APPROXIMATION OF W_p-CONTINUITY SETS BY p-SIDON SETS

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1. INTRODUCTION

For two complex-valued functions x and y defined on the integers Z, we define $x * y$ by

$$x * y(n) = \sum x(n-k)y(k) \qquad (k \in Z),$$

provided the series converges for each $n \in Z$. We let ℓ_p denote the Banach space of complex-valued functions x on Z such that the norm

$$\|x\|_{\ell_p} = \left(\sum |x(n)|^p\right)^{1/p}$$

is finite $\left(\sum\right.$ will always indicate summation over $\left.Z\right)$. Corresponding to each function a in the space J of the complex-valued functions on Z with finite support, we have a trigonometric polynomial

$$f(\theta) = \sum a(n) e^{in\theta}.$$

We let $\|f\|_{W_p}$ and $\|a\|_{W_p}$ denote the norm of the operator $x \to a * x$ on ℓ_p. We are interested in the algebra W_p, which we now define as follows: a complex-valued function f on the circle G (reals modulo 2π) is in W_p if and only if there exists a sequence $\{f_n\}$ $(n = 1, 2, \cdots)$ of trigonometric polynomials that converges uniformly to f and also satisfies the condition

$$\|f_n - f_m\|_{W_p} \to 0 \qquad \text{as } m, n \to \infty.$$

Let $\|f\|_{W_p}$ denote $\lim_{m \to \infty} \|f_n\|_{W_p}$. One can show that with this norm and the pointwise operations, W_p is a Banach algebra with the circle as its maximal ideal space; however, for our purposes we only need to know that $\|f\|_{W_p}$ dominates the supremum norm of f. A short proof of this follows Lemma 2.3. It turns out that W_p is isomorphic to a closed subalgebra of multipliers (bounded operators that commute with all translation operators) on $L_p(Z) = \ell_p$; in fact, this seems to be the natural way to show that W_p is a Banach algebra. There is an extensive literature on multipliers for L_p-spaces over locally compact groups.

Our purpose is to obtain some results concerning W_p-continuity sets, which are defined in Section 4. It is easy to see that the definition is equivalent to saying that a compact subset E of G is a W_p-continuity set if $W_p \mid E = C(E)$, that is, if every continuous function on E is the restriction to E of some element in W_p. Since W_1 is isomorphic to $L_1(Z)$, the W_1-continuity sets are precisely the Helson sets. We shall be interested in the cases $1 < p < 2$ $(W_2 = C(G))$.

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A closely related notion corresponding to multipliers for $L_p(G)$, that of p-Sidon sets, was introduced by Figà-Talamanca in [2]. Our approach to W_p-continuity sets is to prove, roughly speaking, that the W_p-continuity sets are the sets that can be approximated by a sequence $\{E_j\}$ of finite subsets of G such that corresponding sets of integers of the form $(m_j/2\pi)E_j$ are "uniformly" p-Sidon sets. This is the content of Theorems 4.2 and 4.3. Theorem 3.3, which is of interest in itself, is a major step in proving 4.2 and 4.3. It states, approximately, that the norm of a multiplier on $L_p(Z)$ can be determined from the norms of related multipliers on $L_p(G)$.

The value of Theorems 4.2 and 4.3 is that they enable us to reduce questions about compact subsets of G to questions about finite subsets of Z, which can be manipulated more easily; in fact, Rudin [8] has established many properties of the subsets of Z that he calls $\Lambda(p)$-sets. The $\Lambda(p)$-sets are relevant because they are the same as the p-Sidon sets (see Theorem 5.3); this is also true in a more general setting [4, Theorem 6].

In Section 6, we use results of Rudin together with Theorems 4.2 and 4.3 to construct a W_p-continuity set that is not a $W_{p-\varepsilon}$-continuity set.

2. PRELIMINARIES

For $1 \leq p \leq 2$, let L_p denote the space of complex-valued measurable functions g on G with finite norm

$$\|g\|_{L_p} = \left(\frac{1}{2\pi}\int_0^{2\pi} |g(\theta)|^p d\theta\right)^{1/p}.$$

Let P_m denote the space of complex-valued functions x of period m on Z with norm given by

$$\|x\|_{P_m} = \left(\frac{1}{m}\sum_{0\leq n\leq m-1} |x(n)|^p\right)^{1/p}.$$

For a in J, we denote by $|a|_{L_p}$ the operator norm of the operator $g \to h$ on L_p, where g and h have Fourier series

$$\sum c_n e^{-in\theta} \quad \text{and} \quad \sum a(n) c_n e^{-in\theta},$$

respectively. Also, we denote by $|a|_{P_m}$ the norm of the operator $x \to a * x$ on P_m. We shall need the following two lemmas. The first is an immediate consequence of the famous theorem of M. Riesz on the boundedness of the conjugate operator (see, for example, [9, Vol. I, p. 253]).

2.1. LEMMA. *To each real number* p $(1 < p < \infty)$ *there corresponds a positive constant* ρ *with the property that, for every function* $g \in L_p$ *with the Fourier series* $\sum c_n e^{-in\theta}$ *and for every pair of integers* α *and* β $(\alpha < \beta)$, *the polynomial*

$$h(\theta) = \sum_{\alpha \leq n < \beta} c_n e^{-in\theta}$$

satisfies the inequality $\|h\|_{L_p} \leq \rho \|g\|_{L_p}$.

The second lemma is due to Marcinkiewicz and Zygmund [6] (see also [9, Vol. II, p. 30]). We state it in our present notation.

