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APPROXIMATION OF THE STABILITY NUMBER OF A GRAPH VIA COPOSITIVE PROGRAMMING*

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Abstract. Lovász and Schrijver [*SIAM J. Optim.*, 1 (1991), pp. 166–190] showed how to formulate increasingly tight approximations of the stable set polytope of a graph by solving semidefinite programs (SDPs) of increasing size (lift-and-project method). In this paper we present a similar idea. We show how the stability number can be computed as the solution of a conic linear program (LP) over the cone of copositive matrices. Subsequently, we show how to approximate the copositive cone ever more closely via a hierarchy of linear or semidefinite programs of increasing size (liftings). The latter idea is based on recent work by Parrilo [*Structured Semidefinite Programs and Semi-algebraic Geometry Methods in Robustness and Optimization*, Ph.D. thesis, California Institute of Technology, Pasadena, CA, 2000]. In this way we can compute the stability number $\alpha(G)$ of any graph $G(V,E)$ after at most $\alpha(G)^2$ successive liftings for the LP-based approximations. One can compare this to the $n-\alpha(G)-1$ bound for the LP-based lift-and-project scheme of Lovász and Schrijver. Our approach therefore requires fewer liftings for families of graphs where $\alpha(G) < O(\sqrt{n})$. We show that the first SDP-based approximation for $\alpha(G)$ in our series of increasingly tight approximations coincides with the ϑ'-function of Schrijver [*IEEE Trans. Inform. Theory*, 25 (1979), pp. 425–429]. We further show that the second approximation is tight for complements of triangle-free graphs and for odd cycles.

Key words. approximation algorithms, stability number, semidefinite programming, copositive cone, lifting

AMS subject classifications. 90C22, 68R10, 05C69, 90C25

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1. Introduction.

Semidefinite programming has proved to be a useful tool in formulating approximation algorithms for NP-complete problems in combinatorial optimization. The most celebrated example is the 0.878-approximation algorithm for MAX-CUT by Goemans and Williamson [8]. Their ideas have also been extended to obtain improved approximation guarantees for MAX-Bisection, MAX-3-SAT, MAX-k-CUT, and a host of other problems.

For problems which do not allow a fixed approximation guarantee, like the maximum stable set problem, semidefinite programming has also played a role. Lovász and Schrijver [16] showed how to formulate increasingly strong approximations of the maximum stable set of a graph by solving semidefinite programs (SDPs) of increasing size (liftings). They showed that their procedure is finite—the stable set polytope is obtained via a suitable projection.

In this paper we present a similar idea, but from a completely different perspective. We first show how one can compute the stability number by solving a convex conic optimization problem over the cone of copositive matrices.

Nesterov and Nemirovskii [19] showed that conic programming problems can be solved to ϵ-optimality in polynomial time if the cone in question has a *computable*[1] self-concordant barrier. As a consequence, the copositive cone does not allow a computable barrier unless $P = NP$.

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[1]The gradient and Hessian of the barrier must be computable in polynomial time.

Parrilo [20] has recently suggested that the copositive cone may be approximated using linear matrix inequalities (LMIs). The approximation involves matrix variables of size $n^r \times n^r$ after r steps. We will look more closely at this procedure and will also investigate the link with a weaker linear program (LP)-based lifting scheme. Subsequently we will show that $\alpha(G)^2$ liftings are always sufficient to obtain the stability number $\alpha(G)$ of a graph $G(V,E)$ for the LP-based procedure. One can compare this to the result for the Lovász and Schrijver LP-based lift-and-project scheme, which requires $n-\alpha(G)-1$ liftings in the worst case. For families of graphs where $\alpha(G) < O(\sqrt{n})$, our procedure therefore requires fewer liftings in the worst case.

At the first step of our SDP-based lifting scheme, we obtain the Schrijver $\vartheta'(G)$ approximation [25] to $\alpha(G)$, which is already provably stronger than the Lovász ϑ approximation [15] for certain classes of graphs. The approximation after the second lifting is tight for complements of triangle-free graphs and for odd cycles.

1.1. Preliminaries.

The maximum stable set problem. Given a graph $G(V,E)$, a subset $V' \subseteq V$ is called a stable set of G if the induced subgraph on V' contains no edges. The maximum stable set problem is to find the stable set of maximal cardinality. This problem is equivalent to finding the largest clique in the complementary graph and cannot be approximated within a factor $|V|^{\frac{1}{2}-\epsilon}$ for any $\epsilon > 0$ unless $P = NP$, or within a factor $|V|^{1-\epsilon}$ for any $\epsilon > 0$ unless $NP = ZPP$ [10]. The best known approximation guarantee for this problem is $O(|V|/(\log|V|)^2)$ [5]. For a survey of the maximum clique problem, see [3].

Conic programming. We define the following convex cones:

- The $n \times n$ symmetric matrices
  $\mathcal{S}_n = \left\{X \in \mathbb{R}^n \times \mathbb{R}^n,\ X = X^T\right\}$;
- The $n \times n$ symmetric positive semidefinite matrices
  $\mathcal{S}_n^+ = \left\{X \in \mathcal{S}_n,\ y^T X y \geq 0\ \ \forall y \in \mathbb{R}^n\right\}$;
- The $n \times n$ symmetric copositive matrices
  $\mathcal{C}_n = \left\{X \in \mathcal{S}_n,\ y^T X y \geq 0\ \ \forall y \in \mathbb{R}^n,\ y \geq 0\right\}$;
- The $n \times n$ symmetric completely positive matrices
  $\mathcal{C}_n^* = \{X = \sum_{i=1}^k y_i y_i^T,\ y_i \in \mathbb{R}^n,\ y_i \geq 0\ (i = 1, \ldots, k)\}$;
- The $n \times n$ symmetric nonnegative matrices
  $\mathcal{N}_n = \{X \in \mathcal{S}_n,\ X_{ij} \geq 0\ \ (i,j = 1, \ldots, n)\}$.

Recall that the completely positive cone is the dual of the copositive cone, and that the nonnegative and semidefinite cones are self-dual for the inner product $\langle X, Y\rangle := \mathrm{Tr}(XY)$, where "Tr" denotes the trace operator.

For a given cone $\mathcal{K}_n$ and its dual cone $\mathcal{K}_n^*$, we define the primal and dual pair of conic LPs:

$$(P) \qquad p^* := \inf_X \left\{\mathrm{Tr}(CX)\ :\ \mathrm{Tr}(A_i X) = b_i\ (i = 1, \ldots, m),\ X \in \mathcal{K}_n\right\},$$

$$(D) \qquad d^* := \sup_{y \in \mathbb{R}^m} \left\{ b^T y\ :\ \sum_{i=1}^m y_i A_i + S = C,\ S \in \mathcal{K}_n^* \right\}.$$

If $\mathcal{K}_n = \mathcal{S}_n^+$, then we refer to semidefinite programming; if $\mathcal{K}_n = \mathcal{N}_n$, to linear programming; and if $\mathcal{K}_n = \mathcal{C}_n$, to copositive programming.

