because D is a core of A and A_0 is closable.

On the other hand, for some $\omega_1 > \omega_0$(ω_0 being the type of $\{T(t); t > 0\}$), we have

$$R(\xi; \bar{A}_0)x = \int_0^\infty e^{-\xi t} T(t)xdt \qquad \text{for } x \in \Sigma \text{ and } \xi > \omega_1 .$$

Now, let $x \in D$ and $\xi > \omega_1$. Then

$$-(d/dt)[e^{-\xi t} T(t)x] = e^{-\xi t}(\xi - A)T(t)x \qquad \text{for } t > 0 .$$

Integrating this equality with respect to t from 0 to ∞ and using the closedness of A, we get

$$(\xi - A)R(\xi; \bar{A}_0)x = \lim_{\tau\to\infty} [-e^{-\xi\tau} T(\tau)x] + x = x .$$

Obviously, this implies that $(\xi - A)R(\xi; \bar{A}_0)x = x$ for all $x \in X$ and in turn that $D(\bar{A}_0) \subset D(A)$. Therefore, $D(A) = D(\bar{A}_0)$ and $A = \bar{A}_0$. *q.e.d.*

When $U(t)$ is not bounded on D, $\{U(t); t > 0\}$ can not be extended to a semigroup of bounded operators; hence, in general, we can not treat the solution operators in terms of the semigroup of bounded operators. In certain cases, however, we can formulate some kinds of well-posedness of ACP (4.1), using the notion of distribution semigroup which was introduced by J. Lions [4] (also see [7] and [13]).

Let $D(\boldsymbol{R}_+)$ be the Schwartz space corresponding to $\boldsymbol{R}_+ = (0, \infty)$ and let us consider a family $\{U(\phi); \phi \in D(\boldsymbol{R}_+)\}$ of operators, defined on D, by

$$U(\phi)x = \int_0^\infty \phi(t)U(t)xdt , \qquad x \in D .$$

We then say that ACP (4.1) is *R.D.S.G.-well-posed* if there is a regular distribution semigroup (R.D.S.G.) T such that $T(\phi)x = U(\phi)x$ for $x \in D$ and $\phi \in D(\boldsymbol{R}_+)$. If in particular, the R.D.S.G. is an exponential distribution semigroup (E.D.S.G.), then we say that ACP (4.1) is *E.D.S.G.-well-posed.*

A semigroup $\{T(t); t > 0\}$ of bounded operators is *called a strongly continuous distribution semigroup* (*C.D.S.G.*) if there is an R.D.S.G. T such that

$$T(\phi) = \int_0^\infty \phi(t)T(t)xdt \qquad \text{for } x \in X \text{ and } \phi \in D(\boldsymbol{R}_+) .$$

See [14]. We then say that ACP (4.1) is *C.D.S.G.-well-posed* if it is *S.G.*-well-posed and if the semigroup of solution operators of ACP (4.1) is a C.D.S.G.. By definition, if ACP (4.1) is S.G.-well-posed and at the same time R.D.S.G.-well-posed, then it is C.D.S.G.-well-posed.

Remark 4.2. It is proved in [7; Th. 5.15] that a semigroup of bounded operators is a C.D.S.G. if and only if it is a $(C_{(k)})$-semigroup for some k.

This means that if ACP (4.1) is C.D.S.G.-well-posed, then it is $(C_{(k)})$-well-posed for some k, and conversely.

For these kinds of well-posedness, we have the following:

THEOREM 4.3. (a) ACP (4.1) *is R.D.S.G.-well-posed if and only if* A *is the infinitesimal generator of an R.D.S.G.. In this case, there is a real number* ω *such that* $A \in G_1(\omega, k(\tau), \tau)$ *for every* $\tau > 0$ *and for some nonnegative integer* $k(\tau)$.

(b) ACP (4.1) *is E.D.S.G.-well-posed if and only if* A *is the infinitesimal generator of an E.D.S.G.. In this case,* $A \in G_2(\omega, k)$ *for some* ω *and nonnegative integer* k.

(c) ACP (4.1) *is C.D.S.G.-well-posed if and only if* A *is the complete infinitesimal generator of a C.D.S.G.. In this case,* A *is the complete infinitesimal generator of a* $(C_{(k)})$*-semigroup for some nonnegative integer* k.

