GEOPHYSICS, VOL. 64, NO. 6 (NOVEMBER-DECEMBER 1999); P. 1920–1927, 4 FIGS., 1 TABLE.

Approximations to the Zoeppritz equations and their use in AVO analysis

Yanghua Wang*

ABSTRACT

To efficiently invert seismic amplitudes for elastic parameters, pseudoquartic approximations to the Zoeppritz equations are derived to calculate P-P-wave reflection and transmission coefficients as a function of the ray parameter p. These explicit expressions have a compact form in which the coefficients of the p^2 and p^4 terms are given in terms of the vertical slownesses. The amplitude coefficients are also represented as a quadratic function of the elastic contrasts at an interface and are compared to the linear approximation used in conventional amplitude variation with offset (AVO) analysis, which can invert for only two elastic parameters. Numerical analysis with the second-order approximation shows that the condition number of the Fréchet matrix for three elastic parameters is improved significantly from using a linear approximation. Therefore, those quadratic approximations can be used directly with amplitude information to estimate not only two but three parameters: P-wave velocity contrast, S-wave velocity contrast, and the ratio of S-wave and P-wave velocities at an interface.

INTRODUCTION

Amplitude variation with offset (AVO) analysis has been used extensively for lithology and fluid prediction in many regions (e.g., Ostrander, 1984; Rutherford and Williams, 1989). In conventional AVO inversion one assumes that all offset-dependent amplitude effects, other than the reflection coefficient, are corrected. In this case, contrasts in elastic parameters could be estimated theoretically from reflection amplitudes. Apart from the lithology identification, another area of active research focuses on the use of amplitude information in tomographic inversion to determine subsurface structure and elastic properties (e.g., Wang and Houseman, 1995; Wang and Pratt, 1997). In this case, not only the reflection and transmission coefficients at interfaces but also the structural effects must be taken into account.

Both AVO analysis and tomographic amplitude inversion depend on the Zoeppritz equations. In conventional AVO analysis, various linear approximations of the Zoeppritz equations with respect to the elastic contrasts at an interface are used (Bortfeld, 1961; Chapman, 1976; Richards and Frasier, 1976; Aki and Richards, 1980; Shuey, 1985). In tomographic inversion, the exact Zoeppritz equations are usually adopted to calculate reflection and transmission coefficients. To invert efficiently for elastic parameters, however, approximations to the Zoeppritz equations with relatively higher accuracy are desirable in the practice of tomographic amplitude inversion. In this paper I derive quadratic expressions for the P-P-wave reflection and transmission coefficients, R_{PP} and T_{PP}, with respect to the relative contrast in elastic parameters. An immediate advantage of using these quadratic approximations, rather than the exact Zoeppritz equations, in the tomographic inversion is that the sensitivity matrix of Fréchet derivatives of amplitudes with respect to elastic parameters can be computed analytically, not necessary numerically.

The quadratic approximations of amplitude coefficients with respect to the elastic contrasts at an interface are converted from the so-called pseudoquadratic expressions with respect to the ray parameter p. The latter is referred to as pseudo-formulae because the coefficients of the p^2 (and p^4) terms are defined as a function of vertical slownesses, which also depend on the ray parameter. Ursin and Dahl (1992) developed actual quadratic approximations of the Zoeppritz equations as a function of the ray parameter. In their approximations, the coefficients of the Taylor series are computed directly from the medium parameters (not from the vertical slownesses I use). However, an explicit expression of the fourth-order term as a function of the medium parameters is very cumbersome and therefore is implemented only in a computer code. In this paper, I provide alternative approximations to the Zoeppritz equations, with explicit coefficients of both the p^2 and p^4 terms, which are in a fairly compact format. A similar

Manuscript received by the Editor June 30, 1997; revised manuscript received March 30, 1998.
*Formerly Imperial College of Science, Technology and Medicine, University of London; presently Robertson Research Australia Pty. Ltd., 69 Outram Street, West Perth, Western Australia 6005, Australia. E-mail: yanghua@robres.com.au.

pseudoquadratic approximation for the P-P reflection coefficient R_{PP} is also derived by Mallick (1993). This paper not only provides a different derivation of the R_{PP} expression but also extends the derivation to T_{PP}.

The quadratic expressions for the P-P-wave reflection and transmission coefficients as a function of the elastic contrasts at the interface are compared with the previously used linear formulae. Numerical analysis demonstrates that the quadratic approximations are more accurate than the previous linear approximate formulae. In using the quadratic approximation of the P-P-wave reflection coefficient, one can potentially estimate the following three elastic parameters simultaneously: relative P-wave velocity contrast, $\Delta\alpha/\alpha$; relative S-wave velocity contrast, $\Delta\beta/\beta$; and the ratio of average S-wave to average P-wave velocities, β/α.

THE PSEUDO-p^2 EXPRESSIONS

Consider a horizontal interface separating two half-space media, in which the P-wave velocities are, respectively, α_1 and α_2, the SV-wave velocities are β_1 and β_2, and the densities are ρ_1 and ρ_2. For a plane wave propagating through the media, the ray parameter p is constant. In Appendix A, approximations to the Zoeppritz equations are represented in a pseudoquartic form with respect to the ray parameter p, in which the coefficients of the p^2 and p^4 terms are defined in terms of vertical slownesses.

Truncating at the p^2 term, we have the pseudoquadratic formulae of the P-P-wave reflection/transmission coefficients,

$$R_{PP}(p) \approx R_f - 2\frac{\Delta\mu}{\rho}p^2 + (1-R_f)q_\alpha q_\beta\left(\frac{\Delta\mu}{\rho}\right)^2 p^2 \qquad (1)$$

and

$$T_{PP}(p) \approx T_f\frac{q_{\alpha_1}\alpha_1}{q_{\alpha_2}\alpha_2}\left[1 - q_\alpha q_\beta\left(\frac{\Delta\mu}{\rho}\right)^2 p^2\right], \qquad (2)$$

with

$$R_f = \frac{\rho_2 q_{\alpha_1} - \rho_1 q_{\alpha_2}}{\rho_2 q_{\alpha_1} + \rho_1 q_{\alpha_2}} \qquad (3)$$

and

$$T_f = 1 - R_f, \qquad (4)$$

where q_{α_1} and q_{α_2} are the P-wave vertical slownesses; q_α is their average; q_β is correspondingly the average quantity of the SV-wave vertical slownesses; ρ is the average of bulk densities; and $\Delta\mu$ is the difference in the shear moduli. In equation (3) R_f can be understood as the fluid-fluid reflection coefficient, which is the reflection coefficient between two media when the corresponding shear-wave velocities in both media are set to zero; T_f in equation (4) is, correspondingly, the fluid-fluid transmission coefficient. A similar formula for the P-P reflection coefficient [equation (1)] was derived by Mallick (1993), but the $\left[-R_f\, q_\alpha q_\beta(\Delta\mu/\rho)^2\right] p^2$ term was omitted in that derivation. In the following, I refer to the P-P coefficients in equations (1) and (2) as the pseudo-p^2 expressions.

