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Approximation operators in qualitative data analysis*

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Summary. A large part of qualitative data analysis is concerned with approximations of sets on the basis of relational information. In this paper, we present various forms of set approximations via the unifying concept of modal–style operators. Two examples indicate the usefulness of the approach.

1 Introduction

In many instances it is not possible to describe a set precisely, owing to insufficient information or other sources of uncertainty. One way of handling this situation is to assign degrees of belief – probabilities, fuzzy membership assignments etc – to a statement such as "Object x is a member of set X". More cautious approaches consider intervals in which the relevant numerical functions lie, such as upper and lower probabilities or possibility theory. Such (approximations of) "point estimates" are problematic in many cases because the underlying model assumptions are often hard to fulfill. We have argued elsewhere [5] that qualitative tools often give comparative results under much less stringent assumptions. A frequently studied technique of qualitative set description is to determine a lower and an upper approximation of a set using non-numerical techniques. We will call a pair $\langle f, g\rangle$ of functions $2^U \to 2^U$ an *approximation pair* (on U) if

$$f(X) \subseteq X \subseteq g(X) \tag{1}$$

for all $X \subseteq U$. A *weak approximation pair* satisfies

$$f(X) \subseteq g(X). \tag{2}$$

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This seems to be the weakest condition for a sensible concept of set approximation which is internal with respect to U. Stronger structural conditions require that f is an interior operator and g a closure operator, or that they are dual to each other (to be explained below).

A common mathematical basis of this type of set approximation are constructions associated with binary relations on the universe U. A frequently recurring theme is the fact that each binary relation R on U gives rise to a "neighborhood" mapping $f_R : U \to 2^U$ via the assignment

$$x \mapsto \{y \in U : xRy\}. \tag{3}$$

Indeed, according to [14], it was already known to Tarski [30] in 1927 that there is a one-one correspondence between binary relations on U and mappings $f : 2^U \to 2^U$ which satisfy

$$f(\emptyset) = \emptyset,$$
$$f\left(\bigcup_{i\in I} X_i\right) = \bigcup_{i\in I} f(X_i).$$

This observation later formed the basis for the correspondence theory between Kripke frames and modal logics.

Examples are the approximation operators of rough set theory [24], various generalizations [27, 28, 34, 38], the derivation operator of formal concept analysis [33], or the span-content operators of [6].

2 Operators and relations

Much of the mathematical background of qualitative data analysis is concerned with set operators, relational structures, and the interplay among them: Closure and interior operators, (semi–) lattices, polarities, Galois correspondences, duality theory for Kripke frames, and Boolean algebras with operators. Most of the machinery has been developed in the first half of the 20$^{\text{th}}$ century, see for example Birkhoff [1], Jónsson and Tarski [14], McKinsey and Tarski [17, 18], Ore [19]. Many of the early results have been rediscovered by modal logicians [see 13, for a brief discussion], and re-rediscovered by the rough set community [see e.g. 36]. A good overview of the relevant correspondence results is given in [32], and, for rough set theory, in [37, 39].

For unexplained concepts we invite the reader to consult [3] or [9] for order and lattice theory, and [15] for Boolean algebras.

If $\langle A, +, \cdot, -, 0, 1\rangle$ and $\langle B, +, \cdot, -, 0, 1\rangle$ are Boolean algebras, and $f : A \to B$ is a mapping, then the *dual of* f is the function $f^\partial : A \to B$ defined by

$$f^\partial(a) = -f(-a). \tag{1}$$

Clearly, if f preserves $\cdot$, resp. $+$, then its dual preserves $+$, resp. $\cdot$.

If $\langle X, \leq \rangle$ and $\langle Y, \leq \rangle$ are partially ordered sets, a pair $\langle \psi, \varphi \rangle$ is called a *Galois connection* between X and Y, if $\psi : X \to Y$ and $\varphi : Y \to X$ are antitone (i.e. dually order preserving) mappings, and $x \leq x^{\psi\varphi}$, $y \leq y^{\varphi\psi}$ for all $x \in X$, $y \in Y$. $x \in X$ is called *Galois closed* with respect to $\langle \psi, \varphi \rangle$ if $x = x^{\psi\varphi}$.

A mapping $c : 2^U \to 2^U$ is called a *weak closure operator* if it satisfies

Cl1. $c(\emptyset) = \emptyset$,

Cl2. $X \subseteq c(X)$,

Cl3. $X \subseteq Y \Rightarrow c(X) \subseteq c(Y)$.

It is called a *closure operator* if it additionally satisfies

Cl4. $c(c(X)) = c(X)$.

Sets for which $c(X) = X$ are called *closed*. It is well known that the collection of all closed sets of a closure operator can be made into a complete lattice $\mathfrak{L}_c$ by setting

$$\bigvee\{X_i : i \in I\} = c\left(\bigcup\{X_i : i \in I\}\right), \tag{2}$$

$$\bigwedge\{X_i : i \in I\} = \bigcap\{X_i : i \in I\}. \tag{3}$$

Furthermore, each complete lattice is order isomorphic to the complete lattice of closed sets of some closure operator [19]. A closure operator is called *additive*, if it satisfies

Cl5. $c(X \cup Y) = c(X) \cup c(Y)$.

and *completely additive* if it distributes over arbitrary unions.

