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APPROXIMATION PROPERTIES FOR GROUP C^*-ALGEBRAS AND GROUP VON NEUMANN ALGEBRAS

UFFE HAAGERUP AND JON KRAUS

ABSTRACT. Let G be a locally compact group, let $C_r^*(G)$ (resp. $\mathrm{VN}(G)$) be the C^*-algebra (resp. the von Neumann algebra) associated with the left regular representation l of G, let $A(G)$ be the Fourier algebra of G, and let $M_0A(G)$ be the set of completely bounded multipliers of $A(G)$. With the completely bounded norm, $M_0A(G)$ is a dual space, and we say that G has the approximation property (AP) if there is a net $\{u_\alpha\}$ of functions in $A(G)$ (with compact support) such that $u_\alpha \to 1$ in the associated weak *-topology. In particular, G has the AP if G is weakly amenable ($\Leftrightarrow A(G)$ has an approximate identity that is bounded in the completely bounded norm). For a discrete group Γ, we show that Γ has the AP $\Leftrightarrow C_r^*(\Gamma)$ has the slice map property for subspaces of any C^*-algebra $\Leftrightarrow \mathrm{VN}(\Gamma)$ has the slice map property for σ-weakly closed subspaces of any von Neumann algebra (Property S_σ). The semidirect product of weakly amenable groups need not be weakly amenable. We show that the larger class of groups with the AP is stable with respect to semidirect products, and more generally, this class is stable with respect to group extensions. We also obtain some results concerning crossed products. For example, we show that the crossed product $M \otimes_\alpha G$ of a von Neumann algebra M with Property S_σ by a group G with the AP also has Property S_σ.

0. INTRODUCTION

Connes proved in [Co] that if G is a separable connected locally compact group, then $\mathrm{VN}(G)$ is always semidiscrete, and $C_r^*(G)$ is always nuclear. Thus nice approximation properties for $\mathrm{VN}(G)$ or $C_r^*(G)$ give us no information about G in the connected case. However, for discrete groups there is an intimate relation between approximation properties for $\mathrm{VN}(G)$ and $C_r^*(G)$ and approximation properties for G. Lance proved in [Lan] that if Γ is a discrete group, then $C_r^*(\Gamma)$ is nuclear if and only if Γ is amenable, and it was shown in [EL] that $\mathrm{VN}(\Gamma)$ is semidiscrete if and only if Γ is amenable. By definition, a von Neumann algebra M is semidiscrete if and only if the identity map on M can be approximated in the point-weak* (= point-σ-weak) topology by normal finite rank unital completely positive maps (cf. [EL]). Moreover, it was shown by Choi and Effros in [CE] and by Kirchberg in [Ki 1] that a C^*-algebra A is nuclear if and only if the identity map on A can be approximated in the point-

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norm topology by finite rank completely positive contractions. Hence nuclear C^*-algebras satisfy the metric approximation property (MAP) of Grothendieck (which only requires that the identity map can be approximated in the point-norm topology by a net of finite rank contractions, cf. [Gr]). Since the free group on two generators $\mathbf{F}_2$ is not amenable, $C_r^*(\mathbf{F}_2)$ is not nuclear. However, as shown in [Haa 6], $C_r^*(\mathbf{F}_2)$ does have the MAP. In fact, as shown in [DH], there is a net of finite rank complete contractions that converges point-norm to the identity.

A C^*-algebra A is said to have the completely bounded approximation property (CBAP) if there is a positive number C such that the identity map on A can be approximated in the point-norm topology by a net $\{T_\alpha\}$ of finite rank completely bounded maps whose completely bounded norms are bounded by C. The infimum of all values of C for which such constants exist is denoted by $\Lambda(A)$. Similarly, a von Neumann algebra M is said to have the weak* completely bounded approximation property (weak* CBAP) if there is a positive number C such that the identity map on M can be approximated in the point-weak* topology by a net $\{T_\alpha\}$ of normal finite rank completely bounded maps whose completely bounded norms are bounded by C. The infinimum of all values of C for which such constants exist is denoted by $\Lambda(M)$. In both cases the infimum is attained.

In [Haa 7], the first author gave a characterization of those discrete groups whose reduced group C^*-algebras have the CBAP. Before describing this result, we recall some basic definitions and facts concerning Fourier algebras and related topics for the convenience of the reader. More details can be found in [Ey, DH and CH]. Let G be a locally compact group. The group von Neumann algebra $\mathrm{VN}(G)$ of G is the von Neumann algebra generated by $\{l(g): g \in G\}$, where l is the left regular representation of G on $L^2(G)$, and $C_r^*(G)$, the reduced group C^*-algebra of G, is the C^*-algebra generated by $\{l(f): f \in L^1(G)\}$. The Fourier-Stieltjes algebra $B(G)$ of G is the space of coefficients $\langle \pi(g)\zeta, \eta\rangle$ of strongly continuous unitary representations π of G. With the norm

$$\|u\|_{B(G)} = \inf\{\|\zeta\|\,\|\eta\|: u(g) = \langle \pi(g)\zeta, \eta\rangle\}$$

$B(G)$ is a Banach algebra under pointwise multiplication. The Fourier algebra $A(G)$ of G is the closure of $C_c(G)\cap B(G)$ in $B(G)$, where $C_c(G)$ denotes the set of continuous functions on G with compact support. The Fourier algebra is a closed ideal in $B(G)$, and the elements of $A(G)$ are precisely the coefficients of the regular representation. Moreover, $A(G) = \mathrm{VN}(G)_*$. We denote the restriction of $\|\cdot\|_{B(G)}$ to $A(G)$ by $\|\cdot\|_{A(G)}$.

A complex-valued function u on G is a multiplier for $A(G)$ if the linear map $m_u(v) = uv$ maps $A(G)$ into $A(G)$. The set of multipliers of $A(G)$ is denoted by $MA(G)$, and if $u \in MA(G)$, then u is a bounded continuous function, and m_u is a bounded operator. For $u \in MA(G)$, let M_u denote the normal (σ-weakly continuous) linear map from $\mathrm{VN}(G)$ to $\mathrm{VN}(G)$ defined by $M_u = m_u^*$ and let $\overline{M}_u$ denote the restriction of M_u to $C_r^*(G)$. Then u is said to be a completely bounded multiplier if M_u is completely bounded. The space $M_0A(G)$ of completely bounded multipliers, endowed with the norm $\|u\|_{M_0} = \|M_u\|_{cb}$, is a Banach space. Moreover, $B(G) \subset M_0A(G)$, and the $B(G)$ norm dominates the $M_0A(G)$ norm on $B(G)$. The space of completely bounded multipliers has a number of characterizations (cf. [CH, p. 508]). One such characterization that we will make use of in §1 is that the following conditions are equivalent:

(i) u belongs to $M_0A(G)$.
(ii) There exist bounded continuous maps $P, Q\colon G \to \mathscr{H}$ ($\mathscr{H}$ a Hilbert space) such that

$$u(t^{-1}s) = \langle P(s), Q(t)\rangle \quad \forall s, t \in G. \tag{1}$$

(iii) There exist bounded maps $P, Q\colon G \to \mathscr{H}$ ($\mathscr{H}$ a Hilbert space) such that (1) holds.

Moreover, if $u \in M_0A(G)$, then

$$\|u\|_{M_0} = \min\{\|P\|_\infty \|Q\|_\infty\},$$

where the minimum is taken over all (continuous) pairs P, Q for which (1) holds. (For a short proof of the equivalence of (i) and (ii), see [Jo]. A proof that (iii) implies (i) can be found in [Haa 7].)

It is shown in [Le] that a locally compact group G is amenable if and only if $A(G)$ has a bounded approximate identity. A locally compact group is said to be weakly amenable if $A(G)$ has an approximate identity that is bounded in the $M_0A(G)$ norm, i.e., if there is a net $\{u_\alpha\}$ in $A(G)$ and a constant C such that $\|u_\alpha v - v\| \to 0$ for all $v \in A(G)$ and such that $\|u_\alpha\|_{M_0} \le C$ for all α. The infimum (which is attained) of the constants C for which such a net exists is denoted by $\Lambda(G)$. Amenability implies weak amenability but there are weakly amenable groups which are not amenable. For example, it is shown in [DH] that $\Lambda(\mathbf{F}_2) = 1$, but $\mathbf{F}_2$ is not amenable. Moreover, $\Lambda(Sp(1, n)) = 2n - 1$ for $n \ge 2$ [CH], so $\Lambda(G)$ can take on any odd positive integer value. (It is not known whether $\Lambda(G)$ can take on any other values.) Other examples of weakly amenable groups can be found in [DH, CH, Han, Sz, Va 1, Va 2].

It is shown in [Haa 7] that if Γ is a discrete group, then the following conditions are equivalent:

1. Γ is weakly amenable.
2. $C_r^*(\Gamma)$ has the CBAP.
3. $\mathrm{VN}(\Gamma)$ has the weak* CBAP.

Moreover, if any (and hence all) of these conditions hold, then $\Lambda(\Gamma) = \Lambda(\mathrm{VN}(\Gamma)) = \Lambda(C_r^*(\Gamma))$. It is also shown in [Haa 7] that there are discrete groups that are not weakly amenable (see Remark 2.5 below).

In this paper we study a slightly weaker property for locally compact groups. We say that a locally compact group G had the approximation property (AP) if there is a net $\{u_\alpha\}$ in $A(G)$ such that $u_\alpha \to 1$ in the $\sigma(M_0A(G), Q(G))$-topology, where $Q(G)$ denotes the predual of $M_0A(G)$ obtained by completing $L^1(G)$ in the norm

$$\|f\|_Q = \sup\left\{\left|\int_G f(x)u(x)\,dx\right| : u \in M_0A(G),\ \|u\|_{M_0} \le 1\right\}$$

(cf. [He] or [DH].) In analogy with the result quoted above, we prove in §2 that for any discrete group Γ, the following conditions are equivalent:

(a) Γ has the AP.
(b) $C_r^*(\Gamma)$ has the operator approximation property (OAP).
(c) $\mathrm{VN}(\Gamma)$ has the weak* operator approximation property (weak* OAP).

Here the OAP is the operator space version of Grothendieck's approximation property introduced in [ER] under the name AP, and the weak* OAP is the von Neumann algebra version of the OAP introduced in [EKR]. (Definitions of these properties are given in §2.) By [Kr 3], conditions (b) and (c) are equivalent to (respectively) conditions (b′) and (c′) below:

(b′) $C_r^*(\Gamma)$ has the slice map property for subspaces of the compact operators $K(\mathscr{H})$.

(c′) $\mathrm{VN}(\Gamma)$ has the slice map property for σ-weakly closed subspaces of $B(\mathscr{H})$ (Property S_σ).

In fact, it turns out that (b′) is equivalent to the apparently stronger condition

(b″) $C_r^*(\Gamma)$ has the slice map property for subspaces of any C^*-algebra.

On the other hand, it follows from a recent result of Kirchberg in [Ki 4], that (b′) is not equivalent to (b″) for arbitrary C^*-algebras.

It was discovered in [Haa 7], that although $\mathbf{Z}^2$ and $SL(2, \mathbf{Z})$ are two weakly amenable groups, their semidirect product $\mathbf{Z}^2 \times_\rho SL(2, \mathbf{Z})$ (where ρ is the standard action of $SL(2, \mathbf{Z})$ on $\mathbf{Z}^2$) is not weakly amenable. In §1 we prove that the AP property is stable with respect to semidirect products, and more generally group extensions, i.e., if both a normal closed subgroup and the corresponding quotient group have the AP, then the original group also has the AP. In particular, $\mathbf{Z}^2 \times_\rho SL(2, \mathbf{Z})$ is an example of a group with the AP, which is not weakly amenable. Presently, we do not know of an example of a locally compact group without the AP, but we conjecture that $SL(3, \mathbf{Z})$ will fail to have the AP.

In addition to the stability result mentioned above, §1 also contains several sets of equivalent conditions for the AP. In §3 we examine the relationship between the approximation properties of a crossed product von Neumann algebra $N = M \otimes_\alpha G$ and the approximation properties of M. We first show that the approximation properties of M are at least as strong as those of N: if N has Property S_σ (resp. has the weak* CBAP) then M has Property S_σ (resp. has the weak* CBAP). On the other hand, if G is a discrete group without the AP, if $M = \mathbf{C}$, and if α is the trivial action of G on M, then M has Property S_σ (and in fact is semidiscrete), but $\mathrm{VN}(G) = M \otimes_\alpha G$ does not have Property S_σ. Thus the best we could hope for in the converse direction is that if G has the AP and M has Property S_σ, then N has Property S_σ. This turns out to be the case. If we assume that G is weakly amenable and M has the weak* CBAP, then it does not always follow that N has the weak* CBAP, but N does have a "two step" weak* CBAP in the sense that the identity map on N is the limit in the point-weak* topology of a bounded net in $CB_\sigma(N)$ (the space of normal completely bounded maps from N to N with the completely bounded norm) each of whose elements is a limit of a bounded net of finite rank operators in $CB_\sigma(N)$. However, if we assume that G is amenable, then the weak* CBAP for M does imply the weak* CBAP for N. This is in analogy with the known result that if G is amenable and M is semidiscrete, then N is semidiscrete.

1. The approximation property for groups

Definition 1.1. Let G be a locally compact group. Then G is said to have the approximation property (AP) if the constant function 1 is in the $\sigma(M_0A(G), Q(G))$-closure of $A(G)$ in $M_0A(G)$.

Remark 1.2. The inclusion map from $B(G)$ into $M_0A(G)$ is a contraction (cf. [DH, Corollary 1.8]), and so the $\sigma(M_0A(G), Q(G))$-closure of any subset E of $B(G)$ contains the closure of E in the $B(G)$-norm. Hence G has the AP if and only if 1 is in the $\sigma(M_0A(G), Q(G))$-closure of $A_c(G)$ in $M_0A(G)$, where $A_c(G)$ denotes the set of functions in $A(G)$ with compact support. Note that $1 \in A(G)$ if and only if G is compact, in which case $A_c(G) = A(G) = B(G) = M_0(G)$ (cf. [DH, Corollary 1.8]).

