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Approximation of the Fourier Transform and the Dual Gabor Window

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ABSTRACT. Many results and problems in Fourier and Gabor analysis are formulated in the continuous variable case, i.e., for functions on $\mathbb{R}$. In contrast, a suitable setting for practical computations is the finite case, dealing with vectors of finite length. We establish fundamental results for the approximation of the continuous case by finite models, namely, the approximation of the Fourier transform and the approximation of the dual Gabor window of a Gabor frame. The appropriate function space for our approach is the Feichtinger space S_0. It is dense in L^2, much larger than the Schwartz space, and it is a Banach space.

1. Introduction

1.1 Approximating the Fourier Transform

The Fourier transform $\widehat{f}$ of an integrable function f on $\mathbb{R}$ is obtained by the Fourier integral

$$\widehat{f}(\omega) = \int_{\mathbb{R}} f(t) e^{-2\pi i \omega t}\, dt, \qquad \omega \in \mathbb{R}\,. \tag{1.1}$$

For suitable functions f the integral can be approximated as a Riemann sum from sufficiently dense samples. If f decays well, then already a finite number of such samples yields a good approximation of $\widehat{f}(\omega)$, that is, the problem reduces to computations in $\mathbb{C}^n$. In fact, it is essentially the Fourier transform in $\mathbb{C}^n$ which arises in these computations, so fast algorithms based on the Cooley-Tukey FFT can be used. Results on approximating $\widehat{f}$ in this way are found, e.g., in [1, 3, 5, 12].

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In contrast to these established facts, we show how to obtain the approximation of $\widehat{f}$ even as a function on all of $\mathbb{R}$, not just pointwise, from using the Fourier transform in $\mathbb{C}^n$. Our results hold for arbitrary functions from the Feichtinger space S_0, described in Section 1.3, which is dense in the space L^2 of square-integrable functions on $\mathbb{R}$. Indeed, we obtain the convergence to $\widehat{f}$ in S_0, which in particular implies L^p-convergence on both the time and the Fourier side, for all $1 \le p \le \infty$.

1.2 The Dual Window of a Gabor Frame

A Gabor system is a family of functions

$$G(g, a, b) = \{g_{k,l} \colon k, l \in \mathbb{Z}\} \subset L^2 ,$$

obtained from time-frequency shifts of a Gabor window $g \in L^2$ along a time-frequency lattice, i.e.,

$$g_{k,l}(t) = g(t - k\,a)e^{2\pi i\,t\,l\,b}, \qquad t \in \mathbb{R} ,$$

for $k, l \in \mathbb{Z}$, where $a, b > 0$ are the lattice constants. A useful framework in Gabor analysis is the notion of a frame in Hilbert space, which is a more general concept than a Riesz basis, see [9]. The family $\{g_{k,l} \colon k, l \in \mathbb{Z}\}$ is a frame if there exist constants $A, B > 0$ such that

$$A\,\|f\|^2 \le \sum_{k,l\in\mathbb{Z}} |\langle f, g_{k,l}\rangle|^2 \le B\,\|f\|^2, \qquad \text{for all} \quad f \in L^2 . \tag{1.2}$$

For any frame there exists a dual frame, which plays a similar role in frame theory as the biorthogonal system of a Riesz basis. The (canonical) dual frame is given by $\{S^{-1}g_{k,l} \colon k, l \in \mathbb{Z}\}$, where S denotes the frame operator

$$Sf = \sum_{k,l\in\mathbb{Z}} \langle f, g_{k,l}\rangle g_{k,l}, \qquad f \in L^2 .$$

We note that S is bounded and invertible if and only if the frame condition (1.2) holds. The special structure of Gabor frames yields that the dual frame is again a Gabor frame, namely, its elements are of the form $S^{-1}g_{k,l} = \widetilde{g}_{k,l}$, for $k, l \in \mathbb{Z}$, where

$$\widetilde{g} = S^{-1}g .$$

The function $\widetilde{g} \in L^2$ is called the (canonical) dual Gabor window and it is the focus of many results and questions in Gabor analysis. For details on Gabor systems we refer to [24, 25, 28], see also [8, 13, 31, 33, 40], [9, Sections 8-9], and [13, Chapters 3-4]. The analogous notations of a Gabor system are defined for vectors in $\mathbb{C}^n$, see Section 2.4. We are concerned with the approximation of the dual Gabor window $\widetilde{g}$ for a given Gabor frame in L^2 by using only computations in $\mathbb{C}^n$.

Since the dual window $\widetilde{g}$ is obtained from inverting the Gabor frame operator S, one is interested in approximations for S^{-1}. We mention the frame algorithm, based on Neumann series expansions for S^{-1}, see [9, Section 1.2], [28, Algorithm 5.1.1]; iterative methods with higher order of convergence are discussed in [36]. However, our focus is different, we are concerned with approximations by finite-dimensional computations. A brief summary of developments in this direction is included in [15] and we refer to the list of references therein. A fundamental general approach is the Casazza-Christensen double projection method [6, 7], see [9, Section 16.2]. By this technique the inverse frame operator of an arbitrary frame can be approximated from a reduced problem in a finite-dimensional subspace of L^2. For the application of this method to Gabor frames see [9, Sections 16.3 and 16.4], [10, Section 8.3.2].

The Casazza-Christensen method is a technique for general frames. Making use of the special structure of Gabor frames, Strohmer has established a practicable method specific to approximating the dual Gabor window $\widetilde{g}$ [42], see [9, Section 16.5], [10, Section 8.3.3]. The Strohmer method indeed reduces the approximation problem in L^2 to solving a system of linear equations in $\mathbb{C}^n$. We mention that, if g and its Fourier transform $\widehat{g}$ have exponential decay, then this approach also describes the convergence behavior, namely, the approximation error in L^2 decreases also exponentially. Strohmer's method involves a general approximation scheme, the finite section method, and for more on this topic we refer to [11], [30, Section 1.1.3].

Our technique is a new approach, based on the results by Janssen in [34] which describe the transition from continuous to finite Gabor frames. We complement these results by describing the converse direction and, thus, we show how the dual Gabor window can indeed be approximated from algorithms for Gabor frames in $\mathbb{C}^n$. The advantage is that the dual window of a Gabor frame in $\mathbb{C}^n$ can be computed considerably faster than solving a general system of linear equations in $\mathbb{C}^n$, see [41]. These Gabor frames in $\mathbb{C}^n$ are obtained in a simple way from the original Gabor frame. The only assumption for our method is that the Gabor window g belongs to the Feichtinger space $S_0 \subset L^2$, described below. As a bonus, the approximate dual windows converge not just in L^2 but indeed in S_0. We note that convergence of the (dual) Gabor window in S_0 has an important implication which does not hold for just convergence in L^2. Namely, it implies convergence of the corresponding frame operators in the operator norm on L^2, see [19, Corollary 2.3], [26, Corollary 3.3.3 (i) (b)].