Note that Cl5 implies Cl3, but not vice versa: Suppose that U has at least three elements and define

$$c(X) = \begin{cases} X, & \text{if } |X| \leq 1, \\ U, & \text{otherwise.} \end{cases}$$

Then, c satisfies Cl1 – Cl4, but not Cl5. Additive closure operators are called "closure operators" in [17], where their corresponding lattices have been extensively studied. Each such $\mathfrak{L}_c$ is the lattice of closed sets of a topology; conversely, the topological closure operator satisfies Cl1 – Cl5.

We call c a *weak interior operator* if it satisfies

Int1. $i(U) = U$,

Int2. $i(X) \subseteq X$,

Int3. $X \subseteq Y \Rightarrow i(X) \subseteq i(Y)$.

and an *interior operator* if, additionally,

Int4. $i(i(X)) = i(X)$.

An interior operator is called *multiplicative* if it satisfies

Int5. $i(X \cap Y) = i(X) \cap i(Y)$,

and *completely multiplicative* if it distributes over arbitrary intersections. Clearly, the dual of a (weak, additive, completely additive) closure operator is a (weak, multiplicative, completely multiplicative) interior operator and vice versa.

If R is a relation between the elements of U and V, and $x \in U$, we let the *converse of* R be the relation $R^{\smile} = \{\langle y,x\rangle : xRy\}$. The *domain of* R is the set $\mathrm{dom}(R) = \{x \in U : (\exists y \in V)xRy\}$, and $R(x)$ is the set $\{y \in V : xRy\}$; sometimes, $R(x)$ is called a *neighborhood of* x [16, 23].

As mentioned above, the assignment $x \mapsto R(x)$ defines a function $\overline{R} : U \to 2^V$. Conversely, given a function $f : U \to 2^V$, we can define a relation $S_f \subseteq U \times V$ by $xS_f y \Longleftrightarrow y \in f(x)$. Clearly, $\overline{R_f} = f$, and $S_{\overline{R}} = R$.

R can be used to define several operators $2^U \to 2^V$:

$$\langle R\rangle(X) = \{b \in V : (\exists a \in X)aRb\}, \qquad \text{(Possibility operator)} \tag{4}$$

$$[R](X) = \{b \in V : (\forall a \in U)[aRb \Rightarrow a \in X]\}, \qquad \text{(Necessity operator)} \tag{5}$$

$$[[R]](X) = \{b \in V : (\forall a \in U))[a \in X \Rightarrow aRb]\}, \qquad \text{(Sufficiency operator)} \tag{6}$$

$$\langle\langle R\rangle\rangle(X) = \{b \in V : (\exists a \in U)[a \notin X \text{ and } \text{ not } aRb]\}. \qquad \text{(Dual sufficiency operator)} \tag{7}$$

The operators (4) – (7) are generalizations of well known operators used in modal and algebraic logic, see e.g. [7, 21] and also [6, 8, 23].

Clearly, $\langle R\rangle$ and $[R]$, as well as $[[R]]$ and $\langle\langle R\rangle\rangle$ are dual to each other, and for each $\mathfrak{X} \subseteq 2^U$,

$$\langle R\rangle\left(\bigcup_{X\in\mathfrak{X}} X\right) = \bigcup_{X\in\mathfrak{X}} \langle R\rangle(X), \tag{8}$$

$$[R]\left(\bigcap_{X\in\mathfrak{X}} X\right) = \bigcap_{X\in\mathfrak{X}} [R](X), \tag{9}$$

$$[[R]]\left(\bigcup_{X\in\mathfrak{X}} X\right) = \bigcap_{X\in\mathfrak{X}} [[R]](X), \tag{10}$$

$$\langle\langle R\rangle\rangle\left(\bigcap_{X\in\mathfrak{X}} X\right) = \bigcup_{X\in\mathfrak{X}} \langle\langle R\rangle\rangle(X). \tag{11}$$

Furthermore, $\langle R\rangle$ and $[[R]]$ are, respectively, the existential and universal extension of the assignment $x \mapsto R(x)$ to subsets of U, since

$$\langle R\rangle(X) = \bigcup_{x\in X} R(x),\ [[R]](X) = \bigcap_{x\in X} R(x).$$

It is also easily seen that

$$[[R]](X) = [(-R)](U \setminus X),\ \langle\langle R\rangle\rangle(X) = \langle(-R)\rangle(U \setminus X). \tag{12}$$

Here, $-R = \{\langle x,y\rangle \in U \times V : \langle x,y\rangle \notin R\}$ is the relational complement of R. The sufficiency operators $[[R]]$ and $[[R^{\smile}]]$ are intimately connected with Galois connections:

Proposition 1. *[19] The pair $\langle [[R]], [[R^{\smile}]]\rangle$ is Galois connection between $\langle 2^U, \subseteq\rangle$ and $\langle 2^V, \subseteq\rangle$, and each Galois connection between these sets has this form for some $R \subseteq U \times V$.*

The combined operators $[R^{\smile}]\langle R\rangle$ and $\langle R^{\smile}\rangle[R]$ will play a major role in our subsequent discussions. For these, we have

Proposition 2. *1. $[R^{\smile}]\langle R\rangle$ is a closure operator on U.*
2. $[R^{\smile}]\langle R\rangle$ and $\langle R^{\smile}\rangle[R]$ are dual to each other.
3. $\langle R^{\smile}\rangle[R]$ is an interior operator on U.

Proof. 1. Clearly, $[R^{\smile}]\langle R\rangle(\emptyset) = \emptyset$. For Cl2, let $x \in X \subseteq U$. Then, $R(x) \subseteq \langle R\rangle(X)$ by definition of $\langle R\rangle$, and hence, $x \in [R^{\smile}]\langle R\rangle(X)$ by (5).