PROOF OF (a). Assume that ACP (4.1) is R.D.S.G.-well-posed. Let T be the R.D.S.G., $x \in D$ and let $\phi \in D(\boldsymbol{R}_+)$. Then by integration by parts,

$$T(-\delta')T(\phi)x = T(-\delta' * \phi)x = -\int_0^\infty \phi'(t)U(t)xdt = AT(\phi)x , \tag{4.2}$$

where δ is the Dirac function. Now, let $\mathscr{D} = \{T(\phi)x; x \in D, \phi \in D(\boldsymbol{R}_+)\}$. Then from the definition of $T(-\delta')$ (e.g., see [10]) and the denseness of D, it follows that $\mathrm{sp}\,[\mathscr{D}]$ is a core of $\overline{T(-\delta')}$ (note that $\overline{T(-\delta')}$ is the infinitesimal generator of the R.D.S.G. T). Therefore, (4.2) implies that $D(\overline{T(-\delta')}) \subset D(A)$ and $\overline{T(-\delta')}x = Ax$ for $x \in D(\overline{T(-\delta')})$.

Next we demonstrate that $\mathscr{D}$ is also a core of A. To see this, we first take a family $\{\alpha_\varepsilon(t); \varepsilon > 0\} \subset D(\boldsymbol{R}_+)$ such that for each $\varepsilon > 0$, $\alpha_\varepsilon(t) \geqq 0$ for $t > 0$, $\int_0^\infty \alpha_\varepsilon(t)dt = 1$ and such that $\mathrm{Supp}\,[\alpha_\varepsilon] \subset [\varepsilon/2, \varepsilon]$. Let $x \in D(A)$. Then by assumption, we can find a sequence $\{x_n\} \subset D$ such that $x_n \to x$ and $Ax_n \to Ax$ as $n \to \infty$. Therefore, for any $\eta > 0$, there are an integer N and $\varepsilon_0 > 0$ such that $||x_N - x|| < \eta$, $||Ax_N - Ax|| < \eta$ and such that

$$||T(\alpha_{\varepsilon_0})x_N - x_N|| = \left\|\int_0^\infty \alpha_{\varepsilon_0}(t)[U(t)x_N - x_N]dt\right\| \leqq \sup_{0<t\leqq\varepsilon_0} ||U(t)x_N - x_N|| < \eta ,$$

$$||AT(\alpha_{\varepsilon_0})x_N - Ax_N|| = \left\|\int_0^\infty \alpha_{\varepsilon_0}(t)[AU(t)x_N - Ax_N]dt\right\|$$
$$\leqq \sup_{0<t\leqq\varepsilon_0} ||AU(t)x_N - Ax_N|| < \eta .$$

These estimates show that $\mathscr{D}$ is a core of A. Therefore, $D(A) \subset D(\overline{T(-\delta')})$ and hence $A = \overline{T(-\delta')}$.

Conversely, suppose that A is the infinitesimal generator of an R.D.S.G. T. Then, as was stated in Remark 3.1, it is proved in [7; Th. 5.4] and

[8] that there is a real number ω such that $A \in G_1(\omega, k(\tau), \tau)$ for every $\tau > 0$ and for some integer $k(\tau) \geqq 0$. Also, a family $\{U(t); t > 0\}$ of solution operators of ACP_1 for A is constructed on $\bigcap_{n\geqq 1} D(A^n)$ (see Section 2) and the R.D.S.G. T satisfies the relation

$$(4.3) \qquad T(\phi)x = \int_0^\infty \phi(t)U(t)x dt\,, \qquad x \in \bigcap_{n\geqq 1} D(A^n),\ \phi \in D(\boldsymbol{R}_+)\,.$$

Since $\bigcap_{n\geqq 1} D(A^n)$ is a core of A ([7; Lem. 3.6]), this means that the ACP for A is R.D.S.G.-well-posed.

PROOF OF (b). Assume that ACP (5.1) is E.D.S.G.-well-posed. Then by (a), A is the infinitesimal generator of the E.D.S.G.. Conversely, if A is the infinitesimal generator of an E.D.S.G., then by [7; Th. 5.5], $A \in G_2(\omega, k)$ for some ω and integer $k \geqq 0$; hence a family $\{U(t); t > 0\}$ of solution operators ACP_1 for A exists and satisfies (4.3), which means that the ACP for A is E.D.S.G.-well-posed.

