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Approximation of the Viability Kernel

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Abstract

We study recursive inclusions $x^{n+1} \in G(x^n)$. For instance such systems appear for discrete finite difference inclusions $x^{n+1} \in G_\rho(x^n)$ where $G_\rho := \mathbf{1} + \rho F$. The discrete viability kernel of G_ρ, i.e. the largest discrete viability domain, can be an internal approximation of the viability kernel of K under F. We study discrete and finite dynamical systems. In the Lipschitz case we get a generalization to differential inclusions of Euler and Runge-Kutta methods. We prove first that the viability kernel of K under F can be approached by a sequence of discrete viability kernels :associated with $\Gamma_\rho(x) = x + \rho F(x) + \frac{Ml}{2}\rho^2\mathcal{B}$. Secondly, we show that it can be approached by finite viability kernels associated with $\Gamma^\alpha_{h\rho}(x) := x + \rho F(x) : x_h^{n+1} \in (\Gamma_{h\rho}(x_h^n) + \alpha(h)\mathcal{B}) \cap X_h$.

1 Introduction

Let X a finite dimentional vector space and K a compact subset of X.

Let us consider the differential inclusion:

$$(1) \qquad \begin{cases} \dot{x}(t) \in F(x(t)), \text{ for almost all } t \geq 0, \\ x(0) = x_0 \in K, \end{cases}$$

where F is a Marchaud map[1] defined from K to X.

With this inclusion, for a fixed $\rho > 0$, we associate the discrete explicit scheme:

$$(2) \qquad \begin{cases} \frac{x^{n+1} - x^n}{\rho} \in F(x^n), \text{ for all } n \geq 1, \\ x^0 = x_0 \in K, \end{cases}$$

We note G_ρ the set-valued map $G_\rho = \mathbf{1} + \rho F$ and the system (2) can be rewrited as follows:

$$(3) \qquad x^{n+1} \in G_\rho(x^n), \text{ for all } n \geq 0,$$

[1] A set-valued map $F : X \rightsquigarrow Y$ is a Marchaud map if

$$\begin{cases} \mathrm{Dom}(F) \neq \emptyset \\ F \text{ is upper-semicontinuous, convex compact valued} \\ \forall x \in \mathrm{Dom}(F), \ \|F(x)\| := \max_{y \in F(x)} \|y\| \leq c(\|x\| + 1) \end{cases}$$

The Viability Theory allows to study viable solutions of (1) and the subset of elements $x_0 \in K$ such that there exists at least a viable solution starting at x_0. On the other hand, we look for approximation of such solutions and we wonder how the set of initial points from which there exists at least a viable approximation solution to (2) and the set of initial points from which there exists at least a viable solution to (1) are related together.

These sets are called viability kernel of K under F or discrete viability kernel of K under $G\rho$. Byrnes & Isidori [5] and Frankowska & Quincampoix [8] have proposed algorithms which approximate the viability kernel of K under F when F is lipschitzian and K is closed.

We prove that, when F is a Marchaud map, for a good choice of discretizations $G\rho$, the sequence of discrete viability kernels of K under $G\rho$ converges to a subset contained in the viability kernel of K under F. Moreover it converges to the viability kernel if F is lipschitzian.

We show that similar results remain true when we introduce a discretization of the space and consider finite viability kernels.

2 Definitions and General Results

We call **discrete dynamical system** associated with G the following system:

$$x^{n+1} \in G(x^n), \quad \text{for all } n \geq 0, \tag{4}$$

We denote by

- $\mathcal{K}$ the set of all sequences from $I\!N$ to K.

- $\vec{x} := (x^0, ..., x^n, ...) \in \mathcal{X}$ a solution to discrete dynamical system (4)

- $\vec{S}_G(x^0)$ the set of solutions $\vec{x} \in \mathcal{X}$ to the discrete dynamical system starting at x_0.

A solution $\vec{x}$ is viable if and only if $\vec{x} \in \vec{S}_G(x) \cap \mathcal{K}$:

$$\begin{cases} x^{n+1} \in G(x^n), & \forall n \geq 0, \\ x^0 = x \in K \\ x^n \in K, & \forall n \geq 0. \end{cases} \tag{5}$$

It means that there exists a selection of equation (2) which remains in K at each step n.

We study the subset of initial points in K from which there exists at least one viable solution.

Definition 2.1 *Let $G : X \rightsquigarrow X$ be a set-valued map. A subset $D \subset X$ is a* discrete viability domain *of G if*

$$\forall\, x \in D,\ G(x) \cap D \neq \emptyset \tag{6}$$

Let K be a subset of X. The discrete viability kernel *of K under G is the largest closed discrete viability domain contained in K and we denote it $Viab_G(K)$.*

We can point out the following remark and properties:

$$Viab_G(K) = \{x \in K, \text{ such that } \vec{S}_G(x) \cap \mathcal{K} \neq \emptyset\}$$

Since $Viab_G(K)$ is the largest discrete viability domain contained in K, any solution of (5) starting from any initial point $x_0 \in K \backslash Viab_G(K)$ never meets the discrete viability kernel $Viab_G(K)$ while it remains in K.

Moreover any solution of (5) which does not start from $Viab_G(K)$ must leave K in a finite number of steps.

For all closed K_1, K_2 such that $K_1 \subset K_2 \subset X$, then

$$Viab_G(K_1) \subset Viab_G(K_2) \subset X \tag{7}$$

For all setvalued maps G_1, G_2 such that $\forall x \in K$: $G_1(x) \subset G_2(x)$, then

$$Viab_{G_1}(K) \subset Viab_{G_2}(K) \subset X \tag{8}$$

For all subset K' such that $Viab_G(K) \subset K' \subset K$, then

$$Viab_G(K') = Viab_G(K) \tag{9}$$

2.1 A Construction Method for Discrete Viability Kernel

Let us consider the sequence of subsets $K^0 = K,\ K^1, ...,\ K^n, ...$ defined as follows:

$$K^{n+1} := \{x \in K^n \text{ such that: } G(x) \cap K^n \neq \emptyset\}$$

We note

$$K^{\infty} := \bigcap_{n=0}^{+\infty} K^n$$

Proposition 2.1 *Let G: $X \rightsquigarrow X$ a upper semicontinuous set-valued map with closed values and K a compact subset of $Dom(G)$. Then*

$$K^{\infty} = Viab_G(K) \tag{10}$$

Proof — Let us prove that $\forall n \in I\!N$, $Viab_G(K) \subset K^n$. We have $Viab_G(K) \subset K^0$. Since G is upper semicontinuous,

$$K^1 = \{x \in K^0,\ G(x) \cap K^0 \neq \emptyset\}$$

is closed and, for all $x \in K^0 \backslash K^1$, it does not exit any viable solution starting from x. This implies recursively that

$$Viab_G(K) \cap (K^0 \backslash K^1) = \emptyset$$

and then:

$$Viab_G(K) \subset K^1 \subset K.$$

Let us assume that

$$Viab_G(K) = Viab_G(K^{n-1}) \subset K^{n-1}.$$

Since

$$K^n = \{x \in K^{n-1},\ G(x) \cap K^{n-1} \neq \emptyset\},$$

for all $x \in K^{n-1} \backslash K^n$, it does not exist any solution starting from x viable in K^{n-1}, and thus in K. Then $Viab_G(K) \cap (K^{n-1} \backslash K^n) = \emptyset$ and $Viab_G(K) = Viab_G(K^n) \subset K^n$. This implies that

$$Viab_G(K) \subset K^\infty.$$

Conversely, from definition 2.1, K^∞ is a viability domain: indeed for any $x \in K^\infty$, $\forall n \in \mathbb{N}\ :\ x \in K^{n+1}$ and then $G(x) \cap K^n \neq \emptyset$. Since for any fixed x, sets $G(x) \cap K^n$ form a decreasing sequence of non empty compact subsets, then $G(x) \cap K^\infty$ is non empty. We have proved that K^∞ is a viability domain of G and since $Viab_G(K)$ is the largest viability domain:

$$K^\infty \subset Viab_G(K). \qquad \blacksquare$$

Definition 2.2 *Let $G : X \rightsquigarrow X$ a set-valued map and $r > 0$. We call* extension of *G with a ball of radius r the set-valued map $G^r : X \rightsquigarrow X$ defined by :*

$$G^r(x) := G(x) + r\mathcal{B} \tag{11}$$

We consider the sequence of subsets $K^{r,0} = K,\ K^{r,1}, ...,\ K^{r,n}, ...$ defined as follows:

$$K^{r,n+1} := \{x \in K^{r,n} \text{ such that } G^r(x) \cap K^{r,n} \neq \emptyset\},\ K^{r,\infty} := \bigcap_{n=0}^{+\infty} K^{r,n} \tag{12}$$

If G is an upper semicontinuous set-valued map, G^r: $X \rightsquigarrow X$ is also upper semicontinuous and from Proposition 2.1:

$$\forall r > 0,\quad K^{r,\infty} = Viab_{G^r}(K) \tag{13}$$

When r decreases to 0, the viability kernel of K under G^r converges to the viability kernel of K under G:

Proposition 2.2 *Let G be upper semicontinuous and K a compact subset of X. The following property holds:*

$$Viab_G(K) = \bigcap_{r>0} Viab_{G^r}(K) \tag{14}$$

Proof — Let $x_0 \in \bigcap_{r>0} Viab_{G^r}(K)$. For all $r > 0$, Proposition 2.1 implies:

$$G^r(x_0) \cap Viab_{G^r}(K) \neq \emptyset, \ \ \forall r > 0$$

$G^r(x_0)$ is closed, $Viab_{G^r}(K)$ is compact and both are, from (8), decreasing sets when r decreases to zero. Also the intersection $G^r(x_0) \cap Viab_{G^r}(K)$ is a decreasing sequence of nonempty compact sets and

$$\bigcap_{r>0}(G^r(x_0) \cap Viab_{G^r}(K)) = G(x_0) \cap (\bigcap_{r>0} Viab_{G^r}(K)) \neq \emptyset$$

Then $\bigcap_{r>0} Viab_{G^r}(K)$ is a viability domain of G. Since $Viab_G(K)$ is the viability kernel of G, from definition 2.1 it contains $\bigcap_{r>0} Viab_{G^r}(K)$. ■

When G is a k-Lipschitz setvalued map, we have the following result giving an estimation of the growth of the discrete viability kernel when r increases:

Proposition 2.3 *Let $G : X \rightsquigarrow X$ a k-Lipschitz set-valued map, K a closed subset of X. Let $G^r := G + r\mathcal{B}$, $Viab_{G^r}(K)$ and $Viab_G(K)$ the discrete viability kernel of G^r and G respectively.*

Then:

$$Viab_G(K) + \frac{r}{k}\mathcal{B} \subset Viab_{G^r}(K), \ \ \forall r > 0 \tag{15}$$

(see footnote[2])

Proof — Let $r > 0$ given, $x \in Viab_G(K)$ arbitrairely choosen, $\eta < \min(\eta_0, \frac{r}{k})$ and $x' \in (\{x\} + \eta\mathcal{B}) \cap K$. Then

$$\left\{ \begin{array}{lll} i) & \forall x \in Viab_G(K) & G(x) \cap Viab_G(K) \neq \emptyset \\ ii) & \forall x \in Viab_{G^r}(K) & G^r(x) \cap Viab_{G^r}(K) \neq \emptyset \end{array} \right. \tag{16}$$

Since G is k-Lipschitz and $k\eta < r$,

$$G(x) \subset G(x') + k\|x - x'\|\mathcal{B} \subset G^r(x')$$

From (16), we deduce that

$$G^r(x') \cap Viab_G(K) \neq \emptyset$$

and since $Viab_G(K) \subset Viab_{G^r}(K)$

$$G^r(x') \cap Viab_{G^r}(K) \neq \emptyset$$

Then $\forall x \in Viab_G(K)$, $\forall x' \in (\{x\} + \eta\mathcal{B}) \cap K$, there exists a viable solution for the system associated with G^r starting from x' and thus $x' \in Viab_{G^r}(K)$. ■

[2] with the convention: $\emptyset + \eta\mathcal{B} = \emptyset$

3 Approximation of Viability Kernels for Finite Difference Inclusions

Let F a Marchaud map and Γ_ρ a sequence of setvalued maps which correspond to discretizations associated with the initial differential inclusion (1) satisfying:

$$(17) \quad \forall \epsilon > 0,\ \exists \rho_\epsilon > 0,\ \forall \rho \in]0, \rho_\epsilon]\ :\ Graph\left(\frac{\Gamma_\rho - \mathbf{1}}{\rho}\right) \subset Graph(F) + \epsilon\mathcal{B}$$

where $\mathcal{B}$ is the unit ball in $X \times X$.

We note

$$F_\rho := \frac{\Gamma_\rho - \mathbf{1}}{\rho}.$$

Assumption (17) implies that the graph of F contains the graphical upper limit[3] of F_ρ, that is to say that $Graph(F)$ contains the Painlevé-Kuratowski upper limit[4] of $Graph(F_\rho)$:

$$(18) \qquad \limsup_{\rho \to 0} Graph(F_\rho) \subset Graph(F)$$

Let K_ρ a sequence of subsets of X such that $K = \limsup_{\rho > 0} K_\rho$. Possible K_ρ may be constant.

Let $Viab_{\Gamma_\rho}(K_\rho)$ the discrete viability kernel of K_ρ under Γ_ρ.

3.1 The Viability Kernel Convergence Theorem

Theorem 3.1 *Let F a Marchaud map and Γ_ρ a sequence of set-valued maps such that $F = \overline{Co}\mathrm{Lim}^\sharp_{\rho \to 0}\left(\frac{\Gamma_\rho - \mathbf{1}}{\rho}\right)$. Then the upper limit $K^\sharp = \limsup_{\rho \to 0} Viab_{\Gamma_\rho}(K_\rho)$ is a viable subset under F:*

$$\limsup_{\rho \to 0} Viab_{\Gamma_\rho}(K_\rho) \subset Viab_F(K)$$

Proof —- Let us consider $x_0 \in K^\sharp$. There exists a subsequence $x_{\rho,0} \in Viab_{\Gamma_\rho}(K_\rho)$ which converges to x_0 and a K_ρ-viable solution $\vec{x}_\rho := (x^0_\rho, ..., x^n_\rho, ...) \in \vec{S}_{\Gamma_\rho}(x^0_\rho) \cap \mathcal{K}_\rho$ to the discrete system associated with Γ_ρ.