Since both $[R^{\smile}]$ and $\langle R\rangle$ preserve $\subseteq$ by (8) and (9), so does $[R^{\smile}]\langle R\rangle$, and thus, it satisfies Cl3.

Let $Y \subseteq V$. Then,

$$\begin{aligned}
q \in [R^{\smile}]\langle R\rangle[R^{\smile}](Y) &\Rightarrow (\forall s \in V)[qRs \Rightarrow s \in \langle R\rangle[R^{\smile}](Y),\\
&\Rightarrow (\forall s \in V)[qRs \Rightarrow (\exists p \in U)[pRs \wedge (\forall t \in V)[pRt \Rightarrow t \in Y]]],\\
&\Rightarrow (\forall s \in V)[qRs \Rightarrow s \in Y],\\
&\Rightarrow q \in [R^{\smile}](Y),
\end{aligned}$$

which implies Cl4.

2. First, note that $([R^{\smile}]\langle R\rangle)^{\partial}(X) = U \setminus [R^{\smile}]\langle R\rangle(U \setminus X)$. Now,

$$\begin{aligned}
x \in U \setminus [R^{\smile}]\langle R\rangle(-X) &\iff R(x) \not\subseteq \langle R\rangle(U \setminus X),\\
&\iff (\exists z)[xRz \text{ and } z \notin \langle R\rangle(U \setminus X)],\\
&\iff (\exists z)[xRz \text{ and } (\forall y)(y \notin X \Rightarrow y(-R)z)],\\
&\iff (\exists z)[xRz \text{ and } (\forall y)(yRz \Rightarrow y \in X)],\\
&\iff (\exists z)[xRz \text{ and } R^{\smile}(z) \subseteq X],\\
&\iff (\exists z)[xRz \text{ and } z \in [R](X)],\\
&\iff x \in \langle R^{\smile}\rangle[R](X).
\end{aligned}$$

3. follows from 1. and 2. by duality. □

Observe that Proposition 2 is true for arbitrary R. The following result has been known in a different context for some time [33]:

Corollary 1 $[[R^{\smile}]][[R]]$ *is a closure operator.*

Proof. First,

$$\begin{aligned}
[[R^{\smile}]][[R]](X) &= [-R^{\smile}](V \setminus [[R]](X)), && \text{by (12)}\\
&= [-R^{\smile}](V \setminus ([-R](U \setminus X)), && \text{by (12)}\\
&= [-R^{\smile}]\langle -R\rangle(X), && \text{by duality.}
\end{aligned}$$

The claim follows now from Proposition 2. □

This result is also a direct consequence of Proposition 1; we have taken the route above to emphasize the connection with $[R^{\smile}]\langle R\rangle$.

To conclude this Section, let us consider the case where $U = V$. Correspondence theory [32] tells us that

$$X \subseteq \langle R\rangle(X) \iff R \text{ is reflexive,}, \tag{13}$$

$$\langle R\rangle\langle R\rangle(X) \subseteq \langle R\rangle(X) \iff R \text{ is transitive,} \tag{14}$$

$$X \subseteq [R]\langle R\rangle(X) \iff R \text{ is symmetric,} \tag{15}$$

Note that $\langle R\rangle$ is a weak closure operator just in case R is reflexive, and that $\langle R\rangle$ is a completely additive closure operator, if, in addition, R is transitive. In this case, we denote the topology generated by the closed sets by τ_R, i.e.

$$\tau_R = \{U \setminus \langle R\rangle(Y) : Y \subseteq U\} = \{[R](X) : X \subseteq U\}. \tag{16}$$

Since these topologies (and related structures) have been considered in the present context [e.g. 16, 23, 39], we recall some facts about their properties. In the sequel, let R be reflexive and transitive.

Proposition 3. *In τ_R, each $x \in U$ has a smallest open neighborhood.*

Proof. This result has appeared in a different form and context already in [29]. Let $x \in U$, and $\mathfrak{X}$ be the collection of all open neighborhoods of x. By (9) , $\bigcap \mathfrak{X}$ is open, and clearly, it is the smallest open set containing x. □

A topology with the property of Proposition 3 is called *principal*.

Proposition 4. 1. *[29] The collection of all principal topologies on a set U can be made into a lattice which is anti–isomorphic to the lattice of all reflexive and transitive relations on U.*

2. [10] The following are equivalent:

a) R is an equivalence relation.

b) $(\forall X \subseteq U)[X \in \tau_R \iff U \setminus X \in \tau_R]$.

c) τ_R *is regular.*

In this case, a basis for τ_R *are the classes of the partition induced by R, and* $R(x)$ *is the smallest neighborhood of x.*

3 Approximation based on homogenous relations

In our first scenario, we consider a finite set U and various relations and operators on U. The basic idea is that objects are usually not considered in isolation, but are in some way related; these relationships are then used to obtain operators $2^U \to 2^U$ which can approximate subsets of U.