PROOF OF (c) is evident from Theorem 4.1 and Remark 4.2. *q.e.d.*

5. Semi-Discrete Difference Approximation. In [12] and [11] the problem of approximation of an abstract Cauchy problem is proposed and the theorems on convergence of (C_0) and (A)-semigroups are applied to the prablem, respectively. In this section we apply our results to this problem. The results are straightforward extensions of [11]. Let us consider in $L_2 = \prod_{i=1}^N L_2(\boldsymbol{R}^d)$ a Cauchy problem with constant coefficients

$$(5.1) \qquad (d/dt)u(t) = Au(t)\,, \qquad u(0) = u_0(x)\,,$$

where $A = P(D)$ is an $N \times N$ matrix whose elements are formal polinomials of $D_k = i(\partial/\partial x_k)$, $k = 1, 2, \cdots, d$, with complex coefficients and $u(t) = u(x, t)$ is an N-dimensional vector. The operator A is supposed to be closed in L_2.

Let us also consider in L_2 the approximating equations

$$(5.2) \qquad (d/dt)u_h(t) = \Pi(\Delta, h)u_h(t)\,, \qquad u_h(0) = u_0(x)\,,$$

where Δ denotes a semi-discrete finite difference scheme (i.e., a set of divided difference approximations to $D^l = D_1^{l_1} \cdots D_d^{l_d}$), $h = (h_j)$, $h_j > 0$, specifies the mesh spacings and $\Pi(\Delta, h)$ is a linear combinations of translation operators depending on Δ and h. For this formulation, we refer to [11].

We then require that the following *consistency condition* holds:

$$\Pi(\Delta, h)u \to Au \text{ as } |h| = \left(\sum_{j=1}^d h_j^2\right)^{1/2} \to 0 \qquad \text{for sufficiently smooth } u\,.$$

The equation (5.2) is called a *semi-discrete difference scheme*.

The problem of semi-discrete approximation of our interest is to find conditions under which the approximate solution $u_h(t)$ of (5.2) converges to the solution $u(t)$ of (5.1) as the mesh size $|h|$ tends to 0.

We consider this problem by applying Fourier transform: Let us denote by $\hat{u}$ the Fourier transform of $u \in L_2$. Then (5.1) and (5.2) are reduced to the following Cauchy problem, respectively,

$$(5.3) \qquad (d/dt)\hat{u}(\xi, t) = P(\xi)\hat{u}(\xi, t) , \qquad \hat{u}(\xi, 0) = \hat{u}_0(\xi) ,$$

$$(5.4) \qquad (d/dt)\hat{u}_h(\xi, t) = A(\xi, \Pi_h)\hat{u}_h(\xi, t) , \qquad \hat{u}_h(\xi, 0) = \hat{u}_0(\xi) .$$

We note that $(\lambda - A)^{-1}$ corresponds to $(\lambda - P(\xi))^{-1}$ and $\|(\lambda - A)^{-1}\| = \sup_\xi |(\lambda - P(\xi))^{-1}|$, when either side is bounded, and that $\exp(tP(\xi))$ corresponds to the solution operator of (5.1).

In the following, we use the centered difference quotient

$$(5.5) \qquad \delta u_i/\delta x_j = (2h_j)^{-1}[u_i(x + h_j e_j) - u_i(x - h_j e_j)]$$

as an approximation of $\partial u_i/\partial x_j$, where e_j denotes the j-th unit vector, and Δ is understood as a semi-discrete finite difference scheme in the sense of (5.5); we write Δ_0 for this scheme. Then $A(\xi, \Pi_h)$ is exactly same as the given matrix $P(\xi)$ with ξ_j replaced by $\xi_j(h) = h_j^{-1}\sin(h_j\xi_j)$.

For this semi-discrete approximation, we obtain the following:

THEOREM 5.1. (a) *Assume that* ACP (5.1) *is R.D.S.G.-well-posed in* L_2, *and let* $D = \{u \in C^\infty; \hat{u}$ *has a compact support*$\}$. *Then for each* $u \in D$ *and* h, *there exists a solution* $u_h(t)$ *of the equation* (5.2) *for* $\Pi(\Delta_0, h)$ *with the initial value* u *and* $u_h(t)$ *converges to the exact solution of* ACP_1 (5.1) *with the same initial value* u, *where the convergence is uniform with respect to* t *on each compact subinterval of* $[0, \infty)$.

(b) *Assume that* ACP (5.1) *is E.D.S.G.-well-posed in* L_2. *Then there is a positive integer* k *and the conclusion of* (a) *holds with* D *replaced by* $D(A^k)$.