Suppose A, B and C are C^*-algebras. Then for any $a \in A \otimes C$ (where $A \otimes C$ denotes the spatial C^*-tensor product of A and C), and for any $\varphi \in (B \otimes C)^*$, we can define a linear functional $\omega_{a,\varphi}$ on $CB(A, B)$ (the space of completely bounded maps from A to B with the completedly bounded norm) by

$$\omega_{a,\varphi}(T) = \langle T \otimes \mathrm{id}_C(a), \varphi \rangle, \qquad T \in CB(A, B). \tag{2}$$

The right-hand side of equation (2) also makes sense if $\varphi \in (M \overline{\otimes} R)_*$, where M and R are any von Neumann algebras containing B and C respectively. We will use the notation $\omega_{a,\varphi}$ for the linear functional defined by (2) in this case as well. Next suppose M, N, and R are von Neumann algebras. Then for any $a \in M \overline{\otimes} R$ and $\varphi \in (N \overline{\otimes} R)_*$, we can define a linear functional on $CB_\sigma(M, N)$ (the space of normal completely bounded maps from M to N with the completely bounded norm) by

$$\omega_{a,\varphi}(T) = \langle T \otimes \mathrm{id}_R(a), \varphi \rangle, \qquad T \in CB_\sigma(M, N).$$

Note that in all cases we have that

$$\|\omega_{a,\varphi}\| \leq \|a\| \, \|\varphi\|. \tag{3}$$

Since $u \to \overline{M}_u$ and $u \to M_u$ are isometric isomorphisms from $M_0A(G)$ into $CB(C_r^*(G)) = CB(C_r^*(G), C_r^*(G))$ and $CB_\sigma(\mathrm{VN}(G)) = CB_\sigma(\mathrm{VN}(G), \mathrm{VN}(G))$ respectively, we can define linear functionals of the form $\omega_{a,\varphi}$ on $M_0A(G)$. For example, if $a \in C_r^*(G) \otimes K(\mathscr{H})$ and $\varphi \in (\mathrm{VN}(G) \overline{\otimes} B(\mathscr{H}))_*$, then $\omega_{a,\varphi}$ is defined by

$$\omega_{a,\varphi}(u) = \omega_{a,\varphi}(\overline{M}_u) = \langle \overline{M}_u \otimes \mathrm{id}_{K(\mathscr{H})}(a), \varphi \rangle, \quad u \in M_0A(G).$$

Let $PA_c(G)$ denote the set of f in $A_c(G)$ such that $f\,dx$ is a probability measure, i.e., the set of f in $A_c(G)$ such that $f \geq 0$ and $\int_G f(x)\,dx = 1$. Since translation is an isometry in $M_0A(G)$, the map $u \to f * u$ is a contraction in $B(M_0A(G))$ for any $f \in PA_c(G)$. Hence for any $f \in PA_c(G)$ and any linear functional on $M_0A(G)$ of the form $\omega_{a,\varphi}$, we can define a linear functional $\omega_{a,\varphi,f}$ on $M_0A(G)$ by

$$\omega_{a,\varphi,f}(u) = \omega_{a,\varphi}(f * u), \qquad u \in M_0A(G).$$

It is easy to see that we always have

$$\|\omega_{a,\varphi,f}\| \leq \|a\| \, \|\varphi\|. \tag{4}$$

Proposition 1.3. *Suppose that G is a locally compact group, and that $\mathscr{H}$ is a separable infinite dimensional Hilbert space. Then*

(a) $\omega_{a,\varphi,f} \in Q(G)$ *whenever* $a \in \mathrm{VN}(G) \overline{\otimes} B(\mathscr{H})$, $\varphi \in (\mathrm{VN}(G) \overline{\otimes} B(\mathscr{H}))_*$, *and* $f \in PA_c(G)$.

(b) $\omega_{a,\varphi,f} \in Q(G)$ *whenever* $a \in C_r^*(G) \otimes K(\mathscr{H})$, $\varphi \in (C_r^*(G) \otimes K(\mathscr{H}))^*$, *and* $f \in PA_c(G)$.

(c) $\omega_{a,\varphi} \in Q(G)$ *whenever* $a \in C_r^*(G) \otimes K(\mathscr{H})$ *and* $\varphi \in (\mathrm{VN}(G) \overline{\otimes} B(\mathscr{H}))_*$.

Proof. (a) Since the linear span of $\{\varphi_1 \otimes \varphi_2 : \varphi_1 \in A_c(G) \text{ and } \varphi_2 \in B(\mathscr{H})_*\}$ is norm dense in $(\mathrm{VN}(G)\overline{\otimes}B(\mathscr{H}))_*$, using (4) we can reduce to the case when $\varphi = \varphi_1 \otimes \varphi_2$ for some $\varphi_1 \in A_c(G)$ and $\varphi_2 \in B(\mathscr{H})_*$. Moreover, it is easy to show that in this case we have that

$$\omega_{a,\varphi,f}(u) = \langle a, (f*u)\varphi_1 \otimes \varphi_2\rangle = \langle L_{\varphi_2}(a), (f*u)\varphi_1\rangle \quad \forall u \in M_0A(G),$$

where L_{φ_2} is the left slice map associated with φ_2. (See §2 for the definition of slice maps.) Since $L_{\varphi_2}(a) \in \mathrm{VN}(G)$, to prove (a) it suffices to show that if $a \in \mathrm{VN}(G)$, if $\varphi \in A_c(G)$, and if $f \in PA_c(G)$, then $\omega_{a,\varphi,f} \in Q(G)$, where $\omega_{a,\varphi,f}(u) = \langle M_{f*u}(a), \varphi\rangle$ $(u \in M_0A(G))$. We will complete the proof by showing that there is a $g \in L^1(G)$ such that $\omega_{a,\varphi,f}(u) = \int_G u(x)g(x)\,dx$ for all $u \in M_0A(G)$.

So let $a \in \mathrm{VN}(G)$, $\varphi \in A_c(G)$, and $f \in PA_c(G)$, be fixed. Let S denote the compact set $\mathrm{supp}(f)^{-1}\,\mathrm{supp}(\varphi)$ and let 1_S denote its characteristic function. Let $u \in M_0A(G)$. Then the calculations on page 510 of [CH] show that

$$([f*u]\varphi)(x) = ([f*1_S u]\varphi)(x) \quad \forall x \in G. \tag{5}$$

Moreover,

$$([f*1_S u]\varphi)(x) = \left(\int_G f(xy)(1_S u)(y^{-1})\,dy\right)\varphi(x) = \int_G f_y(x)\varphi(x)(1_S u)(y^{-1})\,dy.$$

Define a map $\Phi: G \to A(G)$ by $[\Phi(y)](x) = f_y(x)\varphi(x)$, and define a measure μ on G by $d\mu(y) = (1_S u)(y^{-1})\,dy$. Then Φ is norm continuous and bounded, and $d\mu$ is a bounded Radon measure (since u is bounded and S is compact). Hence there is an element $v = \int_G \Phi(y)\,d\mu(y)$ in $A(G)$ such that

$$\langle b, v\rangle = \int_G \langle b, \Phi(y)\rangle\,d\mu(y) \quad \forall b \in \mathrm{VN}(G). \tag{6}$$

Setting $b = l(x)$ in (6) yields $v(x) = ([f*1_S u]\varphi)(x)$, and thus

$$\begin{aligned}\omega_{a,\varphi,f}(u) &= \langle a, [f*u]\varphi\rangle = \langle a, v\rangle = \int_G \langle a, \Phi(y)\rangle\,d\mu(y)\\ &= \int_G u(y^{-1})\langle a, \Phi(y)\rangle 1_S(y^{-1})\,dy = \int_G u(y)g(y)\,dy,\end{aligned}$$

where $g(y) = \langle a, \Phi(y^{-1})\rangle 1_S(y)\Delta(y^{-1})$. Since

$$\int_G |g(y)|\,dy = \int_G |\langle a, \Phi(y)\rangle| 1_S(y^{-1})\,dy,$$

g is in $L^1(G)$, as required.

(b) Fix a basis for $\mathscr{H}$. Then the elements of $C_r^*(G) \otimes K(\mathscr{H})$ can be viewed as $\infty \times \infty$ matrices with entries in $C_r^*(G)$, and for each n the natural map from $M_n(C_r^*(G))$ into $C_r^*(G) \otimes K(\mathscr{H})$ is an isometry. Moreover, if we view $M_n(C_r^*(G))$ as a subset of $C_r^*(G) \otimes K(\mathscr{H})$, then the union of the $M_n(C_r^*(G))$'s is norm dense in $C_r^*(G) \otimes K(\mathscr{H})$. Hence by (4) we can reduce to the case when $a \in M_n(C_r^*(G))$ for some n and $\varphi \in (M_n(C_r^*(G)))^*$. Then $a = [a_{ij}]$, $a_{ij} \in C_r^*(G)$, $\varphi = [\varphi_{ij}]$, $\varphi_{ij} \in (C_r^*(G))^* = B_\lambda(G)$, and

$$\omega_{a,\varphi,f}(u) = \sum_{i,j} \langle \overline{M}_{f*u}(a_{ij}), \varphi_{ij}\rangle.$$

Hence it suffices to show that the map $\omega_{a,\varphi,f}(u) = \langle \overline{M}_{f*u}(a), \varphi\rangle$ is in $Q(G)$ if $a \in C_r^*(G)$ and $\varphi \in B_\lambda(G)$. Since $\{l(g)\colon g \in C_c(G)\}$ is norm dense in $C_r^*(G)$, we can assume that $a = l(g)$ for some $g \in C_c(G)$. In this case $\overline{M}_{f*u}(a) = l([f*u]g)$ (cf. [DH, p. 460]). Moreover, (5) remains valid if we replace φ by g, and so by the duality between $C_r^*(G)$ and $B_\lambda(G)$ we have

$$\omega_{a,\varphi,f}(u) = \int_G ([f * 1_S u]g)(x)\varphi(x)\,dx = \int_G u(y)h(y)\,dy,$$

where $h(y) = \int_G f(xy^{-1})g(x)\varphi(x)1_S(y)\Delta(y^{-1})\,dx$. A straightforward calculation shows that $h \in L^1(G)$, and hence $\omega_{a,\varphi,f} \in Q(G)$.

(c) By (3) we can reduce to the case where $a = l(f) \otimes b$ for some $f \in L^1(G)$ and some $b \in K(\mathscr{H})$, and where $\varphi = v \otimes \psi$ for some $v \in A(G)$ and some $\psi \in (B(\mathscr{H}))_*$. Then

$$\omega_{a,\varphi}(u) = \langle \overline{M}_u(l(f)), v\rangle\langle b, \psi\rangle = \langle l(uf), v\rangle\langle b, \psi\rangle = \int_G u(x)g(x)\,dx,$$

where $g(x) = f(x)v(x)\langle b, \psi\rangle$ is in $L^1(G)$. Thus $\omega_{a,\varphi} \in Q(G)$. □

For discrete groups, we can improve (a) and (b).

Proposition 1.4. *Suppose that* Γ *is a discrete group, and that* $\mathscr{H}$ *is a Hilbert space. Then*:

(a) $\omega_{a,\varphi} \in Q(\Gamma)$ *whenever* $a \in \mathrm{VN}(\Gamma)\overline{\otimes}B(\mathscr{H})$ *and* $\varphi \in (\mathrm{VN}(\Gamma)\overline{\otimes}B(\mathscr{H}))_*$.
(b) $\omega_{a,\varphi} \in Q(\Gamma)$ *whenever* $a \in C_r^*(\Gamma)\otimes B(\mathscr{H})$ *and* $\varphi \in (C_r^*(\Gamma)\otimes B(\mathscr{H}))^*$.

Proof. (a) Let $f = \delta_e$ be the characteristic function of $\{e\}$. Then $f \in PA_c(\Gamma)$, and $f * u = u$ for all $u \in M_0A(\Gamma)$. Hence $\omega_{a,\varphi} = \omega_{a,\varphi,f} \in Q(\Gamma)$. (Note that the separability of $\mathscr{H}$ was not used in the proof of part (a) of Proposition 1.3.)

(b) Since the closed linear span of $\{l(x)\colon x \in \Gamma\}$ is $C_r^*(\Gamma)$, we can assume that $a = l(x)\otimes b$ for some $x \in \Gamma$ and some $b \in B(\mathscr{H})$. Let δ_x be the characteristic function of $\{x\}$, and let $g(y) = \langle l(y)\otimes b, \varphi\rangle\delta_x(y)$. Then g is in $l^1(\Gamma)$, and

$$\omega_{a,\varphi}(u) = \langle u(x)l(x)\otimes b, \varphi\rangle = u(x)\langle l(x)\otimes b, \varphi\rangle = \sum_{y\in\Gamma} u(y)g(y).$$

Hence $\omega_{a,\varphi} \in Q(\Gamma)$. □

Proposition 1.5. *Suppose that* G *is a locally compact group, and that* $\mathscr{H}$ *is a separable infinite dimensional Hilbert space. Let* $\omega \in Q(G)$. *Then* $\omega = \omega_{a,\varphi}$ *for some* $a \in C_r^*(G)\otimes K(\mathscr{H})$ *and some* $\varphi \in (\mathrm{VN}(G)\overline{\otimes}B(\mathscr{H}))_*$.

The next lemma will be used in the proofs of Proposition 1.5 and Theorem 2.2.

Lemma 1.6. *Suppose that* A *is a* C^**-algebra, that* M *is a von Neumann algebra, and that* $\mathscr{H}$ *is a separable infinite dimensional Hilbert space. Let* X *be a closed subspace of* $CB(A, M)$ *such that* $X = E^*$ *for some Banach space* E. *Let*

$$S = \{\omega_{a,\varphi}\colon a \in (A\otimes K(\mathscr{H}))_1 \text{ and } \varphi \in ((M\overline{\otimes}B(\mathscr{H}))_*)_1\},$$

where $\omega_{a,\varphi}$ *is defined on* X *by*

$$\omega_{a,\varphi}(T) = \langle T\otimes \mathrm{id}_{K(\mathscr{H})}(a), \varphi\rangle, \qquad T \in X.$$

Then S is convex. Moreover, if $S \subset E$, then every $\omega \in E$ is of the form $\omega = \omega_{a,\varphi}$ for some a in $A \otimes K(H)$ and φ in $(M \overline{\otimes} B(\mathscr{H}))_*$.

Proof. We first show that S is convex. Let $\mathscr{H}^{(2)}$ denote the direct sum of two copies of $\mathscr{H}$. Then we can identify $M \overline{\otimes} B(\mathscr{H}^{(2)})$ with $M_2(M \overline{\otimes} B(\mathscr{H}))$ in an obvious way, and we can identify $(M \overline{\otimes} B(\mathscr{H}^{(2)}))_*$ with $M_2((M \overline{\otimes} B(\mathscr{H}))_*)$ in such a way that if $b = [b_{ij}] \in M \overline{\otimes} B(\mathscr{H}^{(2)})$ and $\varphi = [\varphi_{ij}] \in (M \overline{\otimes} B(\mathscr{H}^{(2)}))_*$, then

$$\langle b, \varphi \rangle = \sum_{i,j} \langle b_{ij}, \varphi_{ij} \rangle.$$

Similarly, we can identify $A \otimes K(\mathscr{H}^{(2)})$ with $M_2(A \otimes K(\mathscr{H}))$, and if $T \in CB(A, M)$ then for any $a = [a_{ij}] \in A \otimes K(\mathscr{H}^{(2)})$ we have

$$T \otimes \mathrm{id}_{K(\mathscr{H}^{(2)})}(a) = [T \otimes \mathrm{id}_{K(\mathscr{H})}(a_{ij})].$$

Now let ω_{a_1,φ_1} and ω_{a_2,φ_2} be elements of S, and suppose $0 \leq \lambda \leq 1$. Let a be the diagonal matrix in $A \otimes K(\mathscr{H}^{(2)})$ with diagonal entries a_1 and a_2 and let φ be the diagonal matrix in $(M \overline{\otimes} B(\mathscr{H}^{(2)}))_*$ with diagonal entries $\lambda\varphi_1$ and $(1-\lambda)\varphi_2$. Then

$$\|a\| \leq \max\{\|a_1\|, \|a_2\|\} \leq 1 \quad \text{and} \quad \|\varphi\| \leq \lambda\|\varphi_1\| + (1-\lambda)\|\varphi_2\| \leq 1,$$

and for any $T \in CB(A, M)$ we have

$$\begin{aligned} \omega_{a,\varphi}(T) &= \langle T \otimes \mathrm{id}_{K(\mathscr{H}^{(2)})}(a), \varphi \rangle \\ &= \langle T \otimes \mathrm{id}_{K(\mathscr{H})}(a_1), \lambda\varphi_1 \rangle + \langle T \otimes \mathrm{id}_{K(\mathscr{H})}(a_2), (1-\lambda)\varphi_2 \rangle \\ &= (\lambda\omega_{a_1,\varphi_1} + (1-\lambda)\omega_{a_2,\varphi_2})(T). \end{aligned}$$

Moreover, since $\mathscr{H}$ and $\mathscr{H}^{(2)}$ are unitarily equivalent, $\omega_{a,\varphi} = \omega_{b,\psi}$ for some $b \in (A \otimes K(\mathscr{H}))_1$ and some $\psi \in ((M \overline{\otimes} B(\mathscr{H}))_*)_1$. Hence S is convex.

