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APPROXIMATION ORDER FROM BIVARIATE C^1-CUBICS: A COUNTEREXAMPLE

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ABSTRACT. It is shown that the space of bivariate C^1 piecewise cubic functions on a hexagonal mesh of size h approximates to certain smooth functions only to within $O(h^3)$ even though it contains a local partition of every cubic polynomial.

1. Introduction. We deal with a *scale* (S_h) of approximating spaces, generated from a fixed space S by a simple scaling $S_h := \sigma_h(S)$, with $(\sigma_h f)(x) := f(x/h)$, all f, x, h. We are interested in the *approximation order* obtainable from (S_h), i.e., in $\operatorname{dist}(f, S_h)$, as a function of h and for f sufficiently smooth. Here, $\operatorname{dist}(f, S_h) := \inf_{g\in S_h}\|f-g\|$, and $\|\cdot\|$ is the sup norm on some closed domain $G \subseteq \mathbf{R}^2$, $\|f\| := \sup_{x\in G}|f(x)|$.

It is easy to see that

$$\operatorname{dist}(f, S_h) = O\big(\omega_f(h)\big), \quad \text{all } f \in C(G),$$

in case S contains a local and stable partition of unity. By this we mean that $1 = \sum\phi_i$ on $\mathbf{R}^2$, for some $\phi_i \in S$ with $\sup_i \operatorname{diam}\operatorname{supp}\phi_i < \infty$, and $\sup_i\|\phi_i\| < \infty$. The last condition is automatically satisfied in case the ϕ_i are all nonnegative.

We are interested in suitable conditions on S which insure that

$$\operatorname{dist}(f, S_h) = O(h^k) \quad \text{for all sufficiently smooth } f. \tag{1.1}$$

It is easy to see that (1.1) implies $\mathbf{P}_k \subseteq S$, with $\mathbf{P}_k :=$ polynomials of degree $< k$. On the other hand, this condition is clearly not sufficient for (1.1) since, e.g., $S = \mathbf{P}_k$ implies that $S_h = \mathbf{P}_k$ for all h, hence $\operatorname{dist}(f, S_h)$ is independent of h in this case. What seems to be needed is that $\mathbf{P}_k$ be contained in S "locally", much as $\mathbf{P}_1$ is contained locally in S in case S contains a local partition of unity.

Here is a precise formulation of such a condition.

Condition $\mathbf{P}_k$. *For every* $p \in \mathbf{P}_k$, *there exists a 'sequence'* (ϕ_i) *in* S *so that*

$$p = \sum_i \phi_i \quad \textit{and} \quad \sup_i \operatorname{diam}\operatorname{supp}\phi_i < \infty.$$

At least in case S is "uniform", e.g., S is a space of piecewise polynomial functions on some uniform partition of $\mathbf{R}^2$, one would expect to conclude from this the validity of (1.1), i.e., *global* approximation order $O(h^k)$ (see, e.g., **[BD, D]**).

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It is the purpose of this note to give an example of a piecewise polynomial space S on a uniform partition of $\mathbf{R}^2$ which satisfies Condition $\mathbf{P}_4$, yet gives

$$\operatorname{dist}(f, S_h) \geqslant \operatorname{const}_f h^3, \tag{1.2}$$

for some positive const_f and for the particular function $f\colon x \mapsto (x(1)x(2))^2$. This dashes all hopes that the approximation order from a piecewise polynomial scale (S_h) could be settled by finding out which polynomials are contained locally in S. Presumably, some stability has to be added to Condition $\mathbf{P}_k$ before global approximation order $O(h^k)$ can be deduced.

Here is an outline of this note. In §2, we show that our example space S satisfies Condition $\mathbf{P}_4$ and that (S_h) has approximation order $O(h^3)$ at least. In §3, we identify S within a larger space of piecewise cubic functions as the annihilator of a set Λ of local linear functionals. We also show that there exists a bounded $a \in \mathbf{R}^\Lambda \backslash 0$ so that $\Sigma_{\lambda \in \Lambda} a(\lambda)\lambda f = 0$ for all f with compact support, while $\Sigma_{\operatorname{supp}\lambda \subseteq G} a(\lambda)\lambda = 0$ only if the sum is, in effect, over all or over none of Λ. This is important for the final section, in which we prove that (S_h) has approximation order at most $O(h^3)$.

2. Smooth pp functions and box splines. We consider bivariate pp (:= piecewise polynomial) functions on the partition Δ of $\mathbf{R}^2$ obtained from the three families of meshlines

$$x(1) = n, \quad x(2) = n, \quad x(1) = x(2) + n, \quad \text{all } n \in \mathbf{Z}.$$

We are particularly interested in the space

$$S := \mathbf{P}^1_{4,\Delta} := \mathbf{P}_{4,\Delta} \cap C^1(\mathbf{R}^2)$$

of piecewise cubic functions on the partition Δ and in C^1.