The well-known conic duality theorem (see, e.g., [24]) gives the duality relations between (P) and (D).

THEOREM 1.1 (Conic duality theorem). *If there exists an interior feasible solution $X^0 \in \operatorname{int}(\mathcal{K}_n)$ of (P) and a feasible solution of (D), then $p^* = d^*$ and the supremum in (D) is attained. Similarly, if there exist feasible y^0, S^0 for (D), where $S^0 \in \operatorname{int}(\mathcal{K}_n^*)$, and a feasible solution of (P), then $p^* = d^*$ and the infimum in (P) is attained.*

Optimization over the cones $\mathcal{S}_n^+$ and $\mathcal{N}_n$ can be done in polynomial time (to compute an ϵ-optimal solution), but copositive programming is reducible to some NP-hard problems as we will see in the next section.

2. The stability number via copositive programming. The celebrated sandwich theorem of Lovász relates three characterizing numbers of a graph $G(V, E)$: the chromatic number $\chi(\bar{G})$ of the complementary graph $\bar{G}$, the stability number $\alpha(G)$ of G, and the so-called theta number $\vartheta(G)$. The theta number can be defined as the optimal value of the following semidefinite programming relaxation of the maximum clique problem (see [15, 9]):

$$\vartheta(G) := \max \operatorname{Tr}\left(ee^T X\right) = e^T X e \tag{1}$$

subject to

$$\left.\begin{array}{rcl} X_{ij} & = & 0,\ \{i,j\} \in E\ (i \neq j) \\ \operatorname{Tr}(X) & = & 1 \\ X & \in & \mathcal{S}_n^+ \end{array}\right\}, \tag{2}$$

where e denotes the all-one vector.

The sandwich theorem states the following.

THEOREM 2.1 (Lovász's sandwich theorem). *For any graph $G = (V, E)$, one has*

$$\alpha(G) \leq \vartheta(G) \leq \chi(\bar{G}).$$

In what follows, x_S denotes the incidence vector of a stable set S of size $k = |S|$ in G, i.e.:

$$(x_S)_i = \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that the rank one matrix

$$X := \frac{1}{k} x_S x_S^T$$

is feasible in (2) with objective value

$$e^T X e = \frac{1}{k}\left(e^T x_S\right)^2 = \frac{k^2}{k} = k.$$

We therefore have $\alpha(G) \leq \vartheta(G)$, which proves the relevant part of the sandwich theorem.

We now show that we can actually obtain the stability number $\alpha(G)$ by replacing the semidefinite cone in (2) by the completely positive cone.

THEOREM 2.2. *Let $G(V,E)$ be given with $|E| = n$. The stability number of G is given by*

$$\alpha(G) = \max Tr\left(ee^T X\right) \tag{3}$$

subject to

$$\left.\begin{array}{rcl} X_{ij} & = & 0,\ \{i,j\} \in E\ (i \neq j) \\ Tr(X) & = & 1 \\ X & \in & \mathcal{C}_n^* \end{array}\right\}. \tag{4}$$

Proof. Consider the convex cone:

$$\mathcal{C}_G := \{X \in \mathcal{C}_n^* \ :\ X_{ij} = 0,\ \{i,j\} \in E\}.$$

The extreme rays of this cone are of the form xx^T, where $x \in \mathbb{R}^n$ is nonnegative and its support corresponds to a stable set of G. This follows from the fact that all extreme rays of $\mathcal{C}_n^*$ are of the form xx^T for nonnegative $x \in \mathbb{R}^n$. Therefore, the extreme points of the set defined by (4) are given by the intersection of the extreme rays with the hyperplane defined by $\mathrm{Tr}(X) = 1$.

Since the optimal value of problem (3) is attained at an extreme point, there is an optimal solution of the form:

$$X^* = x^* x^{*T}, \qquad x^* \in \mathbb{R}^n,\ x^* \geq 0,\ \|x^*\| = 1,$$

and where the support of x^* corresponds to a stable set, say S^*. Denoting the optimal value of problem (3) by λ, we therefore have

$$\lambda = \max_{\|x\|=1} (e^T x)^2, \quad x \geq 0, \quad \mathrm{support}(x) = \mathrm{support}(x^*).$$

The optimality conditions of this problem imply

$$x^* = \frac{1}{\sqrt{|S^*|}} x_{S^*},$$

and therefore

$$\lambda = (e^T x^*)^2 = \frac{|S^*|^2}{|S^*|} = |S^*|.$$

This shows that S^* must be the maximum stable set, and consequently $\lambda = \alpha(G)$. ◻

Note that—since $X \in \mathcal{C}_n^*$ is always nonnegative—we can simplify (3) and (4) to

$$\alpha(G) = \max\left\{\mathrm{Tr}\left(ee^T X\right) \ :\ \mathrm{Tr}\left(AX\right) = 0,\ \mathrm{Tr}(X) = 1,\ X \in \mathcal{C}_n^*\right\}, \tag{5}$$

where A is the adjacency matrix of G. The dual problem of (5) is given by

$$\inf_{\lambda, y \in \mathbb{R}} \left\{\lambda \ :\ Q := \lambda I + yA - ee^T \in \mathcal{C}_n\right\}. \tag{6}$$

The primal problem (5) is not strictly feasible (some entries of X must be zero), even though the dual problem (6) is strictly feasible (set $Q = (n+1)I - ee^T$). By the conic duality theorem, we can therefore conclude only that the primal optimal set is nonempty and *not* that the dual optimal set is nonempty. We will now prove, however, that $Q = \alpha(I + A) - ee^T$ is always a dual optimal solution. This result follows from the next lemma.

LEMMA 2.3. *For a given graph $G = (V, E)$, with adjacency matrix A and stability number $\alpha(G)$, and a given parameter $\epsilon \geq 0$, the matrix*

$$Q_\epsilon^* = (1 + \epsilon)\alpha(I + A) - ee^T$$

is copositive.

Proof. Let $\epsilon \geq 0$ be given. We will show that Q_ϵ^* is copositive.

To this end, denote the standard simplex by

$$\Delta := \left\{ x \in \mathbb{R}^n \ : \ \sum_{i=1}^n x_i = 1, \ x \geq 0 \right\}$$

and note that

$$\begin{aligned} \min_{x \in \Delta} x^T Q_\epsilon^* x &= \min_{x \in \Delta} (1 + \epsilon)\alpha \left(x^T x + x^T A x \right) - x^T ee^T x \\ &= (1 + \epsilon)\alpha \min_{x \in \Delta} \left(x^T x + x^T A x \right) - 1. \end{aligned}$$

We now show that the minimum is attained at $x^* = \frac{1}{|S^*|} x_{S^*}$, where S^* denotes the maximum stable set, as before. In other words, we will show that

$$\min_{x \in \Delta} x^T Q_\epsilon^* x = \epsilon. \tag{7}$$

Let $x^* \in \Delta$ be a minimizer of $x^T Q_\epsilon^* x$ over Δ.