1.3 The Feichtinger Space

The Feichtinger space S_0 is the appropriate space for many results in time-frequency analysis. It was introduced in [17] as a new Segal algebra, cf. [39]. S_0 is dense in L^2 and all its members are continuous and integrable functions. In general terms, S_0 is the space of all functions on $\mathbb{R}$ which are represented in the time-frequency domain by an integrable function, cf. [18, Theorem 15]. More precisely, the norm of a function f in S_0 is the L^1-norm of its short-time Fourier transform $V_g f$ with respect to the Gaussian window g, i.e.,

$$\|f\|_{S_0} = \iint_{\mathbb{R}^2} |V_g f(x, \omega)|\, dx\, d\omega \,,$$

where $g(t) = e^{-\pi t^2}$, and the short-time Fourier transform is defined by

$$V_g f(x, \omega) = \int_{\mathbb{R}} f(t)\, \overline{g(t-x)}\, e^{-2\pi i\, \omega\, t}\, dt, \qquad x, \omega \in \mathbb{R} \,.$$

An equivalent norm is obtained when the Gaussian function g is replaced by an arbitrary non-zero function g from S_0. Examples are the triangle function, the trapezoidal function, or any Schwartz function. If $g \in S_0$ generates a Gabor frame for L^2 as described in Section 1.2, then an equivalent discrete norm for S_0 is the ℓ^1-norm of the Gabor coefficients of a given function f,

$$\|f\|'_{S_0} = \sum_{k,l \in \mathbb{Z}} |\langle f, g_{k,l} \rangle| \,.$$

The space S_0 shares several properties with the Schwartz space $\mathcal{S}$, yet S_0 is much larger, it does not rely on differentiability, and it is a Banach space. Time-frequency shifts and the Fourier transform are isometries on S_0. Feichtinger has obtained S_0 from several different concepts, namely, S_0 coincides with the modulation space M^1 and with the Wiener amalgam space $W(A, \ell^1)$, whose local component is the Fourier algebra $A = \mathcal{F}L^1$. For more on S_0, see [18, 19, 26, 28] and the original source [17].

The space S_0 is defined on $\mathbb{R}^d$, for $d = 1, 2, \ldots$, we restrict the presentation of our results to the case $d = 1$; the general case is mentioned in Section 6. Practicable sufficient conditions for membership in S_0 are given in [26, Theorem 3.2.17] and [27]. For instance, if a function $f(t)$ on $\mathbb{R}$ and its Fourier transform decay like $\mathcal{O}(|t|^{-3/2-\varepsilon})$, for some $\varepsilon > 0$, then $f \in S_0$. Another useful sufficient condition for f to be in $S_0 = S_0(\mathbb{R})$ is that f, f' and f'' are in L^1 [37]. Yet a function in S_0 need not be differentiable. For example, a compactly supported function is in S_0 if and only if its Fourier transform is integrable. Correspondingly, because of the Fourier invariance of S_0, any integrable band-limited function is in S_0.

The article is arranged as follows. In Section 2 we state the main results. The relevant properties of the function space S_0 are summarized in Section 3. In Section 4 we formulate crucial steps of our approach as preliminary results. The proofs of the main results are found in Section 5 and, finally, in Section 6 we briefly comment on the case of functions on $\mathbb{R}^d$, for $d \geq 2$.

2. The Main Results

2.1 The Sampling and Reconstruction Operators R_n and L_n

We establish the definitions which are used for formulating our main results. First, given $n \in \mathbb{N}$, we define a set of n points on the real line which we will later use as sampling points. For $t \in \mathbb{R}$, let $\lfloor t \rfloor$ denote the greatest integer less or equal to t, and let $\lceil t \rceil$ denote the least integer greater or equal to t.

Definition 1. Given $n \in \mathbb{N}$, define $\tau_0, \ldots, \tau_{n-1} \in \mathbb{R}$ by

$$\tau_k = \begin{cases} k/\sqrt{n}, & \text{for} \quad k = 0, \ldots, \lfloor \frac{n}{2} \rfloor\,, \\ (k-n)/\sqrt{n}, & \text{for} \quad k = \lfloor \frac{n}{2} \rfloor + 1, \ldots, n-1\,. \end{cases}$$

For example, we have for $n = 16$,

$$(\tau_0, \ldots, \tau_{15}) = \tfrac{1}{4}(0, 1, \ldots, 7, 8, -7, \ldots, -1)\,.$$

Remark 1. The points $\tau_0, \ldots, \tau_{n-1}$ are inside the interval $[-\sqrt{n}/2, \sqrt{n}/2]$ and they are regularly spaced by $1/\sqrt{n}$. Consequently, for $n \to \infty$, the diameter of this set of points and the local density increase simultaneously.

Next, given $n \in \mathbb{N}$, we define a set of n real numbers which we will later use as weights for a set of sampling values.

Definition 2. Given $n \in \mathbb{N}$, define $\lambda_0, \ldots, \lambda_{n-1} \in \mathbb{R}$ by

$$\lambda_k = \begin{cases} 1, & \text{for} \quad k = 0, \ldots, \lfloor \frac{n}{4} \rfloor \quad \text{and} \quad k = \lceil \frac{3n}{4} \rceil, \ldots, n-1\,, \\ |\frac{n}{2} - k|/\frac{n}{4}, & \text{for} \quad \lfloor \frac{n}{4} \rfloor + 1, \ldots, \lceil \frac{3n}{4} \rceil - 1\,. \end{cases}$$

For example, we have for $n = 16$,

$$(\lambda_0, \dots, \lambda_{15}) = \left(1, 1, 1, 1, 1, \tfrac{3}{4}, \tfrac{1}{2}, \tfrac{1}{4}, 0, \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{3}{4}, 1, 1, 1, 1\right) .$$

Remark 2. (i) The numbers $\lambda_0, \dots, \lambda_{n-1}$ can be viewed as the values of a trapezoidal function w_n at the sampling points $\tau_0, \dots, \tau_{n-1}$. In fact, let

$$w(t) = \begin{cases} 1, & |t| \le 1/4 \,, \\ (2 - 4|t|), & 1/4 < |t| < 1/2 \,, \\ 0, & |t| \ge 1/2 \,, \end{cases} \qquad t \in \mathbb{R} \,,$$

and define $w_n(t) = w(t/\sqrt{n})$, for $n \in \mathbb{N}$. Then we have

$$\lambda_k = w_n(\tau_k), \qquad \text{for} \quad k = 0, \dots, n-1 \,. \tag{2.1}$$

For later use we also note that (2.1) implies

$$\sum_{k=0}^{n-1} \lambda_k f(\tau_k) = \sum_{l \in \mathbb{Z}} w_n\big(l/\sqrt{n}\big)\, f\big(l/\sqrt{n}\big), \qquad f \in S_0 \,. \tag{2.2}$$

Observe that the summation is finite also at the right-hand side of (2.2), since w_n is compactly supported.
(ii) Our results also hold, when $\lambda_0, \dots, \lambda_{n-1}$ are defined as the values of a function $w_n(t) = w(t/\sqrt{n})$ as in (i), for an arbitrary function w in S_0 with the properties $w(0) = 1$ and $\operatorname{supp} w \subseteq [-1/2, 1/2]$.

Now we define the key players of our results, an operator R_n which maps a function from S_0 into a vector in $\mathbb{C}^n$ and, for the converse direction, an operator L_n which maps a vector into a function. It will be convenient to consider the vectors v in $\mathbb{C}^n$ as functions on the cyclic group $\mathbb{Z}_n$ and write $v = \big(v(0), \dots, v(n-1)\big)$. In particular, computations with the indices of such a vector will always be understood modulo n. We also mention that all norms on $\mathbb{C}^n$ are equivalent and $L^2(\mathbb{Z}_n) = S_0(\mathbb{Z}_n) = \mathbb{C}^n$.

Definition 3. (i) Define the operator $R_n \colon S_0 \to \mathbb{C}^n$ by

$$R_n f(k) = \lambda_k f(\tau_k), \qquad k = 0, \dots, n-1 \,. \tag{2.3}$$

(ii) Define the operator $L_n \colon \mathbb{C}^n \to S_0$ by

$$L_n v(t) = \sum_{k=0}^{n-1} \lambda_k\, v(k)\, \varphi\big((t - \tau_k)\sqrt{n}\big), \qquad t \in \mathbb{R} \,, \tag{2.4}$$

where $\varphi(t) = (1 - |t|)_+$ is the linear B-spline or roof function.