3.1 Equivalence relations and approximation spaces

As a first example, consider the case of a spatial database. A frequently used key is the *minimum bounding box* which is the smallest aligned rectangle enclosing a spatial object. More generally, given a grid of rectangles (cells) in the plane, one can approximate a spatial object O from above by the smallest set of cells which cover O, and from below by the largest set of cells totally contained in O (Figure 1). This situation is easily captured in a relational setting: Suppose that U is an area in the plane which is covered by a grid of disjoint rectangles; some care must be taken that the boundary of adjacent cells belong to exactly one cell. We now define $R \subseteq U \times U$ by

$$xRy \iff x \text{ and } y \text{ are in the same cell.}$$

Clearly, R is an equivalence relation, and each cell corresponds to exactly one equivalence class of R. If X is a region in U, then the upper approximation $\overline{X}$ of X is the union of all classes of R which intersect X, and the lower approximation is the union of all classes of R totally contained in X. In other words,

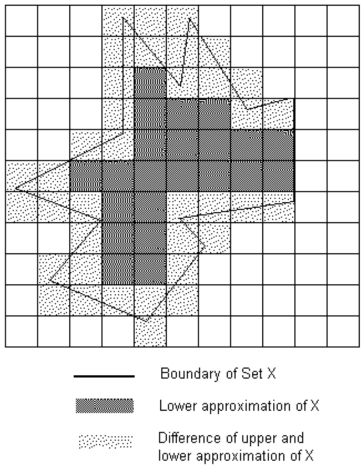
$$\overline{X} = \{x : R(x) \cap X \neq \emptyset\} = \bigcup\{R(x) : x \in X\} = \langle R \rangle(X), \tag{1}$$

$$\underline{X} = \{x : R(x) \subseteq X\} = \{x : R^{\smile}(x) \subseteq X\} = [R](X). \tag{2}$$

More generally, an *approximation space* is a structure $\langle U, R\rangle$, where R is an equivalence relation on the set U [24]. An approximation space tells us the granularity of our knowledge about the world - we can distinguish objects only up to the equivalences classes of R, but not within the classes.

A *rough set* is a pair $\langle A, B\rangle$ such that

Fig. 1. An approximated region



1. A and B are empty or a union of equivalence classes of R,
2. $A \subseteq B$,
3. If C is a singleton class contained in B, then $C \subseteq A$.

Rough sets are approximations of subsets of U in the following sense: Let $X \subseteq U$. Since we cannot distinguish within equivalence classes of R we can say with certainty that some $x \in U$ is a member of X just in case the whole class $R(x)$ is a subset of X. Similarly, we can be sure that $x \notin X$ only if the class $R(x)$ of x is disjoint to X. The rough set approximations are given by (1) and (2). The concepts agree well with the interpretation of these operators in modal logic: $x \in \underline{X}$ if x is certainly a member of X, and $x \in \overline{X}$, if x is possibly a member of X, according to the knowledge delivered by the granularity induced by R. The connection of the rough set approximation operators to modal S5 logic have first been observed by Orlowska [20] and subsequently by many authors; overviews can be found in [4, 40].

Since R is reflexive, the upper approximation is a weak closure operator, and since R is also transitive, it is in fact a completely additive closure operator. Consequently, the lower approximation is a completely multiplicative interior operator. Furthermore, the properties of an equivalence relation imply that

$$\langle R\rangle = \langle R\rangle\langle R\rangle = [R]\langle R\rangle,\ [R] = [R][R] = \langle R\rangle[R]. \tag{3}$$

3.2 More general relations

It was argued by Słowiński and Vanderpooten [27] that for many applications the properties of an equivalence relation are too strong, and that relations with weaker properties should be considered when one wants to express some form of similarity. A first generalization is to require only that R is reflexive and symmetric, and relations with these properties have indeed be called *similarity relations*. This terminology may be somewhat misleading: While a a relation of similarity may be symmetric, calling a reflexive and symmetric relation a “similarity” may cause contextual problems: Suppose that we have agreed on what constitutes similarity and have expressed this by a reflexive and symmetric relation R. Let S be the relation $-R \cup 1'$. Since the complement of a symmetric relation is again symmetric, S is a similarity. Thus, if x is not similar to y according to R, we would say that x is similar to y, according to S.

To suppose that a relation of similarity is symmetric is also not always appropriate. Much of the similarity data used in Computer Science are expert judgments and it is quite reasonable to assume that experts judgments shows some bias. Indeed, the investigations of Tversky [31] show that similarities based on human judgment are often quite asymmetric, and we invite the reader to consult [12] for an example.

It has also been argued that non–reflexive relations can be interpreted as similarity [11], and the following example was given:

> “We may discern persons by comparing photographs taken of them. But it may happen that we are unable to recognize that a same person appears in two different photographs”.

A similar example is the famous experiment by Rothkopf [25]:

> “The S[ubject]s of this experiment were exposed to pairs of aural Morse signals sent at a high tone speed. The signals of each pair were separated by a short temporal interval. The S[ubject]s were asked to indicate whether they thought the signals were the same (or different) by making the appropriate remark on an IBM True–False Answer sheet. Each S[ubject] was asked to respond in this fashion to 351 different pairs of Morse signals.”

We interpret this as

$$xRy \iff x \text{ and } y \text{ are recognized as the same (person, signal).}$$

In both cases, the similarity of x and y is very much in the eye – or the ear – of the beholder, and not necessarily a property of x and y.