If the support of x^* corresponds to a stable set, then the proof is an easy consequence of the inequality:

$$\text{argmax} \left\{ \|x\| \ : x \in \Delta, \ (e^T x)^2, \ x \geq 0, \ \text{support}(x) = S \right\} = \frac{1}{|S|} x_S \quad \forall S \subset V,$$

which can readily be verified via the optimality conditions.

Assume therefore that the support of x^* does not correspond to a stable set, i.e., $x_i^* > 0$ and $x_j^* > 0$, where $\{i, j\} \in E$.

Now we fix all the components of x to the corresponding values of x^*, except for components i and j. Note that, defining $c_0 := \sum_{k \neq i,j} x_k^*$, one can find constants c_1, c_2, and c_3 such that

$$\begin{aligned} {x^*}^T Q_\epsilon^* x^* &= \min_{x_i + x_j = 1 - c_0, x_i \geq 0, x_j \geq 0} (1 + \epsilon)\alpha \left(x_i^2 + 2x_i x_j + x_j^2 \right) + x_i c_1 + x_j c_2 + c_3 \\ &= \min_{x_i + x_j = 1 - c_0, x_i \geq 0, x_j \geq 0} (1 + \epsilon)\alpha \left(x_i + x_j \right)^2 + x_i c_1 + x_j c_2 + c_3 \\ &= \min_{x_i + x_j = 1 - c_0, x_i \geq 0, x_j \geq 0} (1 + \epsilon)\alpha \left(1 - c_0 \right)^2 + x_i c_1 + x_j c_2 + c_3. \end{aligned}$$

The final optimization problem is simply an LP in the two variables x_i and x_j and attains it minimal value in an extremal point at which $x_i = 0$ or $x_j = 0$. We can therefore replace x^* with a vector $\bar{x}$ such that ${x^*}^T Q x^* = \bar{x}^T Q \bar{x}$ and $\bar{x}_i \bar{x}_j = 0$.

By repeating this process, we obtain a minimizer of $x^TQ_\epsilon^*x$ over Δ with support corresponding to a stable set. □

The lemma shows that Q_ϵ^* is copositive and therefore ϵ-optimal in (6). For $\epsilon = 0$ we have the following result.

COROLLARY 2.4. *For any graph $G = (V, E)$ with adjacency matrix A, one has*

$$\alpha(G) = \min_{\lambda} \left\{ \lambda \; : \; \lambda(I + A) - ee^T \in \mathcal{C}_n \right\}.$$

Remark 2.1. The result of Corollary 2.4 is also a consequence of a result by Motzkin and Straus [17], who proved that

$$\frac{1}{\alpha(G)} = \min_{x\in\Delta} x^T(A + I)x,$$

where A is the adjacency matrix of G. To see the relationship between the two results, we also need the known result (see, e.g., [4]) that minimization of a quadratic function over the simplex is equivalent to a copositive programming problem:

$$\min_{x\in\Delta} x^TQx = \min_{X\in(\mathcal{C}_n)^*} \left\{ Tr(QX) \; : \; Tr\left(ee^TX\right) = 1 \right\} = \max_{\lambda\in\mathbb{R}} \left\{ \lambda \; : \; Q - \lambda ee^T \in \mathcal{C}_n \right\}$$

for any $Q \in \mathcal{S}_n$, where the second inequality follows from the strong duality theorem.

Corollary 2.4 implies that we can simplify our conic programs even further to obtain

$$\alpha(G) = \max\left\{ \operatorname{Tr}\left(ee^TX\right) \; : \; \operatorname{Tr}\left((A + I)X\right) = 1, \; X \in \mathcal{C}_n^* \right\}, \tag{8}$$

with associated dual problem:

$$\alpha(G) = \min_{\lambda\in\mathbb{R}} \left\{ \lambda \; : \; Q := \lambda(I + A) - ee^T \in \mathcal{C}_n \right\}. \tag{9}$$

Note that both these problems are strictly feasible, and the conic duality theorem now guarantees complementary primal-dual optimal solutions.

3. Approximations of the copositive cone. The reformulation of the stable set problem as a conic copositive program makes it clear that copositive programming is not tractable (see also [23, 4]). In fact, even the problem of determining whether a matrix is not copositive is NP-complete [18].

Although we have obtained a nice convex reformulation of the stable set problem, there is no obvious way of solving this reformulation. In [4], some ideas from interior point methods for semidefinite programming are adapted for the copositive case, but convergence cannot be proved. The absence of a computable self-concordant barrier for this cone basically precludes the application of interior point methods to copositive programming.

A solution to this problem was recently proposed by Parrilo [20], who showed that one can approximate the copositive cone to any given accuracy by a sufficiently large set of linear matrix inequalities. In other words, each copositive programming problem can be approximated to any given accuracy by a sufficiently large SDP. Of course, the size of the SDP can be exponential in the size of the copositive program.

In the next subsection we will review the approach of Parrilo and subsequently work out the implications for the copositive formulation of the maximum stable set problem. We will also look at a weaker, LP-based approximation scheme.

3.1. Representations as sum-of-squares and polynomials with nonnegative coefficients. We can represent the copositivity requirement for an $(n \times n)$ symmetric matrix M as

$$P(x) := (x \circ x)^T M (x \circ x) = \sum_{i,j=1}^{n} M_{ij} x_i^2 x_j^2 \geq 0 \quad \forall x \in \mathbb{R}^n, \tag{10}$$

where "$\circ$" indicates the componentwise (Hadamard) product. We therefore wish to know whether the polynomial $P(x)$ is nonnegative for all $x \in \mathbb{R}^n$. Although one cannot answer this question in polynomial time in general, one can decide in polynomial time whether $P(x)$ can be written as a sum-of-squares. Before we give a formal exposition of the methodology, we give an example which illustrates the basic idea.

Example 3.1 (see Parrilo [20]). We show how to obtain a sum-of-squares decomposition for the polynomial $2x_1^4 + 2x_1^3 x_2 - x_1^2 x_2^2 + 5x_2^4$.

$$\begin{aligned}
&2x_1^4 + 2x_1^3 x_2 - x_1^2 x_2^2 + 5x_2^4 \\
&= \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_1 x_2 \end{bmatrix}^T \begin{bmatrix} 2 & 0 & 1 \\ 0 & 5 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_1 x_2 \end{bmatrix} \\
&= \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_1 x_2 \end{bmatrix}^T \begin{bmatrix} 2 & -\lambda & 1 \\ -\lambda & 5 & 0 \\ 1 & 0 & -1+2\lambda \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_1 x_2 \end{bmatrix} \quad \forall \lambda \in \mathbb{R}.
\end{aligned}$$

For $\lambda = 3$ the coefficient matrix is positive semidefinite, and we obtain a sum-of-squares decomposition by taking a Choleski decomposition of the coefficient matrix.