QUADRATIC EXPRESSIONS IN TERMS OF ELASTIC CONTRASTS

In this section, I convert the pseudo-p^2 approximations into quadratic expressions with respect to elastic contrasts along the reflection interface. In conventional AVO analysis, one attempts to reveal the contrasts in elastic reflectivities and does not expect to determine absolute values of elastic parameters.

At an interface Snell's law holds:

$$p = \frac{\sin\theta_1}{\alpha_1} = \frac{\sin\theta_2}{\alpha_2}, \qquad (5)$$

where θ_1 and θ_2 are incidence and transmission angles, associated with the P-wave velocities α_1 and α_2. Denoting the average of θ_1 and θ_2 by θ and the difference between them by $\Delta\theta$, we have

$$\tan\frac{\Delta\theta}{2} \approx \frac{1}{2}\frac{\Delta\alpha}{\alpha}\tan\theta, \qquad (6)$$

where α is the average of the P-wave velocities α_1 and α_2, and $\Delta\alpha$ is their difference. Using equation (6), I rewrite the fluid-fluid reflection coefficient [equation (3)] as

$$R_f \approx \frac{1}{2}\left(\frac{\Delta\rho}{\rho} + \sec^2\theta\frac{\Delta\alpha}{\alpha}\right), \qquad (7)$$

which is now in terms of the relative contrasts of density and P-wave velocity. Note that both equations (6) and (7) are accurate up to a second order in the relative contrasts, which are zero valued as shown in Appendix B. In addition, the following expression is approximated up to the p^2 term:

$$\frac{q_{\alpha_1}\alpha_1}{q_{\alpha_2}\alpha_2} \approx 1 + \alpha\Delta\alpha\, p^2. \qquad (8)$$

The contrast in shear modulus can be read as

$$\Delta\mu = \beta^2\Delta\rho + 2\rho\beta\Delta\beta + \frac{1}{4}\Delta\rho(\Delta\beta)^2, \qquad (9)$$

where β and $\Delta\beta$ are the S-wave velocity average and difference, respectively. Substituting equations (6)–(9) into equations (1) and (2), I obtain the P-P reflection/transmission coefficients:

$$\begin{aligned} R_{PP} \approx {} & \left[\frac{1}{2} - 2\left(\frac{\beta}{\alpha}\right)^2\sin^2\theta\right]\frac{\Delta\rho}{\rho} \\ & + \frac{1}{2}\sec^2\theta\frac{\Delta\alpha}{\alpha} - 4\left(\frac{\beta}{\alpha}\right)^2\sin^2\theta\frac{\Delta\beta}{\beta} \\ & + \left(\frac{\beta}{\alpha}\right)^3\cos\theta\sin^2\theta\left(\frac{\Delta\rho}{\rho} + 2\frac{\Delta\beta}{\beta}\right)^2 \end{aligned} \qquad (10)$$

and

$$\begin{aligned} T_{PP} \approx {} & 1 - \frac{1}{2}\frac{\Delta\rho}{\rho} + \frac{1}{2}(\tan^2\theta - 1)\frac{\Delta\alpha}{\alpha} \\ & - \left(\frac{\beta}{\alpha}\right)^3\cos\theta\sin^2\theta\left(\frac{\Delta\rho}{\rho} + 2\frac{\Delta\beta}{\beta}\right)^2, \end{aligned} \qquad (11)$$

where both equations (10) and (11) are quadratic approximations with respect to relative contrasts in three elastic parameters: the bulk density, the P-wave velocity, and the S-wave velocity. In the following section, I compare the accuracy of these approximations with those of previously published linear approximations.

ACCURACY OF THE APPROXIMATIONS

If we ignore the terms containing $(\Delta\rho/\rho + 2\Delta\beta/\beta)^2$ in equations (10) and (11), we obtain the expressions

$$R_{PP}(\theta) \approx \left[\frac{1}{2} - 2\left(\frac{\beta}{\alpha}\right)^2 \sin^2\theta\right]\frac{\Delta\rho}{\rho} + \frac{1}{2}\sec^2\theta\,\frac{\Delta\alpha}{\alpha} - 4\left(\frac{\beta}{\alpha}\right)^2 \sin^2\theta\,\frac{\Delta\beta}{\beta} \tag{12}$$

and

$$T_{PP}(\theta) \approx 1 - \frac{1}{2}\frac{\Delta\rho}{\rho} + \frac{1}{2}\left(\tan^2\theta - 1\right)\frac{\Delta\alpha}{\alpha}, \tag{13}$$

which are linear in each of the three elastic contrast terms. These expressions are equivalent to the approximations previous published in Bortfeld (1961), Richards and Frasier (1976), and Aki and Richards (1980). Shuey (1985) further modifies equation (12) by replacing β and $\Delta\beta$ with σ and $\Delta\sigma$, where σ is Poisson's ratio.

To see the differences between alternative approximations, the *P-P* reflection and transmission coefficients for four examples are shown in Figures 1 and 2. These examples represent shale/sand, shale/limestone (or dolomite), anhydrite/sand, and anhydrite/limestone (or dolomite) interfaces. The parameters for these sedimentary minerals are listed in Table 1. In the examples, some reasonable worst cases, such as shale/limestone (or dolomite) and evaporite (anhydrite)/sandstone, are included to demonstrate the accuracy of the approximations.