From the definition of Γ_ρ, $x^{n+1}_\rho \in \Gamma_\rho(x^n_\rho)$ and then:

$$\forall n > 0,\ \frac{x^{n+1}_\rho - x^n_\rho}{\rho} \in F_\rho(x^n_\rho)$$

[3] The graphical upper limit is the upper limit of the sequence of $Graph(F_\rho)$.

[4] The upper limit of a sequence of subsets D_n of X is

$$D^\sharp = \limsup_{n \to \infty} D_n := \{y \in X \mid \liminf_{n \to \infty} d(y, D_n) = 0\}$$

With this sequence we associate the piecewise linear interpolation $x_\rho(\cdot)$ which coincides to x_ρ^n at nodes $n\rho$:

$$x_\rho(t) = x_\rho^n + \frac{x_\rho^{n+1} - x_\rho^n}{\rho}(t - n\rho),\ \forall t \in [n\rho, (n+1)\rho[,\ \forall n > 0$$

Then

$$\dot{x}_\rho(t) \in F_\rho(x_\rho^n),\ \forall t \in [n\rho, (n+1)\rho[$$

We have

$$d((x_\rho(t), \dot{x}_\rho(t)), Graph(F_\rho)) \le \|x_\rho(t) - x_\rho^n\| \le \rho \|F(x_\rho^n)\|$$

Since F is Marchaud, and from (18), set-valued maps F_ρ satisfy a uniform linear growth:

$$\exists c > 0,\ \|F_\rho(x)\| \le c(\|x\| + 1),\ \forall x \in X$$

As in the proof of the Viability Theorem (see [2],[9]), this implies

$$\begin{cases} \forall t > 0, & \|x_\rho(t)\| \le (\|x_0)\| + 1)e^{ct} \\ \text{for almost all } \ t > 0, & \|x'_\rho(t)\| \le c(\|x_0\| + 1)e^{ct} \end{cases}$$

Then, $\forall \epsilon > 0,\ \forall t > 0$, there exists $\rho_{\epsilon,t} > 0$ such that

$$\forall \rho \in]0, \rho_{\epsilon,t}],\ d((x_\rho(t), \dot{x}_\rho(t)), Graph(F_\rho)) \le c(\|x_0\| + 1)e^{ct} \le \frac{\epsilon}{2}$$

and with (17) we have

$$\exists \rho'_\epsilon > 0,\ \forall \rho \in]0, \rho'_\epsilon]\ :\ Graph(F_\rho) \subset Graph(F) + \frac{\epsilon}{2}\mathcal{B}$$

Let $\rho_{\epsilon,t}^0 := \min(\rho'_\epsilon, \rho_{\epsilon,t}) > 0$ then

$$(x_\rho(t), \dot{x}_\rho(t)) \in Graph(F) + \epsilon\mathcal{B},\ \ \forall \rho \in]0, \rho_{\epsilon,t}]$$

By the Ascoli and Alaoglu Theorems, we derive that there exists $x(\cdot) \in W^{1,1}(0, +\infty; X; e^{-ct}dt)$ and a subsequence (again denoted by) x_ρ which satisfy:

$$(19)\quad \begin{cases} i) & x_\rho(\cdot) \text{ converges uniformly to } x(\cdot) \\ ii) & x'_\rho(\cdot) \text{ converges weakly to } x'(\cdot) \text{ in } L^1(0, +\infty; X; e^{-ct}dt) \end{cases}$$

This implies (see [1] The Convergence Theorem) that $x(\cdot)$ is a solution to the differential inclusion:

$$\begin{cases} \dot{x}(t) \in F(x(t)),\ \text{for almost all } \ t \ge 0 \\ x(0) = x_0 \in K \end{cases}$$

It remains to prove that the limit is a viable solution:

$\forall t > 0$, there exists a sequence $n_t = E(\frac{t}{\rho})$ such that $n_t\rho \to t$ when $\rho \to 0$. Then $x(t) = \lim_{\rho\to 0} x_\rho(n_t\rho)$. Since $\forall \rho$: $x_\rho(n_t\rho) = x_\rho^{n_t} \in Viab_{\Gamma_\rho}(K_\rho)$, $x(t)$ belongs to the upper limit $K^\sharp$ of subsets $Viab_{\Gamma_\rho}(K_\rho) \subset K_\rho$ and then $K^\sharp \subset K$. This implies that $K^\sharp \subset Viab_F(K)$. ■

3.2 Examples of Approximation Processes

1 - The finite difference explicit scheme.

Naturally, the discrete explicit scheme (2)

$$\begin{cases} \frac{x^{n+1}-x^n}{\rho} \in F(x^n), \text{ for all } n \geq 1, \\ x^0 = x_0 \in K, \end{cases}$$

is associated with $F_\rho = F$, $K_\rho = K$ and $G_\rho := \mathbf{1} + \rho F$. It already satisfies property (17) for any ϵ.

2 - The set-valued Runge-Kutta method.

Let us define the set-valued Runge-Kutta scheme Γ_ρ^{RK}:
For any $x \in K$,

$$F_\rho^{RK}(x) := \{y \in X \mid y = \frac{1}{2}(\beta + \gamma) - \frac{\rho}{2}\beta \text{ where } \beta \in F(x),\ \gamma \in F(x + \rho\beta)\}$$

and

(20) $$\Gamma_\rho^{RK}(x) := \mathbf{1} + \rho F_\rho^{RK}(x)$$

Let $(x, y) \in Graph(\Gamma_\rho^{RK})$. From definition (20), there exist $\beta \in F(x)$ and $\gamma \in F(x + \rho\beta)\}$ such that $y = \frac{1}{2}(\beta + \gamma) - \frac{\rho}{2}\beta$. Since $F(x)$ is Marchaud, $\|\beta\|$ is bounded by m and since F is upper semicontinuous,

$$\forall \epsilon > 0,\ \exists \rho_{\epsilon,m}, \forall \rho \in]0, \rho_{\epsilon,m}], F(x + \rho\beta) \subset F(x) + \epsilon\mathcal{B}.$$

Since F is convex valued, $\frac{1}{2}(\beta + \gamma) \in F(x) + \frac{\epsilon}{2}\mathcal{B}$.
Then, choosing $\rho \leq min(\rho_\epsilon, \frac{\epsilon}{2m})$, we have $F_\rho^{RK}(x) \subset F(x) + \epsilon\mathcal{B}$ and then

$$Graph\left(\frac{\Gamma_\rho^{RK}(x) - \mathbf{1}}{\rho}\right) \subset Graph(F(x)) + \epsilon\mathcal{B}$$

Then condition (17) holds and from Theorem 3.1 we deduce the following corollary:

Corollary 3.1 *The upper limit of $Viab_{\Gamma_\rho^{RK}}(K)$ is a viable subset under F:*

$$\limsup_{\rho \to 0} Viab_{\Gamma_\rho^{RK}}(K) \subset Viab_F(K)$$

3 - The thickening process.