Next suppose that $S \subset E$. We first claim that S is norm dense in the unit ball E_1 of E. To see this, suppose that the claim is not true, and let ω be an element of E_1 that is not in the closure of S. Then since S is convex and balanced, there is a $T \in E^* = X$ such that

$$|\langle \omega_{a,\varphi}, T \rangle| \leq 1 < \langle \omega, T \rangle \quad \forall \omega_{a,\varphi} \in S. \tag{7}$$

Since $T \in CB(A, M)$ we have

$$\begin{aligned} \|T\|_{cb} &= \sup\{\|T \otimes \mathrm{id}_{K(\mathscr{H})}(a)\| : a \in (A \otimes K(\mathscr{H}))_1\} \\ &= \sup\{|\langle T \otimes \mathrm{id}_{K(\mathscr{H})}(a), \varphi \rangle| : a \in (A \otimes K(\mathscr{H}))_1 \text{ and } \varphi \in ((M \overline{\otimes} B(\mathscr{H}))_*)_1\} \\ &= \sup\{|\langle \omega_{a,\varphi}, T \rangle| : \omega_{a,\varphi} \in S\} \leq 1. \end{aligned}$$

But then $|\langle \omega, T \rangle| \leq \|\omega\| \, \|T\|_{cb} \leq 1$, which contradicts (7) and proves the claim.

Let $\mathscr{H}^{(\infty)}$ denote the direct sum of a countably infinite number of copies of $\mathscr{H}$. Since $\mathscr{H}$ and $\mathscr{H}^{(\infty)}$ are unitarily equivalent, to complete the proof of the lemma it suffices to show that every $\omega \in E$ is of the form $\omega = \omega_{a,\varphi}$ for some $a \in A \otimes K(\mathscr{H}^{(\infty)})$ and $\varphi \in (M \overline{\otimes} B(\mathscr{H}^{(\infty)}))_*$. We can also assume that $\omega \in E_1$. So let $\omega \in E_1$. Then there is an $\omega_1 \in S$ such that $\|\omega - \omega_1\| < 2^{-1}$. Since $2(\omega - \omega_1) \in E_1$, there is an $\omega_2 \in S$ such that $\|\omega - \omega_1 - 2^{-1}\omega_2\| < 2^{-2}$. Continuing in this fashion, we can find a sequence $\{\omega_n\}$ in S such that

$$\left\| \omega - \sum_{i=1}^{n} 2^{-i+1}\omega_i \right\| < 2^{-n}, \qquad n = 1, 2, \ldots.$$

Thus $\omega = \sum_{i=1}^{\infty} 2^{-i+1}\omega_i$. Since $\omega_i \in S$, there are $b_i \in (A \otimes K(\mathscr{H}))_1$ and $\psi_i \in ((M\overline{\otimes}B(\mathscr{H}))_*)_1$ such that $\omega_i = \omega_{b_i, \psi_i}$. Let $\alpha_i = (2^{-i+1})^{1/2}$, let $a_i = \alpha_i b_i$ and let $\varphi_i = \alpha_i \psi_i$. Let a be the diagonal matrix in $M\overline{\otimes}B(\mathscr{H}^{(\infty)})$ with diagonal entries $a_1, a_2, \ldots$ (where we view the elements of $M\overline{\otimes}B(\mathscr{H}^{(\infty)})$ in the obvious way as $\infty \times \infty$ matrices with entries in $M\overline{\otimes}B(\mathscr{H})$). Then since $a_i \in A \otimes K(\mathscr{H})$ and $\|a_i\| \to 0$, $a \in A \otimes K(\mathscr{H}^{(\infty)})$. Moreover, since $\sum_{i=1}^{\infty}\|\varphi_i\| < \infty$, we can define $\varphi \in (M\overline{\otimes}B(\mathscr{H}^{(\infty)}))_*$ by

$$\varphi([a_{ij}]) = \sum_{i=1}^{\infty}\langle a_{ii}, \varphi_i\rangle, \qquad [a_{ij}] \in M\overline{\otimes}B(\mathscr{H}^{(\infty)}).$$

A straightforward calculation shows that $\omega = \omega_{a,\varphi}$, which completes the proof. □

Proof of Proposition 1.5. Let $A = C_r^*(G)$, $M = \mathrm{VN}(G)$ and let $X = M_0A(G)$, where $M_0A(G)$ is viewed as a subspace of $CB(A, M)$ by identifying $u \in M_0A(G)$ with $\overline{M}_u$. Then $M_0A(G) = Q(G)^*$, and $S \subset Q(G)$ by Proposition 1.3(c), so an application of Lemma 1.6 completes the proof. □

If A is a C^*-algebra, and $\mathscr{H}$ is a separable infinite dimensional Hilbert space, a net $\{T_\alpha\}$ in $CB(A)$ is said to converge in the stable point-norm topology (resp. in the stable point-weak topology) to $T \in CB(A)$ if $T_\alpha \otimes \mathrm{id}_{K(\mathscr{H})}(a) \to T \otimes \mathrm{id}_{K(\mathscr{H})}(a)$ in norm (resp. weakly) for all $a \in A \otimes K(\mathscr{H})$ (cf. [ER]). If M is a von Neumann algebra, a net $\{T_\alpha\}$ in $CB_\sigma(M)$ is said to converge in the stable point-weak* topology to $T \in CB_\sigma(M)$ if $T_\alpha \otimes \mathrm{id}_{B(\mathscr{H})}(a) \to T \otimes \mathrm{id}_{B(\mathscr{H})}(a)$ σ-weakly for all $a \in M\overline{\otimes}B(\mathscr{H})$ (cf. [EKR]). The proof of the next result is similar to the proof of Proposition 2.3 in [Kr 3], and is left to the reader.

Proposition 1.7. *Let M and N be von Nemann algebras, and suppose that the net $\{T_\alpha\}$ in $CB_\sigma(M)$ converges in the stable point-weak* topology to $T \in CB_\sigma(M)$. Then $T_\alpha \otimes \mathrm{id}_N(a) \to T \otimes \mathrm{id}_N(a)$ σ-weakly for all a in $M\overline{\otimes}N$.*

Remark 1.8. Let A be a C^*-algebra, let M be a von Neumann algebra, and let $E = A\hat{\otimes}M_*$, the (completed) operator space projective tensor product of A and M_* (cf. [BP]). Then E is a Banach space, and $CB(A, M) = E^*$. Moreover, a net $\{T_\alpha\}$ in $CB(A, M)$ converges to T in the $\sigma(CB(A, M), E)$-topology if and only if for each a in $A \otimes B(\mathscr{H})$ (where $\mathscr{H}$ is a separable infinite dimensional Hilbert space) the net $\{T_\alpha \otimes \mathrm{id}_{B(H)}(a)\}$ converges to $T \otimes \mathrm{id}_{B(\mathscr{H})}(a)$ in the σ-weak topology of $M\overline{\otimes}B(\mathscr{H})$ (cf. [EKR]). Now suppose that A is contained in M, and that $\{T_\alpha\}$ is a net in $CB(A)$ that converges to $T \in CB(A)$ in the stable point-norm topology. Then it follows from Lemma 1.6 (viewing $\{T_\alpha\}$ as a net $CB(A, M)$) that $\{T_\alpha\}$ converges to T in the $\sigma(CB(A, M), E)$-topology, and hence

$$(8) \qquad \langle T_\alpha \otimes \mathrm{id}_{B(\mathscr{H})}(a), \varphi\rangle \to \langle T \otimes \mathrm{id}_{B(\mathscr{H})}(a), \varphi\rangle \quad \forall a \in A \otimes B(\mathscr{H}), \forall \varphi \in (M\overline{\otimes}B(\mathscr{H}))_*.$$

Moreover, an argument similar to that in the proof of Proposition 2.3 in [Kr 3] shows that we can replace $B(\mathscr{H})$ by any von Neumann algebra N. We will make use of these facts in §2.

Theorem 1.9. *For any locally compact group G, the following three conditions are equivalent*:

(a) G *has the AP*.
(b) *There is a net* $\{u_\alpha\}$ *in* $A_c(G)$ *such that* $\{M_{u_\alpha}\}$ *converges in the stable point-weak* topology to* $\mathrm{id}_{\mathrm{VN}(G)}$.
(c) *There is a net* $\{u_\alpha\}$ *in* $A_c(G)$ *such that* $\{\overline{M}_{u_\alpha}\}$ *converges in the stable point-norm topology to* $\mathrm{id}_{C_r^*(G)}$.

If G *is discrete, then* (a)–(c) *are equivalent to*:

(d) *For any Hilbert space* $\mathscr{H}$, *there is a net* $\{u_\alpha\}$ *in* $A_c(G)$ *such that* $\overline{M}_{u_\alpha} \otimes \mathrm{id}_{B(\mathscr{H})}(a) \to a$ *in norm for every* a *in* $C_r^*(G) \otimes B(\mathscr{H})$.

Proof. (a) $\Rightarrow$ (b) By Remark 1.2, there is a net $\{v_\alpha\}$ in $A_c(G)$ such that $v_\alpha \to 1$ in the $\sigma(M_0A(G), Q(G))$-topology. Let $f \in PA_c(G)$, and let $u_\alpha = f * v_\alpha$. Then since f and v_α have compact support, $u_\alpha \in A_c(G)$. Let $\mathscr{H}$ be a separable infinite dimensional Hilbert space and let $a \in \mathrm{VN}(G)\overline{\otimes} B(\mathscr{H})$. Then it follows from Proposition 1.3(a) and the fact that $f * 1 = 1$ that

$$\begin{aligned}&\langle M_{u_\alpha} \otimes \mathrm{id}_{B(\mathscr{H})}(a), \varphi\rangle \\ &\quad = \omega_{a,\varphi,f}(v_\alpha) \to \omega_{a,\varphi,f}(1) = \langle a, \varphi\rangle \quad \forall \varphi \in (\mathrm{VN}(G)\overline{\otimes} B(\mathscr{H}))_*,\end{aligned}$$

and so $M_{u_\alpha} \to \mathrm{id}_{\mathrm{VN}(G)}$ in the stable point-weak* topology.

(a) $\Rightarrow$ (c) An argument similar to that in the proof of (a) $\Rightarrow$ (b) shows that $\mathrm{id}_{C_r^*(G)}$ is in the closure of $\{\overline{M}_u : u \in A_c(G)\}$ in the stable point-weak topology. But since $\{\overline{M}_u : u \in A_c(G)\}$ is a linear subspace of $CB(A)$, the stable point-weak and stable point-norm closures of $\{\overline{M}_u : u \in A_c(G)\}$ coincide (cf. [EKR]). Hence condition (c) holds.

(a) $\Rightarrow$ (d) Since $\{\overline{M}_u \otimes \mathrm{id}_{B(\mathscr{H})} : u \in A_c(G)\}$ is a linear subspace of $B(C_r^*(G) \otimes B(\mathscr{H}))$, it suffices to show that $\mathrm{id}_{C_r^*(G)} \otimes \mathrm{id}_{B(\mathscr{H})}$ is in the point-weak closure of $\{\overline{M}_u \otimes \mathrm{id}_{B(\mathscr{H})} :\ u \in A_c(G)\}$. But this follows easily from Proposition 1.4(b), and so condition (d) holds.

(b) $\Rightarrow$ (a) Let $\{u_\alpha\}$ be a net in $A_c(G)$ such that $M_{u_\alpha} \to \mathrm{id}_{\mathrm{VN}(G)}$ in the stable point-weak* topology. Then $\omega_{a,\varphi}(u_\alpha) \to \omega_{a,\varphi}(1)$ for all $a \in C_r^*(G) \otimes B(\mathscr{H})$ and $\varphi \in (\mathrm{VN}(G)\overline{\otimes} B(\mathscr{H}))_*$, and hence $u_\alpha \to 1$ in the $\sigma(M_0A(G), Q(G))$-topology by Proposition 1.5. Thus G has the AP.

(c) $\Rightarrow$ (a) and (d) $\Rightarrow$ (a) The proofs of these implications are similar to the proof of (b) $\Rightarrow$ (a) and are left to the reader. □

Remark 1.10. If $A_c(G)$ is replaced by $A(G)$ in conditions (b), (c), and (d) of Theorem 1.9, we obtain three new conditions, which we denote by (b′), (c′), and (d′). It follows from Remark 1.2 that (b′), (c′), and (d′) are equivalent to (b), (c), and (d) respectively, and so (b′), (c′) are equivalent to (a) and (d′) is equivalent to (a) if G is discrete.

As noted in the introduction, a locally compact group G is amenable if and only if $A(G)$ has an approximate identity that is bounded in the $A(G)$-norm, and G is weakly amenable if and only if $A(G)$ has an approximate identity that is bounded in the $M_0A(G)$-norm. Our next result shows that the AP can be characterized by a "stable" approximate identity condition.

Theorem 1.11. *Let G be a locally compact group, and put $K = SU(2)$. The following three conditions are equivalent:*

(a) *G has the AP.*

(b) *For every locally compact group H, there is a net $\{u_\alpha\}$ in $A_c(G)$ such that*

$$\|(u_\alpha \otimes 1)v - v\|_{A(G\times H)} \to 0 \quad \forall v \in A(G \times H). \tag{9}$$

(c) *There is a net $\{u_\alpha\}$ in $A(G)$ such that*

$$\|(u_\alpha \otimes 1)v - v\|_{A(G\times K)} \to 0 \quad \forall v \in A(G \times K). \tag{10}$$

Proof. (a) $\Rightarrow$ (b) Let H be a locally compact group. As shown in the proof of Theorem 1.6 in [DH], if $u \in M_0A(G)$, then

$$M_u \otimes \mathrm{id}_{\mathrm{VN}(H)} = M_{u\otimes 1}, \tag{11}$$

where we identify $\mathrm{VN}(G) \otimes \mathrm{VN}(H)$ with $\mathrm{VN}(G \times H)$ in the usual way. Hence it follows from Theorem 1.9 and Proposition 1.7 that there is a net $\{v_\alpha\}$ in $A_c(G)$ such that

$$\begin{aligned} \langle a, (v_\alpha \otimes 1)v\rangle = \langle M_{v_\alpha\otimes 1}(a), v\rangle \to \langle a, v\rangle \\ \forall a \in \mathrm{VN}(G \times H), \forall v \in A(G \times H). \end{aligned} \tag{12}$$

Thus $\mathrm{id}_{A(G\times H)}$ is in the point-weak closure of the subspace $\{m_{u\otimes 1} : u \in A_c(G)\}$ of $B(A(G \times H))$. Hence $\mathrm{id}_{A(G\times H)}$ is the point-norm closure of $\{m_{u\otimes 1} : u \in A_c(G)\}$ (cf. [DS, Corollary VI.1.5]), and so there is a net $\{u_\alpha\}$ in $A_c(G)$ satisfying (9).

(b) $\Rightarrow$ (c) This implication is trivial.

(c) $\Rightarrow$ (a) Let $\{u_\alpha\}$ be a net in $A(G)$ satisfying (10). Then it follows from (11) and (12) (replacing H by K and v_α by u_α) that

$$M_{u_\alpha} \otimes \mathrm{id}_{\mathrm{VN}(K)}(a) \to a \ \sigma\text{-weakly} \quad \forall a \in \mathrm{VN}(G)\bar{\otimes}\mathrm{VN}(K). \tag{13}$$

It is well known that $\mathrm{VN}(K) \cong \bigoplus_{n=1}^{\infty} M_n(\mathbf{C})$. Moreover, if $\mathscr{H}$ is a separable Hilbert space with basis $\xi_1, \xi_2, \dots$, and if p_n denotes the projection onto $[\xi_1, \xi_2, \dots, \xi_n]$, then the map

$$\Phi: B(\mathscr{H}) \to \bigoplus_{n=1}^{\infty} M_n(\mathbf{C}) : a \to \bigoplus_{n=1}^{\infty} e_n a e_n$$

is a completely isometric σ-weakly continuous map. Thus we can view $B(\mathscr{H})$ as a subspace of $\mathrm{VN}(K)$, and so it follows from (13) that the net $\{M_{u_\alpha}\}$ converges in the stable point-weak* topology to $\mathrm{id}_{\mathrm{VN}(G)}$. Hence G has the AP by Theorem 1.9. □

If G is weakly amenable, then G has the AP. This fact is implied by the next result, which shows the relationship of weak amenability to the definition of the AP.