Remark 3. (i) The operator R_n amounts to sampling the function f at the points $\tau_0, \dots, \tau_{n-1} \in [-\sqrt{n}/2, \sqrt{n}/2]$. The coefficients $\lambda_0, \dots, \lambda_{n-1}$ are used as weights for the sampling values. The use of the weights avoids a sharp jump from $v(n-1)$ to $v(0)$, when v is viewed circularly as a function on $\mathbb{Z}_n$. A priori such a jump is possible since $f(\tau_0)$ need not be close to $f(\tau_{n-1})$. In some cases the weights can be omitted, see Remark 5 (i).
(ii) Applying L_n to a vector $v \in \mathbb{C}^n$ amounts to linear interpolation of the data

$$\{(\tau_0, y_0), \dots, (\tau_{n-1}, y_{n-1})\}, \qquad \text{where} \quad y_k = \lambda_k v(k), \quad k = 0, \dots, n-1 \,.$$

It yields a function on $\mathbb{R}$ with support slightly larger than $[-\sqrt{n}/2, \sqrt{n}/2]$. The use of the weights prevents the interpolator function from having sharp transitions at the endpoints of the supporting interval. For the possibility of using L_n without weights, see Remark 5 (ii).

By $\delta_{k,0}$ we denote the Kronecker delta, equal to one when $k = 0$, and equal to zero otherwise.

Remark 4. The function φ used in the definition of L_n is not differentiable. However, if smoothness of the approximants is desired, φ can be replaced by higher order B-splines. More generally, we can use an arbitrary function $\varphi \in S_0$ which generates a partition of unity, $\sum_{k \in \mathbb{Z}} \varphi(x - k) = 1$, $x \in \mathbb{R}$, or, equivalently, which satisfies $\widehat{\varphi}(k) = \delta_{k,0}$, for $k \in \mathbb{Z}$. In approximation theory this is the basic Strang-Fix condition and we note that for our purpose no higher order Strang-Fix condition is required for φ.

2.2 Approximation from a Vector of Samples

Our first main result describes the reconstruction of a function $f \in S_0$ from the sequence of vectors $\{v_n \in \mathbb{C}^n : n = 1, 2, \ldots\}$, obtained by sampling f according to the definition of the operator $R_n : S_0 \to \mathbb{C}^n$. The reconstruction is described by the operator $L_n : \mathbb{C}^n \to S_0$.

Theorem 1. *Suppose that $f \in S_0$. For $n \in \mathbb{N}$, define the vector $v_n = R_n f \in \mathbb{C}^n$. Then $\|L_n v_n - f\|_{S_0} \to 0$, as $n \to \infty$.*

Remark 5. (i) In the proof of Theorem 1, the coefficients $\lambda_0, \ldots, \lambda_{n-1}$ in the definition of R_n will not be used explicitly. Their use is implicit and corresponds to approximating f by a compactly supported function f_n in S_0. If already f itself is compactly supported, so that $\operatorname{supp} f \subset (-\sqrt{n}/2, \sqrt{n}/2)$ for sufficiently large n, then the coefficients can be omitted for R_n.

(ii) The coefficients $\lambda_0, \ldots, \lambda_{n-1}$ in the definition of L_n are required for obtaining convergence in S_0 in Theorem 1. For just convergence in L^p, $1 \le p \le \infty$, the coefficients can be omitted for L_n. Note that convergence in L^p is considered here for $f \in S_0$ and not for general L^p-functions.

2.3 Approximation of the Fourier Transform

The operators R_n and L_n are defined in such a way that they allow us to obtain our second main result. We show that the Fourier transform of a function from S_0 can be approximated in the S_0-norm from computing the Fourier transform of a vector in $\mathbb{C}^n$. This vector is determined by finitely many samples of f.

Recall the normalization of the Fourier transform, given in (1.1). The Fourier transform on $\mathbb{C}^n$ is normalized as a unitary operator $v \mapsto \widehat{v}$ in $\mathbb{C}^n$,

$$\widehat{v}(k) = \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} v(k) e^{-2\pi i\, k l / n}, \qquad k = 0, \ldots, n-1 .$$

It is usually denoted discrete Fourier transform (DFT) and can be calculated efficiently by fast Fourier transform (FFT) algorithms.

Theorem 2. *Suppose that $f \in S_0$. For $n \in \mathbb{N}$, define the vector $v_n = R_n f \in \mathbb{C}^n$. Let $\widehat{f}$ denote the Fourier transform of f in S_0, and let $\widehat{v_n}$ denote the Fourier transform of v_n in $\mathbb{C}^n$. Then $\|L_n \widehat{v_n} - \widehat{f}\|_{S_0} \to 0$ as $n \to \infty$.*

Recall that convergence in S_0 implies convergence in L^p for all $1 \le p \le \infty$.

Remark 6. The possible simplification of the operators R_n and L_n indicated in Remark 5 can be applied to Theorem 2 as well.

2.4 Approximation of the Dual Gabor Window

Here, we obtain a method for the approximation of the dual window of a Gabor frame using finite Gabor methods. First, we recall the relevant notions for Gabor systems in $\mathbb{C}^n$, see, e.g., [41], or [9, Chapter 10]. Let $n \in \mathbb{N}$. Given a Gabor window vector $v \in \mathbb{C}^n$ and divisors $p, q \in \mathbb{N}$ of n, we define the Gabor system

$$G^{(n)}(v, p, q) = \left\{v_{k,l} \colon k = 0, \ldots, \tfrac{n}{p} - 1,\ l = 0, \ldots, \tfrac{n}{q} - 1\right\} \subset \mathbb{C}^n ,$$

where

$$v_{k,l}(m) = v(m - k\, p)\, e^{2\pi i\, l\, q\, m/n}, \qquad m = 0, \ldots, n-1 ,$$

for $k = 0, \ldots, \frac{n}{p} - 1$ and $l = 0, \ldots, \frac{n}{q} - 1$. The system $G^{(n)}(v, p, q)$ is a Gabor frame for $\mathbb{C}^n$ if it spans $\mathbb{C}^n$. This is the case if and only if the frame operator $S^{(n)}_{v,p,q} \colon \mathbb{C}^n \to \mathbb{C}^n$, defined by

$$S^{(n)}_{v,p,q} u = \sum_{k=0}^{n/p-1} \sum_{l=0}^{n/q-1} \langle u, v_{k,l}\rangle v_{k,l}, \qquad u \in \mathbb{C}^n ,$$

is invertible. Then the dual vector is defined by $\widetilde{v} = \left(S^{(n)}_{v,p,q}\right)^{-1} v \in \mathbb{C}^n$, in analogy to the dual window for a Gabor frame in L^2.

Before we state the theorem, the following remark indicates how we narrow our focus from the case of general lattice constants $a, b > 0$ to the case $a = b < 1$.

Remark 7. (i) It is one of the fundamental results in Gabor analysis that, if the Gabor system $G(g, a, b)$ is a frame, then $ab \le 1$, see [28, Section 7.5]. Furthermore, if the Gabor window g is well time-frequency localized, then we must have $ab < 1$, as implied by the Balian-Low theorem, see [28, Section 8.4]. A variant of this theorem, found in [4], implies that we have $ab < 1$ also for all Gabor frames with a Gabor window g from S_0. Hence, for our results we can assume that $ab < 1$.