PROOF OF (a). Let $\tau > 0$. Then by Theorem 4.3, $A \in G_1(\omega, k(\tau), \tau)$ for some ω and for some nonnegative integer $k(\tau)$. Therefore, it suffices to show that $A(\xi, \Pi_h) = P(\xi(h))$ satisfies the assumptions of Corollary 2.12. Noting that $|\xi(h)| \leqq (\sum_{j=1}^d h_j^{-2})^{1/2}$, we have the estimates

$$(5.6) \qquad \sup_\xi |(\lambda - P(\xi(h)))^{-1}| \leqq \sup_\xi |(\lambda - P(\xi))^{-1}| = \|(\lambda - A)^{-1}\| ,$$

$$(5.7) \qquad \begin{aligned} &\sup_\xi |\lambda^m(\lambda - P(\xi(h)))^{-m}(\lambda_0 - P(\xi(h)))^{-k(\tau)}| \\ &\leqq \sup_\xi |\lambda^m(\lambda - P(\xi))^{-m}(\lambda_0 - P(\xi))^{-k(\tau)}| \\ &= \|\lambda^m(\lambda - A)^{-m}(\lambda_0 - A)^{-k(\tau)}\| \end{aligned}$$

for any real numbers λ, λ_0 and integer $m \geqq 1$ such that the right hand sides are bounded. Since $A \in G_1(\omega, k(\tau), \tau)$ the right hand sides of (5.6) and (5.7) are meaningful and independent of h, for $\lambda, \lambda_0 > \omega$ and $m \geqq 1$ with $m/\lambda \in (0, \tau)$; hence $A(\xi, \Pi_h)$ satisfies conditions (I) and (II) of Corollary 2.12. On the other hand, by the consistency condition and the fact that D is a core of A, it is obvious that $A(\xi, \Pi_h)$ satisfies condition (III′) and the additional assumption of Corollary 2.12. Therefore, Corollary 2.12 is applicable and the conclusion holds.

PROOF OF (b). By Theorem 4.3, there are a real number ω and a nonnegative integer l such that $A \in G_2(\omega, l)$. It is then evident that the conclusion of (a) holds. We see that the space D of initial functions can be extended to $D(A^{l+1})$, by observing that

$$(5.8) \quad \sup_\xi |\exp(tP(\xi(h))(\lambda_0 - P(\xi(h))^{-l}| \leqq \sup_\xi |\exp(tP(\xi))(\lambda_0 - P(\xi))^{-l}|$$

for a fixed $\lambda_0 > \omega$. We have the assertion by setting $k = l + 1$. *q.e.d.*

THEOREM 5.2. *Assume that* ACP (5.1) *is* $(C_{(k)})$*-well-posed in* L_2. *Then for each* $u \in D(A^{k+1})$ *(resp.* $u \in D(A^k)$*) and* h*, there exists a solution* $u_h(t)$ *of the equation* (5.2) *for* $\Pi(\Delta_0, h)$ *with the initial value* u *and* $u_h(t)$ *converges to the exact solution of* ACP_1 (5.1) *(resp.* ACP_2 (5.1)*) with the same initial value* u*, where the convergence is uniform with respect to* t *on each compact subinterval of* $[0, \infty)$ *(resp.* $(0, \infty)$*).*

PROOF. The first assertion is evident from Theorem 5.1 (b). In order to prove the second assertion, it is sufficient to show that $A(\xi, \Pi_h) = P(\xi(h))$ satisfies the assumption of Corollary 3.8. It is proved in the same way as in the proof of Theorem 5.1 (a) that conditions (I) and (III′) are satisfied. Condition (II_{exp}) is proved by (5.8) and condition (IV) is evident from the estimate

$$\sup_\xi |\exp(tP(\xi(h)))| \leqq \sup_\xi |\exp(tP(\xi))| \qquad \text{for } t > 0 . \qquad q.e.d.$$

We conclude this section by exhibiting some simple examples of well-posed Cauchy problems.

EXAMPLE 5.3. Let us consider a Cauchy problem (5.1)

$$(d/dt)u(t) = Au(t) , \qquad u(0) = u_0$$

in $L_2 = L_2(\boldsymbol{R}) \times L_2(\boldsymbol{R})$ such that

$$P(\xi) = \begin{pmatrix} -\xi^2 & \xi^3 \\ 0 & -1 + i\xi \end{pmatrix} .$$

If $\mathrm{Re}(\lambda) > 0$, then

$$(\lambda - P(\xi))^{-1} = \begin{pmatrix} 1/(\lambda + \xi^2) & \xi^3/(\lambda + \xi^2)(\lambda + 1 - i\xi) \\ 0 & 1/(\lambda + 1 - i\xi) \end{pmatrix}$$