Theorem 1.12. *Let G be a locally compact group. The following conditions are equivalent:*

(a) *G is weakly amenable, and $\Lambda(G) \le L$.*

(b) *The function 1 is in the $\sigma(M_0A(G), Q(G))$-closure of $\{u \in A(G) : \|u\|_{M_0} \le L\}$.*

Proof. (a) $\Rightarrow$ (b) Put $S = \{u \in A(G) : \|u\|_{M_0} \le L\}$. Since $\Lambda(G) \le L$, there is a net $\{u_\alpha\}$ in S such that

$$\|u_\alpha v - v\|_{A(G)} \to 0 \quad \forall v \in A(G). \tag{14}$$

Hence

$$\langle M_{u_\alpha}(a), v\rangle = \langle a, u_\alpha v\rangle \to \langle a, v\rangle \quad \forall a \in \mathrm{VN}(G), \forall v \in A(G),$$

and so $M_{u_\alpha} \to \mathrm{id}_{\mathrm{VN}(G)}$ in the point-weak* topology. Since $\{M_{u_\alpha}\}$ is a bounded net in $CB_\sigma(\mathrm{VN}(G))$, $M_{u_\alpha} \to \mathrm{id}_{\mathrm{VN}(G)}$ in the stable point-weak* topology (cf. [Kr 3, Proposition 2.9]), and so $u_\alpha \to 1$ in the $\sigma(M_0A(G), Q(G))$-topology by the proof of the implication (b) $\Rightarrow$ (a) of Theorem 1.9. Thus 1 is in the $\sigma(M_0A(G), Q(G))$-closure of S.

(b) $\Rightarrow$ (a) By the proof of the implication (a) $\Rightarrow$ (b) of Theorem 1.9, there is a net $\{v_\alpha\}$ in S such that $M_{v_\alpha} \to \mathrm{id}_{\mathrm{VN}(G)}$ in the stable point-weak* topology, and hence in the point-weak* topology. Since S is convex, an argument similar to that in the proof of the implication (a) $\Rightarrow$ (b) of Theorem 1.11 shows that there is a net $\{u_\alpha\}$ in S satisfying (14). Hence G is weakly amenable, with $\Lambda(G) \le L$. □

For a locally compact group G, let $P_1(G)$ denote the set of continuous positive-definite functions u on G such that $u(e) = 1$. Note that if $u \in P_1(G)$, then M_u is completely positive (cf. [DH, Proposition 4.2]) and $\|u\|_{B(G)} = \|u\|_{M_0} = u(e) = 1$. If G is amenable, then $A(G)$ has an approximate identity $\{u_\alpha\}$ such that each u_α is in $P_1(G) \cap A(G)$ (cf. [Lau, Lemma 7.2]). Moreover, if $A(G)$ has an approximate identity $\{u_\alpha\}$ such that each u_α is in $(A(G))_1$ (the unit ball of $A(G)$ with respect to the $A(G)$-norm), then G is amenable (cf. [Le]). Using these facts (and the fact that $f * u \in (A(G))_1$ if $f \in PA_c(G)$ and $u \in P_1(G) \cap A(G)$) the proof of Theorem 1.12 can be easily modified to give a proof of the next result. The details are left to the reader.

Theorem 1.13. *Let G be a locally compact group. The following conditions are equivalent*:

(a) *G is amenable.*
(b) *The function 1 is in the $\sigma(M_0A(G), Q(G))$-closure of $P_1(G) \cap A(G)$ in $M_0A(G)$.*

We next discuss the stability properties of the AP.

Proposition 1.14. *If G is a locally compact group with the AP, then every closed subgroup of G also has the AP.*

Proof. Let H be a closed subgroup of G, and put $K = SU(2)$. Then since $H \times K$ is a closed subgroup of $G \times K$, the restriction mapping $u \to u|_{H\times K}$ is a contraction from $A(G \times K)$ onto $A(H \times K)$ (see the proof of Proposition 1.12 in [DH]). Since G has the AP, it follows easily from this and Theorem 1.11 that H has the AP. □

Theorem 1.15. *Let G be a locally compact group, and suppose that H is a closed normal subgroup of G. If H and G/H have the AP, then G has the AP.*

For the proof of Theorem 1.15 we need a lemma.

Lemma 1.16. *Let G be a locally compact group, and suppose that H is a closed subgroup of G such that $\Delta_G(h) = \Delta_H(h)$ for all $h \in H$. For each $f \in C_c(G)$, let Φ_f denote the map defined on $M_0A(H)$ by*

$$\Phi_f(u) = f * u\,dh * \tilde{f} \quad \forall u \in M_0A(H),$$

where dh is a fixed left Haar measure on H. Then Φ_f is a bounded linear map from $M_0A(H)$ into $M_0A(G)$ that is $\sigma(M_0A(H), Q(H))$-$\sigma(M_0A(G), Q(G))$ continuous.

Proof. For any $f \in C_c(G)$ the function defined on G by $x \to \int_H f(xh)\,dh$ is constant on each left coset of H, and hence we can define a function $T_H f$ on G/H by

$$T_H f(\dot{x}) = \int_H f(xh)\,dh, \qquad \dot{x} \in G/H. \tag{15}$$

For any $f \in C_c(G)$, $T_H f \in C_c(G/H)$ (cf. [Re]). In particular, $T_H f$ is always bounded.

Since the modular functions of G and H agree on H, we can choose a measure μ on the quotient space G/H that is invariant under the natural action of G on G/H and satisfies the relation

(16)

$$\int_{G/H} T_H f(\dot{x})\,d\mu(\dot{x}) = \int_{G/H}\int_H f(xh)\,dh\,d\mu(\dot{x}) = \int_G f(x)\,dx \quad \forall f \in C_c(G).$$

(See, e.g., Chapter 8 of [Re].)

Let $f \in C_c(G)$, and let $u \in M_0A(H)$. Then

$$(u\,dh * \tilde{f})(y) = \int_H u(h)\overline{f(y^{-1}h)}\,dh \quad \forall y \in G. \tag{17}$$

It follows from (17) and the lemma on p. 58 of [Re] (using the fact that u is bounded) that the function $u\,dh * \tilde{f}$ is continuous on G, and hence for each x in G the function g_x defined on G by

$$g_x(y) = f(y)(u\,dh * \tilde{f})(y^{-1}x), \qquad y \in G,$$

is in $C_c(G)$. Let $u_f = \Phi_f(u)$. Then, using (16) and (17), we have that

$$\begin{aligned} u_f(x) &= \int_{G/H}\int_H g_x(yh)\,dh\,d\mu(\dot{y}) \\ &= \int_{G/H}\int_H f(yh)\int_H u(k)\overline{f(x^{-1}yhk)}\,dk\,dh\,d\mu(\dot{y}) \\ &= \int_{G/H}\int_H\int_H f(yh)\overline{f(x^{-1}yk)}u(h^{-1}k)\,dk\,dh\,d\mu(\dot{y}) \end{aligned} \tag{18}$$

for all $x \in G$.

To show that $u_f \in M_0A(G)$, it suffices to show that there is a Hilbert space $\mathscr{H}$ and bounded continuous maps $P, Q\colon G \to \mathscr{H}$ such that

$$u_f(y^{-1}x) = \langle P(x), Q(y)\rangle \quad \forall x, y \in G. \tag{19}$$

Since $u \in M_0A(H)$, there is a Hilbert space $\mathscr{K}$ and bounded continuous maps $\zeta, \eta\colon H \to \mathscr{K}$ such that $u(k^{-1}h) = \langle \zeta(h), \eta(k)\rangle$ for all h and k in H, and such that

$$\sup\{\|\zeta(h)\|: h \in H\} = \sup\{\|\eta(h)\|: h \in H\} = (\|u\|_{M_0A(H)})^{1/2}.$$

Since $f \in C_c(G)$, we can define functions $p, q\colon H \to \mathscr{H}$ by

$$p(x) = \int_H \overline{f(xh)}\zeta(h)\,dh, \quad q(x) = \int_H \overline{f(xh)}\eta(h)\,dh, \qquad h \in G.$$

Put $L = (\|u\|_{M_0A(H)})^{1/2}$. Then for any $x \in G$ and $\xi \in \mathscr{H}$ we have that

$$|\langle p(x), \xi\rangle| \le \int_H |\overline{f(xh)}||\langle\zeta(h), \xi\rangle|\,dh \le \left(L\int_H |f(xh)|\,dh\right)\|\xi\|, \tag{20}$$

and hence

$$\|p(x)\| \le L\int_H |f(xh)|\,dh \quad \forall x \in G. \tag{21}$$

Similarly,

$$|\langle p(x) - p(y), \xi\rangle| \le \left(L\int_H |f(xh) - f(yh)|\,dh\right)\|\xi\| \quad \forall x, y \in G, \forall\xi \in \mathscr{H}, \tag{22}$$

from which it follows easily (using the lemma on p. 58 of [Re]) that p is continuous. Note that the estimates (20), (21) and (22) remain valid if we replace p by q and ζ by η. Hence q is also continuous. Moreover, for any x and y in G we have that

$$\begin{aligned} u_f(y^{-1}x) &= \int_{G/H}\int_H\int_H f(zh)\overline{f(x^{-1}yzk)}u(h^{-1}k)\,dk\,dh\,d\mu(\dot z) \\ &= \int_{G/H}\int_H\int_H f(y^{-1}zh)\overline{f(x^{-1}zk)}u(h^{-1}k)\,dk\,dh\,d\mu(\dot z) \\ &= \int_{G/H}\int_H\int_H f(y^{-1}zh)\overline{f(x^{-1}zk)}\langle\zeta(k), \eta(h)\rangle\,dk\,dh\,d\mu(\dot z) \\ &= \int_{G/H}\langle p(x^{-1}z), q(y^{-1}z)\rangle\,d\mu(\dot z), \end{aligned} \tag{23}$$

where the G-invariance of μ is used in the second equality.

If p and q were constant on cosets, we could complete the proof by defining functions P and Q from G to $L^2(G/H, \mathscr{H})$ by $P(x)(\dot z) = p(x^{-1}z)$ and $Q(x)(\dot z) = q(y^{-1}z)$, since then P and Q satisfy equation (19). However, p and q need not be constant on cosets, so we need to make use of cross sections.

It is shown in [Ke] that there is a map $\rho\colon G/H \to G$ which is a locally bounded Baire cross section for the quotient map $\pi_H\colon G \to G/H$, i.e., $\rho^{-1}(B)$ is a Baire set if B is a Baire set, $\rho(C)$ is relatively compact if C is compact, and $\pi_H(\rho(\dot x)) = \dot x$ for all $\dot x \in G/H$. Let ρ be such a cross section, and define a Baire measure ω on G by $\omega(B) = \mu(\rho^{-1}(B))$, B a Baire set. Then if f is any nonnegative continuous function on G, $f \circ \rho$ is a Baire map on G/H, and

$$\int_G f(x)\,d\omega(x) = \int_{G/H} f(\rho(\dot x))\,d\mu(\dot x). \tag{24}$$

In particular, for any $y \in G$,

$$\int_G \|p(y^{-1}x)\|^2\,d\omega(x) = \int_{G/H}\|p(y^{-1}\rho(\dot x))\|^2\,d\mu(\dot x). \tag{25}$$

It follows from (21) and (25) that

$$\begin{aligned}\int_G \|p(y^{-1}x)\|^2\,d\omega(x) &\le \int_{G/H}\left(L\int_H |f(y^{-1}\rho(\dot{x})h)|\,dh\right)^2 d\mu(\dot{x})\\ &= \int_{G/H}\left(L\int_H |f(y^{-1}xh)|\,dh\right)^2 d\mu(\dot{x})\\ &= \int_{G/H}\left(L\int_H |f(xh)|\,dh\right)^2 d\mu(\dot{x}),\end{aligned}$$

where we used the fact that $\pi_H(\rho(\dot{x})) = \dot{x}$ in the first equality, and the G-invariance of μ in the last equality. Put

$$K(f) = \int_{G/H}\left(\int_H |f(xh)|\,dh\right)^2 d\mu(\dot{x})$$

and $\mathscr{H} = L^2(G, \mathscr{K}, \omega)$. Then the function $P\colon G \to \mathscr{H}$ defined by

$$[P(x)](y) = p(x^{-1}y), \quad x, y \in G,$$

satisfies

$$\|P(x)\|^2 \le K(f)L^2 \quad \forall x \in G. \tag{26}$$

Hence P is bounded. Moreover, for any y and z in G,

$$\begin{aligned}\|P(y) - P(z)\|^2 &\le \int_{G/H}\left(L\int_H |f(y^{-1}xh) - f(z^{-1}xh)|\,dh\right)^2 d\mu(\dot{x})\\ &\le 2L^2\|T_H|f|\,\|_\infty \int_{G/H}\int_H |f(y^{-1}xh) - f(z^{-1}xh)|\,dh\,d\mu(\dot{x})\\ &= 2L^2\|T_H|f|\,\|_\infty \int_G |f(y^{-1}x) - f(z^{-1}x)|\,dx,\end{aligned}$$

from which it follows easily that P is continuous.

Similar calculations show that the function $Q\colon G \to \mathscr{H}$ defined by

$$[Q(x)](y) = q(x^{-1}y), \qquad x, y \in G,$$

is bounded and continuous, and it follows easily from (23) and (24) that

$$u_f(y^{-1}x) = \langle P(x), Q(y)\rangle \quad \forall x, y \in G.$$

Hence Φ_f maps $M_0A(H)$ into $M_0A(G)$, and Φ_f is clearly linear. Moreover, since the estimate (26) is also valid when P is replaced by Q, $\|\Phi_f(u)\|_{M_0A(G)} \le K(f)\|u\|_{M_0A(H)}$, and thus Φ_f is bounded.

To show that Φ_f is $\sigma(M_0A(H), Q(H))$-$\sigma(M_0A(G), Q(G))$ continuous, it suffices to show that $(\Phi_f)^*$ maps $Q(G)$ into $Q(H)$, and since $(\Phi_f)^*$ is continuous, the proof will be complete if we show that $(\Phi_f)^*(g) \in Q(H)$ for any $g \in L^1(G)$. So let $g \in L^1(G)$. Then we can define a function g_f on H by

$$g_f(h) = \int_G f(x)(g * \overline{f})(xh)\,dx, \qquad h \in H.$$

It is straightforward to check that $g_f \in L^1(H)$, and that

$$\langle u, (\Phi_f)^*(g)\rangle = \int_H u(h)g_f(h)\,dh = \langle u, g_f\rangle$$

for all $u \in M_0A(H)$. Hence $(\Phi_f)^*(g) = g_f$. Since $L^1(H) \subset Q(H)$, this completes the proof. □

Proof of Theorem 1.15. By Remark 1.2, there is a net $\{u_\alpha\}$ in $A_c(H)$ such that $u_\alpha \to 1_H$ in the $\sigma(M_0A(H), Q(H))$-topology. Then $\Phi_f(u_\alpha) \to \Phi_f(1_H)$ in the $\sigma(M_0A(G), Q(G))$-topology for any $f \in C_c(G)$. Since each u_α is in $L^1(H)$ and $\tilde{f} \in L^2(G)$, $u_\alpha \, dh * \tilde{f} \in L^2(G)$ for all α, and hence $\Phi_f(u_\alpha) \in L^2(G) * L^2(G) = A(G)$ for all α. (In fact, $\Phi_f(u_\alpha) \in A_c(G)$ for all α, since the functions u_α, f and $\tilde{f}$ all have compact support.) Hence to complete the proof, it suffices to show that 1_G is in the $\sigma(M_0A(G), Q(G))$-closed linear span of $\{\Phi_f(1_H) : f \in C_c(G)\}$.

