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APSIDAL MOTION IN BINARY STARS

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1. *Introduction.*—The observed ellipticities and apsidal motions of eclipsing binaries must logically depend upon the distributions of density within the components. It is important to predict as well as possible, for given distributions of density, what the expected ellipticities and rates of apsidal motion should be, in order to secure independent verification or disproof of particular distributions yielded by theories of the internal constitution. In this paper the theory of the apsidal motions * in binary stars (assumed to be constructed of compressible fluids †) will be investigated, in the limiting case where the orbital period is long compared with the free harmonic periods of the component stars. In § 2, the disturbing potential will be given; in § 3, a general treatment will be applied to find the equilibrium tides and the disturbances, in the gravitational fields of the components, caused by those tides; in § 4 the secular perturbations, of the longitude of periastron, caused by the disturbances in the gravitational fields, will be found from the disturbing function by conventional methods. In § 5 the results, although different from Russell's, will be shown to be consistent with them; in § 6 they will be compared with Cowling's. The order of accuracy aimed at will be such that terms of an order up to but not including the tenth power in (a_i/A) will be retained in the analysis. The orbital eccentricity, e, will enter the analysis exactly through functions represented (a) by convergent power series, and alternatively (b) by closed expressions. The mean radii are a_1 and a_2, and the semi-major axis of the relative orbit is A.

2. *The Disturbing Potential.*—Let the components of the binary star be denoted by 1 and 2; let the masses be m_1 and m_2. Then the gravitational potential due to star 2 is m_2G/d at any point whose distance from the centre of mass of star 2 is d. G is the constant of gravitation. The expression is exact when star 2 is spherically constructed; and it can be shown ‡ to be correct to terms of the order of $(m_1/m_2)(a_2/R)^5$ times itself at points near

* The rotational and tidal distortions will be found incidentally.

† The librations of solid stars have been considered by Walter (*Veröff. der Universitäts-Sternwarte*, Königsberg, 2, 1931; 3, 1933); the problem for fluid stars is essentially different, since no "librations" are present except those periodic in the orbital period and its submultiples. The present treatment is essentially along the lines laid down by Russell (*M.N.*, **88**, 641, 1928) except that here (a) the variation of the tidal deformation with distance between centres will be taken into account, and (b) a higher order of accuracy will be aimed at. The present treatment differs greatly from the treatment given by Cowling (*M.N.*, **98**, 734, 1938), with whose results, however, to Cowling's order of accuracy, it agrees.

‡ See equation (9) in § 4; or Chandrasekhar, *M.N.*, **93**, 449, 1933.

star 1, when the tidal distortion of star 2 by star 1 is taken into account. The rotational distortion of star 2, if it has a uniform angular velocity equal to the mean orbital angular velocity, introduces an error in m_2G/d of the order of $(1+m_1/m_2)\,(a_2/R)^5$ times itself at points near 1. Here R is the distance between the centres of mass. Thus to a very high order of accuracy, the simple expression m_2G/d is adequate for finding the distortion of star 1 by star 2, and the consequent disturbance, of the external gravitational potential of star 1, caused by that distortion.

In the vicinity of star 1, $(1/d)$ may be expanded in a series * of spherical harmonics about the centre of mass, Q_1, of star 1. Then the potential due to 2, at a point N whose distance from Q_1 is r, and such that the angle NQ_1Q_2 is θ, is

$$\frac{m_2G}{R}\sum_0^\infty\left(\frac{r}{R}\right)^n P_n(\cos\theta),$$

where $P_n(\cos\theta)$ is the Legendre coefficient of order n. The first term is a mere constant, producing no forces at all; the second causes an acceleration m_2G/R^2, of all the elements of star 1, parallel to the line of centres and towards 2; this acceleration is merely the orbital acceleration of star 1 as a whole. The remaining terms, constituting the tidal potential, are

$$V_t=\frac{m_2G}{R}\sum_2^\infty\left(\frac{r}{R}\right)^n P_n(\cos\theta). \qquad (0)$$