An equivalence relation R on U has the special property that

$$\bigcup\{R(x) : x \in X\} = \{x : R(x) \cap X \neq \emptyset\} = \bigcup\{R(x) : R(x) \cap X \neq \emptyset\}, \tag{4}$$

and each is equal to $\overline{X}$. If $R \subseteq U \times U$ is arbitrary, then (4) need not be true. Thus, one needs to decide what constitutes a lower or upper approximation of X. In the literature, one finds many suggestions for such pairs; in (5) and (6) below, R is assumed to be reflexive:

$$R_*(X) = \{x \in U : R\breve{}(x) \subseteq X\}, \qquad R^*(X) = \bigcup\{R\breve{}(x) : x \in X\}, \qquad [28] \tag{5}$$
$$R_{**}(X) = \bigcup\{R\breve{}(x) : R\breve{}(x) \subseteq X\} \qquad R^{**}(X) = \bigcup\{R\breve{}(x) : R\breve{}(x) \cap X \neq \emptyset\} \qquad [28] \tag{6}$$
$$R_+(X) = \bigcup\{R(x) : R(x) \subseteq X\} \qquad R^+(X) = \bigcup\{R(x) : R(x) \cap X \neq \emptyset\} \qquad [39] \tag{7}$$
$$R^{\blacktriangledown}(X) = \{x \in U : R(x) \subseteq X\}, \qquad R^{\blacktriangle}(X) = \{x \in U : R(x) \cap X \neq \emptyset\} \qquad [11, 35, 36] \tag{8}$$
$$R_\diamond(X) = \bigcup\{R\breve{}(x) : R\breve{}(x) \subseteq X\} \qquad R^\diamond(X) = \{x \in U : R(x) \subseteq \langle R\rangle(X)\} \qquad [6] \tag{9}$$

All of these can be located within the modal operator framework:

$$\begin{array}{ll}
R_*(X) = [R](X) & R^*(X) = \langle R\breve{}\rangle(X), \\
R_{**}(X) = \langle R\breve{}\rangle[R](X) & R^{**}(X) = \langle R\breve{}\rangle\langle R\rangle(X) \\
R_+(X) = \langle R\rangle[R\breve{}](X) & R^+(X) = \langle R\rangle\langle R\breve{}\rangle(X) \\
R^{\blacktriangledown}(X) = [R\breve{}](X) & R^{\blacktriangle}(X) = \langle R\breve{}\rangle(X) \\
R_\diamond(X) = \langle R\breve{}\rangle[R](X) & R^\diamond(X) = [R\breve{}]\langle R\rangle(X)
\end{array}$$

We note without proof some inclusion properties among these relations:

Proposition 5.

$$(\forall X)[X \subseteq \langle R\rangle(X)] \iff (\forall X)[[R](X) \subseteq X] \qquad \iff R \textit{ is reflexive}\,. \tag{10}$$
$$(\forall X)[[R](X) \subseteq \langle R\rangle(X)] \iff (\forall X)[X \subseteq \langle R\rangle\langle R\breve{}\rangle(X)] \iff \mathrm{dom}(R\breve{}) = U. \tag{11}$$
$$(\forall X)[\langle R\rangle[R\breve{}](X) \subseteq \langle R\rangle\langle R\breve{}\rangle(X)]. \tag{12}$$

It can be seen that that the only approximation pair for arbitrary R is $\langle R_\diamond, R^\diamond\rangle$; these functions are also dual to each other in the sense of (1) of Section 2 and a closure, respectively, an interior operator. All other pairs need extra conditions such as reflexivity or totality to satisfy the approximation conditions (1) or (2) on page 1. If R is reflexive, then $\langle R_\diamond, R^\diamond\rangle$ gives the tightest bounds for the approximation pairs $\langle f, g\rangle$ above in the sense that

$$f(X) \subseteq R_\diamond(X) \subseteq X \subseteq R^\diamond \subseteq g(X).$$

4 Approximation based on heterogeneous relations

Another type of approximation arises when we have information about the properties of the elements of the domain. Such information may be given by an information system in the sense of [24], or, more generally, by a binary relation $R \subseteq U \times V$ connecting objects with properties. For this situation we have special names for some of the modal operators: If $X \subseteq U$ and $Y \subseteq V$, we say that

- $\langle R\rangle(X)$ is the *span of* X,

- $[R\breve{}](Y)$ is the *content of* Y,
- $[[R]](X)$ is the *intent of* X,
- $[[R\breve{}]](Y)$ is the *extent of* Y.

The span of X is the set of all properties which are related to some element of X, and the content of Y is the set of those objects which can be completely described by the properties in Y. The intent of X are those properties common to all elements of X, and the extent of Y is the set of all objects which possess all properties in Y. Extent and intent are the basic operators of formal concept analysis (FCA) [33]. Since we know from Corollary 1 that

$$[[R\breve{}]][[R]] = [-R\breve{}]\langle -R\rangle$$

the FCA operators are the content–span operator applied to $-R$, and it depends on the context which one is appropriate to use. We just mention a result from [8] which indicates in another way how the two closures differ:

Proposition 6. *For all $x \in U, X \subseteq U$,*

$$x \in [R\breve{}]\langle R\rangle(X) \iff \bigcap\{R(y) : y \in X\} \subseteq R(x).$$
$$x \in [[R\breve{}]][[R]](X) \iff R(x) \subseteq \bigcup\{R(y) : y \in X\}.$$

For an extensive algebraic and topological view of the connections between FCA and approximation spaces we refer the reader to [22].

In this Section we will investigate more closely the operators $[R\breve{}]\langle R\rangle$ and its dual $\langle R\breve{}\rangle[R]$. We know already that $[R\breve{}]\langle R\rangle$ is a closure operator, and $\langle R\breve{}\rangle[R]$ is an interior operator, so that they can serve as sensible approximations of $X \subseteq U$.