Following the idea in the example, we represent $P(x)$ via

$$P(x) = \tilde{x}^T \tilde{M} \tilde{x}, \tag{11}$$

where $\tilde{x} = [x_1^2, \ldots, x_n^2, x_1 x_2, x_1 x_3, \ldots, x_{n-1} x_n]^T$ and $\tilde{M}$ is a symmetric matrix of order $n + \frac{1}{2} n(n-1)$.

Note that—as in the example—$\tilde{M}$ is not uniquely determined. The nonuniqueness follows from the identities:

$$\begin{aligned}
(x_i x_j)^2 &= (x_i^2)(x_j)^2, \\
(x_i x_j)(x_i x_k) &= (x_i^2)(x_j x_k), \\
(x_i x_j)(x_k x_l) &= (x_i x_k)(x_j x_l) = (x_i x_l)(x_j x_k).
\end{aligned}$$

It is easy to see that the possible choices for $\tilde{M}$ define an affine space.

Condition (10) will certainly hold if at least one of the following two conditions holds:

1. A representation of $P(x) = \tilde{x}^T \tilde{M} \tilde{x}$ exists with $\tilde{M}$ symmetric positive semidefinite. In this case we obtain the sum-of-squares decomposition $P(x) = \|L\tilde{x}\|^2$, where $L^T L = \tilde{M}$ denotes the Choleski factorization of $\tilde{M}$.
2. All the coefficients of $P(x)$ are nonnegative.

Note that the second condition implies the first.

Parrilo showed that $P(x)$ in (10) allows a sum-of-squares decomposition if and only if $M \in \mathcal{S}_n^+ + \mathcal{N}_n$, which is a well-known sufficient condition for copositivity. Let us define the cone $\mathcal{K}_n^0 := \mathcal{S}_n^+ + \mathcal{N}_n$. Similarly, $P(x)$ has only nonnegative coefficients if and only if $M \in \mathcal{N}_n$, which is a weaker sufficient condition for copositivity, and we define $\mathcal{C}_n^0 = \mathcal{N}_n$.

Higher-order sufficient conditions can be derived by considering the polynomial

$$P^{(r)}(x) = P(x)\left(\sum_{i=1}^n x_i^2\right)^r = \left(\sum_{i,j=1}^n M_{ij}x_i^2x_j^2\right)\left(\sum_{i=1}^n x_i^2\right)^r \tag{12}$$

and asking whether $P^{(r)}(x)$—which is a homogeneous polynomial of degree $2(r+2)$—has a sum-of-squares decomposition, or whether it has only nonnegative coefficients.

For $r = 1$, Parrilo showed that a sum-of-squares decomposition exists if and only if[2] the following system of linear matrix inequalities has a solution:

$$M - M^{(i)} \in \mathcal{S}_n^+, \quad i = 1, \dots, n, \tag{13}$$

$$M_{ii}^{(i)} = 0, \quad i = 1, \dots, n, \tag{14}$$

$$M_{jj}^{(i)} + 2M_{ij}^{(j)} = 0, \quad i \neq j, \tag{15}$$

$$M_{jk}^{(i)} + M_{ik}^{(j)} + M_{ij}^{(k)} \geq 0, \quad i < j < k, \tag{16}$$

where $M^{(i)}$ $(i = 1, \dots, n)$ are symmetric matrices.

Similarly, $P^{(1)}(x)$ has only nonnegative coefficients if M satisfies the above system, but with $\mathcal{S}_n^+$ replaced by $\mathcal{N}_n$.

Note that the sets of matrices which satisfy these respective sufficient conditions for copositivity define two respective convex cones. In fact, this is generally true for all r.

DEFINITION 3.1. *Let any integer $r \geq 0$ be given. The convex cone $\mathcal{K}_n^r$ consists of the matrices $M \in \mathcal{S}_n$ for which $P^{(r)}(x)$ in* (12) *has a sum-of-squares decomposition; similarly, we define the cone $\mathcal{C}_n^r$ as the cone of matrices $M \in \mathcal{S}_n$ for which $P^{(r)}(x)$ in* (12) *has only nonnegative coefficients.*

Note that $\mathcal{C}_n^r \subset \mathcal{K}_n^r$ for all $r = 0, 1, \dots$. (If $P(x)$ has only nonnegative coefficients, then it obviously has a sum-of-squares decomposition. The converse is not true in general.)

3.2. Upper bounds on the order of approximation. Every strictly copositive M lies in some cone $\mathcal{C}_n^r$ for r sufficiently large; this follows from the celebrated theorem of Pólya.

THEOREM 3.2 (see Pólya [21]). *Let f be a homogeneous polynomial which is positive on the simplex*

$$\Delta = \left\{ z \in \mathbb{R}^n \; : \; \sum_{i=1}^n z_i = 1, \; z \geq 0 \right\}.$$

For sufficiently large N all the coefficients of the polynomial

$$\left(\sum_{i=1}^n z_i\right)^N f(z)$$

[2] In fact, Parrilo [20] only proved the "if"-part; the proof of the converse is straightforward but tedious and can be done using the proof technique described in section 5.3 of [20].

are positive.

One can apply this theorem to the copositivity test (10) by letting $f(z) = z^T M z$ and associating $x \circ x$ with z.

In summary, we have the following theorem.

THEOREM 3.3. *Let M be strictly copositive. One has*

$$\mathcal{N}_n = \mathcal{C}_n^0 \subset \mathcal{C}_n^1 \subset \cdots \subset \mathcal{C}_n^N \ni M$$

and consequently

$$\mathcal{S}_n^+ + \mathcal{N}_n = \mathcal{K}_n^0 \subset \mathcal{K}_n^1 \subset \cdots \subset \mathcal{K}_n^N \ni M$$

for some sufficiently large N.

A tight upper bound on the size of N in Theorem 3.2 has recently been given by Powers and Reznik [22].

THEOREM 3.4 (see [22]). *Let*

$$f(z) = \sum_j \beta_j \prod_{i=1}^n z_i^{\alpha_{ij}}$$

be a homogeneous polynomial of degree d ($\sum_{i=1}^n \alpha_{ij} = d$ for all j) which is positive on the simplex Δ. The polynomial

$$\left(\sum_{i=1}^n z_i\right)^N f(z)$$

has positive coefficients if

$$N > \frac{d(d-1)L}{2\kappa} - d,$$

where

$$L = \max_j \frac{\alpha_{1j}!\alpha_{2j}!\ldots\alpha_{nj}!}{d!}|\beta_j|$$

and

$$\kappa = \min_{z\in\Delta} f(z).$$

For the problem of checking the copositivity of M we have the following.