In an elliptical orbit, R is a function of the time. The general dynamical theory of the tides (produced on star 1 by the tidal potential (0)) would involve the development of each term of (0) in a series of purely harmonic functions of the mean anomaly, each multiplied by a power of the orbital eccentricity and a function of position. To each such purely harmonic function of the time there would correspond one or more partial tides; the partial tides would sweep around star 1 each with a constant amplitude, speed, and phase; the resultant of all the constantly rotating partial tides would furnish the total distortion. Such a treatment would be exceedingly complicated. In the limit, however, in which the orbital period is long compared with the free tesseral harmonic periods of the star, the results would be given by the equilibrium theory of the tides. In the general dynamical theory, the resultant distortion of star 1 must be periodic in the orbital period. In the equilibrium limit, that must still be true; and the phase differences are zero (since viscosity is small), causing the resultant tidal distortion to have at each instant the equilibrium value appropriate to V_t. The equilibrium limit will here be investigated; the results should be applicable in the absence of approaches to resonance so close as to render necessary the more general dynamical theory; and hence, in view of the expected shortness of the free tesseral harmonic periods, they should probably be applicable in practice.

The stars will here be assumed to rotate with uniform angular velocities

* See, for instance, Jeans, *Electricity and Magnetism*, Chapter VIII, 1925.

ω_1 and ω_2 about axes normal to the orbital plane *. A uniform rotation of star 1 introduces a disturbing potential

$$V_r = -\tfrac{1}{3}\omega_1{}^2 r^2 P_2(\cos\theta'),$$

where θ' is the co-latitude; plus a term, symmetrical about the centre of mass of star 1, which cannot distort † the star from a spherical configuration. The relevant term, V_r, added to V_t, produces a significant disturbing potential $V_r + V_t$.

If U is the gravitational potential of the disturbed gaseous (or liquid) star, and V_d is the complete disturbing potential, so that $\Psi = U + V_d$ is the total potential, then the equation of hydrostatic equilibrium is $dP = \rho d\Psi$ where P is the pressure and ρ the density. It follows immediately that P and ρ must be functions of Ψ alone in a star that is in hydrostatic equilibrium under the action of its own gravity and of a disturbing potential.

3. *The Deformation and the Consequent External Gravitational Potential.*—The elegant and general treatment given by Jeffreys ‡ may be adapted to this problem. Let the disturbing potential, arising from sources other than the attraction of star 1 on itself, be expanded in a series of tesseral harmonic terms of the form

$$c_{n,m} r^n P_n{}^m(\theta, \phi), \tag{A}$$

where $n \geqslant 2$; r is the distance from the centre of mass of star 1; $P_n{}^m$ is a tesseral surface harmonic function of θ and ϕ, polar co-ordinates taken with respect to the centre of mass of star 1; and $c_{n,m}$ is a constant. One considers the form of the surface, over which the density is ρ in the distorted star 1, to be given by the equation

$$r = r_1\Big(1 + \sum_{n,m} Y_n{}^m\Big), \tag{1}$$

where r_1 is a parameter (the mean radius) characterizing the surface in

* Although it is uncertain that actual components so rotate, there is no general agreement as to how they do rotate, so that the assumptions are probably justifiable, at present, on grounds of simplicity. Some assumption has to be made about the variation of ω with depth, and that of uniform rotation is the simplest. The additional complications that would follow from the consideration of inclined axes of rotation would at present, moreover, not be justifiable. See Jeans, *Astronomy and Cosmogony*, Chapter X, 1929; also Rosseland, *Astrophysica Norvegica*, **2**, 173, 1936 and **2**, 249, 1937. Rosseland's solutions are sufficient, not necessary.

† The term can, and in general probably does, produce a swelling or contraction of the star, compared with the equilibrium configuration of an undisturbed star having the same mass and luminosity. Provided that such effects are taken into account in the "model" to which equation (5) of the next section is applied, it is legitimate to ignore such effects here. For a treatment of them, in relation to some particular stellar models, the reader is referred to Chandrasekhar, *M.N.*, **93**, 390, 1933. The reason for its being permissible to regard the term in the present fashion is that our treatment gives information about the expected apsidal motion corresponding to a given distribution of density in the actual, distorted, star; not in some ideal undistorted star.