(ii) By dilating the Gabor window one can reduce the investigation of general Gabor systems $G(g, a, b)$ to the case $a = b$. Namely, given $G(g, a, b)$, let $c = \sqrt{a/b}$ and define the dilated function $g_c(t) = g(c\,t)$, $t \in \mathbb{R}$. Then $G(g, a, b)$ is a Gabor frame if and only if $G(g_c, c, c)$ is a Gabor frame. In this case, the dual window $\widetilde{g}$ for $G(g, a, b)$ is obtained by $\widetilde{g}(t) = \widetilde{g_c}(c\,t)$, $t \in \mathbb{R}$, where $\widetilde{g_c}$ is the dual window for $G(g_c, c, c)$. Consequently, the fact that S_0 is invariant under dilation [26, Theorem 3.2.14] allows us to restrict our attention to the case $a = b$.

(iii) We mention that the reduction in (ii) can be formulated for arbitrary time-frequency lattices $\Lambda \subset \mathbb{R}^2$, more general than $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$, by replacing the dilation in (ii) with the suitable metaplectic transform, see [28, Proposition 9.4.4 and p. 198, Remark 2]. Therefore, since S_0 is invariant under metaplectic transforms [17], our results can also be useful for the case of a hexagonal, or a quincunx lattice, for example. In the setting of functions on $\mathbb{R}^d$, $d \ge 2$, mentioned in Section 6, this remark applies to all lattices $\Lambda \subset \mathbb{R}^{2d}$ which are symplectic [28, Definition 9.4.2].

In view of Remark 7, we restrict the presentation of our results to the case of Gabor frames of the special form $G(g, a, a)$ with $0 < a < 1$, having indicated how they can be used for arbitrary $a, b > 0$.

Theorem 3. *Given $g \in S_0$ and $0 < a < 1$, suppose that $G(g, a, a)$ is a Gabor frame for L^2. For $q \in \mathbb{N}$, let $p = \lceil a^2 q \rceil$, let $n = pq$, and define the vector $v_n = R_n g \in \mathbb{C}^n$. Then the following hold.*

(i) *For all q sufficiently large, the system $G^{(n)}(v_n, p, p)$ is a Gabor frame for $\mathbb{C}^n$.*

(ii) *Denote the dual window of $G(g, a, a)$ by $\widetilde{g}$, which is in S_0, and denote the dual vector of $G^{(n)}(v_n, p, p)$ by $\widetilde{v_n}$ in $\mathbb{C}^n$. Then $\|\sqrt{n} L_n \widetilde{v_n} - \widetilde{g}\|_{S_0} \to 0$ as $q \to \infty$.*

We note that in Theorem 3 the parameters n and p depend on q.

Remark 8. (i) Observe that $p/q \to a^2$ and $n = pq \to \infty$, as $q \to \infty$ in Theorem 3. We mention that for any sequence of integers p and q with these properties, the theorem holds as well. The assignment suggested in the theorem is applicable in practical situations. That is, by varying q one indeed obtains useful values of n for actual computations and with $\sqrt{p/q}$ close to a.

(ii) The possible simplification of the operators R_n and L_n mentioned in Remark 5 can be applied also to Theorem 3.

3. Relevant Properties of S_0

For the definition of the Feichtinger space S_0, see Section 1.3. The space S_0 is the appropriate window class for time-frequency analysis [28, Section 12.1]. Our work utilizes a variety of features of S_0, summarized next.

Lemma 1 ([17, 26]). (i) *The space S_0 is a Banach algebra both under pointwise multiplication and under convolution. In particular, S_0 is closed under these operations.*

(ii) *Time-frequency shifts and the Fourier transform are isometries on S_0.*

(iii) *Given $c > 0$, the restriction mapping $f \mapsto (f(c\,k))_{k \in \mathbb{Z}}$ is a bounded (and surjective) operator from S_0 into ℓ^1.*

A fundamental tool in Fourier analysis is the Poisson summation formula, see [28, Section 1.4]. With the next lemma we recall that the Feichtinger space S_0 is a natural domain for this identity, much larger than the Schwartz space [17], [26, Corollary 3.2.9], [28, Corollary 12.1.15]. Secondly, we include a time-frequency variant of the Poisson summation formula, related with observations in [2, 23, 34, 43].

Lemma 2. (i) ([17, 26]) *For $f \in S_0$, the Poisson summation formula $\sum_{k \in \mathbb{Z}} \widehat{f}(k) = \sum_{k \in \mathbb{Z}} f(k)$ holds with absolute convergence of both series.*

(ii) *For $f, g \in S_0$, the following Poisson summation formula for the short-time Fourier transform holds, $\sum_{k,l \in \mathbb{Z}} V_g f(k, l) = \sum_{k \in \mathbb{Z}} f(k) \overline{\sum_{k \in \mathbb{Z}} g(k)}$, with absolute convergence of these series.*

Proof. (ii) First, denoting $h_x(t) = f(t)\,\overline{g(t - x)}$, for $t, x \in \mathbb{R}$, we have

$$V_g f(x, \omega) = \widehat{h_x}(\omega), \qquad x, \omega \in \mathbb{R},$$

cf. [28, Lemma 3.1.1]. Note that h_x belongs to S_0 since S_0 is invariant under translation and closed under multiplication. Hence, the Poisson summation formula implies that

$\sum_{l\in\mathbb{Z}} \widehat{h_x}(l) = \sum_{l\in\mathbb{Z}} h_x(l)$, for $x \in \mathbb{R}$ and, thus, we obtain

$$\begin{aligned}\sum_{k,l\in\mathbb{Z}} V_g f(k,l) &= \sum_{k\in\mathbb{Z}}\Big(\sum_{l\in\mathbb{Z}} \widehat{h_k}(l)\Big) = \sum_{k\in\mathbb{Z}}\Big(\sum_{l\in\mathbb{Z}} h_k(l)\Big)\\ &= \sum_{k,l\in\mathbb{Z}} f(k)\,\overline{g(k-l)} = \sum_{k\in\mathbb{Z}} f(k)\,\overline{\sum_{k\in\mathbb{Z}} g(k)}\,.\end{aligned}$$

□

Remark 9. (i) Several more general versions of the Poisson summation formula follow from the generic version in Lemma 2 (i) by using standard relations of the Fourier transform. For example, given $a > 0$, we have for $f \in S_0$,

$$\sum_{k\in\mathbb{Z}} \widehat{f}(\omega - k\,a) = \frac{1}{a}\sum_{k\in\mathbb{Z}} f(k/a)e^{2\pi i\,\omega k/a}, \qquad \omega \in \mathbb{R}\,, \tag{3.1}$$

with absolute convergence of both series.

(ii) Using standard properties of the short-time Fourier transform, see [28, Section 3.1], we obtain more general versions of the formula given in Lemma 2 (ii). For example, for $a, b > 0$ and $f, g \in S_0$, we have

$$\begin{aligned}&\sum_{k,l\in\mathbb{Z}} V_g f(x - k\,a, \omega - l\,b)\\ &= \frac{1}{b}\sum_{m\in\mathbb{Z}} f\Big(\frac{m}{b}\Big)\Big(\sum_{k\in\mathbb{Z}} \overline{g\Big(\frac{m}{b} - x - k\,a\Big)}\Big)e^{2\pi i\,\omega m/b}\,,\end{aligned} \tag{3.2}$$

where $x, \omega \in \mathbb{R}$, with absolute convergence of the series.

The space S_0, viewed as a Banach algebra under pointwise multiplication, contains approximate units obtained by dilation.