The *P-P* reflection and transmission coefficients shown in Figures 1 and 2 are obtained from the exact Zoeppritz equations [equations (A-1) and (A-2)], the quadratic approximations [equations (10) and (11)], and the linear approximations [equations (12) and (13)]. As expected, the quadratic approximations in general have better accuracy than linear ones. For the shale/sand and shale/limestone (or dolomite) cases, the quadratic approximations provide very good results. For the worst case with an evaporite (Figures 1c and 1d), the quadratic expression for the reflection coefficient achieves good accuracy for small and intermediate p values. When the incidence angle approaches the critical angle, there is a considerable difference in the reflection coefficient between the exact calculation and the quadratic approximation.

From Figure 1 we see that ignoring the term containing $(\Delta\rho/\rho+2\Delta\beta/\beta)^2$ in equation (10) causes a significant error, and reflection amplitudes become more negative in these examples. From Figure 2 we see that ignoring this term also causes the absolute amplitude of the transmission coefficient to be overestimated. For a multilayered structure, the cumulative error in the amplitude estimate will be significant. The errors in linear approximations arise from assuming that the contrasts of all elastic parameters are small relative to their averages.

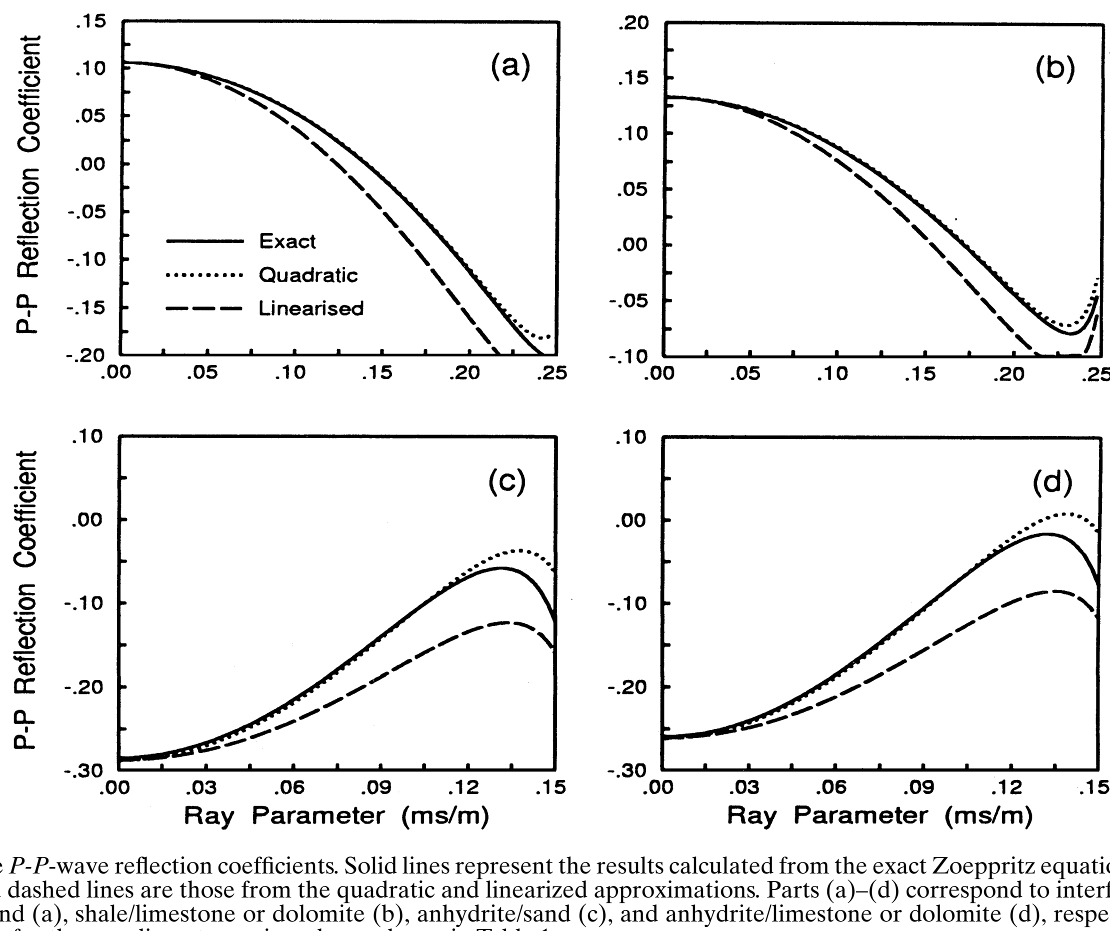

FIG. 1. The *P-P*-wave reflection coefficients. Solid lines represent the results calculated from the exact Zoeppritz equation, whereas dotted and dashed lines are those from the quadratic and linearized approximations. Parts (a)–(d) correspond to interface models of shale/sand (a), shale/limestone or dolomite (b), anhydrite/sand (c), and anhydrite/limestone or dolomite (d), respectively. The parameters for those sedimentary minerals are shown in Table 1.

ELASTIC PARAMETERS FROM AMPLITUDE INVERSION

When using the linearized approximation of the reflection coefficient R_{PP} [equation (12)] in the inversion, it is difficult to estimate more than two elastic parameters (Stolt and Weglein, 1985; de Nicolao et al., 1993; Ursin and Tjåland, 1993). In contrast, Ursin and Tjåland (1996) show that, in using the exact Zoeppritz equations, up to three parameters could be estimated from precritical P-P reflection coefficients. In this section, we will see the capacity of the quadratic approximation [equation (10)] for the estimation of three parameters, which are essential in the description of the elastic properties of a reflector.

Let us first modify equations (10) and (11) to write the coefficients as a function of three elastic parameters. A simple systematic relationship exists between the P-wave velocity and the bulk density of many sedimentary rocks (Gardner et al., 1974). As a result, reflection and transmission coefficients can be estimated satisfactorily from velocity information alone. As shown by Gardner et al. (1974), the density ρ in sedimentary rocks may often be considered proportional to the fourth root of velocity α,

$$\rho \approx 0.31\alpha^{1/4}, \tag{14}$$

where the units for α and ρ are m/s and g/cm^3, respectively. Equation (14) is representative of a large number of laboratory and field observations of different brine-saturated rock types (excluding evaporites). This empirical relationship leads to

$$\frac{\Delta\rho}{\rho} \approx \frac{1}{4}\frac{\Delta\alpha}{\alpha}. \tag{15}$$

Substituting equation (15) into equations (10) and (11) gives

$$\begin{aligned} R_{PP} \approx {} & \left[\frac{5}{8} + \frac{1}{2}\tan^2\theta - \frac{1}{2}\left(\frac{\beta}{\alpha}\right)^2 \sin^2\theta\right]\frac{\Delta\alpha}{\alpha} \\ & - 4\left(\frac{\beta}{\alpha}\right)^2 \sin^2\theta\,\frac{\Delta\beta}{\beta} \\ & + \left(\frac{\beta}{\alpha}\right)^3 \cos\theta\sin^2\theta\left(\frac{1}{4}\frac{\Delta\alpha}{\alpha} + 2\frac{\Delta\beta}{\beta}\right)^2 \end{aligned} \tag{16}$$

Table 1. Model parameters for the calculation of reflection/transmission coefficients.