‡ Jeffreys, *The Earth*, Chapter XIII, 1924. The general procedure appears to have been devised by Laplace, Clairaut, and Radau.

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question, and the Y_n^m's are tesseral surface harmonic functions of θ and ϕ, and are also functions of r_1. Let the distorted outer boundary of star 1 correspond to $r_1 = a$. Then if one ignores small quantities of the order of the squares and cross products of the Y's there holds for Y_n^m the equation *

$$-\frac{Y_n^m}{r_1}\int_0^{r_1}\rho' a'^2 da' + \frac{1}{(2n+1)r_1^{n+1}}\int_0^{r_1}\rho'\frac{\partial}{\partial a'}(a'^{n+3}Y_n^m)da' + \frac{r_1^n}{2n+1}\int_{r_1}^{a}\rho'\frac{\partial}{\partial a'}\left(\frac{Y_n^m}{a'^{n-2}}\right)da' = -\frac{1}{4\pi G}c_{n,\,m}r_1^n P_n^m(\theta,\,\phi). \qquad (2)$$

Here the quantity a' is the value of r_1 for which $\rho = \rho'$. Multiply both sides of equation (2) by r_1^{n+1}, differentiate with respect to r_1, and divide by r_1^{2n}. One obtains

$$-\left[\frac{nY_n^m}{r_1^{n+1}} + r_1^{-n}\frac{\partial Y_n^m}{\partial r_1}\right]\int_0^{r_1}\rho' a'^2 da' + \int_{r_1}^{a}\rho'\frac{\partial}{\partial a'}\left(\frac{Y_n^m}{a'^{n-2}}\right)da' = -\frac{1}{4\pi G}(2n+1)c_{n,\,m}P_n^m(\theta,\,\phi). \qquad (3)$$

Differentiate with respect to r_1:

$$\frac{\partial^2 Y_n^m}{\partial r_1^2} - \frac{n(n+1)}{r_1^2}Y_n^m + \frac{6\rho}{S(r_1)}\left(r_1^2\frac{\partial Y_n^m}{\partial r_1} + r_1 Y_n^m\right) = 0, \qquad (4)$$

where

$$S(r_1) = 3\int_0^{r_1}\rho' a'^2 da'.$$

Now if one introduces a new variable

$$\eta_n = (r_1/Y_n^m)(\partial Y_n^m/\partial r_1),$$

a function of r_1, equation (4) reduces to

$$r\frac{d\eta_n}{dr} + \eta_n^2 - \eta_n - n(n+1) + \frac{6\rho}{\rho_m}(\eta_n + 1) = 0, \qquad (5)$$

where r_1 has been replaced (there should be no danger of confusion) by r. Here ρ is the density at the distance r from the centre, and ρ_m is the mean density interior to r, in a spherical distribution of matter (not necessarily in equilibrium) in which the density is the same function of r that ρ, in the actual distorted star, is of r_1. It is shown by Jeffreys † that at $r = 0$, $\eta_n = n - 2$.

In (3), let $r_1 = a$, the mean radius of the star. Then it is found that the term (A) in the disturbing potential produces a term Y_n^m in (1) whose value at the surface is

$$Y_n^m(a) = \frac{2n+1}{n+\eta_n(a)}\,\frac{1}{m_1 G}c_{n,\,m}\,a^{n+1}P_n^m(\theta,\,\phi), \qquad (6)$$

* Since Ψ must reduce to a function of r_1 alone. See Jeffreys, *ibid.*, equation (2) of § 13.4.