We have foregone the opportunity to make the symmetries in Δ more apparent by having the three families of meshlines intersect each other at an angle of 120° (as is done, e.g., in **[Fr]**). This needlessly complicates the notation. It is sufficient to note that any permutation of the meshline families can be accomplished by some linear map on $\mathbf{R}^2$, and the corresponding change of variables leaves $\mathbf{P}^r_{k,\Delta}$ invariant.

A stable and local partition of unity in S is constructed in **[BD, BH$_1$]** as follows. Consider the linear map $P\colon \mathbf{R}^5 \to \mathbf{R}^2$ characterized by the fact that

$$Pe_i = \begin{Bmatrix} e_1, & i = 1, 4 \\ e_2, & i = 2, 5 \\ e_1 + e_2, & i = 3 \end{Bmatrix},$$

with e_i the ith unit vector (in $\mathbf{R}^5$). Let M be the P-shadow of $B := [0, 1]^5$, i.e., M is the distribution given by the rule

$$M\phi := \int_B \phi \circ P, \quad \text{all } \phi.$$

Since M is the shadow of a box, we call it a *box* spline. It is immediate from the definition that $M \geqslant 0$, $\operatorname{supp} M = P(B)$, and

$$\sum_{v \in V} M(\cdot - j) = 1, \quad \text{with } V := \mathbf{Z}^2,$$

the last because

$$\sum_v M(\cdot - v)\phi = \sum_v \int_{B+v} \phi \circ P = \int_{\mathbf{R}^2 \times [0,1]^3} \phi \circ P = \int_{\mathbf{R}^2} \phi.$$

Further, one verifies that $M \in S$.

In addition, [$\mathbf{BH}_1$] provides an $\mathbf{L}_1$-bounded linear functional λ with support in supp M so that

$$p = \sum_{v \in V} \lambda p(\cdot + v) M(\cdot - v), \quad \text{for all } p \in \mathbf{P}_3, \tag{2.1}$$

and shows how this result leads, in standard quasi-interpolant fashion, to the conclusion that

$$\operatorname{dist}(f, S_h) = O(h^3) \tag{2.2}$$

for all sufficiently smooth f. On the other hand, [$\mathbf{BH}_1$] makes clear that (2.1) is sharp, i.e., that $\mathbf{P}_4 \setminus S_M \neq \varnothing$, with

$$S_M := \operatorname{span}(M(\cdot - v))_{v \in V}.$$

Therefore, (2.2) provides the optimal approximation order from $(S_{M,h})$.

Since S_M is a proper subspace of S, this only provides a lower bound on the approximation order from (S_h). Further, according to Proposition 3.2 of [$\mathbf{BH}_2$], S *satisfies Condition* $\mathbf{P}_4$. This raises the hope that $\operatorname{dist}(f, S_h) = O(h^4)$ for all smooth f.

3. Bernstein coordinates. In this auxiliary section, we identify $S = \mathbf{P}^1_{4,\Delta}$ as a subspace of $\mathbf{P}^0_{4,\Delta}$ satisfying certain homogeneous conditions. We find it most convenient to express these conditions in terms of the Bernstein coordinates for pp functions on a triangulation, as introduced by Farin [**Fa**], following earlier work by de Casteljau [**C**] and Sabin [**S**]. Here is a short explanation of this very useful representation.

On a single triangle τ with vertices U, V and W, we use barycentric coordinates. This means that each point x is associated with the triple (u, v, w) for which

$$x = uU + vV + wW \quad \text{and} \quad u + v + w = 1.$$

In these terms, we describe a polynomial p of degree $\leqslant n$ by

$$p = \sum_{i+j+k=n} b_{ijk} \phi_{ijk},$$

with

$$\phi_{ijk}(x) = \frac{n!}{i!j!k!} u^i v^j w^k.$$

We deal with the 3! choices in this representation by associating b_{ijk} with the point $x_{ijk} := (iU + jV + kW)/n$ for all $i + j + k = n$. The resulting function b: $x_{ijk} \mapsto b_{ijk}$ is called the B(ernstein or -ézier)-net for p (with respect to τ). It is independent of how we associate the vertices of τ with the letters U, V, W. Moreover, if A is an affine change of variables, then the B-net for $p \circ A$ (with respect to $A^{-1}(\tau)$) is $b \circ A$.

This makes it easy to compare polynomial pieces across triangle edges. For example, on an edge of τ, p is entirely determined by b restricted to that edge.

Moreover, if p' is a polynomial of degree $\leqslant n$ on a triangle τ' having that edge in common with τ, then $p = p'$ on that edge iff $b = b'$ on that edge, with b' the B-net for p'.