Material	ρ (g/cm^3)	α (m/s)	β (m/s)	σ
Sand	2.65	3780	2360	0.18
Limestone	2.75	3845	2220	0.25
Shale	2.25	3600	1585	0.38
Anhydrite	2.95	6095	3770	0.19

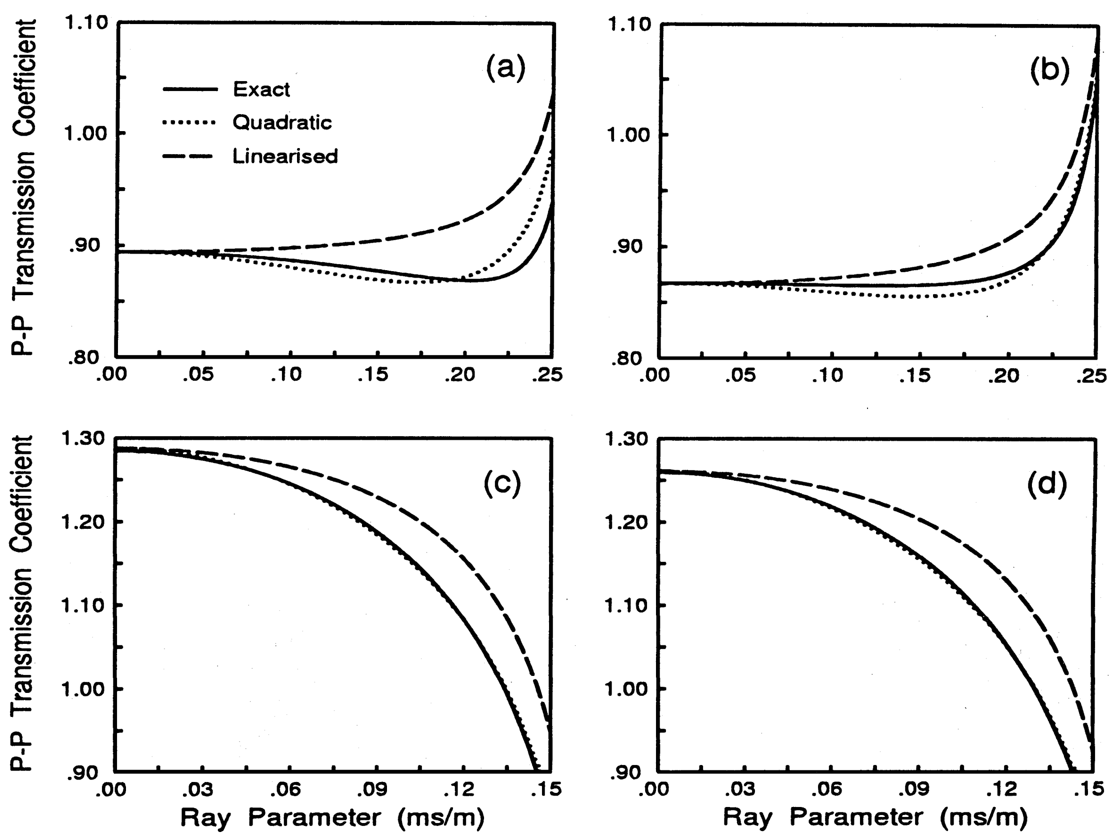

FIG. 2. The P-P-wave transmission coefficients. Solid lines represent the results calculated from the exact Zoeppritz equation, whereas dotted and dashed lines are results obtained from the quadratic approximation and the linearized approximation. Parts (a)–(d) correspond to interface models of shale/sand (a), shale/limestone or dolomite (b), anhydrite/sand (c), and anhydrite/limestone or dolomite (d), respectively. The parameters for those sedimentary minerals are shown in Table 1.

and

$$T_{PP} \approx 1 - \left(\frac{5}{8} - \frac{1}{2}\tan^2\theta\right)\frac{\Delta\alpha}{\alpha} - \left(\frac{\beta}{\alpha}\right)^3 \cos\theta \sin^2\theta \left(\frac{1}{4}\frac{\Delta\alpha}{\alpha} + 2\frac{\Delta\beta}{\beta}\right)^2. \quad (17)$$

We now have the formulae for the reflection and transmission coefficients expressed in terms of three parameters: the contrasts $\Delta\alpha/\alpha$ and $\Delta\beta/\beta$, and the ratio of S-to-P wave average velocities, β/α. In the application of equations (16) and (17), departures from the systematic relationship between velocity and density equations (14) and (15) exist and may in some cases be significant, although the reflection and transmission coefficients in general are not sensitive to the perturbation of the bulk density.

We then consider an inverse problem using reflection amplitude data. A linearized inverse problem can be represented as

$$\delta\mathbf{d} = \mathbf{F}\delta\mathbf{m}, \quad (18)$$

where $\delta\mathbf{d}$ is the data residue, $\mathbf{F}$ is the sensitivity matrix of Fréchet derivatives of the model responses with respect to model parameters, and $\delta\mathbf{m}$ is a small model perturbation so that the linear map between the model and the data perturbations is valid. In the numerical analysis here, the data space ranges from vertical incidence to a maximum ray parameter $p_{\max}$, which is supposed to be smaller than that at the critical incidence angle. The data are sampled uniformly over p; this assumption simplifies computation because the data do not depend on the interface depth in a specified model. The Fréchet matrix is evaluated around the solution point of the shale/sand interface model given in Table 1.

In the inversion with a linearized approximation to the reflection coefficient, the difficulty of estimating more than two elastic parameters arises because of ill conditioning of $\mathbf{F}$. The matrix $\mathbf{F}$ for the three-parameter inverse problem has a very large condition number, where the condition number of a matrix is defined as the ratio of the maximum eigenvalue to the minimum eigenvalue.