2.2. LEMMA. *For each real number* p $(1 < p < \infty)$, *there exist constants* μ_0 *and* μ *such that every trigonometric polynomial*

$$g(x) = \sum_{\alpha \leq n < \alpha+m} c_n e^{-inx}$$

satisfies the condition

$$\mu_0 \|g(2\pi(\cdot)/m\|_{P_m} \leq \|g\|_{L_p} \leq \mu \|g(2\pi(\cdot)/m)\|_{P_m}.$$

Let G_m denote the subset of G consisting of the points $2\pi n/m$ $(n = 0, 1, \cdots, m-1)$. For any complex-valued function f whose domain includes G_m, we define the functions $S_m f$ and $C_m f$ on Z as follows:

$$S_m f(n) = \begin{cases} f(2\pi n/m) & \text{for } 0 \leq n < m, \\ 0 & \text{otherwise,} \end{cases}$$

$$C_m f(k) = \begin{cases} m^{-1} \sum_{0 \leq n < m} f(2\pi n/m) e^{-in(2\pi k/m)} & \text{for } 0 \leq k < m, \\ 0 & \text{otherwise.} \end{cases}$$

These functions will be used later.

2.3. LEMMA. *For each* f *in* W_p,

$$\|f\|_{W_p} \geq \sup\{|f(\theta)|: \theta \in G\}.$$

Proof. For $m = 1, 2, \cdots$ and θ in G, we define the function $U_{m,\theta}$ on Z as follows:

$$U_{m,\theta}(n) = \begin{cases} \frac{1}{m^{1/p}} e^{-in\theta} & \text{for } 0 \leq n < m, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to show that if $f(\theta) = \sum a(n) e^{in\theta}$ is a trigonometric polynomial, then

$$a * U_{m,\theta} = f(\theta) U_{m,\theta} + V_{m,\theta},$$

where $\|V_{m,\theta}\|_{\ell_p} \to 0$ as $m \to \infty$. Since $\|U_{m,\theta}\|_{\ell_p} = 1$ for each m, we see that $\|f\|_{W_p} \geq |f(\theta)|$; and, from this last inequality and the definition of the norm $\| \ \|_{W_p}$ on W_p, we obtain the conclusion of the lemma.

We shall also use the following notation throughout the paper. If $f \in L_1$, then

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx \qquad (n \in Z).$$

We let T denote the space of trigonometric polynomials, and T_m the elements f in T such that $\hat{f}(n) = 0$ if $n \notin [0, m-1]$. For a complex-valued function g defined on a set E, $\|g\|_{E,\infty}$ denotes the supremum of the numbers $|g(x)|$ $(x \in E)$, and $\|g\|_\infty$ denotes the supremum of the numbers $|g(x)|$ (x in the domain of g). For a normed linear space X, $\| \ \|_X$ will be used for the norm, and $| \ |_X$ will denote the operator norm for the space of bounded operators on X.

3. APPROXIMATION TO W_p

3.1. LEMMA. *For* $1 < p < \infty$, *let* ρ, μ, μ_0 *be the constants in Lemmas* 2.1 *and* 2.2. *Then, for each function* f *on* G_m,

$$(\mu/\mu_0)|C_m f|_{P_m} \leq |S_m f|_{L_p} \leq (\rho\mu/\mu_0)|C_m f|_{P_m} \qquad (m = 1, 2, \cdots).$$

Proof. By definition, $|S_m f|_{L_p}$ is the norm of the linear transformation $g \to h$ on L_p, where g has Fourier series $\sum c_n e^{-inx}$ and h has Fourier series $\sum S_m f(n) c_n e^{-inx}$. It follows from Lemma 2.1 that

$$\left\| \sum_{0 \leq n \leq m-1} c_n e^{-in(\cdot)} \right\|_{L_p} \leq \rho \|g\|_{L_p},$$

where ρ is a constant depending only on p. Therefore, we can choose

$$g(x) = \sum c_n e^{-inx} \qquad (n = 0, 1, \cdots, m-1)$$

so that

$$\|g\|_{L_p} \leq \rho \quad \text{and} \quad |s_m f|_{L_p} = \|h\|_{L_p}. \tag{3.1.1}$$

Simple computation shows that

$$h(2\pi(\cdot)/m) = C_m f * (g(2\pi(\cdot)/m)).$$

Combining this with Lemma 2.2 and (3.1.1), we obtain the inequalities

$$|S_m f|_{L_p} = \|h\|_{L_p} \leq \mu\left(\frac{1}{m}\sum_{0 \leq n \leq m-1} |h(2\pi n/m)|^p\right)^{1/p}$$

$$= \mu\, \|C_m f * (g(2\pi(\cdot)/m))\|_{P_m} \leq \mu\, |C_m f|_{P_m} \|g(2\pi(\cdot)/m)\|_{P_m}$$

$$\leq (\mu/\mu_0)|C_m f|_{P_m} \|g\|_{L_p} \leq (\rho\mu/\mu_0)|C_m f|_{P_m}.$$

This establishes the second inequality in Lemma 3.1.

On the other hand, since P_m is m-dimensional, we can choose

$$g(x) = \sum c_n e^{-inx} \qquad (n = 0, 1, \cdots, m-1)$$

so that $\|g(2\pi(\cdot)/m)\|_{P_m} = 1$ and

$$|C_m f|_{P_m} = \|C_m f * g(2\pi(\cdot)/m)\|_{P_m} = \left(\frac{1}{m} \sum_{0 \leq n \leq m-1} |h(2\pi n/m)|^p\right)^{1/p}$$

$$\leq \mu_0^{-1} \|h\|_{L_p} \leq \mu_0^{-1} |S_m f|_{L_p} \|g\|_{L_p} \leq (\mu/\mu_0) |S_m f|_{L_p}.$$

This completes the proof of the lemma.