Lemma 3 ([26]). *Suppose that $\nu \in S_0$ with $\nu(0) = 1$. Given $r > 0$, let $\nu_r(t) = \nu(t/r)$, for $t \in \mathbb{R}$. Then $\|\nu_r f - f\|_{S_0} \to 0$ as $r \to \infty$, for all $f \in S_0$.*

Our main results implicitly use the technique of quasi-interpolation, which we briefly describe next. Schoenberg's quasi-interpolation is a general scheme in approximation theory. Commonly used techniques described by this paradigm are linear interpolation and spline approximation with refining sampling lattices.

Definition 4. Given $\psi \in S_0$ and $h > 0$, let Q_h^{ψ} denote the quasi-interpolation operator, defined for $f \in S_0$ by

$$Q_h^{\psi} f(t) = \sum_{k\in\mathbb{Z}} f(h\,k)\,\psi(t/h - k), \qquad t \in \mathbb{R}\,.$$

Moreover, given $\nu \in S_0$ and $r > 0$, we define the operator $\widehat{Q}_r^{\nu}$ for $f \in S_0$, by

$$\widehat{Q}_r^{\nu} f(t) = \nu(t/r)\sum_{k\in\mathbb{Z}} f(t - r\,k), \qquad t \in \mathbb{R}\,.$$

Remark 10. The operator $\widehat{Q}_r^{\nu}$ is a Fourier transformed version of Q_h^{ψ}. More precisely, assuming that $\nu = \widehat{\psi}$ and $r = 1/h$, we have by [22] that

$$\widehat{Q_h^{\psi} f} = \widehat{Q}_r^{\nu}\widehat{f}, \qquad f \in S_0\,.$$

It is a remarkable feature of S_0 that the quasi-interpolation converges for functions from S_0 indeed in the norm of S_0.

Lemma 4 ([22]). (i) *Suppose that* $\psi \in S_0$ *satisfies* $\widehat{\psi}(k) = \delta_{k,0}$, *for* $k \in \mathbb{Z}$. *Then for all* $f \in S_0$ *we have* $\|Q_h^\psi f - f\|_{S_0} \to 0$ *as* $h \to 0$.

(ii) *Suppose that* $\nu \in S_0$ *satisfies* $\nu(k) = \delta_{k,0}$ *for* $k \in \mathbb{Z}$. *Then for all* $f \in S_0$ *we have* $\|\widehat{Q}_r^\nu f - f\|_{S_0} \to 0$ *as* $r \to \infty$.

Remark 11. In view of Remark 10 we have that the statements (i) and (ii) in Lemma 4 are equivalent, since the Fourier transform is an isometry on S_0.

Next, we describe the Janssen representation of Gabor frame operators [28, Section 7.2], found in [33]. It is the expansion of a Gabor frame operator into a series of time-frequency shifts. The Janssen representation can be understood as an important part of a general duality principle in Gabor analysis, which includes results in [14, 32, 33, 40, 43, 44] and is sometimes called Wexler/Raz-Janssen-Ron/Shen duality. The function space S_0 turns out to be a suitable setting for this general paradigm, see [23, 26]. For example, the Janssen representation cannot be used for general Gabor windows $g \in L^2$, while it always converges absolutely for g from S_0 [26, Theorem 3.5.11 (iii)], as described next.

Lemma 5 ([26, 33]). *Let* $a, b > 0$. *For* $g \in S_0$, *the frame operator* $S_{g,a,b}$ *has an absolutely convergent Janssen representation*

$$S_{g,a,b} f(t) = a^\circ b^\circ \sum_{k,l \in \mathbb{Z}} V_g g(k\,a^\circ, l\,b^\circ) f(t - k\,a^\circ) e^{2\pi i\,t\,l\,b^\circ}, \qquad t \in \mathbb{R}\,, \tag{3.3}$$

where $a^\circ = 1/b$ *and* $b^\circ = 1/a$.

Remark 12. (i) The lattice $\Lambda^\circ = a^\circ\mathbb{Z} \times b^\circ\mathbb{Z} \subset \mathbb{R}^2$ is sometimes called the adjoint lattice and it is a rotation of the orthogonal (dual, reciprocal) lattice $\Lambda^\perp$. See [23, 26] and [21, p. 2014] for a more general setting.

(ii) The normalization factor $a^\circ b^\circ = 1/(ab)$ in (3.3) is usually called the redundancy of the Gabor system $G(g, a, b)$.

The following is an important property of S_0 and a celebrated result in Gabor analysis. It is found in [19] for rational time-frequency lattices $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$ with $ab \in \mathbb{Q}$, see [28, Theorem 13.2.1], and the intricate irrational case $ab \notin \mathbb{Q}$ has been settled in [29, Theorem 4.2].

Lemma 6 ([19, 29]). *Given* $g \in S_0$ *and* $a, b > 0$, *suppose that* $G(g, a, b)$ *is a Gabor frame. Then the dual window* $\widetilde{g}$ *also belongs to* S_0.

The next result is another crucial step in our approach, developed in [21] after preliminary results in [26].

Lemma 7 ([21]). *Given* $g \in S_0$ *and* $a, b > 0$, *let* $G(g, a, b)$ *be a Gabor frame. Let* $g_n \in S_0$ *and* $a_n, b_n > 0$, *for* $n = 1, 2, \ldots$, *and suppose that* $\|g_n - g\|_{S_0} \to 0$ *and* $(a_n, b_n) \to (a, b)$, *as* $n \to \infty$. *Then the following hold.*

(i) *For all n sufficiently large n, we have that* $G(g_n, a_n, b_n)$ *is a Gabor frame for* L^2.

(ii) *Denote the dual window of* $G(g, a, b)$ *by* $\widetilde{g}$ *in* S_0 *and for* $n \in \mathbb{N}$, *denote the dual window of* $G(g_n, a_n, b_n)$ *by* $\widetilde{g_n}$ *in* S_0. *Then* $\|\widetilde{g_n} - \widetilde{g}\|_{S_0} \to 0$ *as* $n \to \infty$.

We note that this result fails for general Gabor windows g from L^2, see [20], or the surprising example of Janssen's tie [35].

4. Preliminary Results

4.1 Periodization and Sampling

In our main results, the transition from functions on $\mathbb{R}$ to vectors in $\mathbb{C}^n$ is given by the operator R_n, i.e., by truncated sampling with weights. Our preliminary results are formulated for a different operator P_n, defined by sampling and periodization, without using weights. We note that the application of R_n is based on just n samples of f, while P_n involves infinitely many samples of f. Our use of P_n is that it satisfies certain properties, described below, which are relevant for our approach. By showing that R_n and P_n are closely related, we will, then obtain similar properties for R_n in our main results.

Definition 5. For $n \in \mathbb{N}$, define the operator $P_n : S_0 \to \mathbb{C}^n$ by

$$P_n f(k) = \sum_{l \in \mathbb{Z}} f\Big(\frac{k}{\sqrt{n}} - l\sqrt{n}\Big), \qquad k = 0, \dots, n-1 \,. \tag{4.1}$$

Remark 13. (i) The operator P_n combines sampling and periodization. The function f is sampled and, then the sequence so obtained is periodized. An equivalent point of view is that first f is periodized and, then the function so obtained is sampled on a fundamental domain. Either way, the sampling density and the length of the period increase simultaneously, as $n \to \infty$.

(ii) P_n is bounded and surjective from S_0 into $\mathbb{C}^n$ and the series in (4.1) converges absolutely. Indeed, sampling is bounded and surjective from S_0 into ℓ^1 by Lemma 1 (iii), and the periodization is bounded and surjective from ℓ^1 into $\mathbb{C}^n$.