COROLLARY 3.5. *If a symmetric $(n \times n)$ matrix M is strictly copositive, then the function*

$$P^N(z) = \left(\sum_{i,j=1}^n M_{ij} z_i z_j\right)\left(\sum_{i=1}^n z_i\right)^N$$

has only nonnegative coefficients if

$$N > L/\kappa - 2,$$

where

$$L = \max_{i,j} |M_{ij}| \tag{17}$$

and

$$\kappa = \min_{z \in \Delta} z^T M z. \tag{18}$$

Proof. The function f in Theorem 3.4 is given by $f(z) = z^T M z$ in this case. The exponents α_{ij} can now take only the values 0, 1, or 2; $d = 2$; and the coefficients β_j correspond to the entries of M. □

Note that κ is a "condition number" of M, which can be arbitrarily small, and cannot be computed in polynomial time in general unless $P = NP$.

COROLLARY 3.6. *If a strictly copositive matrix M satisfies $L/\kappa \le r + 1$, where L and κ are respectively defined in* (17) *and* (18)*, then $M \in \mathcal{C}_n^r \subset \mathcal{K}_n^r$.*

Proof. The proof follows immediately from the definition of $\mathcal{C}_n^r$ and Corollary 3.5. □

4. Application to the maximum stable set problem. We can now define successive approximations to the stability number. In particular, we define successive LP-based approximations via

$$\zeta^{(r)}(G) = \min_{\lambda} \left\{ \lambda \;:\; Q = \lambda(I + A) - ee^T \in \mathcal{C}_n^r \right\} \tag{19}$$

for $r = 0, 1, 2, \ldots$, where we use the convention that $\zeta^{(r)}(G) = \infty$ if the problem is infeasible.

Similarly, we define successive SDP-based approximations via

$$\vartheta^{(r)}(G) = \min_{\lambda} \left\{ \lambda \;:\; Q = \lambda(I + A) - ee^T \in \mathcal{K}_n^r \right\} \tag{20}$$

for $r = 0, 1, 2, \ldots$. Note that we have merely replaced the copositive cone $\mathcal{C}_n$ in (9) by its respective approximations $\mathcal{C}_n^r$ and $\mathcal{K}_n^r$.

The minimum in (20) is always attained. The proof follows directly from the conic duality theorem if we note that $\lambda = n + 1$ always defines a matrix Q in the interior of $\mathcal{K}_n^0$ (and therefore in the interior of $\mathcal{K}_n^r \supset \mathcal{K}_n^0$ for all $r = 1, 2, \ldots$) via (20) and that

$$X^0 := \frac{1}{n^2 + n + |E|}(nI + ee^T)$$

is always strictly feasible in the associated primal problem:

$$\vartheta^{(r)}(G) = \max \left\{ \mathrm{Tr}\left(ee^T X\right) \;:\; \mathrm{Tr}\left((A + I)X\right) = 1,\ X \in (\mathcal{K}_n^r)^* \right\}.$$

The strict feasibility of X^0 follows from the fact that it is in the interior of $\mathcal{C}_n^*$: For any copositive matrix $Y \in \mathcal{C}_n$ we have

$$\mathrm{Tr}(X^0 Y) = \frac{1}{n^2 + n + |E|}\left(n\mathrm{Tr}(Y) + e^T Y e\right).$$

This expression can be zero only if Y is the zero matrix. In other words, $\mathrm{Tr}(X^0 Y) > 0$ for all nonzero $Y \in \mathcal{C}_n$, which means that X^0 is in the interior of $\mathcal{C}_n^*$. Consequently, X^0 is also in the interior of $(\mathcal{K}_n^r)^*$ for all r, since $\mathcal{C}_n^* \subset (\mathcal{K}_n^r)^*$ $(r = 0, 1, \ldots)$.

Note that

$$\alpha(G) \le \vartheta^{(r)}(G) \le \zeta^{(r)}(G), \quad r = 0, 1, \ldots,$$

since $\mathcal{C}_n^r \subset \mathcal{K}_n^r \subset \mathcal{C}_n$.

4.1. An upper bound for the number of liftings. We can now prove our main result.

THEOREM 4.1. *Let a graph $G(V,E)$ be given with stability number $\alpha(G)$, and let $\zeta^{(i)}$ $(i = 0, 1, 2, \dots)$ be defined as in* (19). *One has*

$$\zeta^{(0)} \geq \zeta^{(1)} \geq \cdots \geq \lfloor \zeta^{(r)} \rfloor = \alpha(G)$$

for $r \geq \alpha(G)^2$. Consequently, also $\lfloor \vartheta^{(r)} \rfloor = \alpha(G)$ for $r \geq \alpha(G)^2$.

Proof. Denote, as in the proof of Lemma 2.3,

$$Q^*_\epsilon = (1+\epsilon)\alpha(G)(I + A) - ee^T$$

for a given $\epsilon \geq 0$.

We will now prove that $Q^*_\epsilon \in \mathcal{C}^r_n$ for $r \geq \alpha(G)^2 - \alpha(G) - 2$ if

$$\epsilon := \frac{1}{\alpha(G) + 1/[\alpha(G) - 1]}. \tag{21}$$

Note that if we choose ϵ in this way, then Q^*_ϵ corresponds to a feasible solution of (19), where $\lambda = (1+\epsilon)\alpha(G) < 1 + \alpha(G)$, and we can therefore round down this value of λ to obtain $\alpha(G)$.

We proceed to bound the parameters κ and L in Corollary 3.6 for the matrix Q^*_ϵ.

- The value L is given by $L = (1+\epsilon)\alpha(G) - 1$.
- The condition number κ is given by $\kappa = \epsilon$, by (7).

Now we have

$$L/\kappa = \frac{(1+\epsilon)\alpha(G) - 1}{\epsilon} = \alpha(G)^2 + 1. \tag{22}$$

From Corollary 3.6 it now follows that $Q^*_\epsilon \in \mathcal{C}^r_n$ for $r \geq \alpha(G)^2$. □

Remark 4.1. If we are only interested in computing a $\zeta^{(r)} \leq (1+\epsilon)\alpha(G)$ for a given $\epsilon > 0$, then it is sufficient to choose $r = \alpha(G)/\epsilon$. To see this, note that by (22) we have

$$L/\kappa = \frac{(1+\epsilon)\alpha(G) - 1}{\epsilon} \leq \alpha(G)/\epsilon + 1 \equiv r + 1,$$

so that $Q^*_\epsilon = (1+\epsilon)\alpha(G)(I + A) - ee^T \in \mathcal{C}^r_n$ by Corollary 3.6.

Remark 4.2. The bound $\alpha(G)^2$ in Theorem 4.1 on the number of liftings can be compared to the $n - \alpha(G) - 1$ bound for the LP-based lift-and-project scheme by Lovász–Schrijver [16].[3] For families of graphs where $\alpha(G) < O(\sqrt{n})$, our LP-based lifting scheme requires fewer liftings in the worst case. This bound is satisfied, for example, by random graphs with expected edge density $\frac{1}{2}$, where one almost always has $\alpha(G) \leq 2\log_2 n$ for $n \gg 0$ (see, e.g., p. 148 of [1]).

Example 4.1. Consider the case in which $G(V,E)$ is the 5-cycle (C_5). It is well known that $\alpha(G) = 2$ and $\vartheta(G) = \vartheta'(G) = \sqrt{5}$ in this case.