For each a in J, let lg(a) denote the smallest nonnegative integer s such that a vanishes outside of an interval of the form $k \leq n \leq k + s$ for some integer k.

3.2. LEMMA. *Suppose* s *is a positive integer,* $a \in J$, *and* $\lg(a) \leq s/2$; *then*

$$|a|_{P_s} \leq 2 \|a\|_{W_p}, \tag{3.2.1}$$

$$\|a\|_{W_p} \leq 2 |a|_{P_s}. \tag{3.2.2}$$

Proof. First consider (3.2.1). For convenience, we let X_j $(j = 0, 1)$ denote the functions on Z whose supports lie in the set

$$\bigcup_{k \in Z} ([js/2, (j+1)s/2) + ks).$$

Let $x \in P_s$. Obviously, x can be expressed uniquely as a sum $x_0 + x_1$ $(x_j \in X_j)$. For one of the two possible choices of j, we have the inequality

$$\|a * x\|_{P_s} \leq 2 \|a * x_j\|_{P_s}. \tag{3.2.3}$$

Using this choice of j, we let y denote the function on Z that agrees with x_j on [0, s) and is 0 outside of [0, s). An elementary argument yields the inequalities

$$\|a * x_j\|_{P_s} \leq (s)^{-1/p} \|a * y\|_{\ell_p} \leq (s)^{-1/p} \|a\|_{W_p} \|y\|_{\ell_p} = \|a\|_{W_p} \|x_j\|_{P_s}.$$

Together with (3.2.3), they yield (3.2.1).

To prove (3.2.2) we need the following proposition.

(3.2.4) *If* a *and* x *are elements of* J *such that* $\lg(a) \leq s/2$ *and* $\lg(x) < s/2$, *then* $\|a * x\|_{\ell_p} \leq |a|_{P_s} \|x\|_{\ell_p}$.

Since the two norms in the conclusion of (3.2.4) are translation-invariant, we may assume that a and x are supported in the intervals [0, s/2] and [0, s/2), respectively. Let y be the function in P_s that agrees with x on [0, s/2). Then

$$\|a * x\|_{\ell_p} = (s)^{1/p} \|a * y\|_{P_s} \leq (s)^{1/p} |a|_{P_s} \|y\|_{P_s} = |a|_{P_s} \|x\|_{\ell_p}.$$

This completes the proof of (3.2.4).

Continuing with the proof of (3.2.2), we note that each element u in J can be expressed uniquely as a sum $u_0 + u_1$ ($u_j \in X_j$; j = 0, 1). For one of the two possible choices of j, we must have the inequality

$$\|a * u\|_{\ell_p} \leq 2 \|a * u_j\|_{\ell_p}. \tag{3.2.5}$$

We now fix j so that (3.2.5) is valid. Let x_k denote the element of J that agrees with u_j on $[js/2, (j+1)s/2) + ks$ and is 0 elsewhere. Evidently, (3.2.4) can be applied to a and x_k, for each integer k, and therefore $\|a * x_k\|_{\ell_p} \leq |a|_{P_s} \|x_k\|_{\ell_p}$. Since the functions $a * x_k$ have pairwise disjoint supports and the functions x_k have pairwise disjoint supports, the inequalities

$$(\|a*u_j\|_{\ell_p})^p = \sum_k (\|a*x_k\|_{\ell_p})^p \leq (|a|_{P_s})^p \sum_k (\|x_k\|_{\ell_p})^p = (|a|_{P_s})^p (\|u_j\|_{\ell_p})^p$$

hold. These inequalities together with (3.2.5) and the fact that J is dense in ℓ_p, yield (3.2.2). This completes the proof of the lemma.

3.3. THEOREM. *For* $1 < p < \infty$, *let* ρ, μ, μ_0 *be the constants in Lemmas* 2.1 *and* 2.2. *Then, for each positive integer* m, *we have the inequalities*

$$|s_m f|_{L_p} \leq 2(\rho\mu/\mu_0) \|f\|_{W_p} \qquad (f \in W_p), \tag{3.3.1}$$

$$\|f\|_{W_p} \leq 2(\mu_0/\mu) |S_m f|_{L_p} \tag{3.3.2}$$

for each trigonometric polynomial of the form

$$f(x) = \sum a_n e^{inx} \qquad (n = \alpha, \alpha+1, \cdots, \alpha+m-1),$$

where α *is an integer; and, given* f *in* W_p *and a positive real number* ε, *we can find a positive integer* m *such that*

$$(\mu/2\mu_0) \|f\|_{W_p} < |S_m f|_{L_p} + \varepsilon. \tag{3.3.3}$$

Proof. First we prove (3.3.1) for an arbitrary trigonometric polynomial $f(x) = \sum a_n e^{inx}$. Let w be an element in P_m, and note that w is also in P_{rm} for $r = 1, 2, \cdots$. Since $w \in P_m$ and $C_m f(n) = \sum_j a(n + jm)$ $(0 \leq n \leq m)$, it is clear that $a * w = C_m f * w$. By Lemma 3.2, we can choose r so that

$$|a|_{P_{rm}} \leq 2 \|a\|_{W_p}.$$

With this choice of r and the above observation, we obtain the inequality

$$\|C_m f * w\|_{P_m} = \|a * w\|_{P_m} = \|a*w\|_{P_{rm}} \leq 2 \|f\|_{W_p} \|w\|_{P_{rm}} = 2 \|f\|_{W_p} \|w\|_{P_m}.$$