(iii) For later use we note the following inconspicuous but important relation between P_n and a class of time-frequency shifts. Given $f \in S_0$, let $f_{k,l}(t) = f(t - k/\sqrt{n})e^{2\pi i\, t\, l/\sqrt{n}}$, for $k, l \in \mathbb{Z}$. Then $P_n f_{k,l}(m) = P_n f(m-k)e^{2\pi i\, m\, l/n}$.

We have mentioned that P_n maps S_0 into $\mathbb{C}^n$ in a different way than the operator R_n from (2.3), yet P_n and R_n are closely related. Indeed, we express R_n by using P_n and the weight function w_n given in Remark 2 (i).

Lemma 8. *Given $n \in \mathbb{N}$, we have $R_n f = P_n(w_n f)$, for $f \in S_0$.*

Proof. Since $\operatorname{supp} w_n \subset [-\sqrt{n}/2, \sqrt{n}/2]$, for $n \in \mathbb{N}$, we have by using (2.1) that

$$\begin{aligned} P_n(w_n f)(k) &= \sum_{l \in \mathbb{Z}} (w_n f)\Big(\frac{k}{\sqrt{n}} - l\sqrt{n}\Big) \\ &= \begin{cases} (w_n f)\Big(\frac{k}{\sqrt{n}}\Big), & k = 0, \dots, \lfloor \frac{n}{2} \rfloor \,, \\ (w_n f)\Big(\frac{k}{\sqrt{n}} - \sqrt{n}\Big), & k = \lfloor \frac{n}{2} \rfloor + 1, \dots, n-1 \,, \end{cases} \\ &= (w_n f)(\tau_k) = \lambda_k f(\tau_k) = R_n f(k) \,, \end{aligned}$$

where $k = 0, \dots, n-1$. □

4.2 P_n and the Fourier Transform

The transition from the Fourier transform of functions on $\mathbb{R}$ to the Fourier transform in $\mathbb{C}^n$ can be obtained formally by sampling and periodization [12, Theorem 1] and it is often used in the literature under various conditions on f which ensure its validity. A solid background is given in [2]. The operator P_n has been defined in such a way that it fits into this scheme and, indeed, the transition holds in a precise sense for all functions from S_0. The result follows from the observations in [2, 12], since the required assumptions are always satisfied for functions from S_0. We include a condensed proof.

Proposition 1 (see [2, 12]). *Suppose that $f \in S_0$. Given $n \in \mathbb{N}$, define the vector $u = P_n f \in \mathbb{C}^n$. Let $\widehat{f}$ denote the Fourier transform of f in S_0 and let $\widehat{u}$ denote the Fourier transform of u in $\mathbb{C}^n$. Then $P_n \widehat{f} = \widehat{u}$.*

Proof. Using the Poisson summation formula in the form of (3.1), with $\omega = k/\sqrt{n}$ and $a = 1/\sqrt{n}$, we have

$$\begin{aligned} P_n \widehat{f}(k) &= \sum_{l \in \mathbb{Z}} \widehat{f}\Big(\frac{k}{\sqrt{n}} - l\sqrt{n}\Big) \\ &= \frac{1}{\sqrt{n}} \sum_{l \in \mathbb{Z}} f\Big(\frac{l}{\sqrt{n}}\Big) e^{2\pi i\, k l/n} \\ &= \frac{1}{\sqrt{n}} \sum_{m=0}^{n-1} \sum_{l \in \mathbb{Z}} f\Big(\frac{m}{\sqrt{n}} - l\sqrt{n}\Big) e^{2\pi i\, k m/n} = \widehat{P_n f}(k)\ , \end{aligned}$$

where $k = 0, \dots, n-1$. Note that S_0 is invariant under time-frequency shifts and all series converge absolutely by Lemma 1. Hence, the result is proved, since by Lemma 2 (i) the Poisson summation formula generally holds for $f \in S_0$. □

4.3 P_n and the Dual Gabor Window

The transition from functions to vectors by sampling and periodization has been investigated also for Gabor systems. First structural relations are derived in [44, Appendix D]. For Gabor systems $G(g, a, b)$ with $ab = 1$, more details are given in [38]. Finally, rigorous results for general Gabor frames, $ab \leq 1$, and with attention to frame bounds have been developed in [34]. Here we outline that the operator P_n is suitably defined for this purpose and S_0 is an appropriate function space, see Proposition 2.

Let $n \in \mathbb{N}$ and $v \in \mathbb{C}^n$. The short-time Fourier transform in $\mathbb{C}^n$ with respect to v is defined for $u \in \mathbb{C}^n$, by

$$V_v^{(n)} u(k, l) = \sum_{m=0}^{n-1} u(m)\, v(m-k)\, e^{2\pi i\, l m/n}, \qquad k, l = 0, \dots, n-1\ .$$

The following result describes sampling and periodization of the short-time Fourier transform, it is closely related to observations in [23, 34, 43, 44]. By $P_n^{[2d]}$ we denote the analogue of P_n for functions on $\mathbb{R}^2$, described in (6.1), see Section 6.

Lemma 9. *Let $f, g \in S_0$ and $n \in \mathbb{N}$. Then $P_n^{[2d]} V_g f(k, l) = \frac{1}{\sqrt{n}} V_v^{(n)} u(k, l)$, for $k, l = 0, \dots, n-1$, where $u = P_n f \in \mathbb{C}^n$ and $v = P_n g \in \mathbb{C}^n$.*

Proof. Using the general time-frequency Poisson summation formula (3.2), which is applicable since $f, g \in S_0$, we calculate

$$
\begin{aligned}
&P_n^{[2d]} V_g f(k,l) \\
&= \sum_{r,s\in\mathbb{Z}} V_g f\Big(\frac{k}{\sqrt{n}} - r\sqrt{n}, \frac{l}{\sqrt{n}} - s\sqrt{n}\Big) \\
&= \frac{1}{\sqrt{n}} \sum_{m\in\mathbb{Z}} f\Big(\frac{m}{\sqrt{n}}\Big) \sum_{r\in\mathbb{Z}} g\Big(\frac{m-k}{\sqrt{n}} - r\sqrt{n}\Big) e^{2\pi i\, m\, l/n} \\
&= \frac{1}{\sqrt{n}} \sum_{m=0}^{n-1} \sum_{s\in\mathbb{Z}} \Big(f\Big(\frac{m-s\,n}{\sqrt{n}}\Big) \sum_{r\in\mathbb{Z}} g\Big(\frac{m-s\,n-k}{\sqrt{n}} - r\sqrt{n}\Big) e^{2\pi i\,(m-s\,n) l/n} \Big) \\
&= \frac{1}{\sqrt{n}} \sum_{m=0}^{n-1} \Big(\sum_{s\in\mathbb{Z}} f\Big(\frac{m}{\sqrt{n}} - s\sqrt{n}\Big)\Big) \Big(\sum_{r\in\mathbb{Z}} g\Big(\frac{m-k}{\sqrt{n}} - r\sqrt{n}\Big)\Big) e^{2\pi i\, m\, l/n} \\
&= \frac{1}{\sqrt{n}} \sum_{m=0}^{n-1} u(m)\, v(m-k)\, e^{2\pi i\, m\, l/n} = \frac{1}{\sqrt{n}} V_v^{(n)} u(k,l),
\end{aligned}
$$

where $k, l = 0, \ldots, n-1$. □

Next, we recall that an analogous form of the Janssen representation (3.3) can be formulated for Gabor frame operators on $\mathbb{C}^n$, see [34, Equation (5.4)]; we mention that the result indeed holds, more generally, on groups [23, Corollary 7.7.8]. Given $v \in \mathbb{C}^n$ and $p, q \in \mathbb{N}$ divisors of $n \in \mathbb{N}$, the Gabor frame operator $S_{v,p,q}^{(n)}$ on $\mathbb{C}^n$, defined in (2.4), has the following Janssen representation. For $u \in \mathbb{C}^n$, we have

$$
S_{v,p,q}^{(n)} u(m) = \frac{p^\circ q^\circ}{n} \sum_{k=0}^{q-1} \sum_{l=0}^{p-1} V_v^{(n)} v(k\,p^\circ, l\,q^\circ)\, u(m - k\,p^\circ) e^{2\pi i\, m\, l\, q^\circ/n}\,, \tag{4.2}
$$

with $m = 0, \ldots, n-1$, where $p^\circ = n/q$ and $q^\circ = n/p$.