We first show that the approximation pair $\langle [R\breve{}]\langle R\rangle, \langle R\breve{}\rangle[R]\rangle$ are the original rough set approximation operators (1), (2) defined on page 7, derived from the standard data representation of rough set analysis: An *information system* is a structure

$$\mathrm{i} = \langle U, \Omega, \{V_q : q \in \Omega\}, f\rangle, \tag{1}$$

where

- U is a finite set of objects.
- Ω is a finite set of *attributes*.
- For each $q \in \Omega$,
  - V_q is a set of *attribute values* of attribute q. In the sequel, we let $V = \bigcup_{q\in\Omega} V_q$.
- $f : U \times \Omega \to V$ is a function such that $f(x,q) \in V_q$ for all $x \in U$, $q \in \Omega$, called the We interpret $f(x,q) = a$ as “Object x has the value a at attribute q”.

Furthermore, if $Q = \{q_1, \dots, q_n\} \subseteq \Omega$ we lift f by setting

$$f_Q(x) = \langle f(x, q_1), \dots f(x, q_n) \rangle. \tag{2}$$

Let $R \subseteq U \times V_\Omega$ be the relational version of f_Ω, i.e.

$$xRt \iff f_\Omega = t.$$

Furthermore, let θ be the kernel of f_Ω, i.e.

$$x\theta y \iff f_\Omega(x) = f_\Omega(y).$$

The approximation space which we consider is $\langle U, \theta \rangle$.

Proposition 7. *Let $X \subseteq U$. Then,*

1. $[R\breve{}]\langle R \rangle(X) = \overline{X}$.
2. $\langle R\breve{}\rangle[R](X) = \underline{X}$.

Proof. We only show 1. since 2. follows immediately by duality.

$$\begin{aligned}
z \in [R\breve{}]\langle R \rangle(X) &\iff R(z) \subseteq \langle R \rangle(X), \\
&\iff f_\Omega(z) \in \langle R \rangle(X), \\
&\iff (\exists x \in X) xR f_\Omega(z), \\
&\iff (\exists x \in X) f_\Omega(x) = f_\Omega(z), \\
&\iff (\exists x \in X) x\theta z, \\
&\iff \theta z \cap X \neq \emptyset, \\
&\iff z \in \overline{X},
\end{aligned}$$

which completes the proof. □

4.1 Example: Student assessment

Suppose S is a set of skills and Q is a set of problems, with which the skills in S should be tested. Let $R \subseteq Q \times S$ be a relation such that qRs is interpreted as

Skill s is necessary to solve q. (3)

The skill set $R(q)$ is minimally sufficient to solve q. (4)

This is an assignment given by an expert. The modal operators can be interpreted as follows: Let $P \subseteq Q$, $M \subseteq S$. Then,

$$
\begin{aligned}
s \in \langle R \rangle(P) &\iff s \text{ is necessary to solve some problem in } P.\\
s \in [R](P) &\iff s \text{ is necessary only for problems in } P.\\
s \in [[R]](P) &\iff s \text{ is necessary for all problems in } P.\\
q \in \langle R^{\smile} \rangle(M) &\iff \text{ Some } s \in M \text{ is necessary to solve } q.\\
q \in [R^{\smile}](M) &\iff \text{ } q \text{ can be solved with the skills in } M.\\
q \in [[R^{\smile}]](M) &\iff \text{ All skills in } M \text{ are necessary to solve } q.
\end{aligned}
$$

Suppose that P is a set of problems which student s has been able to solve; we are interested in the true state of knowledge of s. Let us suppose that the student has made no lucky guesses, i.e. we assume that s really possess all the skills required to solve the problems in P and possibly more; in this case, $\langle R \rangle(P)$ is a lower bound for the skills s has, and P is a lower bound for the set of problems s is able to solve. Now,

$$
\begin{aligned}
q \in [R^{\smile}]\langle R \rangle(P) &\iff R(q) \subseteq \langle R \rangle(P)\\
&\iff \langle R \rangle(P) \text{ contains the skills sufficient to solve } q, \qquad \text{by (4).}
\end{aligned}
$$

Thus, q should have been solved, since s has the skills to solve q; if $q \notin P$, it was due to a careless error. Therefore, q can be included in the true knowledge state of s.

For a more detailed description and a test theory based on skill functions we invite the reader to consult [8].

4.2 Example: Morse data

Another example we shall look at is the famous Morse data collected by Rothkopf [25], a flagship of multidimensional scaling (MDS); the experiment has already been described on page 9. Shepard [26] describes the data using the dimensions

1. Length of the signal,
2. Distribution of dots and dashes in the signal, going from only dots to only dashes.

see Figure 4.2 on the following page. The distances between the points in a plane spanned by these dimensions reflect (partially) the ordinal relation among the given proximities.

In the sequel we will present the re–analysis of the data given in [6] which uses the modal operators. Table 1 on the next page shows in each cell the percentage of subjects who regarded the two stimuli as the same.We use upper case letters for first stimuli and lower case letters for second stimuli; the numeric characters are prefixed by a $*$, if they occur as second stimuli. The matrix diagonal corresponds to pairs which are truly the same, the off-diagonal entries correspond to pairs which are truly different.