Hence,

$$|C_m f|_{P_m} \leq 2 \|f\|_{W_p}.$$

This inequality, together with Lemma 3.1, proves (3.3.1) for trigonometric polynomials. We shall now prove (3.3.1) in general. From Lemma 2.3 we know that for each f in W_p,

$$\|f\|_{W_p} \geq \text{the uniform norm of } f;$$

therefore, in particular, $\|f\|_{W_p} \geq |f(x)|$ for each x in G_m. From this observation, the definition of $S_m f$, and the fact that the range of S_m on W_p is finite-dimensional, we see that there exists a constant Q_m, depending only on m, such that

$$|S_m f|_{L_p} \leq Q_m \|f\|_{W_p} \qquad (f \in W_p).$$

Given $f \in W_p$, a positive integer m, and $\varepsilon > 0$, choose a trigonometric polynomial g such that $|f - g| < \varepsilon$. Then

$$|S_m f|_{L_p} \leq \varepsilon Q_m + |S_m g|_{L_p} \leq \varepsilon Q_m + 2(\rho\mu/\mu_0)\|g\|_{W_p}$$

$$\leq \varepsilon(Q_m + 2(\rho\mu/\mu_0)) + 2(\rho\mu/\mu_0)\|f\|_{W_p}.$$

This establishes (3.3.1).

To prove (3.3.2), we let $f(x) = \sum a(n)e^{inx}$, where $\lg(a) \leq m$. From the definition of $C_m f$ it is clear that a is a translate of $C_m f$; therefore, the $|\ |_{P_{2m}}$-norms of $C_m f$ and a are equal. From this observation and from Lemmas 3.2 and 3.1 we conclude that

$$\|f\|_{W_p} \leq 2\,|a|_{P_{2m}} = 2\,|C_m f|_{P_{2m}} \leq 2(\mu_0/\mu)\,|S_m f|_{L_p}.$$

To prove the last assertion of the theorem, let $f \in W_p$, and choose a trigonometric polynomial $g(x) = \sum a_n e^{inx}$ $(n = \alpha, \alpha + 1, \cdots, \alpha + m - 1)$ such that

$$|f - g| < \min(\varepsilon/(4\rho\mu/\mu_0), \varepsilon/(\mu/\mu_0)).$$

Together with this inequality, (3.3.1) and (3.3.2) imply that

$$|S_m f|_{L_p} + \varepsilon/2 \geq |S_m g|_{L_p} \geq (\mu/2\mu_0)\|g\|_{W_p} \geq (\mu/2\mu_0)\|f\|_{W_p} - \varepsilon/2.$$

This completes the proof of Theorem 3.3.

3.4. *Remarks on the theorem.* We now consider the cases $p = 1$ and $p = \infty$. Since both of the norms that appear have the same values for $p = 1, \infty$, we need only consider the case $p = 1$. For each trigonometric polynomial

$$f(x) = \sum a_n e^{inx} \qquad (n = \alpha, \alpha + 1, \cdots, \alpha + q - 1),$$

where α is an integer and q is a positive integer, we consider the related polynomials $F(x) = \sum S_m f(n) e^{-inx}$ $(m = 1, 2, \cdots)$. Since $L_1[0, 2\pi)$ has an approximate identity, it follows that

$$|S_m f|_{L_1} = \frac{1}{2\pi}\int_0^{2\pi} |F(x)|\, dx. \tag{3.4.1}$$

Since

$$F(2\pi n/m) = m \sum_j a_{n+jm} \quad \text{for } m > 0 \text{ and } n = 0, 1, \cdots, m-1,$$

we see that

$$\|f\|_{W_1} = \sum |a_n| = \frac{1}{m}\sum_{n=0}^{m-1} |F(2\pi n/m)| \quad (m \geq q). \tag{3.4.2}$$

From (3.4.1), (3.4.2), and [9, Vol. 2, Chap. 10, Section 8], we see that the constant $2\rho\mu/\mu_0$ in (3.3.1) cannot be replaced by any constant independent of m. From (3.4.1) and (3.4.2) it is clear that $2(\mu_0/\mu)$ can be replaced by 2π in (3.3.2) when $p = 1$, and that $\mu/2\mu_0$ can be replaced by $(2\pi)^{-1}$ in (3.3.3).

4. APPROXIMATION OF W_p-CONTINUITY SETS

Let E be a compact subset of the circle G. We let $H_p(E)$ $(1 \leq p < 2)$ denote the infimum of the numbers K in $[0, \infty]$ (including $+\infty$) that satisfy the following condition: for each continuous function g on E, there is an f in W_p such that $f = g$ on E and $\|f\| \leq K\|g\|_{E,\infty}$. We call E a W_p-*continuity* set if $H_p(E) < \infty$. Note that the W_1-continuity sets are precisely the Helson sets. We shall only deal with the case $1 < p < 2$. For finite subsets F of Z, the number $\sigma_p(F)$ (see the definition after Lemma 5.2) is the analogue of H_p. Our purpose in this section is to compare $H_p(E)$ with $\sigma_p((m/2\pi)E_k)$, where E_k is a sequence of special finite subsets of G that approach E in a certain sense. This comparison is the content of Theorems 4.2 and 4.3. (The constants ρ, μ_0, and μ are defined in Lemmas 2.1 and 2.2.)

4.1. LEMMA. *If* $E \subset G_m$ *and* h *is a complex-valued function on* E, *there exists an* f *in* T_m *such that* $f = h$ *on* E *and*

$$\|f\|_{W_p} \leq 2\rho(\mu_0/\mu)\sigma_p(F)\|h\|_{E,\infty},$$

where $F = (m/2\pi)E$.