[3] Lovász and Schrijver called this bound the *N-index*.

We will show that $\vartheta^{(1)}(G) = 2$; to this end, note that the matrix

$$Q = \begin{pmatrix} 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 \end{pmatrix} \tag{23}$$

corresponds to a feasible solution of (20) for $r = 1$, with $\lambda = 2$. The feasibility follows from the known fact that Q in (23) is in $\mathcal{K}_n^1$ (but not in $\mathcal{K}_n^0 = \mathcal{S}_n^+ + \mathcal{N}_n$); see, e.g., [20].

Example 4.2. Let $G = (V, E)$ be the complement of the graph of an icosahedron (see, e.g., [6]), where $\alpha(G) = 3$. One can solve the relevant semidefinite programming problem to obtain $\vartheta^{(1)}(G) = 1 + \sqrt{5} \approx 3.236068$.

Although $\lfloor \vartheta^{(1)}(G) \rfloor = \alpha(G)$, one has $Q := \alpha(G)(A+I) - ee^T \notin \mathcal{K}_n^1$. Thus Q gives an example of a 12×12 copositive matrix which is not in $\mathcal{K}_n^1$. This gives a partial answer to the following question posed by Parrilo [20]: "Do the copositive cone and $\mathcal{K}_n^1$ coincide for $n \times n$ matrices up to a certain size?" For this size a known lower bound is $n \geq 4$ (for $n \times n$ matrices with $n \leq 4$, $\mathcal{C}_n$ and $\mathcal{K}_n^0$ still coincide), and an upper bound is $n \leq 11$ (by this example).

4.2. Lower bounds on the number of liftings. The following theorem shows that the LP-based approximations always require at least $\alpha(G) - 1$ liftings to compute $\alpha(G)$.

THEOREM 4.2. *Let a graph $G = (V, E)$ with stability number $\alpha(G)$ be given. If $\zeta^{(r)}(G) < \infty$, then $r \geq \alpha(G) - 1$.*

Proof. Let $(1, \ldots, \alpha)$ be a maximum stable set, where $\alpha = \alpha(G)$. Then for $r = \alpha - 2$ the polynomial

$$z^T \left(t(A + I) - ee^T \right) z (e^T z)^r$$

has a monomial $C z_1 z_2 \ldots z_\alpha$ with $C < 0$ for any value of t. This shows that problem (19) is infeasible if $r < \alpha(G) - 1$. ∎

Example 4.3. Here we show that although $\alpha(G) - 1$ liftings are necessary for computing $\alpha(G)$ via the LP-based approximations, this number of liftings is not sufficient in general. For the 4-node graph with one edge we have

$$z^T(\alpha(G)(I + A) - ee^T)z = 3z_1^2 + 3z_2^2 + 6z_1z_2 + 3z_3^2 + 3z_4^2 - (z_1 + z_2 + z_3 + z_4)^2,$$

as $\alpha(G) = 3$ in this case, which clearly has negative coefficients. In order to get only nonnegative coefficients, we have to multiply this quadratic form by $(\sum_{i=1}^4 z_i)^6$.

5. The strength of low-order relaxations. In this section we investigate the strength of the approximations $\vartheta^{(r)}$ and $\zeta^{(r)}$ to $\alpha(G)$ for $r = 0$ and $r = 1$.

THEOREM 5.1. *If $G = (V, E)$ has $\alpha(G) = 2$, then $\zeta^{(1)}(G) \leq 3$.*

Proof. Let A be the adjacency matrix of a graph $G = (V, E)$ with $\alpha(G) = 2$, and let

$$Q(z) = z^T(3A + 3I - ee^T)z(z_1 + \cdots + z_n) := Q_+(z) - Q_-(z),$$

where

$$Q_+(z) := (3z^T(A+I)z)(z_1 + \cdots + z_n), \qquad Q_-(z) = (z_1 + \cdots + z_n)^3.$$

We will show that $Q(z)$ has only nonnegative coefficients, which in turn implies the theorem. The monomials of $Q_+(z)$ can be classified as follows:

$$\begin{array}{ll} 3z_i^3 & \forall\, i, \\ 3z_i z_j^2 & \forall\, i \neq j, \\ 6(A_{ij} + A_{ik} + A_{jk})z_i z_j z_k & \forall i < j < k. \end{array}$$

Note that $A_{ij} + A_{ik} + A_{jk} \geq 1$ if $i < j < k$, since $\alpha(G) = 2$.

The monomials of $Q_-(z)$ are as follows:

$$\begin{array}{ll} z_i^3 & \forall\, i, \\ 3z_i z_j^2 & \forall\, i \neq j, \\ 6z_i z_j z_k & \forall\, i < j < k. \end{array}$$

Hence $Q(z)$ has only nonnegative coefficients, as for every monomial of $Q_-(z)$ there is a monomial with the same variables in $Q_+(z)$ with a coefficient at least as large. □

Next we show that $\vartheta^{(0)}$ coincides with the ϑ'-function of Schrijver [25], which in turn can be seen as a strengthening of the Lovász ϑ-approximation to $\alpha(G)$.

LEMMA 5.2. *Let a graph $G = (V, E)$ be given with adjacency matrix A, and let ϑ' denote the Schrijver ϑ'-function* [25]:

$$\vartheta'(G) = \max\left\{ Tr\left(ee^T X\right) \ : \ Tr(AX) = 0,\ Tr(X) = 1,\ X \in (\mathcal{K}_n^0)^* \right\}.$$

Then

$$\vartheta'(G) = \vartheta^{(0)}(G).$$

Proof. Recall that

$$\vartheta^{(0)}(G) = \min_{\lambda} \left\{ \lambda \ : \ \lambda(I + A) - ee^T \in \mathcal{K}_n^0 \right\}, \tag{24}$$

whereas the dual formulation for $\vartheta'(G)$ is

$$\vartheta'(G) = \min_{\lambda, y} \left\{ \lambda \ : \ \lambda I + yA - ee^T \in \mathcal{K}_n^0 \right\}. \tag{25}$$

Further recall that $\mathcal{K}_n^0 = \mathcal{S}_n^+ + \mathcal{N}_n$, and let

$$\lambda I + yA - ee^T = S + N, \qquad \text{where } S \in \mathcal{S}_n^+ \text{ and } N \in \mathcal{N}_n. \tag{26}$$

Without loss of generality we assume $N_{ii} = 0$ for all $i \in \{1, \ldots, n\}$, as the sum of two positive semidefinite matrices is positive semidefinite, and thus the diagonal part of N can be added to S and subtracted from N.

Assume $A_{ij} \neq 0$. Note that our choice of S and N is such that $S_{ii} = \lambda - 1$. Thus, as $S_{ij} + N_{ij} = y - 1$ and $S_{ii} \geq S_{ij}$,[4] one obtains $\lambda - 1 + N_{ij} \geq y - 1$, and so

[4]Here we use the fact that $S \in \mathcal{S}_n^+$ and has a constant diagonal.