and $\|R(\lambda; A)\| = \sup_\xi |(\lambda - P(\xi))^{-1}| < +\infty$. Therefore, $\{\lambda; \mathrm{Re}(\lambda) > 0\} \subset \rho(A)$. Now this Cauchy problem is not S.G.-well-posed. In fact, we have

$$\exp(tP(\xi)) = \begin{pmatrix} \exp(-t\xi^2) & \xi^3[\exp(t(-1+i\xi)) - \exp(-t\xi^2)]/(\xi^2-1+i\xi) \\ 0 & \exp(t(-1+i\xi)) \end{pmatrix},$$

hence $|\exp(tP(\xi))|$ is not bounded with respect to ξ for any $t > 0$. However, $\exp(tP(\xi))$ is decomposed as

$$\exp(tP(\xi)) = c_1(\xi, t)E + c_2(\xi, t)P(\xi),$$

where E is a 2×2 unit matrix and

$$c_1(\xi, t) = [\xi^2 \exp(t(-1 + i\xi)) + (-1 + i\xi)\exp(-t\xi^2)]/(\xi^2-1 + i\xi),$$
$$c_2(\xi, t) = [\exp(t(-1 + i\xi)) - \exp(-t\xi^2)]/(\xi^2 - 1 + i\xi).$$

It is easy to see that $\sup_\xi |c_j(\xi, t)| \leqq M$, $j = 1, 2$, for some constant $M > 0$. Therefore,

$$\|\exp(tP(\cdot))\hat{u}(\cdot)\| \leqq M(\|\hat{u}(\cdot)\| + \|P(\cdot)\hat{u}(\cdot)\|) = M(\|u\| + \|Au\|) \tag{5.9}$$

for $u \in D(A)$ and $t > 0$. On the other hand, for each ξ,

$$\widehat{R(\lambda; A)u}(\xi) = (\lambda - P(\xi))^{-1}\hat{u}(\xi) = \int_0^\infty e^{-\lambda t} \exp(tP(\xi))\hat{u}(\xi)dt$$

and so,

$$\begin{aligned} \widehat{R(\lambda; A)^m u}(\xi) &= (\lambda - P(\xi))^{-m}\hat{u}(\xi) \\ &= \frac{1}{(m-1)!}\int_0^\infty t^{m-1}e^{-\lambda t} \exp(tP(\xi))\hat{u}(\xi)dt, \end{aligned} \tag{5.10}$$

for $u \in L_2$, $\lambda > 0$ and $m \geqq 1$. Combining (5.9) and (5.10), we obtain

$$\|\lambda^m R(\lambda; A)^m u\| = \|\widehat{\lambda^m R(\lambda; A)^m u}\| = \|\lambda^m(\lambda - P(\cdot))^{-m}\hat{u}(\cdot)\| \leqq M(\|u\| + \|Au\|)$$

for $u \in D(A)$, $\lambda > 0$ and $m \geqq 1$. This means that $A \in G_2(0, 1)$ and hence this Cauchy problem is E.D.S.G.-well-posed.

EXAMPLE 5.4. Let us consider a Cauchy problem (5.1) in $L_2 = L_2(\boldsymbol{R}) \times L_2(\boldsymbol{R})$ such that

$$P(\xi) = \begin{pmatrix} -\xi^2 + i\xi^4 & \xi^q \\ 0 & -\xi^2 + i\xi^4 \end{pmatrix}.$$

This Cauchy problem is the one treated in [11], and it is shown that the Cauchy problem is S.G.-well-posed for any integer $q \geqq 0$. More precisely, it is shown that it is (C_0)-well-posed if $0 \leqq q \leqq 2$, $(1, A)$-well-posed if $q = 3$, $(0, A)$-well-posed if $q = 4$ and is not even (A)-well-posed if $q \geqq 5$. We then show that this Cauchy problem is $(C_{(1)})$-well-posed if $3 \leqq q \leqq 8$. (Note that if $q > 8$, then $\rho(A)$ is empty). Let $\{T(t); t > 0\}$ be a semigroup of the solution operators of the Cauchy problem. Then it is proved in [11] that $\{T(t); t > 0\}$ satisfies conditions (S_1) and (S_2) (which are stated in Section 1) and also that $D(A)$ is contained in the continuity set of $\{T(t); t > 0\}$. Therefore, $\{T(t); t > 0\}$ is a $(C_{(1)})$-semigroup; hence this Cauchy problem is $(C_{(1)})$-well-posed if $3 \leqq q \leqq 8$.