The eigenvalues of the Fréchet matrix versus the maximum ray parameter $p_{\max}$ considered are shown in Figure 3, where (a) depicts the result using the quadratic approximation of the P-P reflection coefficient and (b) corresponds to the result of the linearized R_{PP} approximation. With the linearized approximation, the third eigenvalue is too small, close or less than machine epsilon. In Figure 3b a dashed line represents an example truncating value applied in numerical calculation.

The condition numbers for the cases of the quadratic formula and the linear approximation are given in Figure 4. For all $p_{\max}$, the condition number for the quadratic case is smaller than the condition number for the linear case, and the condition number for the linear case is larger than 80 db (dashed line). In the numerical computation, if the reciprocal condition number (reconum) of a matrix is small enough so that the logical expression $\{1.0 + \text{reconum} == 1.0\}$ is true, then the matrix is regarded as singular to working precision.

Most previously published works (e.g., de Nicolao et al., 1993; Ursin and Tjåland, 1993) in linearized inversions of reflection coefficients only solve for two elastic parameters and often assume that the S-to-P-wave velocity ratio is known a priori. Such a constraint could cause the inverse problem to be inaccurate and biased. From Figure 4 we see that when $p_{\max}$ increases, the condition number for the linear case does not change, but there is a linear decrease in the condition number for the quadratic case. Therefore, when data include reflections with moderate and large offsets, the quadratic equation (16)

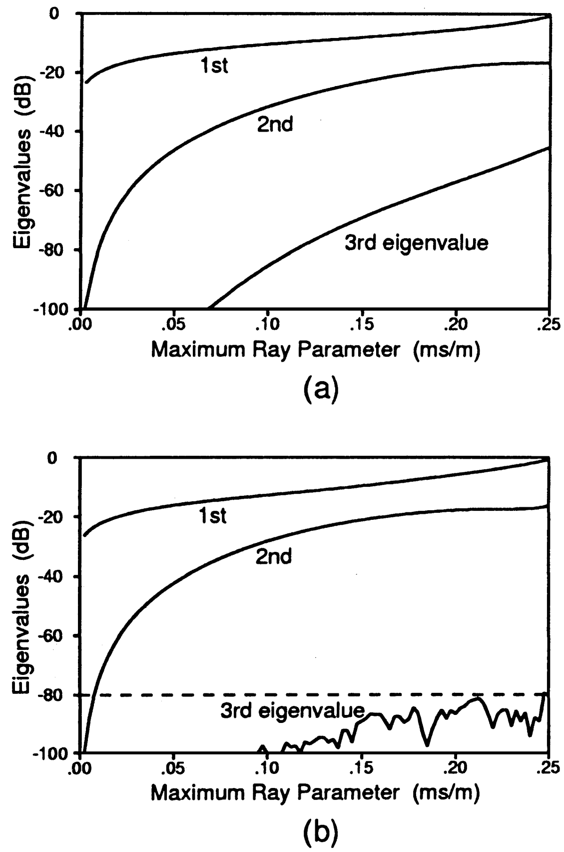

FIG. 3. Eigenvalues of the Fréchet matrix versus the maximum ray parameter $p_{\max}$. Figures (a) and (b) correspond to the inverse problems using, respectively, the quadratic expression and the linearized approximation of the P-P reflection coefficient. The Fréchet matrix is evaluated based on the shale/sand interface model.

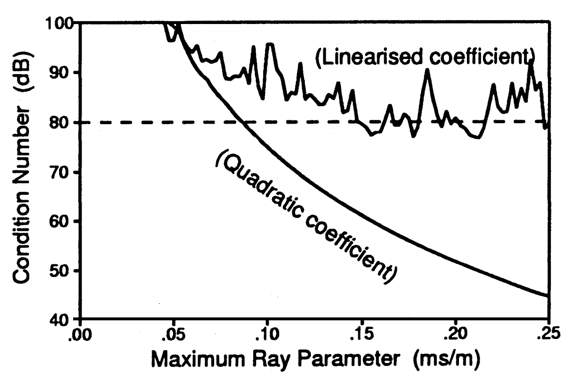

FIG. 4. The condition numbers of the Fréchet matrix in the inverse problem using, respectively, the quadratic expression and the linearized approximation of the P-P reflection coefficient.

could be used in the amplitude inversion, in principle, to estimate three parameters, provided an appropriate inverse algorithm is adopted. The practical implementation of the inversion for those three parameters and the application to real data are presented in Wang (1999), where the reflection coefficient at the reflector is calculated using equation (16) and the transmission coefficients in the overburden medium are evaluated using equation (17).

CONCLUSIONS

This paper introduces pseudoquartic approximations of the *P-P*-wave reflection and transmission coefficients with respect to the ray parameter p. These approximations have a fair compact form, in which the coefficients of the p^2 and p^4 terms are expressed in terms of the vertical slownesses. To compare to the linear formulae used in conventional AVO analysis, I also present quadratic expressions of the *P-P*-wave reflection and transmission coefficients, defined as a function of the elastic contrasts at an interface. Numerical analysis suggests that in using these second-order approximations, the condition number of the sensitivity matrix with respect to three elastic parameters is improved from using a linear approximation. Therefore, one can potentially estimate not only two (in conventional AVO) but three key parameters from an amplitude inversion.

ACKNOWLEDGMENTS

This work was part of a project undertaken by the London University Seismic Research Consortium, sponsored (1995–1997) by Amoco (UK) Exploration Company, Elf UK plc, Enterprise Oil plc, Fina Exploration Ltd., Mobil North Sea Ltd., and Texaco Britain Ltd. Thanks to Dr. Gerhard Pratt and Prof. Michael Worthington for their support of this research. Acknowledgments also to Prof. Greg Houseman, Dr. Ralph Shuey, Prof. Gerard Schuster, Prof. Bjørn Ursin, and anonymous reviewers for their constructive review and to Ata Ilahi for the suggestion regarding example models.

REFERENCES

Aki, K., and Richards, P., 1980, Quantitative seismology—theory and method: W. H. Freeman & Co.

Bortfeld, R., 1961, Approximation to the reflection and transmission coefficients of plane longitudinal and transverse waves: Geophys. Prosp., **9**, 485–502.