Let us define the set-valued map $F_\rho^T : X \to X$ by a thickening of the values of F by balls of radius $\frac{Ml}{2}\rho$:

(21) $$F_\rho^T(x) = F(x) + \frac{Ml}{2}\rho\mathcal{B}$$

and consider the set-valued map associated with the finite differnece scheme for F_ρ^T:

$$\Gamma_\rho^T(x) = x + \rho F_\rho^T(x) \tag{22}$$

When F is Marchaud, we have the following relations between $Viab_{G_\rho}(K)$, $Viab_{\Gamma_\rho^T}(K)$ and $Viab_F(K)$:

Corollary 3.2 *Let F a Marchaud map, G_ρ and Γ_ρ defined by (22).*
Then

$$\limsup_{\rho\to 0} Viab_{G_\rho}(K) \subset \limsup_{\rho\to 0} Viab_{\Gamma_\rho^T}(K) \subset Viab_F(K) \tag{23}$$

Proof —- The first inclusion holds true since $G_\rho(x) \subset \Gamma_\rho^T(x)$ and (8).
On the other hand, since

$$Graph\left(\frac{\Gamma_\rho^T - \mathbf{1}}{\rho}\right) = Graph(F_\rho^T) \subset Graph(F) + \frac{Ml}{2}\rho\mathcal{B}$$

Theorem 3.1 implies the second inclusion. ∎

3.3 Approximation of the Viability Kernel in the Lipschitz case

From now on we use the following notations :

$$\begin{array}{l} G_\rho = \mathbf{1} + \rho F \\ F_\rho = F + \frac{Ml}{2}\rho\mathcal{B} \\ \Gamma_\rho = \mathbf{1} + \rho F_\rho = \mathbf{1} + \rho F + \frac{Ml}{2}\rho^2\mathcal{B} \end{array} \tag{24}$$

When F is l-Lipschitz, we claim that the discrete viability kernel $Viab_{\Gamma_\rho}(K)$ is a good approximation of the viability kernel of K under F.

Theorem 3.2 [5] *Let F a Marchaud and l-Lipschitz set-valued map, let K a closed subset of X satisfying the boundedness condition*

$$M := \sup_{x\in K} \sup_{y\in F(x)} \|y\| < \infty \tag{25}$$

Then

$$\limsup_{\rho\to 0} Viab_{\Gamma_\rho}(K) = Viab_F(K) \tag{26}$$

[5] This result is due to M. Quincampoix and the author when they visit IIASA Institute - Laxenburg, Austria

Proof — Since F is Marchaud, from Corollary 3.2,

$$\limsup_{\rho\to 0} Viab_{\Gamma_\rho}(K) \subset Viab_F(K) \tag{27}$$

We want to check the opposite inclusion.

Let $x_0 \in K$ and consider any solution $x(\cdot) \in \mathcal{S}_F(x_0)$. Let $\rho > 0$ given. We have

$$x(t+\rho) - x(t) = \int_t^{t+\rho} \dot{x}(s)ds, \ \forall t > 0$$

$\dot{x}(s) \in F(x(s))$ and F Lipschitzian imply that

$$x(t+\rho) - x(t) \in \rho F(x(t)) + l\int_t^{t+\rho} \|x(s) - x(t)\| ds\mathcal{B}, \ \forall t > 0$$

But since F is bounded, $\|x(s) - x(t)\| \leq (s-t)M$ and thus

$$x(t+\rho) - x(t) \in \rho F(x(t)) + \frac{Ml}{2}\rho^2\mathcal{B} \tag{28}$$

So, we have proved that if $x(\cdot) \in S_F(x_0)$ then the following sequence

$$\xi_n = x(n\rho), \ \ \forall n \geq 0 \tag{29}$$

is a solution to the discrete dynamical system associated with Γ_ρ:

$$\xi_{n+1} \in \Gamma_\rho(\xi_n), \ \ \forall n \geq 0 \tag{30}$$

Moreover, if $x(\cdot)$ is a viable solution, then $(\xi_n)_n$ is a viable solution to (30). Thus

$$Viab_F(K) \subset Viab_{\Gamma_\rho}(K), \quad \forall \rho > 0 \tag{31}$$

and then

$$Viab_F(K) \subset \limsup_{\rho\to 0^+} Viab_{\Gamma_\rho}(K). \quad \blacksquare \tag{32}$$

4 Approximation by Finite Setvalued Maps

With any $h \in \mathbb{R}$ we associate X_h a countable subset of X, which spans X in the sense that

$$\forall x \in X, \ \exists x_h \in X_h \text{ such that } \ \|x - x_h\| \leq \alpha(h) \tag{33}$$

where $\alpha(h)$ decreases to 0 when $h \to 0$:

$$\lim_{h\to 0} \alpha(h) = 0 \tag{34}$$

4.1 Approximation of discrete and finite viability kernels

Let $G_h : X_h \rightsquigarrow X_h$ a finite set-valued map and a subset $K_h \subset Dom(G_h)$.

We call **finite dynamical system** associated with G_h the following system:

$$x_h^{n+1} \in G_h(x_h^n), \text{ for all } n \geq 0, \tag{35}$$

and we denote by

- $\mathcal{K}_h$ the set of all sequences from $I\!N$ to K_h.

- $\vec{x}_h := (x_h^0, ..., x_h^n, ...) \in \mathcal{X}_h$ a solution to system (35)

- $\vec{S}_{G_h}(x_h^0)$ the set of solutions $\vec{x}_h \in \mathcal{X}_h$ to the finite differential inclusion (35) starting from x_h^0

A solution $\vec{x}_h$ is viable if and only if $\vec{x}_h \in \vec{S}_{G_h}(x_h) \cap \mathcal{K}_h$, that is to say that:

$$\begin{cases} x_h^{n+1} \in G_h(x_h^n), \ \forall n \geq 0, \\ x_h^0 = x_h \in K_h \\ x_h^n \in K_h, \ \forall n \geq 0. \end{cases} \tag{36}$$

Let $K_h^0 = K_h,\ K_h^1, ...,\ K_h^n, ...$ defined recursively as in the second section:

$$K_h^{n+1} := \{x_h \in K_h^n \text{ such that: } G_h(x_h) \cap K_h^n \neq \emptyset\}$$

The viability kernel algorithm and Proposition 2.1 holds true for finite dynamical systems whenever the set-valued map G_h has nonempty values and we have:

$$Viab_{G_h}(K_h) = K_h^\infty := \bigcap_{n=0}^{+\infty} K_h^n \tag{37}$$

Let us notice that K_h^∞ can be emptyset and in any case there exists a finite integer p such that:

$$K_h^\infty = K_h^n = K_h^p, \ \forall n > p$$

What happen when G_h is the reduction to K_h of a set-valued map G ?

We cannot apply no longer more Proposition 2.1 since $G(x_h)$ may not contain any point of the reduction X_h of X and $G_h(x_h)$ be empty.