Higher smoothness across such an edge is also very simply expressible in terms of b and b' (see [**Fa**]). We now describe these conditions only to the extent that we need them, i.e., we describe the conditions which an $f \in \mathbf{P}^0_{4,\Delta}$ must satisfy to belong to $S = \mathbf{P}^1_{4,\Delta}$. Since such an f is continuous, the B-nets of its various pieces must agree at all points of overlap in their domains. We can therefore think of the B-nets of its various pieces as forming one B-net, a function b_f defined on all of

$$V_3 := (\mathbf{Z}/3)^2. \tag{3.1}$$

Let τ, τ' be two triangles of Δ with a common edge ε. There are four points of V_3 on ε. Each of the three pairs x_1, x_2 of such neighboring points has a nearest V_3-neighbor y in τ and a nearest V_3-neighbor y' in τ' so that these four points form the vertices of a parallelogram similar to $\tau \cup \tau'$ and halved by ε. One may verify directly that *f has continuous first derivative across ε if and only if*

$$b_f(x_1) + b_f(x_2) = b_f(y) + b_f(y')$$

for each of these three parallelograms.

Thus, associated with each edge ε in Δ there are three linear functionals λ on $\mathbf{P}^0_{4,\Delta}$, of the form

$$\lambda f := b_f(x_1) + b_f(x_2) - b_f(y) - b_f(y'),$$

with x_1, x_2 neighboring V_3-points on ε and y, y' adjacent V_3-points in the neighboring triangles. Note that we have so normalized λ that the edge points receive weight 1 and the off-edge points receive weight -1.

Since Δ contains three distinct edge types, this gives altogether nine nonoverlapping classes Λ_{ij}, $i, j = 1, 2, 3$, of linear functionals. To be precise, we associate Λ_{ij} with segment i of edge j, and, in particular, Λ_{2j} with the middle segment. We need not be more precise than that.

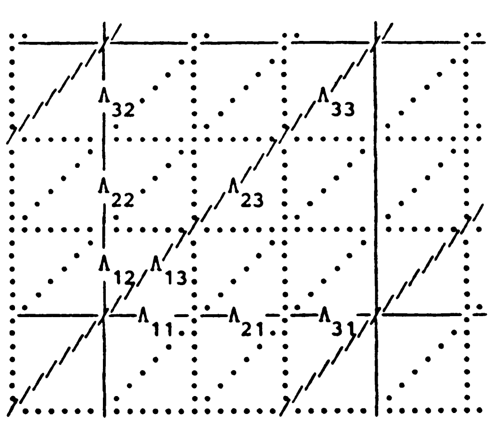

FIGURE 1. The nine classes of local linear functionals

Each class Λ_{ij} is left invariant under translation of the independent variable by any $v \in V$. Their union $\Lambda := \Lambda_{11} \cup \cdots \cup \Lambda_{33}$ characterizes S within $\mathbf{P}^0_{4,\Delta}$ in the sense that

$$S = \ker \Lambda := \bigcap_{\lambda \in \Lambda} \ker \lambda .$$

Next, we seek $a \in \mathbf{R}^\Lambda \backslash 0$ so that $\Sigma_{\lambda \in \Lambda} a(\lambda)\lambda = 0$. By this we mean that $\Sigma a(\lambda)\lambda f = 0$ for every f of compact support. For such an f, the sum has only finitely many nonzero terms since each λ has support only in some pair of adjacent triangles of Δ.

There may be many solutions, but, when we require additionally that a be constant on each Λ_{ij}, then the solution set can be shown to be four-dimensional. There is a two-dimensional set of solutions which vanish on Λ_{2j}, all j. These solutions are of no interest to us since for them we have already

$$\sum_{\operatorname{supp}\lambda \subseteq G} a(\lambda)\lambda = 0, \tag{3.2}$$

for various bounded sets G.

Solutions a for which (3.2) only holds if the sum is, in effect, over all or over none of Λ are obtained as follows.

LEMMA. *Let $a \in \mathbf{R}^\Lambda$ be such that $a_{|\Lambda_{ij}} = A(i, j)$, with*

$$A = \begin{bmatrix} -\alpha & -\beta & -\gamma \\ 2\alpha & 2\beta & 2\gamma \\ -\alpha & -\beta & -\gamma \end{bmatrix}$$

and $\alpha + \beta + \gamma = 0$. Then $\Sigma a(\lambda)\lambda = 0$.

PROOF. By its definition, each λ carries an f to a weighted sum of values of b_f. We can, therefore, understand $\Sigma a(\lambda)\lambda$ by computing the weight it assigns to $b_f(x)$ for each $x \in V_3$. There are three types of points x in V_3, those at a vertex of Δ, those inside an edge, and those inside a triangle. We consider each type in turn.