Proof. For each n in F, let $\alpha(n) = h(2\pi n/m)$. There exists a function β on Z such that $\beta = \alpha$ on F and

$$|\beta|_{L_p} \leq (\sigma_p(F) + \varepsilon)\|\alpha\|_{F,\infty}. \tag{4.1.1}$$

Let γ be the function that is 0 at each integer outside the interval $[0, m)$ and agrees with β at each integer in the interval $[0, m)$. By Lemma 2.1, it is clear that

$$|\gamma|_{L_p} \leq \rho|\beta|_{L_p}. \tag{4.1.2}$$

There exists an f in T_m such that $S_m f = \gamma$, and Theorem 3.3 implies that

$$\|f\|_{W_p} \leq 2(\mu_0/\mu)|\gamma|_{L_p}. \tag{4.1.3}$$

Clearly, $f = h$ on E. Noting that

$$\|\alpha\|_{F,\infty} = \|h\|_{E,\infty},$$

we can combine (4.1.1), (4.1.2), and (4.1.3) to conclude that

$$\|f\|_{W_p} \leq 2\rho(\mu_0/\mu)(\sigma_p(F) + \varepsilon)\|h\|_{E,\infty}.$$

Since T_m is compact, we can remove the ε. This completes the proof.

4.2. THEOREM. *Suppose that* E *is a compact subset of* G, *and that there exists a sequence of sets* E_j $(j = 1, 2, \cdots)$ *such that for each index* j

(i) $E_j \subset G_{m_j}$ *for some* m_j, *where* $m_j \to \infty$,

(ii) *each point in* E *is within a distance* (mod 2π) *of* r/m_j *of some point of* E_j, *and each point of* E_j *is within a distance* (mod 2π) *of* r/m_j *of some point in* E.

Let $K = 2\rho(\mu_0/\mu)\sup_j \sigma_p((m_j/2\pi)E_j)$. *If* $Kr < 1$, *then*

$$H_p(E) \leq K/(1 - rK).$$

Proof. Let g be a continuous function on E. Fix j, and define the function h on E_j as follows. For x in E_j, choose a point t in E that is nearest to x, and let $h(x) = g(t)$. By Lemma 4.1, there exists an f in T_{m_j} such that $f = h$ on E_j and

$$\|f\|_{W_p} \leq K\|h\|_{E_j,\infty} \leq K\|g\|_{E,\infty}. \tag{4.2.1}$$

We now consider how close f and g are on the set E. Let x be a point in E. Let t be a point in E_j that is nearest to x. Let y be a point in E that is nearest to t and for which $h(t) = g(y)$. The choice of y is possible because of the definition of h. Since $f(t) = h(t) = g(y)$, we see that

$$|f(x) - g(x)| \leq |f(x) - f(t)| + |g(y) - g(x)|.$$

Since the distance from x to t does not exceed r/m_j, we conclude from Bernstein's inequality that $|f(x) - f(t)| \leq r\|f\|_\infty$. From this, (4.2.1), the inequality $\|f\|_\infty \leq \|f\|_{W_p}$, the continuity of g on E, and the fact that the distance from y to x does not exceed $2r/m_j$, we conclude that

$$\|f - g\|_{E,\infty} \leq rK\|g\|_{E,\infty} + \omega(g, 2r/m_j),$$

where $\omega(g, \delta)$ is the modulus of continuity. Since g is continuous, $m_j \to \infty$, and K is independent of j, we have proved the following.

For each $\varepsilon > 0$ and each continuous function g on E, there is a trigonometric polynomial f such that

$$\|f\|_{W_p} \leq K\|g\|_{E,\infty} \tag{4.2.2}$$

and

$$\|f - g\|_{E,\infty} \leq (rK + \varepsilon)\|g\|_{E,\infty}. \tag{4.2.3}$$

We can now complete the proof of the theorem by the usual method of successive approximations based on (4.2.2) and (4.2.3). In particular, we can choose a sequence of trigonometric polynomials f_n $(n = 1, 2, \cdots)$ such that

$$\|g - (f_1 + \cdots + f_{n+1})\|_{E,\infty} \leq (rK + \varepsilon)^{n+1} \|g\|_{E,\infty},$$

$$\|f_{n+1}\|_{W_p} \leq K(rK + \varepsilon)^n \|g\|_{E,\infty}.$$

From this we conclude that if ε is chosen so that $rK + \varepsilon < 1$, then $f \left(= \sum f_n \right)$ is in W_p, $f = g$ on E, and

$$\|f\|_{W_p} \leq (K/(1 - rK - \varepsilon)) \|g\|_{E,\infty}.$$

Therefore, by definition, $H_p(E) \leq K/(1 - rK)$, and the proof is complete.

4.3. THEOREM. *If* E *is a* W_p-*continuity set and* $D \subset E \cap G_m$, *then* $\sigma_p(F) \leq 2(\rho\mu/\mu_0)H_p(E)$, *where* $F = (m/2\pi)D$.

Proof. Let α be a complex-valued function defined on F. Choose a continuous function f on E such that $\|f\|_{E,\infty} = \|\alpha\|_{F,\infty}$ and $f(2\pi n/m) = \alpha(n)$ for each n in F. For each $\varepsilon > 0$, there is an element g in W_p such that $g = f$ on E and

$$\|g\|_{W_p} \leq (\varepsilon + H_p(E)) \|f\|_{E,\infty}. \tag{4.3.1}$$

We now define the function β on Z by the rule

$$\beta(n) = \begin{cases} g(2\pi n/m) & \text{if } 0 \leq n < m, \\ 0 & \text{otherwise.} \end{cases}$$

From the choice of f, g, and β, it is clear that $\alpha = \beta$ on F. By Theorem 3.3,

$$|\beta|_{L_p} \leq 2(\rho\mu/\mu_0) \|g\|_{W_p}. \tag{4.3.2}$$

Since ε is arbitrary, the conclusion of the lemma follows from (4.3.1) and (4.3.2).