To turnover this difficulty, we will consider greater set-valued maps G^r which still approximate G. But the choice of such approximations is subjet to two opposite considerations: on one hand, they have to be large enough in order that the reductions to X_h of such approximations have their domain containing K_h (have nonempty values on K_h), and so, it will be possible to apply again Proposition 2.1. On the other hand, the enlargement is limited as far as the graphical assumption (1) of Theorem 3.1 still holds so as to the viability kernel of K under G contains the upper limit of finite viability kernels of K_h under the finite set-valued approximations G_h^r.

In case of upper semicontinuous set-valued maps, we bring in the fore some discretization process which leads to approach a subset of a the viability kernel.

In the Lipschitz case, these process enables us to approach the viability kernel completely.

Notations:

the reduction to the finite subset X_h of any subset D will be noted by a lower index h: $D_h := D \cap X_h$;

the extension of a set-valued map G with a ball of radius r by an upper index: $\forall x \in X,\ G^r(x) := G(x) + r\mathcal{B}$.

Thus the reduction to X_h of the extension of a set-valued map G with a ball of radius r will be noted G^r_h.

Let us notice that the extension operation has to be done before the reduction one otherwise it could be empty even for $r > \alpha(h)$.

From property (33) which defines $\alpha(h)$, we consider now the extension with $r = \alpha(h)$. We observe that $G_h^{\alpha(h)}$ satisfies the non emptyness property:

$$\forall x_h \in Dom(G) \cap X_h,\ G_h^{\alpha(h)}(x_h) := G^{\alpha(h)}(x_h) \cap X_h \neq \emptyset \tag{38}$$

and the decreasing sequence of finite subsets $K_h^{\alpha(h),0} = K_h,\ K_h^{\alpha(h),1}, ...,\ K_h^{\alpha(h),n}, ...$ defined by

$$K_h^{\alpha(h),n+1} := \{x \in K_h^{\alpha(h),n} \text{ such that } G_h^{\alpha(h)}(x) \cap K_h^{\alpha(h),n} \neq \emptyset\}$$

satisfies property (37):

$$K_h^{\alpha(h),\infty} := \bigcap_{n=0}^{+\infty} K_h^{\alpha(h),n} = Viab_{G_h^{\alpha(h)}}(K_h)$$

As a partial conclusion, we are able to approximate the discrete viability kernel of K under G: first we extend G such that for all $x \in K$, images of G^r encounters X_h, in other words such that $Dom(G^r_h) = Dom(G) \cap X_h$. To be sure of this, without loss of generality, we can choose $r = \alpha(h)$. Secondly we look after the discrete viability kernel of K under G^r, the finite viability kernel of K_h under G^r_h and at last we let h decreasing to 0.

What relations link together the discrete viability kernels $Viab_G(K)$ or $Viab_{G^{\alpha(h)}}(K)$ and the finite viability kernels $Viab_{G_h}(K_h)$ or $Viab_{G_h^{\alpha(h)}}(K_h)$ whenever K_h is the reduction of K to X_h: are the latters the reduction to X_h of the formers ? Does the upper limit of the latters, when h goes to 0, coincide with the former ?

4.2 Properties of the finite viability kernel

A first answer is given by applying Proposition 2.2: since $\lim_{h\to 0} \alpha(h) = 0$, we have

$$\bigcap_{h>0} Viab_{G^{\alpha(h)}}(K) = Viab_G(K)$$

The following result gives a necessary and sufficient condition for $Viab_{G_h^{\alpha(h)}}(K_h)$ to be the reduction of $Viab_{G^{\alpha(h)}}(K)$ to X_h:

Let $G^{\alpha(h)} : X \rightsquigarrow X$, $G_h^{\alpha(h)} : X_h \rightsquigarrow X_h$ and K_h a finite subset of $Dom(G_h^{\alpha(h)})$ defined as follows:

$$\begin{cases} G^{\alpha(h)}(x) := G(x) + \alpha(h)\mathcal{B} \\ K_h := K \cap X_h \\ \forall x_h \in Dom(G) \cap X_h \ : \ G_h^{\alpha(h)}(x_h) := G^{\alpha(h)}(x_h) \cap X_h \end{cases}$$

From definition of $\alpha(h)$, $\forall x_h \in K_h, \ \ G_h^{\alpha(h)}(x_h) \neq \emptyset$.

Proposition 4.1 *Let G: $X \rightsquigarrow X$ an upper semicontinuous set-valued map with closed values and K a closed subset of $Dom(G)$.*

Let r such that $\forall x \in Dom(G^r) \cap X_h, \ G^r(x) \cap X_h \neq \emptyset$:

$$(39) \qquad Dom(G_h^r) = Dom(G^r) \cap X_h$$

Then

$$(40) \qquad Viab_{G_h^r}(K_h) \subset Viab_{G^r}(K) \cap X_h$$

It coincides if and only if $Viab_{G^r}(K) \cap X_h$ is a discrete viability domain of K under G^r:

$$(41) \qquad \forall x_h \in Viab_{G^r}(K) \cap X_h, \ G^r(x_h) \cap (Viab_{G^r}(K) \cap X_h) \neq \emptyset$$

Proof — From (37) we have to check that the two following statements

$$(42) \qquad \forall x_h \in Viab_{G^r}(K) \cap X_h, \ \ G^r(x_h) \cap (Viab_{G^r}(K) \cap X_h) \neq \emptyset$$

and

$$(43) \qquad Viab_{G_h^r}(K_h) = Viab_{G^r}(K) \cap X_h$$

are equivalent.

Assume that (42) holds. Let $x_h \in Viab_{G_h^r}(K_h)$. There exists $\vec{x}_h \in \vec{S}_{G_h^r}(x_h) \cap \mathcal{K}_h$ viable in $K_h \subset K$. Since $G_h^r(x_h) \subset G^r(x_h)$, $\vec{x}_h \in \vec{S}_{G^r}(x_h) \cap \mathcal{K}$ is viable in K. Then $x_h \in Viab_G^r(K)$ and we obtain inclusion:

$$(44) \qquad Viab_{G_h^r}(K_h) \subset Viab_{G^r}(K) \cap X_h.$$

On the other hand, $Viab_{G^r}(K) \cap X_h$ is a discrete viability domain of G^r. Then from definition 2.1 and definition of G_h^r it is also a finite viability domain of G_h^r contained in K_h and thus is contained in the finite viability kernel of K_h under G_h^r: $Viab_{G^r}(K) \cap X_h \subset Viab_{G_h^r}(K_h)$. We obtain the opposite inclusion and prove that (43) is true.

Conversely, if (43) holds, $Viab_{G^r}(K) \cap X_h$ is the finite viability kernel of K_h under G_h^r, it is obviously a finite viability domain of G^r. ■

Remarks

1 - Inclusion (40) is always true.

Let call $A := \{x_h \in Viab_{G^r}(K) \cap X_h,\ x_h \notin Viab_{G^r_h}\}$ and $B := \{x_h \in Viab_{G^r}(K) \cap X_h,\ \mid G^r(x_h) \cap (Viab_{G^r}(K) \cap X_h) = \emptyset\}$.

2 - It is easy to prove that $B \subset A$ and Proposition 4.1 says that if B is empty, A is empty too.