$N_{ij} \geq y - \lambda$. Hence $N + (\lambda - y)A \in \mathcal{N}_n$. Therefore $\lambda(I + A) - ee^T \in \mathcal{K}_n^0$ as long as (26) holds. Hence we can always assume that $y = \lambda$. ∎

Remark 5.1. As far as we know, our simplified formulation of the Schrijver ϑ'-function, namely,

$$\vartheta'(G) = \max \left\{ Tr\left(ee^T X\right) \ : \ Tr\left((A+I)X\right) = 1, \ X \in (\mathcal{K}_n^0)^* \right\},$$

is not mentioned in the literature.

Let us restate the definition of $\vartheta^{(1)}(G)$ by using (13)–(16) as follows:

$$\vartheta^{(1)}(G) := \min \beta \qquad \text{subject to} \tag{27}$$

$$\beta(I + A) - ee^T - M^{(i)} \in \mathcal{S}_n^+, \quad i = 1, \dots, n, \tag{28}$$

$$M_{ii}^{(i)} = 0, \quad i = 1, \dots, n, \tag{29}$$

$$M_{jj}^{(i)} + 2M_{ij}^{(j)} = 0, \quad i \neq j, \tag{30}$$

$$M_{jk}^{(i)} + M_{ik}^{(j)} + M_{ij}^{(k)} \geq 0, \quad i < j < k, \tag{31}$$

where $M^{(i)}$ $(i = 1, \dots, n)$ are symmetric matrices.

For $v \in V$, denote by $v^\perp$ the the union of the neighborhood[5] of v with v itself, and for $D \subseteq V$ denote by $G(D)$ the subgraph of G induced on D (that is, $G(D) = (D, \{(x,y) \in E \mid x, y \in D\})$). Also, $A(D)$ will denote the adjacency matrix of $G(D)$.

Theorem 5.3. *The system of LMIs* (27)–(31) *has a feasible solution with* $\beta = 1 + \max_{k \in V}(\vartheta'(G(V - k^\perp)))$ *and* $M_{ij}^{(i)} = 0$ *for all* i, j. *Thus*

$$\vartheta^{(1)}(G) \leq 1 + \max_{k \in V}(\vartheta'(G(V - k^\perp))).$$

In particular, if $G(V - k^\perp)$ *is perfect for all* $k \in V$, *where* $k^\perp \neq V$, *then* $\vartheta^{(1)}(G) = \beta = \alpha(G)$.

Proof. Define $M = \beta(I + A) - ee^T$, and set $M_{ij}^{(i)} = 0$ for all i, j. We now apply the Schur lemma with respect to the ith row and ith column to the matrix $M - M^{(i)}$ for each $i = 1, \dots, n$. This transforms (28) into

$$\beta I_{n-1} + \beta A(V - \{i\}) - e_{n-1}e_{n-1}^T - \Lambda^{(i)} - \frac{1}{\beta - 1} m_i m_i^T \in \mathcal{S}_n^+, \quad i = 1, \dots, n, \tag{32}$$

where $\Lambda^{(i)}$ is obtained from $M^{(i)}$ by removing the ith row and column, and m_i is the ith row of M with the ith entry removed. In other words, $(m_i)_j = \beta - 1$ if $(i,j) \in E$, and $(m_i)_j = -1$, otherwise.

By (30), the matrix $\Lambda^{(i)}$ has zero diagonal. Thus the jth diagonal entry of the matrix on the left-hand side of (32) is zero if $(i,j) \in E$. This means that the corresponding row and column of this matrix must be zero.

Having fixed some variables as indicated, we now work out the implications from the constraint (31). There are several cases to distinguish for $\Lambda_{jk}^{(i)}$ with $j < k$ and $(i,j) \in E$, as follows:

1. $(i,k) \in E$; here $(m_i m_i^T)_{jk} = (\beta - 1)^2$.
   (a) $(j,k) \in E$; here $\Lambda_{jk}^{(i)} = 0$.
   (b) $(j,k) \notin E$; here $\Lambda_{jk}^{(i)} = -\beta$.

[5]By neighborhood of v, we mean the set of vertices adjacent to v in G.

2. $(i,k) \notin E$; here $(m_i m_i^T)_{jk} = 1 - \beta$.
   (a) $(j,k) \in E$; here $\Lambda^{(i)}_{jk} = \beta$.
   (b) $(j,k) \notin E$; here $\Lambda^{(i)}_{jk} = 0$.

Note that in case 1(b), the choice $\Lambda^{(i)}_{jk} = -\beta < 0$ does not violate (31), as $\Lambda^{(j)}_{ki}$ and $\Lambda^{(k)}_{ij}$ fall under case 2(a), and thus $\Lambda^{(i)}_{jk} + \Lambda^{(j)}_{ki} + \Lambda^{(k)}_{ij} = \beta > 0$. In case 1(a), all the Λ's where the indices i, j, k appear are set to 0.

Finally, in case 2(b), one has that $\Lambda^{(i)}_{jk} = 0$, and $\Lambda^{(j)}_{ki} = 0$ together with (31) imply $\Lambda^{(k)}_{ij} \geq 0$.

For each i, denote $n_i = |V - i^{\perp}|$, and define $\Delta^{(i)}$ as the $n_i \times n_i$ matrix of variables that is obtained from $\Lambda^{(i)}$ after all the variables have been fixed as indicated. In other words, $\Lambda^{(i)}_{jk}$ corresponds to $\Delta^{(i)}_{jk}$ if and only if neither j nor k are adjacent to i in G.

We arrive at the following SDP:

$$\begin{gathered}\beta^* := \min \beta \qquad \text{subject to} \\ \beta(I_{n_i} + A(V - i^{\perp})) - \left(1 + \frac{1}{\beta - 1}\right) e_{n_i} e_{n_i}^T - \Delta^{(i)} \in \mathcal{S}_n^+ \qquad \forall i \in V, \\ \Delta^{(i)}_{jk} \geq 0 \qquad \forall i \in V,\ j,k \in V - i^{\perp},\ (j,k) \in E, \\ \Delta^{(i)}_{jk} + \Delta^{(j)}_{ki} + \Delta^{(k)}_{ij} \geq 0 \qquad \forall i \in V,\ j,k \in V - i^{\perp}.\end{gathered}$$

Note that $\beta^* \geq \vartheta^{(1)}(G)$. Multiplying both sides of all the constraints by $1 - 1/\beta$ and setting $(1 - 1/\beta)\Delta^{(i)} = \Omega^{(i)}$, one obtains

$$\beta^* = \min \beta \qquad \text{subject to} \tag{33}$$

$$(\beta - 1)(I_{n_i} + A(V - i^{\perp})) - e_{n_i} e_{n_i}^T - \Omega^{(i)} \in \mathcal{S}_n^+ \qquad \forall i \in V, \tag{34}$$

$$\Omega^{(i)}_{jk} \geq 0 \qquad \forall i \in V,\ j,k \in V - i^{\perp},\ (j,k) \in E, \tag{35}$$

$$\Omega^{(i)}_{jk} + \Omega^{(j)}_{ki} + \Omega^{(k)}_{ij} \geq 0 \qquad \forall i \in V,\ j,k \in V - i^{\perp}. \tag{36}$$