† Jeffreys, *ibid.*, § 13.52.

where $\eta_n(a)$ is the value of η_n at the outer surface. Then set $r_1 = a$ in (2) and make use of the appropriate expression * for the gravitational potential (due to the distortion) at external points. One then finds that the distortion of star 1 produced by the term (A) in the disturbing potential gives rise to a contribution

$$U_{n,\,m} = \frac{n+1-\eta_n(a)}{n+\eta_n(a)} c_{n,\,m} \frac{a^{2n+1}}{r^{n+1}} P_n{}^m(\theta, \phi), \tag{7}$$

to the gravitational potential, at external points, of star 1.

The external shape is given by

$$r = a\Big(1 + \sum_{n,\,m} Y_n{}^m(a)\Big), \tag{6 a}$$

where the separate terms are given by (6). The total contribution of the distortion of star 1 to the external gravitational potential of star 1 is the sum of the separate terms $U_{n,\,m}$ given by (7).

For any model, equation (5) may be solved, numerically or otherwise, so as to find the value $\eta_n(a)$ of η_n at the boundary $r = a$. Then equations (6) and (6 *a*) give the superficial distortion of star 1; and the equations (7) give the contribution of the whole "internal" distortion of star 1 to its external gravitational potential.

We have introduced tesseral harmonics explicity in order to make simpler the practical applications of these results later on. For the preceding results show that to any term, in the disturbing potential, of the form

$$cr^n P_n, \tag{B}$$

where P_n is a surface harmonic of order $n \geqslant 2$, there corresponds a contribution to the superficial distortion given by

$$Y_n(a) = c\frac{2n+1}{n+\eta_n(a)} \frac{a^{n+1}}{m_1 G} P_n, \tag{6 b}$$

and a contribution to the external gravitational potential of star 1

$$U_n = c\frac{a^{2n+1}}{r^{n+1}} \frac{n+1-\eta_n(a)}{n+\eta_n(a)} P_n. \tag{7 b}$$

Different P_n's need not be referred to the same axes; and the Y_n and U_n always have the same character as the surface harmonic, P_n, present in the disturbing term (B) that causes them.

Although but little use will be made in this paper of the equations (6) and (6 *b*) that furnish the superficial distortion, they follow immediately from the analysis and have been given for completeness.

The results obtained in this section will now be applied to the disturbing potential $V_t + V_r$.

* Obtained, simply, by replacing each Y_n', in Jeffreys' equation (3) of § 13.12, by a linear combination of terms $Y_n{}^m$ in different m's.

4. *The Secular Motion of the Apse.*—The terms of order $n=2$ in the disturbing potential V_r+V_t that acts on star 1 have been found to be

$$\frac{m_2 G}{R^3}\, r^2 P_2(\cos\theta) - \tfrac{1}{3}\omega_1{}^2 r^2 P_2(\cos\theta'). \tag{8}$$

From equation (7 *b*), these terms so distort * star 1 as to contribute to its external gravitational potential at all points the terms

$$2\frac{a_1{}^5}{r^3}k_{2,1}\left[\frac{m_2 G}{R^3}P_2(\cos\theta) - \frac{\omega_1{}^2}{3}P_2(\cos\theta')\right]. \tag{9}$$

For the centre of mass of star 2, $r=R$, $\theta=0$ and $\theta'=\pi/2$; the acceleration of star 2 thus contains the terms, since $P_2(x)=\frac{1}{2}(3x^2-1)$,

$$6\frac{a_1{}^5}{R^4}k_{2,1}\left[\frac{m_2 G}{R^3}+\frac{\omega_1{}^2}{6}\right],$$

directed towards the centre of mass of star 1. These last terms correspond to a potential function

$$\psi_2 = a_1{}^5 k_{2,1}\left[\frac{m_2 G}{R^6}+\frac{\omega_1{}^2}{3R^3}\right]. \tag{9 a}$$

In equations (9) and (9 *a*), $k_{2,1}$ (denoted by k_1 in Russell's paper) is the quantity

$$\frac{3-\eta_2(a)}{4+2\eta_2(a)}$$

computed for star 1. Here, by equation (5), $\eta_2(a)$ is the value at $r=a$ of the variable η_2 that is zero at $r=0$ and that satisfies the first order differential equation

$$r\frac{d\eta_2}{dr}+\eta_2{}^2-\eta_2-6+\frac{6\rho}{\rho_m}(\eta_2+1)=0. \tag{10}$$