Let $f \in C_c(G)$, and put $g = T_H f$. Then it follows from (18) that

$$\Phi_f(1_H)(x) = (g * \tilde{g})(\dot{x}) \quad \forall x \in G.$$

Since H is a closed normal subgroup of G, the function T_H maps $C_c(G)$ onto $C_c(G/H)$ (cf. [Re]). Hence

$$\{\Phi_f(1_H) : f \in C_c(G)\} = \{(g * \tilde{g}) \circ \pi_H : g \in C_c(G/H)\}. \tag{27}$$

Moreover, the map T_H extends to a map (still denoted by T_H) from $L^1(G)$ onto $L^1(G/H)$ such that (15) is valid a.e. for each $f \in L^1(G)$, and such that (16) is valid for all $f \in L^1$.

Define a map Ψ on $M_0A(G/H)$ by $\Psi(u) = u \circ \pi_H$, $u \in M_0A(G/H)$. Then Ψ is an isometry from $M_0A(G/H)$ onto the subspace of $M_0A(G)$ consisting of functions that are constant on the left cosets of H. For any $f \in L^1(G)$ and for any $u \in M_0A(G/H)$ we have

$$\begin{aligned} \langle u, T_H f\rangle &= \int_{G/H} u(\dot{x}) \int_H f(xh)\, dh\, d\mu(\dot{x}) \\ &= \int_{G/H} \int_H [\Psi(u)](xh) f(xh)\, dh\, d\mu(\dot{x}) \\ &= \int_G [\Psi(u)](x) f(x)\, dx = \langle \Psi(u), f\rangle. \end{aligned} \tag{28}$$

It follows easily from (28) that Ψ is $\sigma(M_0A(G/H), Q(G/H))$-$\sigma(M_0A(G), Q(G))$ continuous.

Let E denote the linear span of $\{g * \tilde{g} : g \in C_c(G/H)\}$. Then E is dense in $A(G/H)$ in the $B(G/H)$-norm, and so it follows from Remark 1.2 that $1_{G/H}$ is in the $\sigma(M_0A(G), Q(G))$-closure of E. Hence if follows from (27) that 1_G is in the $\sigma(M_0A(G), Q(G))$-closed linear span of $\{\Phi_f(1_H) : f \in C_c(G)\}$, and so G has the AP. □

Corollary 1.17. *If H and K are locally compact groups with the AP, and if $\rho : K \to \operatorname{Aut} H$ is a continuous homomorphism, then the semidirect product $H \times_\rho K$ has the AP.*

If G is amenable, and H is a closed normal subgroup of G, then G/H is amenable. It is an open question whether G/H has the AP if G has the AP. However, since free groups are weakly amenable (cf. [DH]), and since any discrete group is the quotient of a free group, a positive answer to this question would imply that all discrete groups have the AP, which seems unlikely.

We conclude this section with a discussion of a weaker version of the AP. It is shown in [DH] that if we let X denote the completion of $L^1(G)$ with respect to the norm

$$\|f\|_X = \sup\left\{\left|\int_G f(x)u(x)\,dx\right| : u \in MA(G),\ \|u\|_M \le 1\right\},$$

then $MA(G) = X^*$.

Definition 1.18. Let G be a locally compact group. Then G is said to have the AP$'$ if the function 1 is in the $\sigma(MA(G), X)$-closure of $A(G)$ in $MA(G)$.

If $a \in \mathrm{VN}(G)$ and $\varphi \in (\mathrm{VN}(G))_*$, we can define a linear functional $\omega_{a,\varphi}$ on $MA(G)$ by setting $\omega_{a,\varphi}(u) = \omega_{a,\varphi}(M_u)$, and for any $f \in PA_c(G)$, we can define a linear functional $\omega_{a,\varphi,f}$ on $MA(G)$ by $\omega_{a,\varphi,f}(u) = \omega_{a,\varphi}(f * u)$. It follows easily from the proof of Proposition 1.3 that the linear functionals $\omega_{a,\varphi,f}$ are all in X, that (with the obvious definitions) $\omega_{a,\varphi,f}$ is in X whenever $a \in C_r^*(G)$, $\varphi \in (C_r^*(G))^*$, and $f \in PA_c(G)$, and that $\omega_{a,\varphi}$ is in X whenever $a \in C_r^*(G)$ and $\varphi \in (\mathrm{VN}(G))_*$.

Proposition 1.19. *Let G be a locally compact group, and consider the following conditions:*

(a) *G has the AP$'$.*
(b) *There is a net $\{u_\alpha\}$ in $A_c(G)$ such that $\{M_{u_\alpha}\}$ converges in the point-weak* topology to $\mathrm{id}_{\mathrm{VN}(G)}$.*
(c) *There is a net $\{u_\alpha\}$ in $A_c(G)$ such that $\{\overline{M}_{u_\alpha}\}$ converges in the point-norm topology to $\mathrm{id}_{C_r^*(G)}$.*
(d) *There is a net $\{u_\alpha\}$ in $A_c(G)$ such that $\|u_\alpha v - v\|_{A(G)} \to 0\ \ \forall v \in A(G)$.*
(e) *Every a in $\mathrm{VN}(G)$ satisfies condition (H) of Eymard.*

Then (a) *implies* (b), (c) *and* (d), *and* (d) *implies* (e).

Proof. The proofs that (a) implies (b), (c) and (d) are similar to the proofs of the corresponding implications in Theorems 1.9 and 1.11, and are left to the reader. The proof that (d) implies (e) is the same as the proof of (4.16) in [Ey]. □

Clearly the AP implies the AP$'$. For connected semisimple Lie groups with finite center, the AP and the AP$'$ are equivalent. This follows by the method of proof used in [CH, Proposition 1.6] to show that $\Lambda(G) = \Lambda'(G)$ for these groups. We do not know whether the AP and the AP$'$ are equivalent in general.

2. The discrete case

If M is a von Neumann algebra, we let $F_\sigma(M)$ denote the set of normal finite rank maps from M to M, and if A is a C^*-algebra, we let $F(A)$ denote the set of bounded finite rank maps from A to A. Following [ER] and [EKR] we say that a von Neumann algebra M has the weak* operator approximation property (weak* OAP) if the identity map from M to M is in the stable point-weak* closure of $F_\sigma(M)$, and that a C^*-algebra A has the operator approximation property (OAP) if the identity map from A to A is in the stable point-norm closure of $F(A)$. We say that A has the strong operator approximation property if for any C^*-algebra B there is a net $\{T_\alpha\}$ in $F(A)$ such that $T_\alpha \otimes \mathrm{id}_B(a) \to a$ in norm for every a in $A \otimes B$. Note that since

$\{T \otimes \mathrm{id}_B : T \in F(A)\}$ is always a subspace of $B(A \otimes B)$, A has the strong if for any C^*-algebra B there is a net $\{T_\alpha\}$ in $F(A)$ such that $T_\alpha \otimes \mathrm{id}_B \to \mathrm{id}_{A\otimes B}$ in the point-weak topology of $B(A \otimes B)$.

If M and N are von Neumann algebras, and if φ is in the predual M_* of M, then the right slice map R_φ is the unique normal linear map from $M\overline{\otimes}N$ to N such that $R_\varphi(a \otimes b) = \varphi(a)b$ $(a \in M, b \in N)$. Left slice maps are similarly defined. If $V \subset M$ and $W \subset N$ are σ-weakly closed subspaces, the Fubini product $F(V, W)$ of V and W is the set of all $x \in M\overline{\otimes}N$ all of whose right slices $R_\varphi(x)$ are in W and all of whose left slices $L_\psi(x)$ $(\psi \in N_*)$ are in V. The Fubini product is a σ-weakly closed subspace of $M\overline{\otimes}N$, and so we always have that $F(V, W)$ contains the σ-weakly closed linear span $V\overline{\otimes}W$ of the elementary tensors $v \otimes w$ $(v \in V, w \in W)$. However, as shown in [Kr 3], $F(V, W)$ can be strictly larger than $V\overline{\otimes}W$. (This can be viewed as saying that there are elements in $F(V, W)$ that cannot be "synthesized" from their slices.) If R is a von Neumann algebra, a σ-weakly closed subspace V of $B(\mathscr{H})$ is said to have Property S_σ for R if $F(V, W) = V\overline{\otimes}W$ for every σ-weakly closed subspace W of R. A σ-weakly closed subspace V of $B(\mathscr{H})$ is said to have Property S_σ if it has Property S_σ for R for every von Neumann algebra R. It is shown in [Kr 3] that the weak* OAP is equivalent to Property S_σ.

Slice maps can also be defined for C^*-algebras by making the obvious modifications of the definitions (cf. [To]). For example, if A and B are C^*-algebras and if $\varphi \in A^*$, the right slice map R_φ is the unique bounded map from $A \otimes B$ to B such that $R_\varphi(a \otimes b) = \varphi(a)b$ $(a \in A, b \in B)$. If A and B are C^*-algebras, and if W is a closed subspace of B, the triple (A, B, W) is said to have the slice map property if $\{x \in A \otimes B : R_\varphi(x) \in W \ \ \forall \varphi \in A^*\} = A \otimes W$, and A is said to have the slice map property for B if (A, B, W) has the slice map property for every closed subspace W of B. A C^*-algebra A is said to have the general slice map property if it has the slice map property for every C^*-algebra B. It is shown in [Kr 3] that A has the OAP if and only if A has the slice map property for $K(\mathscr{H})$ ($\mathscr{H}$ a separable infinite dimensional Hilbert space), and that A has the strong OAP if and only if A has the general slice map property.

Theorem 2.1. *Let Γ be a discrete group. The following conditions are equivalent:*

(a) *Γ has the AP.*
(b) $\mathrm{VN}(\Gamma)$ *has Property S_σ* ($\mathrm{VN}(\Gamma)$ *has the weak*OAP*).
(c) $C_r^*(\Gamma)$ *has Property S for subspaces of $K(\mathscr{H})$* ($C_r^*(\Gamma)$ *has the OAP*).
(d) $C_r^*(\Gamma)$ *has Property S for subspaces* ($C_r^*(\Gamma)$ *has the strong OAP*).

Moreover, if any (and hence all) of conditions (a)–(d) *are satisfied, then $A(\Gamma)$ has the Banach space approximation property.*

Proof. Since Γ is a discrete group, if we let δ_x denote the characteristic function of $\{x\}$ $(x \in \Gamma)$, then $\{\delta_x : x \in \Gamma\}$ is a basis for $l^2(\Gamma)$, and

$$\mathrm{tr}(a) = \langle a\delta_e, \delta_e\rangle, \qquad a \in \mathrm{VN}(\Gamma),$$

is a faithful normal trace on $\mathrm{VN}(\Gamma)$. Moreover if $u \in A_c(\Gamma)$ (so u has finite support), then since $C_r^*(\Gamma)$ is the closed linear span of $\{l(x) : u(x) \in \Gamma\}$, both M_u and $\overline{M}_u$ are finite rank operators (with range the linear span of $\{l(x) : u(x) \neq 0\}$). Hence it follows immediately from Theorem 1.9 that (a) implies (b), (c), and (d). Moreover, if condition (b) holds, it follows from Theorem 3.1 in [Kr 3] that $A(\Gamma) = \mathrm{VN}(\Gamma)_*$ has the Banach space approximation property. Since (d) implies (c) is trivial, it suffices to prove that (b) $\Rightarrow$ (a) and that (c) $\Rightarrow$ (a).

(b) $\Rightarrow$ (a) Let $M = \mathrm{VN}(\Gamma)$. Then there exist a normal *-isomorphism π from M onto a von Neumann subalgebra N of $M\overline{\otimes}M$ such that $\pi(l(x)) = l(x)\otimes l(x)$ for all x in Γ (cf. [St, 18.7]). Let ε be the normal conditional expectation from $M\overline{\otimes}M$ onto N that leaves $\mathrm{tr}\otimes\mathrm{tr}$ invariant, and put $\rho = \pi^{-1}\circ\varepsilon$. For $T \in F_\sigma(M)$, let u_T be the function defined on Γ by

$$u_T(x) = \mathrm{tr}(l(x)^* T(l(x))), \qquad x \in \Gamma. \tag{29}$$

It is shown in [Haa 7] that u_T is in $l^2(\Gamma)$ (and so is in $A(\Gamma)$), and that

$$M_{u_T} = \rho\circ(T\otimes\mathrm{id}_M)\circ\pi. \tag{30}$$

Let $\{T_\alpha\}$ be a net in $F_\sigma(M)$ that converges in the stable point-weak* topology to id_M, and let $u_\alpha = u_{T_\alpha}$. Then since ρ and π are normal *-isomorphisms such that $\rho\circ\pi = \mathrm{id}_M$, it follows easily from (30) and Proposition 1.7 that $\{M_{u_\alpha}\}$ also converges to id_M in the stable point-weak* topology. Hence G has the AP by Remark 1.10.

(c) $\Rightarrow$ (a) Let $A = C_r^*(\Gamma)$. It is shown in [Haa 7] that for any $T \in F(A)$, (29) again defines a function in $l^2(\Gamma)$, and that (30) remains valid if we replace M_{u_T} by $\overline{M}_{u_T}$. Since A has the OAP, there is a net $\{T_\alpha\}$ in $F(A)$ such that $T_\alpha \to \mathrm{id}_A$ in the stable point-norm topology. Hence by Remark 1.8, for any von Neumann algebra N we have that

$$\langle T_\alpha\otimes\mathrm{id}_N(a), \varphi\rangle \to \langle a, \varphi\rangle \quad \forall\varphi \in (M\overline{\otimes}N)_*. \tag{31}$$

Setting $u_\alpha = u_{T_\alpha}$, it follows from the modified (30) and from (31) (applied to $N = M\overline{\otimes}B(\mathscr{H})$) that $\omega_{a,\varphi}(u_\alpha) \to \omega_{a,\varphi}(1)$ $\forall a \in A\otimes K(\mathscr{H})$ and $\forall\varphi \in (M\overline{\otimes}B(\mathscr{H}))_*$. Hence, by Proposition 1.5, $u_\alpha \to 1$ in the $\sigma(M_0A(\Gamma), Q(\Gamma))$-topology, and so Γ has the AP. □

A C^*-algebra A is said to be exact (cf. [Ki 2]) if for any exact sequence of C^*-algebras $0\to C\to D\to E\to 0$, the sequence of C^*-algebras $0\to A\otimes C\to A\otimes D\to A\otimes E\to 0$ is also exact. It is shown in [Ki 2] that a C^*-algebra is exact if and only if $(A, B(\mathscr{H}), K(\mathscr{H}))$ has the slice map property. Hence the general slice map property implies exactness, and so it follows from Theorem 2.1 that for C^*-algebras A of the form $A = C_r^*(\Gamma)$, the OAP implies exactness. On the other hand, Kirchberg has shown that if $A = \mathrm{cone}(C_r^*(SL(2,\mathbf{Z})))$, then A has an extension B by $K(\mathscr{H})$ which is not exact (cf. [Ki 4]). Since $SL(2,\mathbf{Z})$ is weakly amenable (cf. [DH]), $C_r^*(SL(2,\mathbf{Z}))$ has the general slice map property, and hence so does A. Thus A has the OAP. Since the OAP is preserved by extensions [Ki 5], B is an example of a C^*-algebra which has the OAP but is not exact. In particular, B does not have the strong OAP.

A class of C^*-algebras for which the OAP implies the general slice map property (and hence exactness) is the class of locally reflexive C^*-algebras, introduced in [EH]. For completeness, we give a proof of this result, which was announced in [ER]. (See also [Ki 5] for a different proof of Theorem 2.2.)