Remark 14. The normalization factor $p^\circ q^\circ/n = n/(pq)$ in (4.2) is called the redundancy of the Gabor system $G^{(n)}(v, p, q)$, cf. Remark 12 (ii).

The next lemma relates sampling and periodization with the action of Gabor frame operators. The result follows from [34, Proposition 3 and Section 5], the proof is based on the Janssen representation; we condense the proof by making use of Lemma 9 and properties of S_0.

Lemma 10 (see [34]). *Given $f, g \in S_0$ and $0 < a < 1$, suppose that $a^2 = p/q \in \mathbb{Q}$ and let $n = pq$. Then $P_n S_{g,a,a} f = \frac{1}{\sqrt{n}} S_{v,p,p}^{(n)} u$, where $u = P_n f \in \mathbb{C}^n$ and $v = P_n g \in \mathbb{C}^n$.*

Proof. Since $g \in S_0$, by Lemma 5 the frame operator $S_{g,a,a}$ has an absolutely convergent

Janssen representation (3.3). Hence, using $a^2 = n/q^2$ and $p/a = \sqrt{n}$ we calculate

$$
\begin{aligned}
&S_{g,a,a} f(t) \\
&= \frac{1}{a^2} \sum_{r,s\in\mathbb{Z}} V_g g(r/a, s/a) f(t - r/a) e^{2\pi i t s/a} \\
&= \frac{1}{a^2} \sum_{k,l=0}^{p-1} \sum_{r,s\in\mathbb{Z}} V_g g\Big(\frac{k - r p}{a}, \frac{l - s p}{a}\Big) f\Big(t - \frac{k - r p}{a}\Big) e^{2\pi i t (l-s p)/a} \\
&= \frac{q^2}{n} \sum_{k,l=0}^{p-1} \sum_{r,s\in\mathbb{Z}} V_g g\Big(\frac{k q - r n}{\sqrt{n}}, \frac{l q - s n}{\sqrt{n}}\Big) f\Big(t - \frac{k q - r n}{\sqrt{n}}\Big) e^{2\pi i t (l q-s n)/\sqrt{n}} ,
\end{aligned}
\tag{4.3}
$$

where $t \in \mathbb{R}$. Next, we apply P_n to (4.3). Thus, using Lemma 9, Remark 13 (iii) and (4.2) we obtain

$$
\begin{aligned}
&P_n S_{g,a,a} f(m) \\
&= \frac{q^2}{n} \sum_{k,l=0}^{p-1} \Big(\sum_{r,s\in\mathbb{Z}} V_g g\Big(\frac{k q}{\sqrt{n}} - r\sqrt{n}, \frac{l q}{\sqrt{n}} - s\sqrt{n}\Big)\Big) u(m - k q) e^{2\pi i m l q/n} , \\
&= \frac{q^2}{n} \sum_{k,l=0}^{p-1} P_n^{[2d]} V_g g(k q, l q)\, u(m - k q) e^{2\pi i m l q/n} , \\
&= \frac{1}{\sqrt{n}} \frac{q^2}{n} \sum_{k,l=0}^{p-1} V_v^{(n)} v(k q, l q)\, u(m - k q) e^{2\pi i m l q/n} \\
&= \frac{1}{\sqrt{n}} S_{v,p,p}^{(n)} u(m), \qquad m = 0, \ldots, n-1 .
\end{aligned}
$$

□

The next proposition is concerned with sampling and periodization of a Gabor window. The result follows essentially from [34, Proposition 4 and Section 5]; we formulate the result specifically for our purposes, using the Feichtinger space S_0, and include the proof.

Proposition 2 (see [34]). *Given $g \in S_0$ and $0 < a < 1$, suppose that $G(g, a, a)$ is a Gabor frame for L^2 with $a^2 = p/q \in \mathbb{Q}$. Let $n = pq$ and define the vector $v = P_n g$ in $\mathbb{C}^n$. Then the following hold.*

(i) *The system $G^{(n)}(v, p, p)$ is a Gabor frame for $\mathbb{C}^n$.*

(ii) *Denote the dual window of $G(g, a, a)$ by $\widetilde{g}$, which is in S_0, and denote the dual vector of $G^{(n)}(v, p, p)$ by $\widetilde{v}$ in $\mathbb{C}^n$. Then $\widetilde{v} = \frac{1}{\sqrt{n}} P_n \widetilde{g}$.*

Proof. First, for $S = S_{g,a,a}$ and $S^{(n)} = S^{(n)}_{v,p,p}$, we have by Lemma 10 that

$$
P_n S = \tfrac{1}{\sqrt{n}} S^{(n)} P_n . \tag{4.4}
$$

Since by assumption $G(g, a, a)$ is a frame we have that $S\colon S_0 \to S_0$ is surjective. Since by Remark 13 (ii) also $P_n\colon S_0 \to \mathbb{C}^n$ is surjective, we conclude from (4.4) that $S^{(n)}\colon \mathbb{C}^n \to \mathbb{C}^n$ is surjective. That is, $S^{(n)}$ is invertible and $G^{(n)}(v, p, p)$ is a frame for $\mathbb{C}^n$. Consequently, the dual window is of the form

$$\widetilde{v} = \left(S^{(n)}\right)^{-1} v = \left(S^{(n)}\right)^{-1} P_n g = \left(S^{(n)}\right)^{-1} P_n S \widetilde{g}$$
$$= \tfrac{1}{\sqrt{n}} \left(S^{(n)}\right)^{-1} S^{(n)} P_n \widetilde{g} = \tfrac{1}{\sqrt{n}} P_n \widetilde{g} \,. \qquad \square$$

4.4 Reconstruction from P_n

As a crucial result for our approach we show how a function in S_0 can be reconstructed from the vectors obtained by applying P_n, that is, from sampling and periodization.