The terms (9 *a*) correspond to a contribution $\psi_2(1+m_2/m_1)$ to the ordinary disturbing function S of perturbation theory, when the elements of the relative orbit are under consideration. The rate of motion of the apse is known to be given by the equation, linear in S,

$$\dot{\varpi}=(1-e^2)^{1/2}\frac{1}{e}\frac{\partial S}{\partial e}(\mu A)^{-1/2}, \tag{11}$$

* As an illustration of the exceedingly great generality of the analysis described in the preceding section, equation (6 *b*) may be used to compute the distortion of a polytrope of index three under the action of the terms (8). The value $k_2=0{\cdot}0144$ for that model was computed by Russell (*ibid.*) from equation (5). Equation (6 *b*) yields a value of Δ_2, a coefficient of P_2 in Chandrasekhar's special treatment of the tidal distortion (*M.N.*, **93**, 456, 1933) of the model in question, equal to $1{\cdot}0288$; whereas Chandrasekhar's own value, reached by his special analysis, is $1{\cdot}0289$. Chandrasekhar's Δ_2 should equal $2k_2+1$. Likewise, equation (6 *b*) leads, with Russell's value of k_2, to a rotational oblateness $41{\cdot}810\ v$, where Chandrasekhar's special treatment leads to the value $41{\cdot}811\ v$ (*M.N.*, **93**, 399, 1933).

where e is the orbital eccentricity, A is the semi-major axis of the relative orbit, $\mu = G(m_1 + m_2)$, and S is supposed to be defined * as a function of the mean anomaly M, of e, and of A.

The quantities R^{-3} and R^{-6}, appearing in ψ_2, and thus in S, must be developed in terms of M, e, and A. Although $(A/R)^j$ can be expanded in terms of Hansen's coefficients, it is perhaps simpler to consider the expansion, involving † Bessel's coefficients,

$$\frac{A}{R} = 1 + 2\sum_{p=1}^{\infty} J_p(pe) \cos pM.$$

If this expansion is raised to the j'th power, an infinite series in powers of e will be obtained, plus a set of such series each multiplied by a cosine of a multiple of M. The whole expression is to be operated on by $(1/e)(\partial/\partial e)$, and it is seen that the periodic terms will contribute merely periodic terms to the velocity $\dot{\varpi}$ of the apse. The secular motion of the apse arises from the series in e^2 (odd powers are lacking) alone, and the lowest terms of importance are those in e^2 itself, which contribute pure constants to the secular motion. The series in e^2 can be easily found, for

$$\begin{aligned}(A/R)^j &= (1 - e\cos E)^{-j}\\ &= \sum_0^{\infty} e^g \frac{(j+g-1)!}{(j-1)!\,g!} \cos^g E,\end{aligned}$$

where E is the eccentric anomaly. When g is even, the expression for $\cos^g E$ in terms of cosines of multiples of E is a constant term $g!/2^g[(g/2)!]^2$ plus cosine terms in even multiples of E; when g is odd, the expression is $g!\cos E/2^{g-1}[(g-1)/2]!\,[(g+1)/2]!$ plus cosine terms in odd multiples of E. Since the development of $\cos mE$ in cosines of multiples of M has as its only constant term 1, $-\frac{1}{2}e$, or 0 according to whether $m=0$, $m=1$, or $m>1$, it follows that the coefficient of e^k in the non-periodic part of $(A/R)^j$ is

$$b(j, k) = \frac{(j+k-2)!}{(j-2)!\,2^k(k/2)!^2}$$

when k, which must be even, is larger than 0; the coefficient of e^0 is unity. The non-periodic part of $(A/R)^j$ can thus be written down at once as a series, in e^2, convergent for all values of e less than unity. One finds, thus,

$$\left(\frac{A}{R}\right)^3 = 1 + \frac{3}{2}e^2 + \frac{15}{8}e^4 + \frac{35}{16}e^6 + \frac{315}{128}e^8 + \ \ldots\ + \text{periodic terms},$$

$$\left(\frac{A}{R}\right)^6 = 1 + \frac{15}{2}e^2 + \frac{105}{4}e^4 + \frac{525}{8}e^6 + \frac{17325}{128}e^8 + \ \ldots\ + \text{periodic terms}.$$

