Tôhoku Math. Journ.
27 (1975), 129-134.

APPROXIMATION PROPERTY OF FUNCTIONS AND ABSOLUTE CONVERGENCE OF FOURIER SERIES

(Dedicated to Professor G. Sunouchi in honor of his professorship)

CHINAMI WATARI AND YASUO OKUYAMA

(Received March 28, 1974)

Introduction. There are several classical criteria for the absolute convergence of (trigonometric) Fourier series. However, they have been proposed more or less independently, and their interrelations do not seem clear enough. The work of McLaughlin [1] is a trial in this direction, but, in spite of its length, it still leaves some gaps to be filled up. Our purpose here is to clarify the predominance of Bernstein's criterion (generalized by Steckin [3]); many other criteria then follow as its corollaries.

Our results are stated in terms of the best approximations; there is a rather complete parallelism between the modulus of continuity and the best approximation, and there is no limitation to the "goodness" of the behavior of a function.

1. Comparison Theorem. Consider an (L^1-) complete orthonormal system $\Phi = \{\varphi_n\}$ of bounded functions over a set of finite measure, and denote by Φ_n the linear space spanned by the first n elements of Φ. The system under consideration is postulated to have the following properties:

1° (Nikolsky property) For $1 \leqq p < q \leqq \infty$, and $P \in \Phi_n$ we have $\| P \|_q \leqq An^\alpha \| P \|_p$, $\alpha = (1/p) - (1/q)$. We denote by A a constant which may be different in different contexts.

2° (de la Vallee Poussin property) There exists a sequence of linear operators G_n: $L^1 \to \Phi_{2n}$ such that (i) bounded uniformly in n and p, (ii) G_n leaves the element of Φ_n invariant, (iii) For $1 \leqq p \leqq \infty$ we have

$$\| f - G_n f \|_p \leqq AE_n^{(p)}(f) = A \inf \{\| f - P \|_p ; P \in \Phi_n\} .$$

Observe that these properties are held by the system of trigonometric functions as well as that of Walsh functions (see for examples [3] and [9] for the former, [4] and [7] for the latter).

Our theorem now reads, indicating the conjugate exponent;

THEOREM 1. *Let Φ have the properties* 1° *and* 2° *and let* $1 \leqq p <$

$q < \infty$. *Then* $\sum_{n=1}^{\infty} n^{-1/q'} E_n^{(q)}(f) \leqq A \sum_{n=1}^{\infty} n^{-1/p'} E_n^{(p)}(f)$.

PROOF. By Cauchy's condensation principle, what we have to prove is $\sum_{n=0}^{\infty} 2^{n/q} E_{2^n}^{(q)}(f) \leqq A \sum_{n=0}^{\infty} 2^{n/p} E_{2^n}^{(p)}(f)$. This is reduced, by property 2°, to

$$\sum_{n=0}^{\infty} 2^{n/q} \| f - G_{2^n} f \|_q \leqq A \sum_{n=0}^{\infty} 2^{n/p} \| f - G_{2^n} f \|_p .$$

Or, by the subadditivity and the property 1°, we see

$$\| f - G_{2^n} f \|_q \leqq \sum_{k=n}^{\infty} \| G_{2^{k+1}} f - G_{2^k} f \|_q$$
$$\leqq A \sum_{k=n}^{\infty} 2^{k\alpha} \| G_{2^{k+1}} f - G_{2^k} f \|_p \leqq A \sum_{k=n}^{\infty} 2^{k\alpha} \| f - G_{2^k} f \|_p .$$

Multiply both sides by $2^{n/q}$ and sum up; the result follows upon interchanging the order of summations.

2. Absolute Convergence of Fourier Series. The well-known Bernstein-Steckin criterion says

$$\text{(B)} \qquad \sum_{n=1}^{\infty} n^{-1/2} E_n^{(2)}(f) < \infty \quad \text{implies} \quad \sum_{n=1}^{\infty} | c_n | < \infty$$

where c_n is the Fourier coefficient of f with respect to Φ. The result corresponding to the theorem of Zygmund (generalized by Salem) reads, with slight generalization and in our context,

$$\text{(Z)} \qquad E_n^{(1)}(f) = O(1/n) \quad \text{and} \quad \sum_{n=1}^{\infty} n^{-1} (E_n^{(\infty)}(f))^{1/2} < \infty$$
$$\text{together imply} \quad \sum_{n=1}^{\infty} | c_n | < \infty .$$