As the length of the signal is one of the dimension identified in [26] (and also in [2]), we are interested in the behavior of the modal–style operators on the sets

Fig. 2. MDS interpretation of the Morse data [26]

Table 1. Morse data

	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z	*1	*2	*3	*4	*5	*6	*7	*8	*9	*0
A	92	4	6	13	3	14	10	13	46	5	22	3	25	34	6	6	9	35	23	6	37	13	17	12	7	3	2	7	5	5	8	6	5	6	2	3
B	5	84	37	31	5	28	17	21	5	19	34	40	6	10	12	22	25	16	18	2	18	34	8	84	30	42	12	17	14	40	32	74	43	17	4	4
C	4	38	87	17	4	29	13	7	11	19	24	35	14	3	9	51	34	24	14	6	6	11	14	32	82	38	13	15	31	14	10	30	28	24	18	12
D	8	62	17	88	7	23	40	36	9	13	81	56	8	7	9	27	9	45	29	6	17	20	27	40	15	33	3	9	6	11	9	19	8	10	5	6
E	6	13	14	6	97	2	4	4	17	1	5	6	4	4	5	1	5	10	7	67	3	3	2	5	6	5	4	3	5	3	5	2	4	2	3	3
F	4	51	33	19	2	90	10	29	5	33	16	50	7	6	10	42	12	35	14	2	21	27	25	19	27	13	8	16	47	25	26	24	21	5	5	5
G	9	18	27	38	1	14	90	6	5	22	33	16	14	13	62	52	23	21	5	3	15	14	32	21	23	39	15	14	5	10	4	10	17	23	20	11
H	3	45	23	25	9	32	8	87	10	10	9	29	5	8	8	14	8	17	37	4	36	59	9	33	14	11	3	9	15	43	70	35	17	4	3	3
I	64	7	7	13	10	8	6	12	93	3	5	16	13	30	7	3	5	19	35	16	10	5	8	2	5	7	2	5	8	9	6	8	5	2	4	5
J	7	9	38	9	2	24	18	5	4	85	22	31	8	3	21	63	47	11	2	7	9	9	9	22	32	28	67	66	33	15	7	11	28	29	26	23
K	5	24	38	73	1	17	25	11	5	27	91	33	10	12	31	14	31	22	2	2	23	17	33	63	16	18	5	9	17	8	8	18	14	13	5	6
L	2	69	43	45	10	24	12	26	9	30	27	86	6	2	9	37	36	28	12	5	16	19	20	31	25	59	12	13	17	15	26	29	36	16	7	3
M	24	12	5	14	7	17	29	8	8	11	23	8	96	62	11	10	15	20	7	9	13	4	21	9	18	8	5	7	6	6	5	7	11	7	10	4
N	31	4	13	30	8	12	10	16	13	3	16	8	59	93	5	9	5	28	12	10	16	4	12	4	6	11	5	2	3	4	4	6	2	2	10	2
O	7	7	20	6	5	9	76	7	2	39	26	10	4	8	86	37	35	10	3	4	11	14	25	35	27	27	19	17	7	7	6	18	14	11	20	12
P	5	22	33	12	5	36	22	12	3	78	14	46	5	6	21	83	43	23	9	4	12	19	19	19	41	30	34	44	24	11	15	17	24	23	25	13
Q	8	20	38	11	4	15	10	5	2	27	23	26	7	6	22	51	91	11	2	3	6	14	12	37	50	63	34	32	17	12	9	27	40	58	37	24
R	13	14	16	23	5	34	26	15	7	12	21	33	14	12	12	29	8	87	16	2	23	23	62	14	12	13	7	10	13	4	7	12	7	9	1	2
S	17	24	5	30	11	26	5	59	16	3	13	10	5	17	6	6	3	18	96	9	56	24	12	10	6	7	8	2	2	15	28	9	5	5	5	2
T	13	10	1	5	46	3	6	6	14	6	14	7	6	5	6	11	4	4	7	96	8	5	4	2	2	6	5	5	3	3	3	8	7	6	14	6
U	14	29	12	32	4	32	11	34	21	7	44	32	11	13	6	20	12	40	51	6	93	57	34	17	9	11	6	6	16	34	10	9	9	7	4	3
V	5	17	24	16	9	29	6	39	5	11	26	43	4	1	9	17	10	17	11	6	32	92	17	57	35	10	10	14	28	79	44	36	25	10	1	5
W	9	21	30	22	9	36	25	15	4	25	29	18	15	6	26	20	25	61	12	4	19	20	86	22	25	22	10	22	19	16	5	9	11	6	3	7
X	7	64	45	19	3	28	11	6	1	35	50	42	10	8	24	32	61	10	12	3	12	17	21	91	48	26	12	20	24	27	16	57	29	16	17	6
Y	9	23	62	15	4	26	22	9	1	30	12	14	5	6	14	30	52	5	7	4	6	13	21	44	86	23	26	44	40	15	11	26	22	33	23	16
Z	3	46	45	18	2	22	17	10	7	23	21	51	11	2	15	59	72	14	4	3	9	11	12	36	42	87	16	21	27	9	10	25	66	47	15	15
1	2	5	10	3	3	5	13	4	2	29	5	14	9	7	14	30	28	9	4	2	3	12	14	17	19	22	84	63	13	8	10	8	19	32	57	55
2	7	14	22	5	4	20	13	3	25	26	9	14	2	3	17	37	28	6	5	3	6	10	11	17	30	13	62	89	54	20	5	14	20	21	16	11
3	3	8	21	5	4	32	6	12	2	23	6	13	5	2	5	37	19	9	7	6	4	16	6	22	25	12	18	64	86	31	23	41	16	17	8	10
4	6	19	19	12	8	25	14	16	7	21	13	19	3	3	2	17	29	11	9	3	17	55	8	37	24	3	5	26	44	89	42	44	32	10	3	3
5	8	45	15	14	2	45	4	67	7	14	4	41	2	0	4	13	7	9	27	2	14	45	7	45	10	10	14	10	30	69	90	42	24	10	6	5
6	7	80	30	17	4	23	4	14	2	11	11	27	6	2	7	16	30	11	14	3	12	30	9	58	38	39	15	14	26	24	17	88	69	14	5	14
7	6	33	22	14	5	25	6	4	6	24	13	32	7	6	7	36	39	12	6	2	3	13	9	30	30	50	22	29	18	15	12	61	85	70	20	13
8	3	23	40	6	3	15	15	6	2	33	10	14	3	6	14	12	45	2	6	4	6	7	5	24	35	50	42	29	16	16	9	30	60	89	61	26
9	3	14	23	3	1	6	14	5	2	30	6	7	16	11	10	31	32	5	6	7	6	3	8	11	21	24	57	39	9	12	4	11	42	56	91	78
0	9	3	11	2	5	7	14	4	5	30	8	3	2	3	25	21	29	2	3	4	5	3	2	12	15	20	50	26	9	11	5	22	17	52	81	94