* The equation for $\dot{\varpi}$ given above may be obtained readily by considering variations of the canonical constants of an orbit in a plane. Generally, in such an orbit, S is a function of ϵ (the mean longitude at the epoch), of ϖ (the longitude of periastron), of A, and of e, as well as of the time. For present purposes, since only $(\partial S/\partial e)$ is involved, one may regard S as a function simply of M, e, and A.

† See Plummer's *Dynamical Astronomy*, Chapter IV.

It follows from these expansions, and from equations (9 *a*) and (11), that distortions of order 2 of star 1 contribute to the secular motion $\dot{\varpi}_s$, of the apse terms

$$k_{2,1}a_1^5\frac{1}{m_1}\frac{(m_1+m_2)^{1/2}}{(GA)^{1/2}}\left[15\frac{m_2G}{A^6}f_2(e)+\frac{\omega_1^2}{A^3}g_2(e)\right], \tag{12}$$

where $f_2(e)$ and $g_2(e)$ are series, convergent for any value of e less than unity,

$$f_2(e)=1+\frac{13}{2}e^2+\frac{181}{8}e^4+\frac{465}{8}e^6+\ \ldots \tag{12 a}$$

$$g_2(e)=1+2e^2+3e^4+4e^6+\ \ldots \tag{12 b}$$

Making use of the relation $\nu^2A^3=\mu$, where ν is the mean orbital motion, one finds for the contribution to $\dot{\varpi}_s/\nu$

$$k_{2,1}\frac{a_1^5}{A^5}\left[15\frac{m_2}{m_1}f_2(e)+\frac{\omega_1^2A^3}{m_1G}g_2(e)\right]. \tag{13}$$

If one now considers the perturbations due to the action on star 1 of the distortion (caused by the attraction of star 1 and by the rotation of star 2) of star 2, one sees that the second order harmonic distortions give rise to a secular apsidal motion (a precession) $\dot{\varpi}_2$ such that

$$\begin{aligned}\frac{\dot{\varpi}_2}{\nu}=&k_{2,1}\frac{a_1^5}{A^5}\left[15\frac{m_2}{m_1}f_2(e)+\frac{\omega_1^2A^3}{m_1G}g_2(e)\right]\\&+k_{2,2}\frac{a_2^5}{A^5}\left[15\frac{m_1}{m_2}f_2(e)+\frac{\omega_2^2A^3}{m_2G}g_2(e)\right],\end{aligned} \tag{14}$$

where, as usual, the subscripts 1 and 2 refer to stars 1 and 2 respectively.

Should one wish to set the ω's equal to ν, one obtains the less general result

$$\begin{aligned}\frac{\dot{\varpi}_2}{\nu}=&k_{2,1}\frac{a_1^5}{A^5}\left[\frac{m_2}{m_1}(15f_2(e)+g_2(e))+g_2(e)\right]\\&+k_{2,2}\frac{a_2^5}{A^5}\left[\frac{m_1}{m_2}(15f_2(e)+g_2(e))+g_2(e)\right].\end{aligned} \tag{14 a}$$

The contributions arising from the third and fourth order harmonic distortions may be dealt with similarly. The terms of these orders in the disturbing potential are purely tidal, arising from the tidal potential (0). The disturbing term of order $n=3$ is

$$\frac{m_2G}{R^4}r^3P_3(\cos\theta)$$

and by (7 *b*)