Though $E_n^{(1)}(f) = O(1/n)$ is weaker than the assumption that f be of bounded variation (which amounts to $f \in \mathrm{Lip}^{(1)} 1$) in trigonometric case, (Z) is included in (B) as seen through the following simple observation:

$$(E_{2n}^{(2)}(f))^2 \leqq \| f - G_n f \|_2^2 \leqq \| f - G_n f \|_1 \cdot \| f - G_n f \|_\infty$$
$$\leqq A E_n^{(1)}(f) E_n^{(\infty)}(f) \leqq (A/n) E_n^{(\infty)}(f) .$$

Consequently $\sum_{n=1}^{\infty} n^{-1/2} E_n^{(2)}(f) \leqq A \sum_{n=1}^{\infty} (1/n)(E_n^{(\infty)}(f))^{1/2} < \infty$.

The following proposition corresponds to the generalization of (B) proposed by Szász:

$$\text{(S)} \quad \sum_{n=1}^{\infty} n^{-1/p'} E_n^{(p)}(f) < \infty \quad \text{for some} \quad p \in [1, 2] \quad \text{implies} \quad \sum_{n=1}^{\infty} | c_n | < \infty .$$

But, our theorem given in § 1 shows that the case $p = 2$ is of widest applicability; that is, (S) is included in (B).

3. Weighted Absolute Convergence. We gather, with slight generalizations, some results corresponding to [8] 6.6.6. Besides the properties 1° and 2°, Φ is assumed to have the following property:

3° (Bernstein property) (i) For $P = \sum_{k=1}^{n} c_k\varphi_k \in \Phi_n$, write $P^{[\alpha]} = \sum_{k=1}^{n} k^\alpha c_k\varphi_k$. Then $\| P^{[\alpha]}\|_p \leqq An^\alpha \| P\|_p$ for $\alpha > 0$. (ii) For $f = \sum_{k=1}^{\infty} c_k\varphi_k \in L^p$ and $\alpha > 0$, we have $f^{[-\alpha]} = \sum_{k=1}^{\infty} k^{-\alpha}c_k\varphi_k \in L^p$ and $E_n^{(p)}(f^{[-\alpha]}) \leqq An^{-\alpha}E_n^{(p)}(f)$.

Observe that the property 3° also is the case for trigonometric system or the Walsh system ([5], [6]). The argument which follows is typical.

LEMMA 2. *Let* $f = \sum_{n=1}^{\infty} c_n\varphi_n \in L^p$ *and let* $E_n^{(p)}(f) = O(n^{-\alpha})$ *for some* $\alpha > 0$. *Then, for any* $\beta < \alpha$, *the series*

$$\sum_{k=0}^{\infty} (G_{2^{k+1}}f - G_{2^k}f)^{[\beta]}$$

converges in L^p *to a function (which is denoted by* $f^{[\beta]}$*) whose n-th Fourier coefficient is* $n^\beta c_n$ *and*

$$E_n^{(p)}(f^{[\beta]}) = O(n^{-\alpha+\beta}) .$$

PROOF. It suffices to estimate $E_{2^n}^{(p)}(f^{[\beta]})$. 2° (iii) gives

$$\|f - G_{2^k}f\|_p \leqq AE_{2^k}^{(p)}(f) \leqq A2^{-k\alpha} .$$

Applying the operator $G_{2^{k+1}}$ to the function $f - G_{2^k}f$, we see

$$\| G_{2^{k+1}}f - G_{2^k}f\|_p \leqq A \|f - G_{2^k}f\|_p .$$

Now $G_{2^{k+1}}f - G_{2^k}f$ is in $\Phi_{2^{k+2}}$ and property 3° yields (we may assume $\beta > 0$, for otherwise there is nothing to prove)

$$\| (G_{2^{k+1}}f - G_{2^k}f)^{[\beta]} \|_p \leqq A2^{k\beta} \| G_{2^{k+1}}f - G_{2^k}f\|_p \leqq A2^{-k(\alpha-\beta)} .$$

Thus the sequence $(G_{2^k}f)^{[\beta]}$; $k = 1, 2, \cdots$ converges in L^p (to $f^{[\beta]}$ say) and

$$E_{2^n}^{(p)}(f^{[\beta]}) \leqq A \|f^{[\beta]} - (G_{2^{n-1}}f)^{[\beta]}\|_p \leqq A2^{-n(\alpha-\beta)} .$$

The Fourier coefficients of $f^{[\beta]}$ is easily calculated by writing

$$f = (f - S_n) + S_n ,$$

where S_n is the n-th partial sum of the Fourier series of f.