$$X_n = \{p : \text{The length of the Morse code for first stimulus } p \text{ is } n\},$$
$$Y_n = \{q : \text{The length of the Morse code for second stimulus } q \text{ is } n\}.$$

which are given in Table 2.

Table 2. Distinguished sets

Stimulus (first position)	Stimulus (second position)
$X_1 = \{E,T\}$	$Y_1 = \{e,t\}$
$X_2 = \{A,I,M,N\}$	$Y_2 = \{a,i,m,n\}$
$X_3 = \{D,G,K,O,R,S,U,W\}$	$Y_3 = \{d,g,k,o,r,s,u,w\}$
$X_4 = \{B,C,F,H,J,L,P,Q,V,X,Y,Z\}$	$Y_4 = \{b,c,f,h,j,l,p,q,v,x,y,z\}$
$X_5 = \{0,1,2,3,4,5,6,7,8,9\}$	$Y_5 = \{*0,*1,*2,*3,*4,*5,*6,*7,*8,*9\}$

We are aiming at a description of similarity dependencies among these four sets of stimuli and their elements.

Let U be the set of first stimuli, V be the set of second stimuli, and pRq if a (fixed) subject regards them as the same. The operators can be interpreted as follows:

$q \in \langle R \rangle (X_n) \iff$ q was gauged to be the same as some first stimulus of length n.
$q \in [R](X_n) \iff$ q was gauged to be the same only as first stimuli of length n.
$q \in [[R]](X_n) \iff$ q was gauged to be the same to all first stimuli of length n, and possibly others.
$p \in [R\breve{}]\langle R \rangle (X_n) \iff$ Every signal, which cannot be distinguished from p cannot be distinguished from some stimulus of length n.
$p \in \langle R\breve{} \rangle [R](X_n) \iff$ Some signals, which cannot be distinguished from p were gauged to be the same only to stimuli of length n.
$p \in [[R\breve{}]][[R]](X_n) \iff$ Whenever q cannot be distinguished from all stimuli of length n, then q cannot be distinguished from p.

In order to consider the aggregated data given in Table 1, we need to consider "cut–off" points, and set

$$R_s = \{\langle p,q \rangle : \text{At least } s\% \text{ of the subjects responded "same"}, \text{ when } \langle p,q \rangle \text{ was presented}\}. \quad (5)$$

Observe that $R_s \subseteq R_t$ in case $t \leq s$. For first stimuli, the approximation operators now are interpreted as

$p \in [R_s\breve{}]\langle R_t \rangle (X_n) \iff$ Every second stimulus which could not be distinguished from p by at least s% of all subjects could not be distinguished from some first stimulus of length n by at least t% of all subjects.
$p \in \langle R_s\breve{} \rangle [R_t](X_n) \iff$ There is a second stimulus q such that at least s% of subjects gauged q to be the same as p, and at least t% of subjects gauged q to be the same only as stimuli of length n.

We have analyzed the data for various cut–points, and have found, among other results, that

- The signal length is the first determining factor for the discrimination of the stimuli, because:
  - Signals of length 1 or 2 are easy to discriminate from other stimuli.
  - Signals of length 3 are easy to discriminate from other stimuli, if they are located at the first position.
  - Signals of length 3 in the second position overlap with signals of length 4. Signals of length 4 overlap mainly with signals of length 5.
- The character of the impulses is of less effect because a signal must contain mainly short Morse impulses, and should contain at least 4 (first stimuli) or 3 (second stimuli) Morse impulses to be hard to discriminate.

We invite the reader to consult [6] for the details.

5 Conclusion and outlook

In this paper we have explored various tools for set approximation based on a relational connection of two "universes" U and V, or a relational connection within one "universe" U. The intention was to find a proper extension of rough sets in case of any binary relation. There is a list of proposals for set approximations based on certain binary relations such as similarity relations or dominance relations. We have shown that all these proposals can be well expressed in terms of modal style operators and that the (new) operator $[R\breve{}]\langle R\rangle$ (content-span operator) exhibits some kind of optimality because it gives the tightest bounds among the proposed operators based on a reflexive relation.

In case of any binary relation the content-span operator is interesting as well, because applying $[R\breve{}]\langle R\rangle$ is a complementary approach to the one taken by formal concept analysis – with exactly the same expressive power.

Two examples from diverse application fields indicate that the operators $[R\breve{}]\langle R\rangle$ are not only well suited for approximating sets, but that the resulting approximations offer meaningful interpretations. More applications, however, are needed to delineate the situations in which either of these can be applied.

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