Chapman, C. H., 1976, Exact and approximate generalised ray theory in vertically inhomogeneous media: Geophys. J. Roy. Astr. Soc., **46**, 201–233.

de Nicolao, A., Drufuca, G., and Rocca, F., 1993, Eigenvectors and eigenvalues of linearized elastic inversion: Geophysics, **58**, 670–679.

Gardner, G. H. F., Gardner, L. W., and Gregory, A. R., 1974, Formation velocity and density—the diagnostic basics for stratigraphic traps: Geophysics, **39**, 770–780.

Mallick, S., 1993, A simple approximation to the P-wave reflection coefficient and its implication in the inversion of amplitude variation with offset data: Geophysics, **58**, 544–552.

Ostrander, W. J., 1984, Plane-wave reflection coefficients for gas sands at nonnormal angles of incidence: Geophysics, **49**, 1637–1648.

Richards, P. G., and Frasier, C. W., 1976, Scattering of elastic waves from depth-dependent inhomogeneities: Geophysics, **41**, 441–458.

Rutherford, S. R., and Williams, R. H., 1989, Amplitude-versus-offset variations in gas sands: Geophysics, **54**, 680–688.

Shuey, R. T., 1985, A simplification of the Zoeppritz equations: Geophysics, **50**, 609–614.

Stolt, R. H., and Weglein, A. B., 1985, Migration and inversion of seismic data: Geophysics, **50**, 2458–2472.

Ursin, B., and Dahl, T., 1992, Seismic reflection amplitudes: Geophys. Prosp., **40**, 483–512.

Ursin, B., and Tjåland, E., 1993, The accuracy of linearised elastic parameter estimation: J. Seismic Expl., **2**, 349–363.

——— 1996, The information content of the elastic reflection matrix: Geophys. J. Internat., **125**, 214–228.

Wang, Y., 1999, Simultaneous inversion for model geometry and elastic parameters: Geophysics, **64**, 182–190.

Wang, Y., and Houseman, G. A., 1995, Tomographic inversion of reflection seismic amplitude data for velocity variation: Geophys. J. Internat., **123**, 355–372.

Wang, Y., and Pratt, R. G., 1997, Sensitivities of seismic traveltimes and amplitudes in reflection tomography: Geophys. J. Internat., **131**, 618–642.

APPENDIX A

APPROXIMATION OF *P-P* REFLECTION AND TRANSMISSION COEFFICIENTS

The exact formulae for *P-P* reflection and transmission coefficients can be expressed in terms of the ray parameter p as (Aki and Richards, 1980)

$$R_{PP}(p) = \frac{E + Fp^2 + Gp^4 - Dp^6}{A + Bp^2 + Cp^4 + Dp^6} \tag{A-1}$$

and

$$T_{PP}(p) = \frac{H + Ip^2}{A + Bp^2 + Cp^4 + Dp^6}, \tag{A-2}$$

where

$$\begin{aligned}
A &= (\rho_2 q_{\alpha_1} + \rho_1 q_{\alpha_2})(\rho_2 q_{\beta_1} + \rho_1 q_{\beta_2}),\\
B &= -4\Delta\mu(\rho_2 q_{\alpha_1} q_{\beta_1} - \rho_1 q_{\alpha_2} q_{\beta_2}) + (\Delta\rho)^2\\
&\quad + 4(\Delta\mu)^2 q_{\alpha_1} q_{\alpha_2} q_{\beta_1} q_{\beta_2},\\
C &= 4(\Delta\mu)^2(q_{\alpha_1} q_{\beta_1} + q_{\alpha_2} q_{\beta_2}) - 4\Delta\mu\Delta\rho,\\
D &= 4(\Delta\mu)^2,\\
E &= (\rho_2 q_{\alpha_1} - \rho_1 q_{\alpha_2})(\rho_2 q_{\beta_1} + \rho_1 q_{\beta_2}),\\
F &= -4\Delta\mu(\rho_2 q_{\alpha_1} q_{\beta_1} + \rho_1 q_{\alpha_2} q_{\beta_2}) - (\Delta\rho)^2\\
&\quad + 4(\Delta\mu)^2 q_{\alpha_1} q_{\alpha_2} q_{\beta_1} q_{\beta_2},\\
G &= 4(\Delta\mu)^2(q_{\alpha_1} q_{\beta_1} - q_{\alpha_2} q_{\beta_2}) + 4\Delta\mu\Delta\rho,\\
H &= 2(\rho_2 q_{\beta_1} + \rho_1 q_{\beta_2})\,\rho_1 q_{\alpha_1}(\alpha_1/\alpha_2),
\end{aligned}$$

and

$$I = -4\Delta\mu(q_{\beta_1} - q_{\beta_2})\,\rho_1 q_{\alpha_1}(\alpha_1/\alpha_2).$$

These values are defined in terms of the P-wave velocities α_1 and α_2; the P-wave and SV-wave vertical slownesses q_{α_1}, q_{α_2}, q_{β_1}, and q_{β_2}; the contrast in density $\Delta\rho$; and the contrast in shear moduli $\Delta\mu = \rho_2\beta_2^2 - \rho_1\beta_1^2$, where ρ_1 and ρ_2 are the bulk densities and β_1 and β_2 are the SV-wave velocities.

Using the Taylor expansion,

$$(A + Bp^2 + Cp^4 + Dp^6)^{-1} \approx \frac{1}{A} - \frac{B}{A^2}p^2 - \left(\frac{C}{A^2} - \frac{B^2}{A^3}\right)p^4, \quad \text{(A-3)}$$

I rewrite equations (A-1) and (A-2) as

$$R_{PP}(p) \approx \frac{E}{A} + \left(\frac{F}{A} - \frac{BE}{A^2}\right)p^2 + \left(\frac{G}{A} - \frac{BF}{A^2} - \frac{CE}{A^2} + \frac{B^2E}{A^3}\right)p^4 \quad \text{(A-4)}$$

and

$$T_{PP}(p) \approx \frac{H}{A} + \left(\frac{I}{A} - \frac{BH}{A^2}\right)p^2 - \left(\frac{BI}{A^2} + \frac{CH}{A^2} - \frac{B^2H}{A^3}\right)p^4. \quad \text{(A-5)}$$

The coefficients A, B, C, etc., defined after equation (A-2) also depend on p through the vertical slowness. Thus, I refer to equations (A-4) and (A-5) as the pseudoquartic approximations with respect to the ray parameter p.