Replacing (36) by a stronger constraint $\Omega^{(i)}_{jk} \geq 0$ $(i \in V)$, we obtain n problems

$$\begin{gathered}\beta_i^* := \min \beta_i \qquad \text{subject to} \\ (\beta_i - 1)(I_{n_i} + A(V - i^{\perp})) - e_{n_i} e_{n_i}^T - \Omega \in \mathcal{S}_n^+, \\ \Omega_{jk} \geq 0 \qquad \forall j,k \in V,\end{gathered}$$

so that

$$\max_{i \in V} \beta_i^* \geq \beta^* \geq \vartheta^{(1)}(G) \geq \alpha(G). \tag{37}$$

By the definition of $\vartheta^{(0)}$, one has $\beta_i^* - 1 = \vartheta^{(0)}(G(V - i^{\perp}))$, which equals $\vartheta'(G(V - i^{\perp}))$ by Lemma 5.2. If $G(V - i^{\perp})$ is perfect for all $i \in V$, then

$$\max_{i \in V} \beta_i^* = \max_{i \in V} \vartheta^{(0)}(G(V - i^{\perp})) + 1 = \max_{i \in V} \alpha(G(V - i^{\perp})) + 1 = \alpha(G).$$

Thus $\vartheta^{(1)}(G) = \alpha(G)$ by (37). $\square$

Thus, for instance, the 5-cycle example of the previous section can be generalized to all cycles.

COROLLARY 5.4. *Let $G(V, E)$ be a cycle of length n. One has $\vartheta^{(1)}(G) = \alpha(G)$. Similarly, $\alpha(G) = \vartheta^{(1)}(G)$ if G is a wheel.*

Proof. Let $G = (V, E)$ be a cycle of length n. The required result now immediately follows from Theorem 5.3 by observing that $G(V - v^{\perp})$ is an $(n-3)$-path for all $v \in V$. The proof for wheels is similar. □

Also, complements of triangle-free graphs are recognized.

COROLLARY 5.5. *If $G = (V, E)$ has stability number $\alpha(G) = 2$, then $\vartheta^{(1)}(G) = 2$.*

Proof. The proof immediately follows from Theorem 5.3 by observing that $G(V - v^{\perp})$ is a clique (or the empty graph) for all $v \in V$. □

As a consequence, the complements of cycles or wheels are also recognized. The proof proceeds in the same way as before and is therefore omitted.

COROLLARY 5.6. *Let $G(V, E)$ be the complement of a cycle or of a wheel. In both cases one has $\vartheta^{(1)}(G) = \alpha(G)$.*

Remark 5.2. It is worth mentioning that neither the upper bound $\beta = 1 + \max_{k \in V}(\vartheta'(G(V - k^{\perp})))$ on $\vartheta^{(1)}$ given in Theorem 5.3 nor the upper bound β^* used in its proof is sharp. This is demonstrated by the example of the 7-vertex graph G obtained by taking an isolated node and a pentagon and joining these six nodes with an extra node. (The result can be viewed as a pentagon "umbrella.") Then $\beta^* = 3.068$, while $\vartheta^{(1)}(G) = \alpha(G) = 3$, and the bound given by Theorem 5.3 is $\beta = 1 + \max_{k \in V}(\vartheta'(G(V - k^{\perp}))) = 1 + \sqrt{5} \approx 3.23$.

We conjecture that the result of Corollary 5.5 can be extended to include all values of α.

CONJECTURE 5.1. *If $G = (V, E)$ has stability number $\alpha(G)$, then $\vartheta^{(\alpha(G)-1)}(G) = \alpha(G)$.*

Note that we have proven the conjecture for $\alpha(G) \leq 2$.

6. Conclusions and future work. We have introduced two successive lifting procedures for computing the stability number $\alpha(G)$ of a graph. The first procedure involves generalizations of the Schrijver ϑ'-function, which in turn is a generalization of the well-known Lovász ϑ-function. These generalized ϑ-functions were denoted by $\vartheta^{(r)}$ $(r = 0, 1, \dots)$, where $\vartheta^{(0)}(G) = \vartheta'(G)$ for all $G = (V, E)$, and $\vartheta^{(0)}(G) \geq \vartheta^{(1)}(G) \geq \dots \geq \lfloor \vartheta^{(N)} \rfloor = \alpha(G)$ for some sufficiently large N. We have also introduced related LP-based approximations to $\alpha(G)$, namely, the numbers $\zeta^{(r)} \geq \vartheta^{(r)}$, which satisfy $\lfloor \zeta^{(N)} \rfloor = \alpha(G)$ if $N \geq \alpha^2(G)$. This can be compared to the $n - \alpha(G) - 1$ bound for the LP-based lift-and-project scheme by Lovász and Schrijver [16]. For classes of graphs where $\alpha(G) < O(\sqrt{n})$, our procedure therefore requires fewer liftings in the worst case. At step r of the respective procedures, an SDP (respectively, LP) problem involving matrix variables of size $n^{r+1} \times n^{r+1}$ is solved.

The underlying idea for these approximations was to write the maximum stable set problem as a conic linear program over the cone of copositive matrices, and to subsequently perform successive approximations of this cone by using linear (matrix) inequalities. This link between copositive matrices and the maximum stable set has also allowed us to give a partial answer to a question posed by Parrilo [20] concerning a class of copositive matrices (see Example 4.2).

There have been several—seemingly different—lift-and-project strategies for approximating combinatorial optimization problems. Apart from the approach of Lovász and Schrijver [16] (see also [7, 11]) for the stable set polytope, Anjos and Wolkowicz [2] have introduced a technique of successive Lagrangian relaxations for the MAX-CUT problem, which also leads to semidefinite programming relaxations of size $(n^r \times n^r)$ after r relaxations. Most recently, Laserre [13, 14] has introduced yet another lift-and-project approach, based on the properties of moment matrices. Laurent [11, 12] has recently shown the relationship between these approaches. In the same vein, it would be very interesting to explore possible links between the approach of Lovász and Schrijver and the lifting scheme introduced in this paper. In particular, it seems unlikely that the bound on the number of liftings ($r = \alpha(G)^2$) is tight: The Lovász–Schrijver SDP-based procedure only requires $\alpha(G)$ liftings in the worst case. We conjecture that the proof of Theorem 5.3 can be extended to show that $\alpha(G) - 1$ liftings always suffice for our SDP-based lifting scheme.

Another interesting line of research is to further investigate the theoretical properties of the $\vartheta^{(1)}$ number introduced in this paper. Actual computation of this number involves SDPs with $n^2 \times n^2$ matrices having an $n \times n$ block diagonal structure, and it can still be done for graphs of small size (say $n \leq 30$) with current interior point technology.

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