A vertex of Δ serves as an edge point for six λ's, one from each of the classes Λ_{ij} with $i \neq 2$. Thus, the total weight at a vertex point is

$$\sum_{i \neq 2} A(i, j) = -2(\alpha + \beta + \gamma) = 0.$$

As to inside edge points, consider without loss of generality one on an edge of type 1. Such a point serves as an edge point for one $\lambda \in \Lambda_{21}$ and one $\lambda \in \Lambda_{11} \cup \Lambda_{31}$, and as an off-edge point for one $\lambda \in \Lambda_{12} \cup \Lambda_{32}$ and one $\lambda \in \Lambda_{13} \cup \Lambda_{33}$. Its total weight is therefore

$$\begin{aligned} A(2,1) + A(1,1) - A(1,2) - A(1,3) &= 2\alpha + (-\alpha) - (-\beta) - (-\gamma) \\ &= \alpha + \beta + \gamma = 0. \end{aligned}$$

Finally, an interior point is an off-edge point for three λ's, one each from Λ_{2j}, $j = 1, 2, 3$. Its weight is therefore $-2\alpha - 2\beta - 2\gamma = 0$. □

We extend each λ to the continuous linear functional λI on $C(\mathbf{R}^2)$ with the aid of the local linear map I which associates f with the unique element If of $\mathbf{P}^0_{4,\Delta}$ which agrees with f on J_3. It is then a simple matter to check that, for every $f \in \mathbf{P}_5$, the map $\lambda \mapsto \lambda If$ is constant on each Λ_{ij} (since each λ vanishes on $\mathbf{P}_4$ and, for any j, f and $f(\cdot + j)$ differ only by some cubic polynomial). In particular, for $f(x) := (x(1)x(2))^2$, we have (using the association of edge types indicated in Figure 1) that

$$(\Lambda_{ij} If) = \begin{bmatrix} -6 & -6 & 6 \\ 3 & 3 & -12 \\ -6 & -6 & 6 \end{bmatrix}.$$

Therefore,

$$\kappa := \sum_{i,j=1}^{3} a(\Lambda_{ij}) \Lambda_{ij} If = 18(\alpha + \beta) - 36\gamma.$$

The number κ is nonzero for many choices of α, β, γ for which $\alpha + \beta + \gamma = 0$. Make such a choice. Then, for a square Q with sides parallel to the axes,

$$\sum_{\operatorname{supp}\lambda \subseteq Q} a(\lambda)\lambda If = \kappa \cdot \operatorname{area}(Q) + O(\operatorname{perimeter}(Q)) \tag{3.3}$$

while $\sum_{\operatorname{supp}\lambda \subseteq Q} a(\lambda)\lambda I$ has support only on triangles near the boundary of Q, hence

$$\Bigl\| \sum_{\operatorname{supp}\lambda} a(\lambda)\lambda I \Bigr\| = O(\operatorname{perimeter}(Q)). \tag{3.4}$$

This is the essential fact required in the next and final section.

4. An upper bound for the approximation order. In this section, we establish the main point of this note. With $S = \mathbf{P}^1_{4,\Delta}$ as defined earlier, we show that, *for all small enough* h, $\operatorname{dist}(f, S_h) \geqslant \operatorname{const} h^3$ *for some positive* const *and for the particular function* $f: x \mapsto (x(1)x(2))^2$.

For the proof, we pick some axis-oriented square Q in G and consider

$$\mu_h := \sum_{\operatorname{supp}\lambda \subseteq Q_h} a(\lambda) I\sigma_{1/h},$$

with $Q_h := \{x/h \colon x \in Q\}$. Since $S_h \subseteq \ker \mu_h$, we have

$$\operatorname{dist}(f, S_h) \geqslant \operatorname{dist}_Q(f, S_h) \geqslant \operatorname{dist}_Q(f, \ker \mu_h) = |\mu_h f| / \|\mu_h\|_{C(Q)}.$$

Further, $\sigma_{1/h} f = h^4 f$, while $\operatorname{area}(Q_h) = \operatorname{area}(Q)/h^2$ and

$$\operatorname{perimeter}(Q_h) = \operatorname{perimeter}(Q)/h.$$

Therefore, from (3.3),

$$|\mu_h f| = \operatorname{const} h^2 + O(h^3)$$

with const $:= |\kappa| \operatorname{area}(Q) > 0$, while, from (3.4), $\|\mu_h\|_{C(Q)} = O(1/h)$. This shows that

$$\operatorname{dist}(f, S_h) \geqslant \operatorname{const} h^3 + O(h^4),$$

for some positive constant, as asserted.

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