Theorem 2.2. *Let A be a locally reflexive C^*-algebra with the OAP. Then A has the strong OAP, and so has the general slice map property.*

Proof. Since A has the OAP, there is a net $\{T_\alpha\}$ in $F(A)$ such that $T_\alpha \to \mathrm{id}_A$ in the stable point-norm topology. Let B be a C^*-algebra. It suffices to show that

$$(32) \qquad \langle T_\alpha \otimes \mathrm{id}_B(a), \varphi\rangle \to \langle a, \varphi\rangle \quad \forall a \in A \otimes B, \forall \varphi \in (A \otimes B)^*.$$

Let $\varphi \in (A \otimes B)^*$. Since A is locally reflexive, φ has a unique extension to a continuous linear functional $\tilde{\varphi}$ on $A^{**} \otimes B$ (where we view A as a subalgebra of A^{**} in the usual way) which is σ-weakly continuous in the first variable (cf. [EH]). Let $M = A^{**}$, and let $E = A \hat{\otimes} M_*$. For each $c \in A \otimes B$, let $\tilde{\varphi}_c$ denote the linear functional on $CB(A, M)$ defined by

$$\langle T, \tilde{\varphi}_c\rangle = \langle T \otimes \mathrm{id}_B(c), \tilde{\varphi}\rangle \quad \forall T \in CB(A, M).$$

If $\tilde{\varphi}_c \in E$, then if follows from Remark 1.8 that

$$\langle T_\alpha \otimes \mathrm{id}_B(c), \varphi\rangle = \langle T_\alpha \otimes \mathrm{id}_B(c), \tilde{\varphi}\rangle = \langle T_\alpha, \tilde{\varphi}_c\rangle \to \langle \mathrm{id}_A, \tilde{\varphi}_c\rangle = \langle c, \varphi\rangle,$$

where we view id_A as an element of $CB(A, M)$. Hence it suffices to show that $\tilde{\varphi}_c \in E$ for all $c \in A \otimes B$. By the Krein-Smulian theorem, to show that $\tilde{\varphi}_c \in E$, it is enough to show that $\tilde{\varphi}_c$ is $\sigma(CB(A, M), E)$ continuous on the closed unit ball of $CB(A, M)$. So let $\{S_\beta\}$ be a net in $(CB(A, M))_1$ that converges in the $\sigma(CB(A, M), E)$-topology to an element S of $(CB(A, M))_1$. Then it follows from Remark 1.8 that $S_\beta(a) \to S(a)$ σ-weakly for all a in A. Hence, since $\tilde{\varphi}$ is σ-weakly continuous in the first variable,

$$\langle (S_\beta \otimes \mathrm{id}_B)(a \otimes b), \tilde{\varphi}\rangle = \langle S_\beta(a) \otimes b, \tilde{\varphi}\rangle \to \langle S(a) \otimes b, \tilde{\varphi}\rangle \quad \forall a \in A, \forall b \in B.$$

Since the net $\{S_\beta \otimes \mathrm{id}_B\}$ is bounded, it follows easily from this and from the definition of $\tilde{\varphi}_c$ that $\langle S_\beta, \tilde{\varphi}_c\rangle \to \langle S, \tilde{\varphi}_c\rangle$ for all $c \in A \otimes B$. This completes the proof. □

Remark 2.3. It is an open problem whether exactness implies the OAP. Since exactness implies locally reflexivity (cf. [Ki 3]), it follows from Theorem 2.2 that this problem is equivalent to the problem of whether exactness implies the general slice map property. It is a result due to Connes that if Γ is a discrete closed subgroup of a connected Lie group (in particular, if $\Gamma = SL(3, \mathbf{Z})$), then $C_r^*(\Gamma)$ is a subalgebra of a nuclear C^*-algebra, and so is exact (cf. [Ki 4]). (It is an open problem whether $C_r^*(\Gamma)$ is exact for all discrete groups Γ.) Hence if $SL(3, \mathbf{Z})$ fails to have the AP, then by Theorem 2.1 $C_r^*(SL(3, \mathbf{Z}))$ would provide an example of an exact C^*-algebra which does not have the OAP.

Now suppose that G is a second countable locally compact group, and that Γ is a lattice in G, i.e., a closed discrete subgroup of G for which G/Γ has a bounded G-invariant measure. Then the quotient map $\rho: G \to G/\Gamma$ has a Borel cross section. Let Ω be the range of a Borel cross section, and let μ_Γ be the counting measure on Γ. Define a function Φ on $M_0A(\Gamma)$ by

$$\Phi(u) = 1_\Omega * u\mu_\Gamma * \check{1}_\Omega, \qquad u \in M_0A(\Gamma),$$

where 1_Ω denotes the characteristic function of Ω. It is shown in [Haa 7] that Φ is a contraction from $M_0A(\Gamma)$ into $M_0A(G)$, and that Φ maps $A(\Gamma)$ into $A(G)$. Moreover, $\Phi(1_\Gamma) = 1_G$, and an argument similar to that in the last paragraph of the proof of Lemma 1.16 shows that Φ is $\sigma(M_0A(\Gamma), Q(\Gamma))$-$\sigma(M_0A(G), Q(G))$ continuous. Hence if Γ has the AP, then G has the AP. The converse is also true, by Proposition 1.14. Combining the results, we obtain:

Theorem 2.4. *Suppose G is a second countable locally compact group, and that Γ is a lattice in G. Then G has the AP if and only if Γ has the AP.*

Remark 2.5. As noted in the introduction, we do not know of any examples of locally compact groups without the AP. It is shown in [Haa 7] that if G is a noncompact simple Lie group with finite center and real rank ≥ 2, then G is not weakly amenable. The proof of this result is based on the following results: (1) every such Lie group has a closed subgroup with finite center and locally isomorphic to either $SL(3, \mathbf{R})$ or $Sp(2, \mathbf{R})$, (2) closed subgroups of weakly amenable groups are weakly amenable, (3) if G and H are locally isomorphic simple Lie groups with finite center then G is weakly amenable if and only if H is weakly amenable, and (4) $SL(3, \mathbf{R})$ and $Sp(2, \mathbf{R})$ are not weakly amenable. Closed subgroups of groups with the AP have the AP (Proposition 1.14) and it is easily verified that (3) remains valid if weak amenability is replaced by the AP. Hence if both $SL(3, \mathbf{R})$ and $Sp(2, \mathbf{R})$ fail to have the AP, then no noncompact simple Lie group with finite center and real rank ≥ 2 has the AP. Also note that it follows from Theorem 2.4 that $SL(3, \mathbf{R})$ fails to have the AP if and only if $SL(3, \mathbf{Z})$ fails to have the AP.

3. Crossed products and the approximation property

Let M be a von Neumann algebra, and let G be a locally compact group. If there is a homomorphism α from G to the group $\mathrm{Aut}(M)$ of *-automorphisms of M that is continuous with respect to the point-weak* topology on $\mathrm{Aut}(M)$, the triple (M, G, α) is called a W^*-dynamical system. If (M, G, α) is a W^*-dynamical system, we denote the associated crossed product von Neumann algebra by $M \otimes_\alpha G$.

The main results of this section are the following pair of theorems.

Theorem 3.1. *Let (M, G, α) be a W^*-dynamical system, and let $N = M \otimes_\alpha G$.*

(a) *If N has Property S_σ for a von Neumann algebra R, then M has Property S_σ for R.*
(b) *If N has the weak* CBAP, then so does M, and $\Lambda(M) \leq \Lambda(N)$.*

Theorem 3.2. *Let (M, G, α) be a W^*-dynamical system, and let $N = M \otimes_\alpha G$.*

(a) *If G has the AP and M has Property S_σ for a von Neumann algebra R, then N has Property S_σ for R.*
(b) *If G is weakly amenable and M has the weak* CBAP, then id_N is the limit in the point-weak* topology of a bounded net in $CB_\sigma(N)$ each of whose elements is the limit in the point-weak* topology of a bounded net in $F_\sigma(N)$.*
(c) *If G is amenable and M has the weak* CBAP, then N has the weak* CBAP and $\Lambda(N) \leq \Lambda(M)$.*

Theorems 3.1 and 3.2 are natural generalizations of the following two known results:

(1) If $N = M \otimes_\alpha G$ is semidiscrete, then so is M.

(2) If M is semidiscrete and G is amenable, then $N = M \otimes_\alpha G$ is semidiscrete.

Statement (2) follows from a result of Connes [Co, Proposition 6.8] combined with Wassermann's result in [Wa] that semidiscreteness is equivalent to injectivity for general von Neumann algebras. In the same paper Wassermann proved statement (1) in the case $G = \mathbf{R}$, but his proof (cf. [Wa, p. 46]) works without any changes for an arbitrary locally compact group G. Note that statements (1) and (2) can also be proved by making trivial modifications in the proofs of Theorem 3.1(b) and Theorem 3.2(c) given below.

The proofs of Theorems 3.1 and 3.2 make use of operator valued weights (cf. [Haa 4 and Haa 5]). If M is a von Neumann algebra, the extended positive part $\widehat{M}_+$ of M is the set of homogeneous, additive and lower semicontinuous functions on M_*^+ with values in $[0, \infty]$. M_+ can be regarded as a subset of $\widehat{M}_+$ in the obvious way. Suppose N is a von Neumann subalgebra of M. An operator valued weight T from M to N is a map of M_+ into $\widehat{N}_+$ that satisfies

(i) $T(\lambda x) = \lambda T(x)$, $\lambda \geq 0$, $x \in M_+$,
(ii) $T(x+y) = T(x) + T(y)$, $x, y \in M_+$,
(iii) $T(a^*xa) = a^*T(x)a$, $x \in M_+$, $a \in N$.

Moreover, T is said to be normal if

(iv) $x_i \nearrow x \Rightarrow T(x_i) \nearrow T(x)$, $x_i, x \in M_+$.

Put

$$n_T = \{x \in M \colon T(x^*x) \in N_+\},$$
$$m_T = n_T^* n_T = \operatorname{span}\{x^*y \colon x, y \in n_T\}.$$

It is easily verified that T has a unique linear extension $\dot{T}\colon m_T \to N$ which satisfies

(v) $\dot{T}(axb) = a\dot{T}(x)b$, $x \in m_T$, $a, b \in N$.

Note that if $T(1) = 1$, then $\dot{T}$ is a conditional expectation from M to N.

An operator valued weight from M to N is said to be faithful if $T(x^*x) = 0 \Rightarrow x = 0$, and is said to be semifinite if n_T is σ-weakly dense in M. The set of faithful, normal, semifinite weights from M to N is denoted by $P(M, N)$.

Lemma 3.3. *Let M and N be von Neumann algebras with $N \subset M$, let $T \in P(M, N)$, and let $a \in n_T$. Then there is a unique normal completely positive map S from M to N satisfying*

$$S(x) = T(a^*xa), \qquad x \in M_+.$$

Moreover,

$$S(bxc) = bS(x)c, \qquad b \in N \cap \{a^*\}', c \in N \cap \{a\}', x \in M, \tag{33}$$

and

$$\|S\|_{cb} = \|T(a^*a)\|.$$

Proof. Since $a \in n_T$, $a^*Ma \subset m_T$. Hence we can define S on M by $S(x) = \dot{T}(a^*xa)$, $x \in M$. Then S is normal and positive, (33) follows from (v), and the uniqueness of S is clear. Let $\mathscr{H}$ be a separable infinite dimensional Hilbert space. Then there is unique operator valued weight $T \otimes \mathrm{id}_{B(\mathscr{H})} \in P(M \overline{\otimes} B(\mathscr{H}), N \overline{\otimes} B(\mathscr{H}))$ satisfying

$$(\phi_1 \otimes \phi_2) \circ (T \otimes \mathrm{id}_{B(\mathscr{H})}) = (\phi_1 \circ T) \otimes (\phi_2 \circ \mathrm{id}_{B(\mathscr{H})}) \tag{34}$$

for all pairs of faithful, normal, semifinite weights ϕ_1 and ϕ_2 on N and $B(\mathscr{H})$ respectively (cf. [Haa 5, Theorem 5.5]). It follows easily from (34) that

$$T \otimes \mathrm{id}_{B(\mathscr{H})}((a \otimes 1)^*(a \otimes 1)) = T(a^*a) \otimes 1 \in N\overline{\otimes}B(\mathscr{H}),$$

and that if $\widetilde{S}$ is the unique normal positive map from $M\overline{\otimes}B(\mathscr{H})$ to $N\overline{\otimes}B(\mathscr{H})$ satisfying

$$\widetilde{S}(x) = T \otimes \mathrm{id}_{B(\mathscr{H})}((a \otimes 1)^* x (a \otimes 1)), \qquad x \in (M\overline{\otimes}B(\mathscr{H}))_+,$$

then

$$\widetilde{S}(x \otimes y) = S(x) \otimes y, \qquad x \in M, y \in B(\mathscr{H}). \tag{35}$$

Since $\widetilde{S}$ is positive, it follows from (35) that S is completely positive. Since S is completely positive, $\|S\|_{cb} = \|S(1)\| = \|T(a^*a)\|$. □

If M and R are von Neumann algebras, if E is a subset of $CB_\sigma(M)$, and if $x \in M\overline{\otimes}R$, then we denote the σ-weakly closed convex hull of $\{T \otimes \mathrm{id}_R(x) \colon T \in E\}$ by $C(x; E; R)$, and we denote the σ-weakly closed linear span of $\{T \otimes \mathrm{id}_R(x) \colon T \in E\}$ by $S(x; E; R)$. Note that if E is convex (resp., if E is a subspace), then $C(x; E; R)$ (resp., $S(x; E; R)$) is the σ-weak closure of $\{T \otimes \mathrm{id}_R(x) \colon T \in E\}$ for all x in $M\overline{\otimes}R$. The proof of the next lemma is similar to the proof of Proposition 2.1 in [Kr 3], and so is left to the reader.

Lemma 3.4. *Let M be a von Neumann algebra, let $\mathscr{H}$ be a separable infinite dimensional Hilbert space, and let E be a convex subset of $CB_\sigma(M)$. Suppose that $x \in C(x; E; B(\mathscr{H}))$ for all x in $M\overline{\otimes}B(\mathscr{H})$. Then there is a net $\{T_i\}$ in E such that $T_i \otimes \mathrm{id}_{B(\mathscr{H})}(x) \to x$ σ-weakly for all x in $M\overline{\otimes}B(\mathscr{H})$.*

Before proving Theorem 3.1, we recall some basic facts about crossed products, mainly to establish notation. Let (M, G, α) be a W^*-dynamical system. We may assume that M (acting on $\mathscr{H}$) is in standard form. Let $\alpha \to u_\alpha$ be the canonical unitary implementation of $\mathrm{Aut}(M)$ on $\mathscr{H}$ (cf. [Haa 1]), and put $u(t) = u_{\alpha_t}$ for all t in G. Then $t \to u(t)$ is a strongly continuous unitary representation of G on $\mathscr{H}$, and

$$\alpha_t(a) = u(t)au(t)^*, \qquad a \in M, t \in G.$$

We will identify $L^2(G, \mathscr{H})$ with $\mathscr{H} \otimes L^2(G)$ in the usual way. Let λ be the unitary representation of G on $L^2(G, \mathscr{H})$ defined by

$$(\lambda(s)\xi)(t) = \xi(s^{-1}t), \qquad \xi \in L^2(G, \mathscr{H}), s, t \in G,$$

and let π be the *-representation of M on $L^2(G, \mathscr{H})$ defined by

$$(\pi(a)\xi)(t) = \alpha_t^{-1}(a)\xi(t), \qquad a \in M, \xi \in L^2(G, \mathscr{H}), t \in G.$$

Then

$$\lambda(s) = 1 \otimes l(s), \qquad s \in G$$

(where $s \to l(s)$ is the left regular representation of G on $L^2(G)$),

$$\pi(\alpha_t(a)) = \lambda(t)\pi(a)\lambda(t^{-1}), \qquad a \in M, t \in G,$$

and $N = M \otimes_\alpha G$ is generated by $\pi(M)$ and $\lambda(G)$. Moreover,

$$\pi(a) = U^*(a \otimes 1)U, \qquad a \in M, \tag{36}$$

where U is the unitary operator on $L^2(G, \mathscr{H})$ defined by

$$(U\xi)(t) = ((u(t) \otimes 1)\xi)(t) = u(t)\xi(t), \qquad \xi \in L^2(G, \mathscr{H}), t \in G.$$

Note that since π is a *-isomorphism, in order to proof part (a) of Theorem 3.1 it suffices to show that if N has Property S_σ for R, then $\pi(M)$ has Property S_σ for R. It is shown in [Haa 3] that there is a faithful normal semifinite operator valued weight T from N to $\pi(M)$. When G is discrete this weight is a normal conditional expectation onto $\pi(M)$ (cf. [Haa 3, Corollary 3.7]) and hence $\pi(M)$ has Property S_σ (cf. [Kr 1, Proposition 1.19]). However, in the general case, $T(1)$ need not be in N, and more work is needed.