Combining the criterion (B) with Lemma 2, we obtain the following propositions:

I. $E_n^{(2)}(f) = O(n^{-\alpha})$, $\alpha > 0$ *and* $\beta < \alpha$ *imply* $\sum n^{\beta-1/2} |c_n| < \infty$.

In fact, if $\beta > 1/2$, Lemma 2 gives $E_n^{(2)}(f^{[\beta-1/2]}) \leqq An^{-\alpha+\beta-1/2}$ while for $\beta \leqq 1/2$ the same estimate is assured by 3° (ii).

II. $E_n^{(1)}(f) \leqq A/n$ *and* $E_n^{(\infty)}(f) \leqq An^{-\alpha}$ *together imply, for* $(0 <)\beta < \alpha$, $\sum n^{\beta/2} |c_n| < \infty$.

For the proof, we estimate the best approximation to $f^{[\beta/2]}$ in two norms, $p=1$ and $p=\infty$, by means of Lemma 2. $E_n^{(2)}(f^{[\beta/2]})$ is then estimated as in § 2 and the criterion (B) applies.

III. *$E_n^{(p)}(f) \leqq An^{-\alpha}(\alpha>0)$, $1 \leqq p \leqq 2$, and $\beta < \alpha-(1/p)$ imply $\sum n^\beta |c_n| < \infty$.*

For (S) (and (B)) applies to $f^{[\beta]}$ through Lemma 2.

4. **Convergence of $\sum |c_n|^\beta$.** We are interested in the case of "small" β, that is, $\beta<2$ for $f\in L^2$ and $\beta<q'$ for $f\in L^q$ with $1\leqq q<2$. Theorem 1 is then generalized (partially but sufficiently for later applications) as follows:

THEOREM 3. *Let $1\leqq p<q\leqq 2$, $0<\beta<q'$. Then*

$$\sum_{n=1}^{\infty}(n^{-1/q'}E_n^{(q)}(f))^\beta \leqq A\sum_{n=1}^{\infty}(n^{-1/p'}E_n^{(p)}(f))^\beta ,$$

or equivalently by condensation,

$$\sum_{n=0}^{\infty}2^{n(1-\beta/q')}(E_{2^n}^{(q)}(f))^\beta \leqq A\sum_{n=0}^{\infty}2^{n(1-\beta/p')}(E_{2^n}^{(p)}(f))^\beta .$$

The proof of this theorem is based on the following lemma on scalar series of non-negative terms:

LEMMA 4. *Let $a_k\geqq 0$, $R_n\leqq\sum_{k=n}^{\infty}a_k$, $\rho>1$ and $0<\beta<\infty$. Then,*

$$\sum_{n=1}^{\infty}R_n^\beta\rho^n \leqq A\sum_{k=1}^{\infty}a_k^\beta\rho^k .$$

PROOF. The case $\beta<1$ is reduced to the simple case $\beta=1$ by Jensen's inequality. If $\beta>1$, Hölder's inequality with indices β, β' gives

$$R_n \leqq \sum_{k=n}^{\infty}a_k = \sum_{k=n}^{\infty}\rho^{-k/2\beta}\rho^{k/2\beta}a_k \leqq \Big(\sum_{k=n}^{\infty}\rho^{-k\beta'/2\beta}\Big)^{1/\beta'}\Big(\sum_{k=n}^{\infty}\rho^{k/2}a_k^\beta\Big)^{1/\beta}$$
$$\leqq A\rho^{-n/2\beta}\Big(\sum_{k=n}^{\infty}\rho^{k/2}a_k^\beta\Big)^{1/\beta} .$$

Thus

$$R_n^\beta \leqq A\rho^{-n/2}\sum_{k=n}^{\infty}\rho^{k/2}a_k^\beta .$$

Multiply both sides by ρ^n and sum up, and the result follows.