3 - If $A \neq \emptyset$, then all solutions $\vec{x}_h$ to the finite dynamical system starting from any point $x_h^0 \in A$, must leaves K_h after a finite number of steps, although x_0 belongs to the dicrete viability kernel of K under G^r.

4 - If $B \neq \emptyset$, then all solutions $\vec{x}_h$ to the finite dynamical system starting from any point $x_h^0 \in B$, leaves K_h at the first step.

5 - If $x_h^0 \in A$ and x_h^n is the last element of a solution to the finite dynamical system, starting at x_h^0, which is still in K_h, then $x_h^n \in B$.

6 - If $G^r(x) \subset K,\ \forall x \in K$ then for all $x_h \in K_h$, $G^r(x_h) \cap K_h \neq \emptyset$. Then inclusion (40) becomes an equality :

$$K_h = Viab_{G^r_h}(K_h) = Viab_{G^r}(K) \cap X_h.$$

4.3 Approximation of the viability kernel of K under F by finite viability kernel in the Lipschitz Case

When G is a k-Lipschitz set-valued map, we cannot prove that in (40), the inclusion becomes an equality. Nevertheless we have in the Lipschitz case an immediate and interesting information about points $x_h \in K_h$ which do not satisfy (41):

Proposition 4.2 *Let $G : X \to X$ a k-Lipschitz set-valued map. Let $r \geq \max(k,1)\alpha(h)$. For all $x_h \in Viab_{G^r}(K) \cap X_h$ such that*

$$G^r(x_h) \cap (Viab_{G^r}(K) \cap X_h) = \emptyset$$

then

$$x_h \notin Viab_G(K).$$

Proof — Let $x_h \in Viab_G(K) \cap X_h$. From definition of the viability kernel, we have

$$G(x_h) \cap Viab_G(K) \neq \emptyset$$

From definition of $\alpha(h)$,

$$(G(x_h) \cap Viab_G(K) + \alpha(h)\mathcal{B}) \cap X_h) \neq \emptyset$$

this implies that

$$(G(x_h)+\alpha(h)\mathcal{B})\cap(Viab_G(K)+\alpha(h)\mathcal{B})\cap X_h\neq\emptyset$$

and for any $r\geq\max(k,1)\alpha(h)$,

$$(G(x_h)+r\mathcal{B})\cap(Viab_G(K)+\frac{r}{k}\mathcal{B})\cap X_h\neq\emptyset$$

Since G is k-Lipschitz, we can apply Lemma 2.3, and thenwe obtain:

$$G^r(x_h)\cap(Viab_{G^r}(K)\cap X_h)\neq\emptyset. \quad \blacksquare$$

In particular, if we apply this Proposition for $G=\Gamma_\rho$ we have:
$\forall x_h\in Viab_{\Gamma_\rho^{\alpha(h)}}(K)\cap X_h$ such that $\Gamma_\rho^{\alpha(h)}(x_h)\cap(Viab_{\Gamma_\rho^{\alpha(h)}}(K)\cap X_h)=\emptyset$, then

$$x_h\notin Viab_{\Gamma_\rho}(K)$$

and since from (31) $Viab_F(K)\subset Viab_{\Gamma_\rho}(K)$

$$x_h\notin Viab_F(K)$$

We can deduce the following approximation result when K is a viability domain:

Corollary 4.1 *Let $G:X\to X$ a k-Lipschitz set-valued map and K a viability domain of G. Then*

$$\forall r\geq\max(k,1)\alpha(h),\ \ Viab_{G_h^r}(K_h)=K_h.$$

Proof — Let $X_h\in K_h$. x_h belongs to $Viab_G(K)\cap X_h$ and from Proposition 4.2 $G^r(x_h)\cap(Viab_{G^r}(K)\cap X_h)\neq\emptyset$. Since $Viab_G(K)\subset Viab_{G^r}(K)=K$, we replace $Viab_{G^r}(K)$ by K and then we obtain:

$$G^r(x_h)\cap K_h)\neq\emptyset$$

that is to say that K_h is a viability domain of G_h^r. $\blacksquare$

However, we prove that when h goes to 0, if ρ goes to 0 "slower" than $a(h)$, we can approximate the viability kernel of K under F by a sequence of finite viability kernels of reduction to X_h of some larger extensions of $\mathbf{1}+\rho F$.

We look now for extension G^r of G such that any solution $\vec{\xi}\in\vec{S}_G(\xi^0)$ can be approached by solution $\vec{\xi_h}\in\vec{S}_{G_h^r}(\xi_h^0)$

Lemma 4.1 *Let $G : X \rightsquigarrow X$ a k-Lipschitz set-valued map. Let $r \geq k\alpha(h)$. Let $G^r : X \rightsquigarrow X$ the extension of G:*

$$\forall x \in X,\ G^r(x) := G(x) + r\mathcal{B}.$$

and consider $G^r_h : X_h \rightsquigarrow X_h$ the reduction of G^r to X_h:

$$G^r_h(x_h) := G^r(x_h)) \cap X_h,\ \ \forall x_h \in X_h.$$

If the following property holds true:

$$\text{(45)} \qquad \forall \xi \in G(x),\ \exists \xi_h \in G(x) \cap X_h \text{ such that } \ \|\xi - \xi_h\| \leq \frac{r}{k}$$

Then with any solution $\vec{\xi} := (\xi^n)_n \in \vec{S}_G(\xi^0)$ to the discrete dynamical system:

$$\text{(46)} \qquad \xi^{n+1} \in G(\xi^n),\ \forall n \geq 0$$

we can associate a solution $\vec{\xi}_h := (\xi^n_h)_n \in \vec{S}_{G^r_h}(\xi^0_h)$ to the finite dynamical system:

$$\text{(47)} \qquad \xi^{n+1}_h \in G^r_h(\xi^n_h),\ \forall n \geq 0$$

such that

$$\text{(48)} \qquad \|\xi^n_h - \xi^n\| \leq \frac{r}{k},\ \forall n \geq 0.$$

Proof — Let $\xi^0 \in X$ and $\vec{\xi} \in \vec{S}_G(\xi^0)$. From definition of $\alpha(h)$, since $r \geq k\alpha(h)$, $\exists \xi^0_h \in (\{\xi^0\} + \frac{r}{k}\mathcal{B}) \cap X_h$. Assume that we found a sequence ξ^k_h satisfying (47) and (48) until $k = n$.

Since G is k-Lipschitz,

$$G(\xi^n) \subset G(\xi^n_h) + k\|\xi^n - \xi^n_h\|\mathcal{B}$$

and then, from (48),

$$G(\xi^n) \subset G^r(\xi^n_h)$$

Since $\xi^{n+1} \in G(\xi^n)$, from (45), there exists $\xi^{n+1}_h \in G(\xi^n) \cap X_h$ such that

$$\|\xi^{n+1} - \xi^{n+1}_h\| \leq \frac{r}{k}$$

On the other hand we have

$$\xi^{n+1}_h \in G(\xi^n_h) + k\|\xi^n - \xi^n_h\|\mathcal{B} \subset G^r_h(\xi^n_h).$$

Since $G(\xi^n_h) \cap X_h \subset G^r_h(\xi^n_h)$, $\ \forall n \geq 0$, then $\vec{\xi}_h \in \vec{S}_{G^r_h}(\xi^0_h)$.