Let $K(G, M)$ denote the space of σ-strong* continuous functions from G to M with compact support. With the product

$$(x * y)(s) = \int_G \alpha_t(x(st))y(t^{-1})\,dt, \qquad x, y \in K(G, M), s \in G,$$

and involution

$$x^\sharp(t) = \Delta(t)^{-1}\alpha_t^{-1}(x(t^{-1})^*), \qquad x \in K(G, M), t \in G,$$

$K(G, M)$ is an involutive algebra, and the formula

$$\mu(x) = \int_G \lambda(t)\pi(x(t))\,dt$$

defines an involutive representation of $K(G, M)$ on $L^2(G, \mathscr{H})$ (cf. [Haa 2, Lemma 2.3]).

For $f \in L^1(G)$, let

$$f^*(t) = \Delta(t)^{-1}\overline{f(t^{-1})}, \qquad t \in G$$

(this is denoted by $f^\sharp$ in [Haa 2]). With this involution and the usual convolution product, $L^1(G)$ is an involutive algebra, and the formula

$$\lambda(f) = \int_G f(t)\lambda(t)\,dt, \qquad f \in L^1(G),$$

defines an involutive representation of $L^1(G)$ on $L^2(G, \mathscr{H})$ (cf. [Di, Chapter 13]). Moreover, $\lambda(f) \in N$ for all $f \in L^1(G)$, since $\lambda(t) \in N$ for all $t \in G$.

For $f \in C_c(G)$, let $x(f)$ denote the function from G to M given by

$$(x(f))(t) = f(t)1, \qquad t \in G.$$

Then it is easily checked that $x(f) \in K(G, M)$, that

$$\mu(x(f)) = \lambda(f), \tag{37}$$

and that

$$x(f^*) = x(f)^\sharp. \tag{38}$$

Note that it follows from (37) and (38) that

$$\lambda(f^*) = \mu(x(f)^\sharp). \tag{39}$$

Lemma 3.5. *Let T be the operator valued weight from $N = M \otimes_\alpha G$ to M constructed in* [Haa 3, *Theorem* 3.1]. *Then $\lambda(f) \in n_T$ for all $f \in C_c(G)$. Let T_f denote the unique completely positive map from N to M satisfying*

$$T_f(x) = T(\lambda(f)^* x \lambda(f)), \qquad x \in N_+.$$

Then

$$T_f(\pi(a)) = \int_G |f(t)|^2 \pi(\alpha_t^{-1}(a))\,dt \quad \forall a \in M\,, \forall f \in C_c(G). \tag{40}$$

Proof. Let $f \in C_c(G)$. It follows from (37) and [Haa 3, Theorem 3.1(c)] that $T(\lambda(f)^*\lambda(f)) = T(\mu(x(f))^*\mu(x(f))) = T(\mu(x(f)^\sharp * x(f))) = \pi((x(f)^\sharp * x(f))(e))$. Moreover, using [Haa 2, Lemma 2.3(b)] we get that

$$\begin{aligned}(x(f)^\sharp * x(f))(e) &= \int_G (x(f)(t))^* x(f)(t)\,dt = \int_G (f(t)1)^* f(t)1\,dt \\ &= \left(\int_G |f(t)|^2\,dt\right) 1 = \|f\|_2^2 1.\end{aligned}$$

Hence $\lambda(f) \in n_T$ and

$$\|T_f\|_{cb} = \|f\|_2^2. \tag{41}$$

Next fix $a \in M$ and $f \in C_c(G)$, and put $x(t) = f(t)\alpha_t^{-1}(a)$, $t \in G$. It is easily checked that $x \in K(G, M)$. Moreover,

$$\begin{aligned}\pi(a)\lambda(f) &= \int_G f(t)\pi(a)\lambda(t)\,dt = \int_G f(t)\lambda(t)\pi(\alpha_t^{-1}(a))\,dt \\ &= \int_G \lambda(t)\pi(x(t))\,dt = \mu(x).\end{aligned}$$

Hence (using [Haa 3, Theorem 3.1(c) and Haa 2, Lemma 2.3(b)]) we have that

(42)

$$\begin{aligned}T_f(\pi(a^*a)) &= T(\lambda(f)^*\pi(a)^*\pi(a)\lambda(f)) = T(\mu(x)^*\mu(x)) \\ &= T(\mu(x^\sharp * x)) = \pi((x^\sharp * x)(e)) = \pi\left(\int_G x(t)^* x(t)\,dt\right) \\ &= \pi\left(\int_G \overline{f(t)}\alpha_t^{-1}(a^*) f(t)\alpha_t^{-1}(a)\,dt\right) = \pi\left(\int_G |f(t)|^2 \alpha_t^{-1}(a^*a)\,dt\right).\end{aligned}$$

Since M is the linear span of its positive elements, (40) follows from (42). □

Proof of Theorem 3.1. (a) As noted above, it suffices to show that $\pi(M)$ has Property S_σ for R. Moreover, in order to show that $\pi(M)$ has Property S_σ for R, it suffices to show that $x \in S(x\,;F_\sigma(\pi(M))\,;R)$ $\forall x \in \pi(M)\overline{\otimes}R$ (cf. [Kr 3, Theorem 2.8]).

Let $\{f_i\}_{i\in I}$ be a net of nonnegative functions in $C_c(G)$ satisfying

$$\int_G f_i(t)\,dt = 1 \tag{43}$$

whose supports shrink to zero. Then

$$\lim_i \int_G f_i(t)g(t)\,dt = g(e) \quad \forall g \in C(G).$$

For each $i \in I$, let g_i denote the square root of f_i, and put $T_i = T_{g_i}$. Then using (40) we get that

$$\langle T_i(\pi(a))\,,\varphi\rangle = \int_G f_i(t)\langle \pi(\alpha_t^{-1}(a))\,,\varphi\rangle\,dt \quad \forall a \in M\,, \forall \varphi \in \pi(M)_*.$$

Hence

(44) $\lim_i \langle T_i(\pi(a)), \varphi\rangle = \langle \pi(\alpha_e^{-1}(a)), \varphi\rangle = \langle \pi(a), \varphi\rangle \quad \forall a \in M, \forall \varphi \in \pi(M)_*$.

It follows from (41) and (43) that $\|T_i\|_{cb} = 1$ for all $i \in I$, and so (44) implies that

(45) $$T_i \otimes \mathrm{id}_R(x) \to x \ \sigma\text{-weakly} \quad \forall x \in \pi(M)\overline{\otimes}R$$

(cf. [Kr 3, Proposition 2.9]).

Now let $x \in \pi(M)\overline{\otimes}R$. Then since $x \in N\overline{\otimes}R$, and since N has Property S_σ for R, there is a net $\{S_\lambda\}_{\lambda\in\Lambda}$ in $F_\sigma(N)$ such that

(46) $$\lim_\lambda \langle S_\lambda \otimes \mathrm{id}_R(x), \psi\rangle = \langle x, \psi\rangle \quad \forall \psi \in (N\overline{\otimes}R)_*$$

(cf. [Kr 3, Theorem 2.8]). Since for each i in I, $T_i \otimes \mathrm{id}_R$ is a σ-weakly continuous map from $N\overline{\otimes}R$ to $\pi(M)\overline{\otimes}R$, it follows from (46) that for each i in I we have that

(47) $$\lim_\lambda \langle (T_i \circ S_\lambda) \otimes \mathrm{id}_R(x), \varphi\rangle = \langle T_i \otimes \mathrm{id}_R(x), \varphi\rangle \quad \forall \varphi \in (\pi(M)\overline{\otimes}R)_*.$$

Finally, since the restriction of $T_i \circ S_\lambda$ to $\pi(M)$ is in $F_\sigma(\pi(M))$ for all $i \in I$ and for all $\lambda \in \Lambda$, (47) and (45) together imply that $x \in S(x; F_\sigma(\pi(M)); R)$. Since x is an arbitrary element of $\pi(M)\overline{\otimes}R$, this completes the proof of part (a).

(b) Since π is a *-isomorphism, $\Lambda(M) = \Lambda(\pi(M))$, and so it suffices to show that $\Lambda(\pi(M)) \le \Lambda(N)$. Let $E = \{S \in F_\sigma(\pi(M)) \colon \|S\|_{cb} \le \Lambda(N)\}$, and let $R = B(\mathscr{K})$, where $\mathscr{K}$ is a separable infinite dimensional Hilbert space. Since N has the weak* CBAP, it has Property S_σ for R, and it follows from the proof of part (a) (using the fact that the net $\{S_\lambda\}$ can be chosen such that $\|S_\lambda\|_{cb} \le \Lambda(N)$ for all $\lambda \in \Lambda$) that

(48) $$x \in C(x; E; R) \quad \forall x \in \pi(M)\overline{\otimes}R.$$

Since E is convex, it follows from (48) and Lemma 3.4 that $\Lambda(\pi(M)) \le \Lambda(N)$. □

The proof of Theorem 3.2 is along the same lines as that of Theorem 3.1, making use of the existence of an operator valued weight from $M\overline{\otimes}B(L^2(G))$ to N. Before proving Theorem 3.2, we establish some preliminary results.

Let ρ be the unitary representation of G on $L^2(G, \mathscr{H})$ defined by

$$(\rho(s)\xi)(t) = \Delta^{1/2}(s)\xi(ts), \qquad \xi \in L^2(G, \mathscr{H}), s, t \in G,$$

and let r be the right regular representation of G on $L^2(G)$, defined by

$$(r(s)f)(t) = \Delta^{1/2}(s)f(ts), \qquad f \in L^2(G), s, t \in G.$$

Then $\rho(s) = 1 \otimes r(s)$, $s \in G$.

Put $M_0 = M\overline{\otimes}B(L^2(G))$. Since we are identifying $L^2(G, \mathscr{H})$ with $\mathscr{H} \otimes L^2(G)$, M_0 can be viewed as a von Neumann subalgebra of $B(L^2(G, \mathscr{H}))$. Moreover, since $M'\overline{\otimes}1 \subset N'$ (cf. [Haa 2, Theorem 2.1]), $N \subset M_0$.

For each t in G, put $w(t) = u(t) \otimes r(t)$, and let β_t be the *-automorphism of M_0 defined by

$$\beta_t(x) = w(t)xw(t)^*, \qquad x \in M_0.$$

Then (M_0, G, β) is a W^*-dynamical system.

For each f in $L^\infty(G)$ and each s in G, let ${}_sf$ and f_s be the functions in $L^\infty(G)$ defined by

$$ {}_sf(t) = f(st) \quad \text{and} \quad f_s(t) = f(ts), \qquad t \in G. $$

Let ν be the *-representation of $L^\infty(G)$ on $L^2(G, \mathscr{H})$ defined by

$$ \nu(f) = 1 \otimes M_f, \qquad f \in L^\infty(G), $$

where M_f is the operator of multiplication by f on $L^2(G)$. Note that $\nu(f) \in M_0$ for all f in $L^\infty(G)$, and that

$$ (\nu(f)\xi)(t) = f(t)\xi(t) \quad \forall f \in L^\infty(G), \forall \xi \in L^2(G, \mathscr{H}), \forall t \in G. $$

The proof of the next lemma is straightforward, and is left to the reader.

Lemma 3.6. (a) *For any* $t \in G$

$$ w(t) = U^*\rho(t)U \quad \text{and} \quad U\lambda(t)U^* = u(t) \otimes l(t). $$

(b) *For any* $f \in L^\infty(G)$ *and any* $a \in M$

$$ \pi(a)\nu(f) = \nu(f)\pi(a). $$

(c) *For any* $f \in L^\infty(G)$ *and any* $t \in G$

$$ \lambda(t^{-1})\nu(f)\lambda(t) = \nu({}_tf) \quad \text{and} \quad \beta_t(\nu(f)) = \nu(f_t). $$

Lemma 3.7. $N = \{a \in M_0 \colon \beta_t(a) = a \ \forall t \in G\}$.

Proof. Put $(M_0)^\beta = \{a \in M_0 \colon \beta_t(a) = a \ \forall t \in G\}$. By Theorem 2.1 in [Haa 2], N' is generated by $M'\overline{\otimes}1$ and $\{U^*\rho(t)U \colon t \in G\} = \{w(t) \colon t \in G\}$. Hence

$$ \begin{aligned} N = N'' &= (M'\overline{\otimes}1)' \cap \{w(t) \colon t \in G\}' \\ &= M_0 \cap \{a \in B(L^2(G, \mathscr{H})) \colon w(t)aw(t)^* = a \ \forall t \in G\} \\ &= (M_0)^\beta. \quad \square \end{aligned} $$

Lemma 3.8. *For* $x \in (M_0)_+$, *let* $S(x)$ *be the function on* $(M_0)_*^+$ *defined by*

$$ (S(x))(\varphi) = \int_G \langle \beta_t(x), \varphi \rangle \, dt, \qquad \varphi \in (M_0)_*^+. $$

Then $S(x) \in \widehat{N}_+$ *for all* $x \in (M_0)_+$, *and* $S \in P(M_0, N)$. *Moreover,* $\nu(f) \in m_S$ *for all* f *in* $L^1(G) \cap L^\infty(G)$, *and*

$$ \dot{S}(\nu(f)) = \left(\int_G f(t)\, dt \right) 1 \quad \forall f \in L^1(G) \cap L^\infty(G). \tag{49} $$

Proof. The proof that $S(x) \in \widehat{N}_+$ for all $x \in (M_0)_+$, and that S is a faithful, normal operator valued weight from M_0 to $(M_0)^\beta = N$ is similar to the proof of Lemma 5.2 in [Haa 5], and so is left to the reader. Also note that (49) implies that $\nu(f) \in n_S$ for all $f \in C_c(G)$, and hence that S is semifinite, since the identity operator on $L^2(G, \mathscr{H})$ is in the strong closure of $\{\nu(f) \colon f \in C_c(G)\}$. Since $L^1(G) \cap L^\infty(G)$ is the linear span of its nonnegative elements, to complete the proof of the lemma it suffices to show that $\nu(f) \in m_S$ and that (49) is valid whenever f is a nonnegative function in $L^1(G) \cap L^\infty(G)$. So suppose $f \in L^1(G) \cap L^\infty(G)$, $f \geq 0$. Then M_f is a positive operator, and so

$\nu(f) = 1 \otimes M_f \in (M_0)_+$. For $\xi \in L^2(G, \mathscr{H})$, let ω_ξ be the positive vector functional in $(M_0)_*^+$ defined by

$$\langle a, \omega_\xi \rangle = \langle a\xi, \xi \rangle = \int_G \langle (a\xi)(t), \xi(t) \rangle \, dt, \qquad a \in M_0.$$

Then

$$\begin{aligned}(S(\nu(f)))(\omega_\xi) &= \int_G \langle \beta_t(\nu(f)), \omega_\xi \rangle \, dt = \int_G \langle \nu(f_t), \omega_\xi \rangle \, dt \\ &= \int_G \left(\int_G \langle (\nu(f_t)\xi)(s), \xi(s) \rangle \, ds \right) dt \\ &= \int_G \left(\int_G \langle f(st)\xi(s), \xi(s) \rangle \, ds \right) dt \\ &= \int_G \left(\int_G f(st) \langle \xi(s), \xi(s) \rangle \, dt \right) ds \\ &= \int_G \left\langle \left(\int_G f(st) \, dt \right) \xi(s), \xi(s) \right\rangle ds \\ &= \int_G \left\langle \left(\int_G f(t) \, dt \right) \xi(s), \xi(s) \right\rangle ds = \left\langle \left(\int_G f(t) \, dt \right) 1, \omega_\xi \right\rangle \\ &= \left(\left(\int_G f(t) \, dt \right) 1 \right) (\omega_\xi).\end{aligned}$$

It follows from the proof of Theorem 1.5 in [Haa 4] that two elements of $\widehat{N}_+$ that agree on the set of positive vector functional have the same spectral resolution, and so are equal. Hence $\nu(f) \in m_S$ and (49) holds. □

Lemma 3.9. *Let $f \in C_c(G)$. Then $\nu(f)^* \in n_S$. Let S_f denote the unique completely positive map from M_0 to N satisfying*

$$S_f(x) = S(\nu(f) x \nu(f)^*), \qquad x \in (M_0)_+,$$

and put $v = f * \tilde{f}$. Then

(a) $\|S_f\|_{cb} = \|f\|_2^2$.
(b) $S_f(\pi(a) x \pi(b)) = \pi(a) S_f(x) \pi(b) \ \ \forall x \in M_0, \ \forall a, b \in M$.
(c) $S_f(\pi(a)\lambda(t)) = v(t)\pi(a)\lambda(t) \ \ \forall a \in M, \ \forall t \in G$.