$$\psi_3=k_{3,1}\frac{m_2G}{R^8}a_1^7,$$

where $k_{3,1}$ is the quantity

$$\frac{4-\eta_3(a)}{6+2\eta_3(a)}$$

and where $\eta_3(a)$ is obtained by solving equation (5), with n set equal to 3, and subject to the boundary condition $\eta_3 = 1$ at $r = 0$, for the value of η_3 at the boundary of star 1. The contribution to the disturbing function is $\psi_3(1 + m_2/m_1)$, and if one makes use of the expansion of $(A/R)^8$ one finds for the contribution $\dot{\varpi}_3$ of the third order deformations to the secular motion of the apse

$$\frac{\dot{\varpi}_3}{\nu} = 28\left[k_{3,1}\frac{a_1^7}{A^7}\frac{m_2}{m_1} + k_{3,2}\frac{a_2^7}{A^7}\frac{m_1}{m_2}\right]f_3(e), \tag{15}$$

where

$$f_3(e) = 1 + \tfrac{43}{4}e^2 + \tfrac{449}{8}e^4 + \tfrac{12941}{64}e^6 + \ . \ . \ . \tag{15 a}$$

The disturbing term of order $n = 4$ is

$$\frac{m_2 G}{R^5} r^4 P_4(\cos\theta)$$

and by (7 *b*)

$$\psi_4 = k_{4,1}\frac{m_2 G}{R^{10}} a_1^9,$$

where $k_{4,1}$ is the quantity

$$\frac{5 - \eta_4(a)}{8 + 2\eta_4(a)},$$

and where $\eta_4(a)$ is given by equation (5), with n set equal to 4, subject to the boundary condition that $\eta_4 = 2$ at $r = 0$. From the expansion of $(A/R)^{10}$ one finds for the contribution $\dot{\varpi}_4$ of the fourth order deformations to the secular motion of the apse

$$\frac{\dot{\varpi}_4}{\nu} = 45\left[k_{4,1}\frac{a_1^9}{A^9}\frac{m_2}{m_1} + k_{4,2}\frac{a_2^9}{A^9}\frac{m_1}{m_2}\right]f_4(e), \tag{16}$$

where

$$f_4(e) = 1 + 16e^2 + \tfrac{467}{4}e^4 + \tfrac{8975}{16}e^6 + \ . \ . \ . \tag{16 a}$$

Still higher orders in (0) could be taken into account; but one would then have to take into account also the effect upon the apsidal motion of the distortion of each star produced by the distortion of the other star; these effects should introduce terms in $\dot{\varpi}_s/\nu$ of the order of $(a_i/A)^{10}$. The results given here, for $\dot{\varpi}_s/\nu$ in the equilibrium limit, should be correct up to but not including terms of the tenth order in (a_i/A). To the present order of accuracy, since $\dot{\varpi}_s/\nu = P/P'$,

$$\frac{P}{P'} = \frac{\dot{\varpi}_2}{\nu} + \frac{\dot{\varpi}_3}{\nu} + \frac{\dot{\varpi}_4}{\nu}, \tag{17}$$

where P and P' are the orbital and apsidal periods, respectively, and where the $\dot{\varpi}/\nu$'s are given by equations (14), (15), and (16) if no assumptions are made about the ω's, or by (14 *a*), (15), and (16) if the ω's are set equal to ν. For well separated components, the term $\dot{\varpi}_2/\nu$ alone, given by (14) or (14 *a*), should yield a sufficiently accurate value of P/P'. The present theory always predicts a precession.

Values of k_2 have been tabulated for a series of models by Russell *. For homogeneous stars k_2 has the value $3/4$; for polytropes of index three, the value 0·0144; for completely concentrated stars all the k's are zero. For homogeneous stars, k_3 and k_4 have the values $3/8$ and $1/4$, respectively. All the k's can readily be found for any model by numerical integration of equation (5).