This ends the proof of Lemma 4.1. ∎

We deduce the following result:

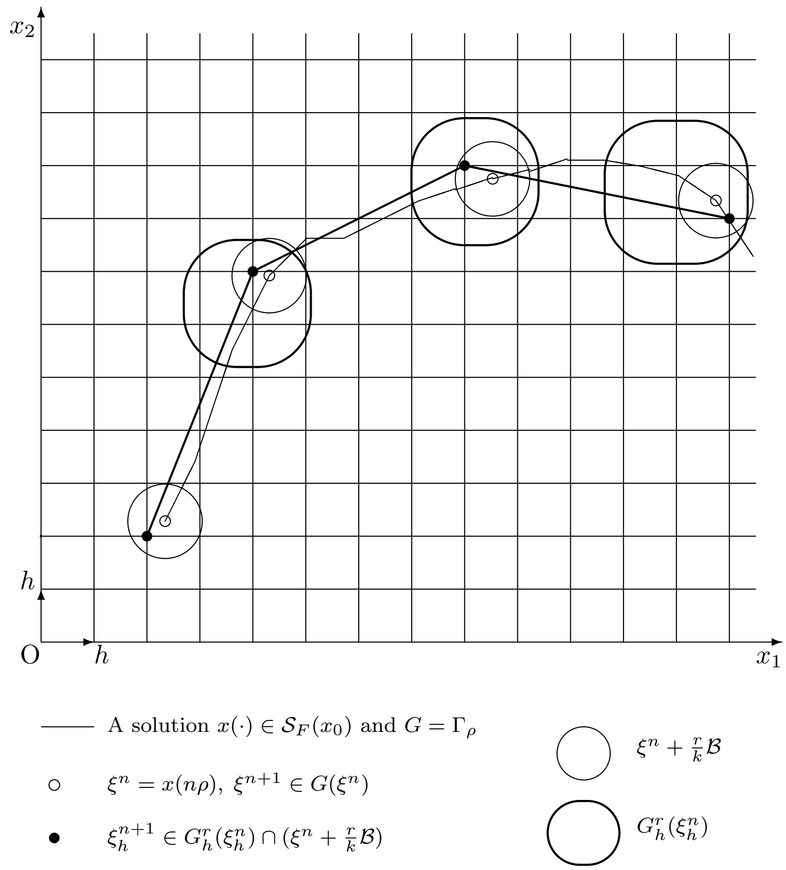

Figure 1: The Extension-Reduction Process

Corollary 4.2 *Let $G : X \rightsquigarrow X$ a k-Lipschitz set-valued map satisfying property (45). Let K a closed subset of X Then, for all $r \geq k\alpha(h)$ we have:*

$$Viab_G(K) \subset Viab_{G_h^r}(K_h^{\frac{r}{k}}) + \frac{r}{k}\mathcal{B}$$

Proof — From Lemma 4.1, for all $\xi^0 \in Viab_G(K)$, there exists $\vec{\xi} \in \vec{S}_G(\xi^0)$ viable in K, $\xi_h^0 \in K_h^{\frac{r}{k}}$ and $\vec{\xi}_h \in \vec{S}_{G_h^r}(\xi_h^0)$ viable in $(K_h^{\frac{r}{k}}))$. By definition of the discrete viability kernel,

$$\xi_h^0 \in Viab_{G_h^r}(K_h^{\frac{r}{k}})$$

and since $\|\xi^0 - \xi_h^0\| \leq \frac{r}{k}$, we have

$$Viab_G(K) \subset Viab_{G_h^r}(K_h^{\frac{r}{k}}) + \frac{r}{k}\mathcal{B}$$

The reduction process satisfies the following property:

Lemma 4.2 *For any closed subset $D \subset X$, and any decreasing sequence of closed subsets D_ρ such that $D = \bigcap_{\rho>0} D_\rho$, we have:*

$$D = \limsup_{\rho,h\to 0} ((D_\rho + \alpha(h)\mathcal{B}) \cap X_h) \tag{49}$$

If D is satisfies the property: $\forall x \in D, \exists x_h \in D \cap X_h \ : \ \|x - x_h\| \leq \alpha(h)$, then

$$D = \limsup_{h\to 0} (D \cap X_h) \tag{50}$$

Proof — Proof of second statement is immediate. We just prove the first equality.

Let $x \in \limsup_{\rho,h\to 0} (D_\rho + \alpha(h)\mathcal{B}) \cap X_h$. There exists ρ_n, h_n converging to zero and $x_{\rho_n h_n} \in (D_{\rho_n} + \alpha(h_n)\mathcal{B}) \cap X_{h_n} \subset D_{\rho_n} + \alpha(h_n)\mathcal{B}$ which converges to x when n converge to ∞. Then $x \in \bigcap_{\rho\to 0} \bigcap_{h\to 0}(D_\rho + \alpha(h)\mathcal{B}) = D$.

Conversely, let $x \in D$. Since $D = \bigcap_{\rho\to 0} D_\rho$, there exists a sequence x_ρ which converges to x. From definition of $\alpha(h)$, there exists $y_{\rho h} \in (D_\rho + \alpha(h)\mathcal{B}) \cap X_h$ such that $\|y_{\rho h} - x_\rho\| \leq \alpha(h)$ and $\lim_{\rho,h\to 0} y_{\rho h} = x$. ■

4.4 Approximation of $Viab_F(K)$ by Viability Kernel of finite subsets $Viab_{\Gamma_h^r}(K_h^r)$

Now we can state the following result:

Theorem 4.1 *Let $F : X \rightsquigarrow X$ a Marchaud and l-Lipschitz set-valued map, K a closed subset of $Dom(F)$ satisfying the boundedness condition:*

$$M := \sup_{x\in K} \sup_{y\in F(x)} \|y\| < \infty \tag{51}$$

Let $G_\rho := \mathbf{1} + \rho F$, $\Gamma_\rho := \mathbf{1} + \rho F + \frac{Ml}{2}\rho^2\mathcal{B}$ and we note $k = 1 + \rho l$.
Let $h > 0$, X_h a reduction of X and $\alpha(h)$ defined by (33).
Assume that ρ and h are choosen such that:

$$\alpha(h) \leq \frac{Ml}{2}\rho^2 \tag{52}$$

Let $\Gamma_\rho^{kMl\rho^2} : X \to X$ and $\Gamma_{\rho h}^{kMl\rho^2} : X_h \to X_h$ defined as follows :

$$\Gamma_\rho^{kMl\rho^2}(x) := \Gamma_\rho(x) + kMl\rho^2\mathcal{B}$$

$$\Gamma_{\rho h}^{kMl\rho^2}(x_h) := \Gamma_\rho^{kMl\rho^2}(x_h) \cap X_h$$

Then:

$$Viab_F(K) = \limsup_{\rho,h\to 0}(Viab_{\Gamma_\rho}(K) + \alpha(h)\mathcal{B}) \cap X_h \tag{53}$$

and

$$Viab_F(K) = \limsup_{\rho,h\to 0} Viab_{\Gamma_{\rho h}^{kMl\rho^2}}(K_h^{Ml\rho^2}) \tag{54}$$

Proof — From Theorem 3.2

$$Viab_F(K) = \limsup_{\rho \to 0} Viab_{\Gamma_\rho}(K)$$

The sequence of embeded subsets $Viab_{\Gamma_\rho}(K)$ converges to $Viab_F(K)$ when ρ decreases to zero. Then applying Lemma 4.2, we obtain the first equality (53):

$$Viab_F(K) = \limsup_{\rho, h \to 0} (Viab_{\Gamma_\rho}(K) + \alpha(h)\mathcal{B}) \cap X_h$$

To prove the second equality (54), we apply Corollary 4.2 with $G = \Gamma_\rho$.