Proof. Since $\nu(f)\nu(f)^* = \nu(|f|^2)$ and since $|f|^2 \in L^1(G) \cap L^\infty(G)$, it follows from Lemma 3.8 that $\nu(f)^* \in n_S$ and that $\|S_f\|_{cb} = \|S_f(1)\| = \|f\|_2^2$. Moreover, (b) follows immediately from (33) and Lemma 3.6(b).

Next observe that for any $t \in G$ we have that

$$\nu(f)\lambda(t)\nu(f)^* = \nu(f)\nu({}_{t^{-1}}\overline{f})\lambda(t) = \nu(f({}_{t^{-1}}\overline{f}))\lambda(t). \tag{50}$$

Since $\lambda(t) \in N$, it follows from (50) and (49) that

$$\begin{aligned}S_f(\lambda(t)) &= \dot{S}(\nu(f({}_{t^{-1}}\overline{f}))\lambda(t)) = \dot{S}(\nu(f({}_{t^{-1}}\overline{f})))\lambda(t) \\ &= \left(\int_G f(s)\overline{f(t^{-1}s)} \, ds \right) \lambda(t) = \left(\int_G f(s)\tilde{f}(s^{-1}t) \, ds \right) \lambda(t) \\ &= v(t)\lambda(t).\end{aligned}$$

Hence for any $a \in M$ and any $t \in G$,

$$S_f(\pi(a)\lambda(t)) = \pi(a) S_f(\lambda(t)) = v(t)\pi(a)\lambda(t). \quad \square$$

Proof of Theorem 3.2. (a) Since Property S_σ for R is preserved by *-isomorphisms, it suffices to show that UNU^* has Property S_σ for R. Since N is the σ-weakly closed linear span of $\{\pi(a)\lambda(t)\colon a \in M,\ t \in G\}$, UNU^* is the σ-weakly closed linear span of $\{U\pi(a)\lambda(t)U^*\colon a \in M,\ t \in G\} = \{au(t)\otimes l(t)\colon a \in M,\ t \in G\}$ (where for the last equality we used (36) and Lemma 3.6(a)). Hence $UNU^* \subset B(\mathscr{H})\bar{\otimes}\mathrm{VN}(G)$.

For $f \in C_c(G)$, let $S_{f,U}$ be the normal completely positive map from UM_0U^* to UNU^* defined by

$$S_{f,U}(x) = US_f(U^*xU)U^*, \qquad x \in UM_0U^*.$$

Put $v = f * \tilde{f}$. Then $v \in A(G)$, and for all $a \in M$ and $t \in G$ we have that

$$\begin{aligned} S_{f,U}(au(t)\otimes l(t)) &= US_f(U^*(U\pi(a)\lambda(t)U^*)U)U^* \\ &= US_f(\pi(a)\lambda(t))U^* = Uv(t)\pi(a)\lambda(t)U^* \\ &= au(t)\otimes v(t)l(t) = \mathrm{id}_{B(\mathscr{H})}\otimes M_v(au(t)\otimes l(t)). \end{aligned} \tag{51}$$

Since UNU^* is the σ-weakly closed linear span of $\{au(t)\otimes l(t)\colon a \in M,\ t \in G\}$, it follows from (51) that

$$S_{f,U}(x) = \mathrm{id}_{B(\mathscr{H})}\otimes M_v(x) \quad \forall x \in UNU^*. \tag{52}$$

Let Φ be the unique *-isomorphism from $B(\mathscr{H})\bar{\otimes}\mathrm{VN}(G)\bar{\otimes}R$ to $\mathrm{VN}(G)\bar{\otimes} B(\mathscr{H})\bar{\otimes}R$ such that

$$\Phi(a\otimes b\otimes c) = b\otimes a\otimes c, \qquad a \in B(\mathscr{H}),\ b \in \mathrm{VN}(G),\ c \in R.$$

Since G has the AP, and since the linear span of $\{v\colon v = f*\tilde{f}$ for some $f \in C_c(G)\}$ is dense in $A(G)$ in the $B(G)$-norm, it follows easily from Remarks 1.10 and 1.2 and from Proposition 1.7 that for every $x \in B(\mathscr{H})\bar{\otimes}\mathrm{VN}(G)\bar{\otimes}R$ we have that

$$\Phi(x) \in S(\Phi(x)\,;\,\{M_v : v = f*\tilde{f} \text{ for some } f \in C_c(G)\}\,;\,B(\mathscr{H})\bar{\otimes}R),$$

and hence

$$x \in S(x\,;\,\{\mathrm{id}_{B(\mathscr{H})}\otimes M_v : v = f*\tilde{f} \text{ for some } f \in C_c(G)\},\,R) \tag{53}$$

for every $x \in B(\mathscr{H})\bar{\otimes}\mathrm{VN}(G)\bar{\otimes}R$. Combining (52) and (53) we get that

$$x \in S(x\,;\,\{S_{f,U} : f \in C_c(G)\}\,;\,R) \quad \forall x \in UNU^*\bar{\otimes}R. \tag{54}$$

Since M has Property S_σ for R, so does M_0 (cf. [Kr3, Proposition 3.4] for the case $\dim(R) = \infty$. The case $\dim(R)$ less than infinity is trivial, because in this case any von Neumann algebra has Property S_σ for R). Hence UM_0U^* also has Property S_σ for R. Thus

$$x \in S(x\,;\,F_\sigma(UM_0U^*)\,;\,R) \quad \forall x \in UM_0U^*\bar{\otimes}R \tag{55}$$

(cf. [Kr 3, Theorem 2.8]).

Let $x \in UNU^*\bar{\otimes}R$, and let $f \in C_c(G)$. Then since $S_{f,U}\otimes\mathrm{id}_R$ is σ-weakly continuous, and since the restriction of $S_{f,U}\circ T$ to UNU^* is in $F_\sigma(UNU^*)$ whenever $T \in F_\sigma(UM_0U^*)$, it follows from (55) that

$$S_{f,U}\otimes\mathrm{id}_R(x) \in S(x\,;\,F_\sigma(UNU^*)\,;\,R). \tag{56}$$

Combining (54) and (56) we get that $x \in S(x\,;\,F_\sigma(UNU^*)\,;\,R)\ \forall x \in UNU^*\bar{\otimes}R$, which implies (by Theorem 2.8 in [Kr 3]) that UNU^* has Property S_σ for R.

(b) Let $L = \Lambda(G)$, and let E denote the linear span of $\{f * \tilde{f} : f \in C_c(G)\}$. Since E is dense in $A(G)$ in the $B(G)$-norm, the argument in the third paragraph of the proof of Proposition 1.1 in [CH] (with $A_c(G)$ replaced by E) shows that there is a net $\{v_i\}_{i\in I}$ in E such that

$$\|v_i u - u\|_{A(G)} \to 0 \quad \forall u \in A(G), \tag{57}$$

and

$$\|v_i\|_{M_0} \leq L \quad \forall i \in I. \tag{58}$$

It follows from (57) and (58) that

$$\mathrm{id}_{B(\mathscr{H})} \otimes M_{v_i}(x) \to x \ \sigma\text{-weakly} \quad \forall x \in B(\mathscr{H}) \overline{\otimes} \mathrm{VN}(G) \tag{59}$$

(see the proof of the implication (a) $\Rightarrow$ (b) of Theorem 1.12).

Since each v_i is a linear combination of functions in $\{f * \tilde{f} : f \in C_c(G)\}$, it follows from (52) that for each $i \in I$ there is a normal completely bounded map T_i from UM_0U^* to UNU^* such that

$$T_i(x) = \mathrm{id}_{B(\mathscr{H})} \otimes M_{v_i}(x) \quad \forall x \in UNU^*. \tag{60}$$

For each $i \in I$, let S_i be the operator in $CB_\sigma(N)$ defined by

$$S_i(x) = U^* T_i(UxU^*)U, \qquad x \in N.$$

Then it follows from (60) that

$$\|S_i\|_{cb} \leq L \quad \forall i \in I,$$

and it follows from (59) (and the fact that $UNU^* \subset B(\mathscr{H}) \overline{\otimes} \mathrm{VN}(G)$) that

$$S_i(x) \to x \ \sigma\text{-weakly} \quad \forall x \in N. \tag{61}$$

Since M has the weak* CBAP, it follows from straightforward modifications of the proofs of Proposition 3.4 and Lemma 2.5(ii) in [EL] (replacing Stinespring's Theorem by the representation theorem for normal completely bounded maps in the proof of Lemma 2.5(ii)) that $\Lambda(M) \leq \Lambda(M_0) \leq \Lambda(M)\Lambda(B(L^2(G)))$. Since $\Lambda(B(L^2(G))) = 1$, $\Lambda(M) = \Lambda(M_0)$. Hence there is a net $\{T_\lambda\}_{\lambda\in\Lambda}$ in $F_\sigma(M_0)$ bounded by $\Lambda(M)$ such that

$$T_\lambda(x) \to x \ \sigma\text{-weakly} \quad \forall x \in M_0.$$

For each $i \in I$ and $\lambda \in \Lambda$, let $S_{i,\lambda}$ be the operator in $F_\sigma(N)$ defined by

$$S_{i,\lambda}(x) = U^* T_i(UT_\lambda(x)U^*)U, \qquad x \in N.$$

Then for each $i \in I$ we have that

$$\|S_{i,\lambda}\|_{cb} \leq \Lambda(M)\|T_i\|_{cb} \quad \forall \lambda \in \Lambda \tag{62}$$

and

$$S_{i,\lambda}(x) \to S_i(x) \ \sigma\text{-weakly} \quad \forall x \in N. \tag{63}$$

Hence id_N is the limit in the point-weak* topology of a bounded net $\{S_i\}$ in $CB_\sigma(N)$ each of whose elements is the limit in the point-weak* topology of a bounded net in $F_\sigma(N)$.

(c) If, in the proof of part (b), we make the stronger assumption that G is amenable, then it follows from the proof of Lemma 7.2 in [Lau], the density of

$C_c(G)$ in $L^2(G)$, Lemma (3.1) in [Ey], and the argument in the third paragraph of the proof of Proposition 1.1 in [CH] (with $A_c(G)$ replaced by $\{f * \tilde{f}: f \in C_c(G)$ and $\|f\|_2 = 1\})$ that the net $\{v_i\}$ can be chosen to be in $\{f * \tilde{f}: f \in C_c(G)$ and $\|f\|_2 = 1\}$. Hence (using Lemma 3.9), we can assume that the operators T_i are completely positive unital maps. Then

$$\|S_i\|_{cb} = 1 \quad \forall i \in I, \tag{64}$$

and it follows from (62) that

$$\|S_{i,\lambda}\|_{cb} \leq \Lambda(M) \quad \forall i \in I, \forall \lambda \in \Lambda. \tag{65}$$

Let $\mathscr{H}$ be a separable infinite dimensional Hilbert space. Then it follows from (64) and (61) that

$$S_i \otimes \mathrm{id}_{B(\mathscr{H})}(x) \to x \ \sigma\text{-weakly} \quad \forall x \in N\bar{\otimes}B(\mathscr{H})$$

and it follows from (65) and (63) that

$$S_{i,\lambda} \otimes \mathrm{id}_{B(\mathscr{H})}(x) \to S_i \otimes \mathrm{id}_{B(\mathscr{H})}(x) \ \sigma\text{-weakly} \quad \forall x \in N\bar{\otimes}B(\mathscr{H})$$

(cf. [Kr 3, Proposition 2.9]). Hence

$$x \in C(x; \{T \in F_\sigma(N): \|T\|_{cb} \leq \Lambda(M)\}; B(\mathscr{H})) \quad \forall x \in N\bar{\otimes}B(\mathscr{H}),$$

and so $\Lambda(N) \leq \Lambda(M)$ by Lemma 3.4. □

Remark 3.10. Suppose that G and H are locally compact groups, and that $\rho: G \to \operatorname{Aut} H$ is a continuous homomorphism. Then ρ induces an action α of G on $M = \mathrm{VN}(H)$ such that $M \otimes_\alpha G$ is unitarily equivalent to $\mathrm{VN}(H \times_\rho G)$ (cf. [Su, Proposition 2.2]). Thus combining Theorem 3.2(a) with Theorem 2.1 we get another proof of the fact that the semidirect product of two discrete groups with the AP has the AP. Moreover, it follows from Theorem 3.2(c) and the first author's characterization of weakly amenable discrete groups that if G is an amenable discrete group, and H is a weakly amenable discrete group, then $H \times_\rho G$ is weakly amenable. On the other hand, if $G = SL(2, \mathbf{Z})$, if $H = \mathbf{Z}^2$, and if ρ is the natural action of $SL(2, \mathbf{Z})$ on $\mathbf{Z}^2$, then G is weakly amenable (with $\Lambda(G) = 1$), $M = \mathrm{VN}(H)$ is semidiscrete, but $M \otimes_\alpha G$ does not have the weak* CBAP, since, as noted above, $\mathbf{Z}^2 \times_\rho SL(2, \mathbf{Z})$ is not weakly amenable. Thus the "two step weak* CBAP" in Theorem 3.2(b) cannot always be replaced by the weak* CBAP in the nonamenable case, even if we assume that $\Lambda(G) = 1$ and that M is semidiscrete. Note that it follows from the proof of Theorem 3.2(c) that if the net $\{T_i\}$ in the proof of Theorem 3.2(b) can be chosen to be bounded, then N has the weak* CBAP. Hence we can not always choose this net to be bounded, even though the net $\{S_i\}$ is always bounded.

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, ODENSE UNIVERSITY, CAMPUSVEG 55, DK-5230 ODENSE M, DENMARK
E-mail address: haagerup@imada.ou.dk

DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF NEW YORK AT BUFFALO, BUFFALO, NEW YORK, 14214-3093
E-mail address: mthjek@ubvms