5. COMPARISON OF CERTAIN CONSTANTS

The purpose of this section is to give three results, Theorem 5.3, and Lemmas 5.4 and 5.5, that will be used in the next section. Lemmas 5.4 and 5.5 are due to Rudin, and they appear in [8] as Theorems 3.5 and 4.5, respectively. Theorem 5.3 would be a special case of [4, Theorem 6], except that we state explicitly the relation of certain constants. [4, Theorem 6] is a general result for all compact groups, and the proof requires knowledge of representation theory. For this reason we have included a short proof of Theorem 5.3. Our proof depends on Theorem 5.1, which is due to Figà-Talamanca [3], and Lemma 5.2. Lemma 5.2 is essentially a part of [5, Theorem 1]. A short proof of Lemma 5.2 is given in [7, p. 48]. Further discussion of trigonometric series with random coefficients can be found in Section 8 of Chapter V in [9].

Let M_p denote the space of functions h on Z such that $h\hat{f}$ is the Fourier transform of a function in L_p for each f in L_p. It is well known that each h in M_p

determines a bounded operator U_h on L_p, where $U_h f$ is the function in L_p with Fourier transform $h\hat{f}$. Evidently, M_p is a Banach algebra with this operator norm. For each subset F of Z, let $M_p | F$ denote the quotient space of M_p with respect to the subspace of all functions h_1 in M_p that vanish on F, and give $M_p | F$ the usual quotient norm; that is, if $h^\sim$ is the coset containing h, let

$$\|h^\sim\|_{M_p|F} = \inf\{\|h - h_1\|_{M_p}: h_1 \in M_p,\ h_1 \equiv 0 \text{ on } F\}.$$

It will also be convenient to let $\|\ \|_{M_p|F}$ denote the pseudonorm on M_p that has the value $\|h^\sim\|_{M_p|F}$ at h in M_p.

Let A_p denote the linear space of functions k of the form

$$k = \sum f_j * g_j \quad (j = 1, 2, \cdots), \tag{5.0.1}$$

where $f_j \in L_p$, $g_j \in L_q$, and

$$\sum \|f_j\|_{L_p} \|g_j\|_{L_q} < \infty \quad (j = 1, 2, \cdots). \tag{5.0.2}$$

If we let $\|k\|_{A_p}$ denote the infimum of the numbers in (5.0.2), where f_j and g_j satisfy (5.0.1), then A_p with this norm is a Banach space. One of the important features of the Banach space A_p is described in the following theorem due to Figà-Talamanca [3, Theorem 1], who first introduced the space A_p.

5.1. THEOREM. *For each* h *in* M_p, *the function* ϕ_h *defined by*

$$\phi_h(k) = \sum_j (U_h f_j) * g_j(0),$$

where k *is represented as in* (5.0.1), *is a well-defined bounded linear functional on* A_p. *Furthermore, the map* $h \to \phi_h$ *is an isometric linear map from* M_p *onto the dual of* A_p.

5.2. LEMMA. *If* g *is a trigonometric polynomial and* q *is a real number* $(2 < q < \infty)$, *there exists a function* r_n *on* Z *such that* $r_n = 1$ *or* -1 *for each* n *and*

$$\|g_1\|_{L_q} \leq R\|g\|_{L_2},$$

where $\hat{g}_1(n) = r_n \hat{g}(n)$ $(n \in Z)$ *and* R *is a constant that depends only on* q.

Suppose $F \subset Z$, $1 < p < 2$, and $p^{-1} + q^{-1} = 1$. A trigonometric polynomial f will be called an *F-polynomial* if $\hat{f}(n) = 0$ for $n \notin F$. We now define the three constants $\beta_p(F)$, $\lambda_p(F)$, and $\sigma_p(F)$. We define $\beta_p(F)$ as the infimum of all real numbers K (possibly $+\infty$) such that

$$\sum |\hat{f}(n)\hat{g}(n)| \leq K\|f\|_{L_p}\|g\|_{L_q}$$

for all trigonometric polynomials f and all F-polynomials g. Let $\lambda_p(F)$ denote the infimum of all real numbers K (possibly $+\infty$) such that

$$\|g\|_{L_q} \leq K \|g\|_{L_2}$$

for all F-polynomials g. Finally, let $\sigma_p(F)$ denote the infimum of all real numbers K (possibly $+\infty$) such that

$$\|h\|_{M_p|F} \leq K \|h\|_{F,\infty}$$

for all h in M_p.

5.3. THEOREM. *To each* p $(1 < 1 < 2)$ *there corresponds a positive number* R_p *such that*

$$R_p \lambda_p(F) \leq \sigma_p(F) \leq \lambda_p(F)$$

for each subset F *of* Z.

Proof. To prove the theorem, it clearly suffices to establish the three inequalities

(5.3.1) $$R_p \lambda_p(F) \leq \beta_p(F),$$

(5.3.2) $$\sigma_p(F) \leq \lambda_p(F),$$

(5.3.3) $$\beta_p(F) \leq \sigma_p(F).$$

First consider (5.3.1). Suppose g is an F-polynomial and $\varepsilon > 0$. Choose a trigonometric polynomial f such that $\|f\|_{L_p} = 1$ and

(5.3.4) $$\|g\|_{L_q} \leq \varepsilon + \left|\sum \hat{f}(n)\hat{g}(n)\right|.$$

By Lemma 5.2, we can choose $r_n = \pm 1$ so that $\|g_1\|_{L_q} \leq R \|g\|_{L_2}$, where $\hat{g}_1(n) = r_n \hat{g}$. Thus,

$$\left|\sum \hat{f}(n)\hat{g}(n)\right| \leq \sum |r_n \hat{f}(n)\hat{g}(n)| \leq \beta_p \|f\|_{L_p} \|g_1\|_{L_q} \leq \beta_p R \|g\|_{L_2}.$$

The assertion (5.3.1) follows from these inequalities and (5.3.4).