We have to check first that assumption (52) implies the thickness condition (45) of Corollary 4.2:

indeed

$$\forall \xi \in \Gamma_\rho(x),\ \exists \xi' \in x + \rho F(x) \text{ such that } \|\xi' - \xi\| \leq \frac{Ml}{2}\rho^2$$

From the definition of $\alpha(h)$

$$\exists \xi'_h \in X_h \text{ such that } \|\xi'_h - \xi'\| \leq \alpha(h)$$

Since from (52),

$$(x + \rho F(x) + \alpha(h)\mathcal{B}) \cap X_h \subset \left(x + \rho F(x) + \frac{Ml}{2}\rho^2\mathcal{B}\right) \cap X_h = \Gamma_\rho(x) \cap X_h$$

Then we proved that $\forall \xi \in \Gamma_\rho(x),\ \exists \xi'_h \in \Gamma_\rho(x) \cap X_h$ such that $\|\xi'_h - \xi\| \leq \|\xi'_h - \xi'\| + \|\xi' - \xi\| \leq Ml\rho^2$

We are able now to apply Corollary 4.2 with $r = kMl\rho^2$ and thus we obtain:

$$Viab_{\Gamma_\rho}(K) \subset Viab_{\Gamma_{\rho h}^{kMl\rho^2}}(K_h^{Ml\rho^2}) + Ml\rho^2\mathcal{B}$$

which implies that

$$Viab_F(K) \subset \limsup_{\rho, h \to 0} \left(Viab_{\Gamma_{\rho h}^{kMl\rho^2}}(K_h^{Ml\rho^2}) + Ml\rho^2\mathcal{B}\right)$$

and then

(55) $$Viab_F(K) \subset \limsup_{\rho, h \to 0} Viab_{\Gamma_{\rho h}^{kMl\rho^2}}(K_h^{Ml\rho^2})$$

To prove the opposite inclusion, we observe that

$$\Gamma_{\rho h}^{kMl\rho^2} = \mathbf{1} + \rho F + \left(\frac{Ml}{2}\rho^2 + kMl\rho^2\right)\mathcal{B}$$

and then we have

$$Graph\left(\frac{\Gamma_{\rho h}^{kMl\rho^2} - \mathbf{1}}{\rho}\right) \subset Graph(F) + (\frac{3}{2} + \rho l)Ml\rho\mathcal{B}$$

and if we assume for instance that $\rho l \leq \frac{1}{2}$,

$$(\frac{3}{2} + \rho l)Ml\rho \leq 2Ml\rho$$

Then,

$$\forall \epsilon > 0,\ \exists \rho_\epsilon > 0 \text{ such that } \forall \rho \in]0, \rho_\epsilon[\ :\ 2Ml\rho \leq \epsilon$$

and

$$\exists h_{\epsilon,\rho} > 0 \text{ such that } \ \forall h \in]0, h_{\epsilon,\rho}[,\ :\ \alpha(h) \leq \frac{Ml}{2}\rho^2$$

The Convergence Theorem 3.1 implies that:

$$\limsup_{\rho,h \to 0} Viab_{\Gamma_{\rho h}^{kMl\rho^2}}(K_h^{Ml\rho^2}) \subset Viab_F(K).$$

and with (55) we proved the equality

$$\limsup_{\rho,h \to 0} Viab_{\Gamma_{\rho h}^{kMl\rho^2}}(K_h^{Ml\rho^2}) = Viab_F(K) \quad \blacksquare$$

4.5 Conclusion : a numerical method for computing viability kernel

These results allow us to look for numerical approximation of the viability kernel of K under F associated with the initial differential inclusion (1):

$$\dot{x}(t) \in F(x(t)), \text{ for almost all } t \geq 0.$$

We consider the discrete explicit scheme:

$$\begin{cases} x^{n+1} \in x^n + \rho F(x^n) + 2Ml\rho^2\mathcal{B},\ \forall n \geq 0, \\ x^0 = x_0 \in K, \end{cases}$$

We recall that the condition

(56) $$(x_h + \rho F(x_h) + r\mathcal{B} \cap X_h \neq \emptyset$$

will be true if ρ and h satisfy the condition:

(57) $$r = 2Ml\rho^2 \geq \alpha(h)$$

witch is a stability condition meaning that the space discretization step has to be "smaller" than the time's one.

We set

$$\Gamma_\rho(x) := x + \rho F(x) + \frac{Ml}{2}\rho^2\mathcal{B}$$

$$G_\rho^{2Ml\rho^2}(x) := x + \rho F(x) + 2Ml\rho^2\mathcal{B}$$

From (13) in the discrete case and (37) in the finite case we obtain:

$$K_\rho^{\frac{Ml}{2}\rho^2,\infty} := Viab_{\Gamma_\rho}(K).$$

$$K_{\rho h}^{\frac{Ml}{2}\rho^2,\infty} := Viab_{\Gamma_{\rho h}}(K_h).$$

but $K_{\rho h}^{\frac{Ml}{2}\rho^2,\infty}$ can be empty, and

$$K_\rho^{2Ml\rho^2,\infty} := Viab_{G_\rho^{2Ml\rho^2}}(K).$$

$$K_{\rho h}^{2Ml\rho^2,\infty} := Viab_{G_{\rho h}^{2Ml\rho^2}}(K_h^{Ml\rho^2}).$$

but now, if $h > 0$ and $\rho > 0$ satisfy the condition (57), the finite viability kernel $K_{\rho h}^{2Ml\rho^2,\infty}$ is non empty.

Gathering general results we proved in preceeding sections, we have the following convergence properties of approximations of viability kernel of K under F with finite viability kernels computable in a finite number of steps:

Theorem 4.2 *If F is a Marchaud setvalued map, K a compact subset of X, $h > 0$ and $\rho > 0$ satisfying the condition (57).*
Then

$$\begin{cases} K_{\rho,h}^{\frac{Ml}{2}\rho^2,\infty} \subset K_\rho^{\frac{Ml}{2}\rho^2,\infty} \cap X_h \\ \lim_{h\to 0} K_\rho^{\frac{Ml}{2}\rho^2,\infty} = K_\rho^\infty \\ \limsup_{\rho\to 0} K_\rho^\infty \subset Viab_F(K) \end{cases}$$

Moreover, if F is l-Lipschitz, then

$$\limsup_{\rho,h\to 0} K_{\rho,h}^{2Ml\rho^2,\infty} = Viab_F(K)$$

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