Proposition 3. *Suppose that $f \in S_0$. For $n \in \mathbb{N}$, define the vector $u_n = P_n f \in \mathbb{C}^n$. Then $\|L_n u_n - f\|_{S_0} \to 0$ as $n \to \infty$. In particular, $\{L_n P_n : n \in \mathbb{N}\}$ is uniformly bounded.*

Proof. By direct computation we find that

$$L_n P_n f(t) = \sum_{k=0}^{n-1} \lambda_k \sum_{l \in \mathbb{Z}} f\Big(\frac{k}{\sqrt{n}} - l\sqrt{n}\Big)\, \varphi\big((t-\tau_k)\sqrt{n}\big), \qquad t \in \mathbb{R}\,, \tag{4.5}$$

and also obtain

$$Q^{\varphi}_{1/\sqrt{n}}\, \widehat{Q}^{w}_{\sqrt{n}} f(t) = \sum_{k \in \mathbb{Z}} w_n(k/\sqrt{n}) \sum_{l \in \mathbb{Z}} f\Big(\frac{k}{\sqrt{n}} - l\sqrt{n}\Big)\, \varphi\big(t\sqrt{n} - k\big)\,, \tag{4.6}$$

where $t \in \mathbb{R}$. Now using (2.2) we conclude that (4.5) and (4.6) coincide, that is,

$$L_n P_n = Q^{\varphi}_{1/\sqrt{n}}\, \widehat{Q}^{w}_{\sqrt{n}}\,. \tag{4.7}$$

Next, we have $\widehat{\varphi}(k) = \delta_{k,0}$ and $w(k) = \delta_{k,0}$, so (4.7) and Lemma 4 imply

$$\|L_n P_n f - f\|_{S_0} = \|Q^{\varphi}_{1/\sqrt{n}}\, \widehat{Q}^{w}_{\sqrt{n}} f - f\|_{S_0} \to 0, \qquad \text{as} \quad n \to \infty\,. \qquad \square$$

5. Proofs of Theorems 1–3

Proof of Theorem 1. Suppose that f is in S_0, let $v_n = R_n f$ and define $f_n = w_n f$. By Lemma 3 we have that

$$\|f_n - f\|_{S_0} \to 0, \qquad \text{as} \quad n \to \infty\,. \tag{5.1}$$

Next, by Lemma 8 we obtain

$$v_n = R_n f = P_n(w_n f) = P_n f_n\,, \qquad \text{for} \quad n \in \mathbb{N}. \tag{5.2}$$

Thus, using (5.1), (5.2), and Proposition 3 we conclude that

$$\begin{aligned} \|L_n v_n - f\|_{S_0} &= \|L_n P_n f_n - f\|_{S_0} \\ &\le \|L_n P_n (f_n - f)\|_{S_0} + \|L_n P_n f - f\|_{S_0} \\ &\le C\,\|f_n - f\|_{S_0} + \|L_n P_n f - f\|_{S_0} \to 0, \qquad \text{as} \quad n \to \infty\,. \qquad \square \end{aligned}$$

Proof of Theorem 2. Suppose that f is in S_0, let $v_n = R_n f$ and define $f_n = w_n f$. From Lemma 1 (ii) and Lemma 3 we obtain

$$\|\widehat{f_n} - \widehat{f}\|_{S_0} = \|f_n - f\|_{S_0} \to 0, \qquad \text{as} \quad n \to \infty\,. \tag{5.3}$$

Next, by Lemma 8 we have $v_n = R_n f = P_n(w_n f) = P_n f_n$, for $n \in \mathbb{N}$, so Proposition 1 implies

$$\widehat{v_n} = P_n \widehat{f_n}, \qquad \text{for} \quad n \in \mathbb{N} . \tag{5.4}$$

Thus, using (5.3), (5.4), and Proposition 3 we conclude that

$$\begin{aligned}\|L_n \widehat{v_n} - \widehat{f}\|_{S_0} &= \|L_n P_n \widehat{f_n} - \widehat{f}\|_{S_0} \\ &\le \|L_n P_n\big(\widehat{f_n} - \widehat{f}\big)\|_{S_0} + \|L_n P_n \widehat{f} - \widehat{f}\|_{S_0} \\ &\le C\, \|\widehat{f_n} - \widehat{f}\|_{S_0} + \|L_n P_n \widehat{f} - \widehat{f}\|_{S_0} \to 0, \qquad \text{as} \quad n \to \infty .\end{aligned}$$ □

Proof of Theorem 3. For p, q, n defined in the theorem, denote $g_n = w_n g$ and let $a_n = \sqrt{p/q}$. By Lemma 3 we have $\|g_n - g\|_{S_0} \to 0$ as $n \to \infty$. Since also

$$a_n = \sqrt{p/q} = \sqrt{\lceil a^2 q \rceil / q} \to a, \qquad \text{as} \quad n \to \infty ,$$

we obtain from Lemma 7 that $G(g_n, a_n, a_n)$ is a Gabor frame for L^2, for all n sufficiently large, and the dual window $\widetilde{g_n}$ satisfies

$$\|\widetilde{g_n} - \widetilde{g}\|_{S_0} \to 0, \qquad \text{as} \quad n \to \infty . \tag{5.5}$$

By Lemma 8 we have $v_n = R_n g = P_n(w_n g) = P_n g_n$, for $n \in \mathbb{N}$. Hence, Proposition 2 implies that $G^{(n)}(v_n, p, p)$ is a Gabor frame for $\mathbb{C}^n$ and the dual vector satisfies

$$\widetilde{v_n} = \tfrac{1}{\sqrt{n}} P_n \widetilde{g_n}, \qquad \text{for} \quad n \in \mathbb{N} . \tag{5.6}$$

Next, using (5.5), (5.6), and Proposition 3 we conclude that

$$\begin{aligned}\|\sqrt{n} L_n \widetilde{v_n} - \widetilde{g}\|_{S_0} &= \|L_n P_n \widetilde{g_n} - \widetilde{g}\|_{S_0} \\ &\le \|L_n P_n(\widetilde{g_n} - \widetilde{g})\|_{S_0} + \|L_n P_n \widetilde{g} - \widetilde{g}\|_{S_0} \\ &\le C\, \|\widetilde{g_n} - \widetilde{g}\|_{S_0} + \|L_n P_n \widetilde{g} - \widetilde{g}\|_{S_0} \to 0, \qquad \text{as} \quad n \to \infty .\end{aligned}$$

Thus, the desired statement follows since $q \to \infty$ yields $n \to \infty$. □

6. Further Comments

Our results are formulated for functions on $\mathbb{R}$. We note that the corresponding results hold for functions on $\mathbb{R}^d$, for any $d = 1, 2, \dots$ For example, let $d = 2$ and by $M_{n,n}(\mathbb{C})$ denote the complex $n \times n$-matrices. Then the definition of R_n in (2.3) is modified to $R_n^{[2d]} \colon S_0(\mathbb{R}^2) \to M_{n,n}(\mathbb{C})$,

$$R_n^{[2d]} f(k) = \lambda_k\, f(\tau_k), \qquad k = (k_1, k_2), \quad k_1, k_2 = 1, \dots, n ,$$

where $\lambda_k = \lambda_{k_1} \lambda_{k_2} \in \mathbb{R}$ and $\tau_k = (\tau_{k_1}, \tau_{k_2}) \in \mathbb{R}^2$. Correspondingly, L_n in (2.4) is replaced by $L_n^{[2d]} \colon M_{n,n}(\mathbb{C}) \to S_0(\mathbb{R}^2)$,

$$L_n^{[2d]} v(t) = \sum_{k_1, k_2 = 1}^{n} \lambda_k\, v(k)\, \Phi\big((t - \tau_k)\sqrt{n}\big), \qquad k = (k_1, k_2), \quad t \in \mathbb{R}^2 ,$$

where $\Phi(t) = \varphi(t_1)\varphi(t_2)$, for $t = (t_1, t_2) \in \mathbb{R}^2$, and φ is given in Definition 3. Finally, P_n in (4.1) takes the form $P_n^{[2d]}\colon S_0(\mathbb{R}^2) \to M_{n,n}(\mathbb{C})$,

$$P_n^{[2d]} f(k) = \sum_{l_1,l_2\in\mathbb{Z}} f\Big(\frac{k}{\sqrt{n}} - l\sqrt{n}\Big), \qquad k = (k_1, k_2),\ l = (l_1, l_2)\ , \tag{6.1}$$

where $k_1, k_2 = 0, \ldots, n-1$. With these and analogous modifications all our results hold for higher dimensions.

Finally, we point out that our work is closely related with the finite approximation of quantum systems, see [16]. We believe that our results can be useful also in this context.

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