In order to establish (5.3.2), it clearly suffices to show that

(5.3.5) $$\|\xi h\|_{M_p} \leq \lambda_p \|h\|_{F,\infty},$$

where ξ is the characteristic function of F as a subset of Z. Each multiplier h in M_p defines a linear transformation on the space T by restricting U_h to T; furthermore, from the definition of U_h and the fact that L_p and L_q are dual spaces, it is clear that $|U_h|_X = |U_h|_Y$, where X and Y denote T with norms $\| \; \|_{L_p}$ and $\| \; \|_{L_q}$, respectively. For any trigonometric polynomial f, $U_{\xi h} f$ is an F-polynomial; therefore

$$\|U_{\xi h} f\|_{L_q} \leq \lambda_p(F) \|U_{\xi h} f\|_{L_2} \leq \lambda_p(F) \|\xi h\|_{Z,\infty} \|f\|_{L_2} \leq \lambda_p(F) \|h\|_{F,\infty} \|f\|_{L_q}$$

These inequalities together with our remarks prove (5.3.5), and this completes the proof of (5.3.2).

Now consider (5.3.3). Suppose f is a trigonometric polynomial and g is an F-polynomial. Let F_1 be the set of integers n such that $\hat{g}(n) \neq 0$. Let h be a complex-valued function on Z such that h is 0 on the complement of F_1, $\|h\|_{Z,\infty} = 1$, and

(5.3.6) $$\sum h(n)\hat{f}(n)\hat{g}(n) = \sum |\hat{f}(n)\hat{g}(n)| .$$

By hypothesis, we can choose $h_1 \in M_p$ so that $h_1 = h$ on F_1 and

(5.3.7) $$\|h_1\|_{M_p} \leq \sigma_p(F) + \varepsilon ,$$

where ε is an arbitrary positive number. From the definition of U_{h_1} and the fact that $h = h_1$ on F it is clear that

(5.3.8) $$\sum h(n)\hat{f}(n)\hat{g}(n) = (U_{h_1} f) * g(0) .$$

Since $f * g \in A_p$ and $\|f * g\|_{A_p} \leq \|f\|_{L_p} \|g\|_{L_q}$, it follows from Theorem 5.1 that

(5.3.9) $$|(U_{h_1} f) * g(0)| \leq \|h_1\|_{M_p} \|f\|_{L_p} \|g\|_{L_q} .$$

The inequality (5.3.3) now follows if we combine (5.3.6), (5.3.7), (5.3.8), and (5.3.9). This completes the proof of Theorem 5.3.

We now state two lemmas due to Rudin, [8, Theorem 3.5] and [8, Theorem 4.5]. For each subset F of Z and each positive integer n, let $\alpha(F, n)$ denote the maximum number of elements that can occur in the intersection of F and an arithmetic progression of n terms. Corresponding to each set of nonnegative integers F, each positive integer s, and each nonnegative integer n, we let $r_s(F, n)$ denote the number of different s-tuples $(n_1, \cdots, n_s)$ with $n_j \in F$ $(j = 1, 2, \cdots, s)$ and $n_1 + n_2 + \cdots + n_s = n$.

5.4. LEMMA. *If* $F \subset Z$, $n = 1, 2, \cdots$, $1 < p < 2$, *and* $p^{-1} + q^{-1} = 1$, *then*

$$\alpha(F, n) \leq 4(\lambda_p(F))^2 n^{2/q} .$$

5.5. LEMMA. *If* F *is a set of nonnegative integers and* s *is an integer* $(s \geq 2)$, *then*

$$\lambda_p(F) \leq \max_{n \in Z} (r_s(F, n))^{1/s} ,$$

where $p^{-1} + (2s)^{-1} = 1$.

6. A W_p-CONTINUITY SET THAT IS NOT A $W_{p-\varepsilon}$-CONTINUITY SET

In this section, we describe a set E that is a $W_{4/3}$-continuity set, but is not a W_p-continuity set for $p < 4/3$ (Theorem 6.1). Let $2 = p_1 < p_2 < \cdots$ denote the prime numbers, and for $m = 1, 2, \cdots$, let D_m denote the set of numbers

$$\nu_{p_m}(k^2) + k(2p_m) \qquad (k = 1, 2, \cdots, p_m - 1),$$

where $\nu_q(n) \equiv n \pmod q$ and $0 \leq \nu_q(n) < q$ for each positive integer q and integer n. Let N_m denote the maximum element in D_m. Since

$$\alpha(D_m, N_m) \geq p_m - 1 \geq p_m/2 = \frac{(2p_m^2)^{1/2}}{2\sqrt{2}}$$

and $N_m < 2p_m^2$, we have the inequality

$$\alpha(D_m, N_m) \geq 2^{-3/2} N_m^{1/2}. \tag{6.0.1}$$

This, together with the following result due to Erdös [1], will be used in the proof of the theorem.

(6.0.2) If m_1, m_2, n_1, and n_2 are all in some set D_k and $m_1 + m_2 = n_1 + n_2$, then $\{m_1, m_2\} = \{n_1, n_2\}$.