PROOF OF THEOREM 3. From the property 2°,

$$E_{2^n+1}^{(q)}(f) \leqq A\,\|f-G_{2^n}f\|_q \leqq A\sum_{k=n}^{\infty}\|G_{2^{k+1}}f-G_{2^k}f\|_q .$$

Take

$$a_k=\|G_{2^{k+1}}f-G_{2^k}f\|_q , \qquad R_n=\|f-G_{2^n}f\|_q .$$

We obtain, for $\rho = 2^{(1-\beta/q')} > 1$,

$$\sum_{n=0}^{\infty} 2^{n(1-\beta/q')}(E_{2^n}^{(q)}(f))^\beta \leqq A \sum_{n=0}^{\infty} 2^{n(1-\beta/q')} \| G_{2^{n+1}}f - G_{2^n}f \|_q^\beta .$$

Property 1° now shows that the last expression does not exceed

$$A \sum_{n=0}^{\infty} 2^{n(1-\beta/q')} 2^{n\beta((1/p)-(1/q))} \| G_{2^{n+1}}f - G_{2^n}f \|_p$$

$$\leqq A \sum_{n=0}^{\infty} 2^{n(1-\beta/p')} \| G_{2^{n+1}}f - G_{2^n}f \|_p$$

$$\leqq A \sum_{n=0}^{\infty} 2^{n(1-\beta/p')} \| f - G_{2^n}f \|_p \quad \text{again by property } 2°. \qquad \text{q.e.d.}$$

The simple argument leading to the Bernstein criterion (B) actually gives more:

(B^β) For $0 < \beta < 2$, $\sum_{n=1}^{\infty} (n^{-1/2} E_n^{(2)}(f))^\beta < \infty$ implies $\sum_{n=1}^{\infty} | c_n |^\beta < \infty$.

The only change needed in the proof is Hölder's inequality with indices $2/\beta$ and $2/(2-\beta)$ in place of Cauchy-Schwarz inequality.

Combining (B^β) with Theorem 3, we obtain

COROLLARY 5. *Let* $1 \leqq p \leqq 2$ *and* $0 < \beta < 2$. *Then*

$$\sum_{n=1}^{\infty} (n^{-1/p'} E_n^{(p)}(f))^\beta < \infty \quad \textit{implies} \quad \sum_{n=1}^{\infty} | c_n |^\beta < \infty .$$

REMARK. The usual assumption $E_n^{(p)}(f) = O(n^{-\alpha})$ (corresponding to $f \in \mathrm{Lip}^{(p)}\alpha$) with $\beta > p/(p(1+\alpha) - 1)$, that is, $\alpha > (1/\beta) - (1/p')$ is more stringent than that of Corollary 5. With this strong hypothesis, the conclusion may be derived, by Hölder's inequality, from III of § 3.

The convergence of the series $\sum_{n=1}^{\infty} n^\gamma | c_n |^\beta$ may be treated along the same line; for example, for $1 \leqq p \leqq 2$ and $0 < \beta < p'$,

$$\sum_{n=1}^{\infty} (n^{\delta-1/p'} E_n^{(p)}(f))^\beta < \infty \quad \text{implies} \quad \sum_{n=1}^{\infty} (n^{-1/p'} E_n^{(p)}(f^{[\delta]}))^\beta < \infty ,$$

and Corollary 5 assures, for $0 < \beta < 2$, the convergence of $\sum_{n=1}^{\infty} (n^\delta | c_n |)^\beta$.

REFERENCES

[1] J. R. McLAUGHLIN, Absolute convergence of series of Fourier coefficients, Trans. Amer. Math. Soc., 184 (1973), 291-316.

[2] S. M. NIKOLSKY, Inequalities for entire functions of finite type and their applications in the theory of differentiable functions of several variables, Trudy Mat. Inst. Steklov, 38 (1951), 244-278.

[3] S. B. STECKIN, Absolute convergence of orthogonal series, Uspehi Mat. Nauk, 2 (1947), 177-178.
[4] C. WATARI, Best approximation by Walsh polynomials, Tôhoku Math. J., 15 (1963), 1-5.
[5] C. WATARI, A note on saturation and best approximation, Tôhoku Math. J., 15 (1963), 273-276.
[6] C. WATARI, Multipliers for Walsh Fourier series, Tôhoku Math. J., 16 (1964), 239-251.
[7] C. WATARI, Approximation of functions by a Walsh-Fourier series, Proceedings of applications of Walsh functions, (1970), 166-169.
[8] A. ZYGMUND, Trigonometrical series, Warszawa, 1935.
[9] A. ZYGMUND, Trigonometric series, vol. I, (pp. 115 and 154) Cambridge, 1959.

DEPARTMENT OF MATHEMATICS
COLLEGE OF GENERAL EDUCATION
TÔHOKU UNIVERSITY
KAWAUCHI, SENDAI, JAPAN.
AND
DEPARTMENT OF MATHEMATICS
FACULTY OF ENGINEERING
SHINSHU UNIVERSITY
NAGANO, JAPAN.