Denoting the first term on the right-hand side of equation (A-4) as $R_f \equiv E/A$, I have

$$R_f = \frac{\rho_2 q_{\alpha_1} - \rho_1 q_{\alpha_2}}{\rho_2 q_{\alpha_1} + \rho_1 q_{\alpha_2}}, \quad \text{(A-6)}$$

which is the fluid-fluid reflection coefficient, i.e., the reflection coefficient between the two media when the S-wave velocities in both media are set to zero. The fluid-fluid transmission coefficient, $T_f \equiv 1 - R_f$, is defined by

$$T_f = \frac{2\rho_1 q_{\alpha_2}}{\rho_2 q_{\alpha_1} + \rho_1 q_{\alpha_2}}. \quad \text{(A-7)}$$

The first term on the right-hand side of equation (A-5) can be expressed as

$$\frac{H}{A} = T_f \frac{q_{\alpha_1}\alpha_1}{q_{\alpha_2}\alpha_2}. \quad \text{(A-8)}$$

The coefficient of the p^2 term in equation (A-4) is approximated by

$$\begin{aligned}\frac{F}{A} - \frac{BE}{A^2} = &-\frac{1}{A}\big[4\Delta\mu\rho_2 q_{\alpha_1} q_{\beta_1}(1 - R_f) \\ &+ 4\Delta\mu\rho_1 q_{\alpha_2} q_{\beta_2}(1 + R_f) + (\Delta\rho)^2(1 + R_f) \\ &- 4(\Delta\mu)^2 q_{\alpha_1} q_{\alpha_2} q_{\beta_1} q_{\beta_2}(1 - R_f)\big] \\ \approx &-2\frac{\Delta\mu}{\rho} + (1 - R_f)\left(\frac{\Delta\mu}{\rho}\right)^2 q_\alpha q_\beta, \end{aligned} \quad \text{(A-9)}$$

where q_α and q_β are the average values of the vertical slownesses of the P-wave and the SV-wave, respectively. In the approximation above, I assume that

$$\left(\frac{\Delta\rho}{\rho}\right)^2 \approx 0 \quad \text{(A-10)}$$

and

$$\frac{\rho_2 q_{\alpha_1} q_{\beta_1} - \rho_1 q_{\alpha_2} q_{\beta_2}}{(\rho_2 q_{\alpha_1} + \rho_1 q_{\alpha_2})(\rho_2 q_{\beta_1} + \rho_1 q_{\beta_2})} \approx 0. \quad \text{(A-11)}$$

For the p^2 term in the transmission coefficient (A-5), considering

$$\frac{I}{A} = -\frac{2\Delta\mu}{\rho_2 q_{\beta_1} - \rho_1 q_{\beta_2}}\frac{H}{A} \quad \text{(A-12)}$$

and applying approximations (A-10) and (A-11), I have the coefficient

$$\frac{I}{A} - \frac{BH}{A^2} \approx -\left(\frac{\Delta\mu}{\rho}\right)^2 q_\alpha q_\beta \frac{H}{A}. \quad \text{(A-13)}$$

Similar approximations to the coefficients of the p^4 terms are made. I finally obtain the following approximations to the P-P-wave reflection and transmission coefficients:

$$\begin{aligned} R_{PP} \approx R_f &- \left[2\frac{\Delta\mu}{\rho} - (1 - R_f) q_\alpha q_\beta \left(\frac{\Delta\mu}{\rho}\right)^2\right]p^2 \\ &- \left[2R_f\left(\frac{\Delta\mu}{\rho}\right)^2 - 2q_\alpha q_\beta\left(\frac{\Delta\mu}{\rho}\right)^3 \right. \\ &\left. + (1 - R_f) q_\alpha^2 q_\beta^2 \left(\frac{\Delta\mu}{\rho}\right)^4\right]p^4 \end{aligned} \quad \text{(A-14)}$$

and

$$\begin{aligned} T_{PP} \approx T_f \frac{q_{\alpha_1}\alpha_1}{q_{\alpha_2}\alpha_2}\Bigg\{&1 - q_\alpha q_\beta \left(\frac{\Delta\mu}{\rho}\right)^2 p^2 \\ &- \left[2\left(\frac{\Delta\mu}{\rho}\right)^2 - q_\alpha^2 q_\beta^2\left(\frac{\Delta\mu}{\rho}\right)^4\right]p^4\Bigg\}. \end{aligned} \quad \text{(A-15)}$$

In this paper, I approximate the P-P-wave reflection and transmission coefficients by truncating the expressions at the p^2 term, namely the pseudoquadratic expressions.

APPENDIX B

DERIVATION OF EQUATIONS (6)–(8)

Following Snell's law,

$$\frac{\alpha_1}{\alpha_2} = \frac{\sin\theta_1}{\sin\theta_2}, \tag{B-1}$$

and using Taylor series expansions, accurate up to second order—

$$\frac{\alpha_1}{\alpha_2} \approx 1 - \frac{\Delta\alpha}{\alpha} + \frac{1}{2}\left(\frac{\Delta\alpha}{\alpha}\right)^2 \tag{B-2}$$

and

$$\frac{\sin\theta_1}{\sin\theta_2} \approx 1 - 2\frac{\tan\dfrac{\Delta\theta}{2}}{\tan\theta} + 2\left(\frac{\tan\dfrac{\Delta\theta}{2}}{\tan\theta}\right)^2, \tag{B-3}$$

where $\Delta\theta = \theta_2 - \theta_1$, $\theta = (\theta_1 + \theta_2)/2$, $\Delta\alpha = \alpha_2 - \alpha_1$, and $\alpha = (\alpha_1 + \alpha_2)/2$—I have the quadratic equation

$$\left(\frac{\tan\dfrac{\Delta\theta}{2}}{\tan\theta}\right)^2 - \frac{\tan\dfrac{\Delta\theta}{2}}{\tan\theta} + \frac{1}{2}\frac{\Delta\alpha}{\alpha} - \frac{1}{4}\left(\frac{\Delta\alpha}{\alpha}\right)^2 \approx 0. \tag{B-4}$$

Solving this equation, I obtain formula (6):

$$\tan\frac{\Delta\theta}{2} \approx \frac{1}{2}\frac{\Delta\alpha}{\alpha}\tan\theta. \tag{B-5}$$