In order to define the set E, let t denote a real number subject to the condition

$$t > \max\{3, 4\rho(\mu_0/\mu)\} \tag{6.0.3}$$

where ρ, μ_0, and μ (see Lemmas 2.1 and 2.2) correspond to the value $p = 4/3$. Let $r_1, r_2, \cdots$ be integers such that $tN_m < r_m$ for $m = 1, 2, \cdots$. For each real number q and each set S of real numbers, we let qS denote the set $\{qx: x \in S\}$. Let E denote the set

$$\{0\} \cup \bigcup_{1 \leq k < \infty} D_k/r_1 \cdots r_k.$$

It is convenient to let F_m denote the set

$$\{0\} \cup r_2 \cdots r_m D_1 \cup r_3 \cdots r_m D_2 \cup \cdots \cup D_m.$$

Recall that $H_p(E)$ is defined at the beginning of Section 4.

6.1. THEOREM. *For the set* E *defined above,*

$$H_{4/3}(E) < \infty \quad \textit{and} \quad H_p(E) = \infty \ \textit{for} \ 1 \leq p < 4/3.$$

Proof. Let $E_m = (2\pi/r_1 \cdots r_m)F_m$. The main idea of the proof is to show that the sequence of sets E_m is related to E in such a manner that Theorems 4.2 and 4.3 can be applied. We fix m at an arbitrary value, and for convenience, we let

$$B_1 = D_m, \quad B_2 = r_m D_{m-1}, \cdots, \quad B_m = r_2 r_3 \cdots r_m D_1.$$

Since $3N_k < r_k$ for each positive integer k, it is clear that the largest integer in F_m, namely $r_2 \cdots r_m N_1$, is less than $r_1 \cdots r_m$. We shall now complete the proof, assuming that

$$r_2(F_m, n) \leq 4 \quad (m = 1, 2, \cdots; n \in Z), \tag{6.1.1}$$

and then we shall prove (6.1.1).

From Theorem 5.3, Lemma 5.5 (with $s = 2$), and (6.1.1), we see that

$$\sigma_{4/3}(F_m) \leq \lambda_{4/3}(F_m) \leq 4^{1/2}. \tag{6.1.2}$$

Furthermore, $E_m \subset E$, and $N_{m+1}/r_1 \cdots r_{m+1}$ is the element in E that is farther from E_m than any other element in E. If we let $m_j = r_1 \cdots r_j$ in Theorem 4.2, then it is clear from the inequality $tN_{m+1} < r_{m+1}$ that the number r in Theorem 4.2 does not exceed 1/t. From (6.1.2) we see that the constant K in Theorem 4.2 corresponding to $p = 4/3$ does not exceed $4\rho(\mu_0/\mu)$. Therefore it follows from our choice of t (see (6.0.3)) that $Kr < 1$. We conclude from Theorem 4.2 that

$$H_{4/3}(E) \leq K/(1 - rK).$$

We now consider the second assertion in the theorem. Suppose that $1 \leq p < 4/3$ and $p^{-1} + q^{-1} = 1$, so that $q > 4$. From (6.0.1), the inclusion $D_m \subset F_m$, Lemma 5.4, and Theorems 5.3 and 4.3, respectively, we obtain the inequalities

$$2^{-3/2}N^{1/2} \leq \alpha(D_m, N_m) \leq \alpha(F_m, N_m) \leq 4(\lambda_p(F_m))^2 N_m^{2/q}$$
$$\leq 4R_p^{-2}(\sigma_p(F_m))^2 N_m^{2/q} \leq 4R_p^{-2}(2\rho\mu/\mu_0)^2(H_p(E))^2 N_m^{2/q}.$$

Since $q > 4$ and $N_m \to \infty$ as $m \to \infty$, it is now clear that $H_p(E) = \infty$.

We now complete the proof of the theorem by establishing (6.1.1). The following fact is needed, and it is easily proved.

(6.1.3) *If* q_1, q_2, *and* q_3 *are all in some set* B_k *and* $\varepsilon_j = -1, 0,$ *or* 1 *for* $j = 0, 1, 2$, *then either*

$$\left|\sum \varepsilon_j q_j\right| = 0 \quad \textit{or else}$$

$$\left|\sum \varepsilon_j q_j\right| > 3|q| \quad \textit{for each } q \textit{ and } i \ (q \in B_i,\ i < j).$$

In order to prove (6.1.1) it clearly suffices to prove that $r_2(F_m \setminus \{0\}, n) \leq 2$. Let m_1, m_2, n_1, and n_2 be elements in $F_m \setminus \{0\}$ such that

$$m_1 + m_2 = n_1 + n_2. \tag{6.1.4}$$

Let k be the largest integer such that B_k contains one of these four numbers. We may clearly assume, by changing notation if necessary, that $m_1 \leq m_2$, $n_1 \leq n_2$, and $m_2 \leq n_2$. If we rearrange equation (6.1.4) so that precisely the numbers in B_k appear on the right side of the resulting equation, we have the six possibilities

(1) $m_1 + m_2 = n_1 + n_2$,

(2) $m_1 = n_1 + n_2 - m_2$,

(3) $0 = n_1 + n_2 - m_1 - m_2$,

(4) $m_1 + m_2 - n_1 = n_2$,

(5) $m_1 - n_1 = n_2 - m_2$,

(6) $-n_1 = n_2 - m_2 - m_2$.

From (6.1.3) and the fact that none of the four numbers is 0, we see that (1), (2), (4), and (6) are not possible. In the case of (3), all four numbers belong to B_k, and after factoring out a product of r_j's, we conclude from (6.0.2) that $\{m_1, m_2\} = \{n_1, n_2\}$. In the case of (5), it follows from (6.1.3) that $m_1 = n_1$ and $n_2 = m_2$. We have therefore shown that (6.1.4) can happen only if $\{m_1, m_2\} = \{n_1, n_2\}$. This shows that $r_2(F_m \setminus \{0\}, n) \leq 2$.

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