EXAMPLE 5.5. Finally, let us consider a Cauchy problem (5.1) in $L_2 = L_2(\boldsymbol{R}) \times L_2(\boldsymbol{R}) \times L_2(\boldsymbol{R})$ such that

$$P(\xi) = pE + qF\,, \qquad p = -\xi^2 + i\xi^8, q = \xi^8 + 1\,,$$

where E denotes a 3×3 unit matrix and F denotes a 3×3 nilpotent matrix such that only upper off-diagonal elements are 1.

By simple calculations,

$$(\lambda - P(\xi))^{-1} = \sum_{i=0}^{2} (\lambda - p)^{-i-1} q^i F^i\,, \tag{5.11}$$

$$\exp(tP(\xi)) = e^{pt} \sum_{i=0}^{2} (i!)^{-1} t^i q^i F^i\,, \tag{5.12}$$

where $F^0 = E$. It is easy to see that $\{\lambda; \operatorname{Re}(\lambda) > 0\} \subset \rho(A)$ and $\|R(\lambda; A)\| \leqq M$ for λ with $\operatorname{Re}(\lambda) > 0$ and for some constant $M > 0$. Also, it is seen from (5.12) that $\sup_\xi |\exp(tP(\xi))| < +\infty$ for $t > 0$, so that this Cauchy problem is S.G.-well-posed.

Next we shall show that this Cauchy problem is neither $(0, A)$- nor $(1, A)$-well-posed, but (A)-well-posed. In order to show (A)-well-posedness, it is sufficient to show that $\|\lambda R(\lambda; A)\| \leqq M$ as $\lambda \to +\infty$, for some constant $M > 0$; other conditions for (A)-well-posedness can be shown by the same argument as in [11]. But this is evident from (5.11) and this Cauchy problem is (A)-well-posed. By the way, let us consider an element $u_0 \in L_2$ such that

$$\hat{u}_0 = (p-1)^{-1} r^{-1} e_3\,, \qquad r = \xi^4 + 1\,,$$

where $e_j (j = 1, 2, 3)$ denotes a unit vector whose j-th element is 1. Then

$$[P(\xi) - E]\hat{u}_0 = (p-1)^{-1} r^{-1} q e_2 + r^{-1} e_3\,,$$
$$[P(\xi) - E]^2 \hat{u}_0 = (p-1)^{-1} r^{-1} q^2 e_1 + 2r^{-1} q e_2 + (p-1) r^{-1} e_3\,.$$

This means that $u_0 \in D(A) \backslash D(A^2)$, where $D(A^k) = \{u \in L_2; P(\cdot)^k \hat{u}(\cdot) \in L_2\}$.

Noting that for some constant $C > 0$,

$$|q^2(p-1)^{-1}r^{-1}|^2 = (\xi^8+1)^4(\xi^4+1)^{-2}((\xi^2+1)^2+\xi^{16})^{-1} \geqq C\xi^8 \,,$$

we obtain

$$\begin{aligned} &\|\exp(tP(\cdot))\hat{u}_0(\cdot)\|^2 \\ &\geqq \frac{t^4}{4}\int_{-\infty}^{+\infty}(\xi^8+1)^4(\xi^4+1)^{-2}((\xi^2+1)^2+\xi^{16})^{-1}\exp(-2t\xi^2)d\xi \\ &\geqq \text{const. } t^4\int_{-\infty}^{+\infty}\xi^8\exp(-2t\xi^2)d\xi \,. \end{aligned}$$

Setting $2t\xi^2 = \sigma^2$, we see that the above right hand side is equal to

$$\text{const. } t^4\int_{-\infty}^{+\infty}(2t)^{-4-(1/2)}\sigma^8\exp(-\sigma^2)d\sigma = \text{const. } t^{-(1/2)} \,.$$

This means that $\|\exp(tP(\cdot)\hat{u}(\cdot)\|$ blows up at $t=0$. Therefore, the semigroup $\{T(t); t>0\}$ of the solution operators of this Cauchy problem is not a $(C_{(1)})$-semigroup, a fortiori, is neither $(0, A)$- nor $(1, A)$-semigroup. Hence, this Cauchy problem is $(C_{(2)})$-well-posed (in fact, (A)-well-posed), though it is not $(C_{(1)})$-well-posed. (Recall that $(1, A) \subset (0, A) \subset (C_{(1)})$ and $(A) \subset (C_{(2)})$.)

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