Using this formula, I make approximation to the fluid-fluid reflection coefficient (equation 7),

$$\begin{aligned} R_f &= \frac{\rho_2\alpha_2\cos\theta_1 - \rho_1\alpha_1\cos\theta_2}{\rho_2\alpha_2\cos\theta_1 + \rho_1\alpha_1\cos\theta_2} \\ &= \frac{\dfrac{\Delta\rho}{\rho} + \dfrac{\Delta\alpha}{\alpha} + \left(2 + \dfrac{1}{2}\dfrac{\Delta\rho}{\rho}\dfrac{\Delta\alpha}{\alpha}\right)\tan\theta\tan\dfrac{\Delta\theta}{2}}{2 + \dfrac{1}{2}\dfrac{\Delta\rho}{\rho}\dfrac{\Delta\alpha}{\alpha} + \left(\dfrac{\Delta\rho}{\rho} + \dfrac{\Delta\alpha}{\alpha}\right)\tan\theta\tan\dfrac{\Delta\theta}{2}} \\ &\approx \frac{1}{2}\left(\frac{\Delta\rho}{\rho} + \frac{\Delta\alpha}{\alpha}\sec^2\theta\right) + O(\chi^3), \end{aligned} \tag{B-6}$$

where χ represents the relative contrast of an elastic parameter at the interface.

For nonevanescent waves, p is small and the vertical slownesses are real: $q_{\alpha_1} = \sqrt{1/\alpha_1^2 - p^2}$ and $q_{\alpha_2} = \sqrt{1/\alpha_2^2 - p^2}$. Equation (8) is finally derived by

$$\begin{aligned} \frac{q_{\alpha_1}\alpha_1}{q_{\alpha_2}\alpha_2} &\approx \frac{1 - \frac{1}{2}\alpha_1^2 p^2 - \frac{1}{8}\alpha_1^4 p^4}{1 - \frac{1}{2}\alpha_2^2 p^2 - \frac{1}{8}\alpha_2^4 p^4} \\ &\approx 1 + \alpha\,\Delta\alpha\, p^2 + O(p^4). \end{aligned} \tag{B-7}$$

APPENDIX C

APPROXIMATION OF *P-SV* REFLECTION AND TRANSMISSION COEFFICIENTS

Good approximation to *P-SV*-wave reflection and transmission coefficients, R_{PS} and T_{PS} are also important for AVO analysis, especially for potential multicomponent seismic AVO analysis. They are given here for completeness.

The exact formulae for *P-SV* reflection and transmission coefficients can be expressed in terms of the ray parameter p as (Aki and Richards, 1980)

$$R_{PS}(p) = \left(2q_{\alpha_1}\frac{\alpha_1}{\beta_1}p\right)\frac{J + Kp^2 - Dp^4}{A + Bp^2 + Cp^4 + Dp^6} \tag{C-1}$$

and

$$T_{PS}(p) = \left(2q_{\alpha_1}\frac{\alpha_1}{\beta_2}p\right)\frac{L + Mp^2}{A + Bp^2 + Cp^4 + Dp^6}, \tag{C-2}$$

where

$$\begin{aligned} J &= -\rho_2\Delta\rho - 2\rho_1 q_{\alpha_2}q_{\beta_2}\Delta\mu, \\ K &= 2(\rho_2 + \Delta\rho)\Delta\mu - 4q_{\alpha_2}q_{\beta_2}(\Delta\mu)^2, \\ L &= \rho_1\Delta\rho - 2\rho_1 q_{\alpha_2}q_{\beta_1}\Delta\mu, \end{aligned}$$

and

$$M = -2\rho_1\Delta\mu.$$

The values for A, B, C, and D are given in Appendix A. Using the Taylor expansion (A-3), we have

$$\begin{aligned} R_{PS}(p) \approx \left(2q_{\alpha_1}\frac{\alpha_1}{\beta_1}p\right)\Bigg[\frac{J}{A} &+ \left(\frac{K}{A} - \frac{BJ}{A^2}\right)p^2 \\ &- \left(\frac{D}{A} + \frac{BK}{A^2} + \frac{CJ}{A^2} - \frac{B^2J}{A^3}\right)p^4\Bigg] \end{aligned} \tag{C-3}$$

and

$$\begin{aligned} T_{PS}(p) \approx \left(2q_{\alpha_1}\frac{\alpha_1}{\beta_2}p\right)\Bigg[\frac{L}{A} &+ \left(\frac{M}{A} - \frac{BL}{A^2}\right)p^2 \\ &- \left(\frac{BM}{A^2} + \frac{CL}{A^2} - \frac{B^2L}{A^3}\right)p^4\Bigg]. \end{aligned} \tag{C-4}$$

The *P-SV*-wave reflection and transmission coefficients are then approximated as

$$\begin{aligned} R_{PS}(p) = -\frac{\alpha_1}{\beta_1}\frac{\Delta\mu}{\rho}q_\alpha p\Bigg\{1 &- \left[\frac{1}{q_\alpha q_\beta} - 2\frac{\Delta\mu}{\rho} + q_\alpha q_\beta\left(\frac{\Delta\mu}{\rho}\right)^2\right]p^2 + \left[\frac{2}{q_\alpha q_\beta}\frac{\Delta\mu}{\rho} - \left(\frac{\Delta\mu}{\rho}\right)^2\right. \\ &\left. - 2q_\alpha q_\beta\left(\frac{\Delta\mu}{\rho}\right)^3 + q_\alpha^2 q_\beta^2\left(\frac{\Delta\mu}{\rho}\right)^4\right]p^4\Bigg\} \end{aligned} \tag{C-5}$$

and

$$\begin{aligned} T_{PS}(p) = -\frac{\alpha_1}{\beta_2}\frac{\Delta\mu}{\rho}q_\alpha p\Bigg\{1 &+ \left[\frac{1}{q_\alpha q_\beta} - q_\alpha q_\beta\left(\frac{\Delta\mu}{\rho}\right)^2\right]p^2 \\ &- \left[3\left(\frac{\Delta\mu}{\rho}\right)^2 - q_\alpha^2 q_\beta^2\left(\frac{\Delta\mu}{\rho}\right)^4\right]p^4\Bigg\}, \end{aligned} \tag{C-6}$$

which are referred to as the pseudoquartic expressions, with respect to the ray parameter p.