In the case of certain polytropic models, however, the integrations need not be carried out, for the k's have in effect been computed, under a different guise, by Chandrasekhar †. If equations (6 a) and (6 b) are applied to the tidal potential (0), one finds directly, for the equilibrium shape of star 1 produced by the tidal ‡ action,

$$\frac{r-a_1}{a_1}=\frac{m_2}{m_1}\left[\frac{5}{2+\eta_2(a)}z^3P_2+\frac{7}{3+\eta_3(a)}z^4P_3+\frac{9}{4+\eta_4(a)}z^5P_4\right],$$

where P_j denotes $P_j(\cos\theta)$ and where $z=(a_1/R)$. In the present notation, Chandrasekhar's equation (34) is

$$\frac{r-a_1}{a_1}=\frac{m_2}{m_1}\sum_2^4\Delta_j z^{j+1}P_j.$$

The corresponding coefficients must be equal; from this it follows that $k_j=\frac{1}{2}(\Delta_j-1)$. Since Chandrasekhar lists the Δ_j's for the polytropes of indices 1, 3/2, 2, 3 and 4, one has at once the values

Polytropic index	1	1·5	2	3	4
k_2	0·25992	0·1446	0·0741	0·0144	0·00134
k_3	0·1064	0·0540	0·0244	0·00368	0·00024
k_4	0·0602	0·0281	0·0116	0·00140	0·00007

The value of k_2 obtained in this way, for the polytrope of index three, agrees exactly with the value computed by Russell directly from equation (5).

The series expansions (12 a), (12 b), (15 a) and (16 a) do not converge very rapidly for values of e larger than a few tenths. Closed expressions may be employed instead. For $g_2(e)=(1/3)F_3(e)$; $f_2(e)=(1/15)F_6(e)$; $f_3(e)=(1/28)F_8(e)$; and $f_4(e)=(1/45)F_{10}(e)$ where

$$F_j(e)=\frac{j}{2\pi e}(1-e^2)^{1/2}\int_0^{2\pi}\frac{\cos E-e}{(1-e\cos E)^{j+1}}dE.$$

The integral may be evaluated by the substitution

$$\cos\theta=(\cos E-e)/(1-e\cos E);$$

* Russell, *ibid.*

† Chandrasekhar, *M.N.*, 93, 456, 1933.

‡ The rotational distortion

$$\frac{r-a_1}{a_1}=-\frac{5}{2+\eta_2(a)}\frac{\omega_1{}^2a_1{}^3}{3m_1G}P_2(\cos\theta'),$$

given by (6 b), is not needed here but is given for completeness.

and it is then found, exactly for all values of e less than unity, that

$$\left.\begin{aligned} g_2(e) &= (1-e^2)^{-2}, \\ f_2(e) &= (1-e^2)^{-5}(1+\tfrac{3}{2}e^2+\tfrac{1}{8}e^4), \\ f_3(e) &= (1-e^2)^{-7}(1+\tfrac{15}{4}e^2+\tfrac{15}{8}e^4+\tfrac{5}{64}e^6), \\ f_4(e) &= (1-e^2)^{-9}(1+7e^2+\tfrac{35}{4}e^4+\tfrac{35}{16}e^6+\tfrac{7}{128}e^8). \end{aligned}\right\} \qquad (18)$$

The functions $g_2(e)$ and the $f(e)$'s have the values unity when $e=0$; are rapidly increasing functions of e; and approach infinity, thereby causing the secular precession of the apse to approach infinity, as e approaches unity.

5. *Comparison with Russell's Results.*—Russell set the ω's equal to ν, considered the limit of vanishing e, ignored tidal deformations of higher order than $n=2$, and considered the amplitude of the tidal deformation to be fixed at the value corresponding to a distance between centres equal to the semi-major axis A. His equation for P/P' should therefore be comparable to, although not identical with, equation (14 a) when the f_2 and g_2 in that equation are set equal to unity, since (14 a) then carries the analysis to Russell's order of accuracy. Russell's equation then in fact differs from (14 a) only in containing the numerical factor 7 where the factor 16 appears in (14 a). The amount of this discrepancy is exactly what one should expect. A comparison of equations (14) and (14 a) shows that 16 is the sum of unity, arising from the rotation, and 15 arising from the tides; Russell's 7 is likewise the sum of unity, from the rotation, and 6 from the tidal deformation. Russell's treatment and the present treatment yield, as they should, the same rotational contributions to the apsidal motion; the present treatment yields a tidal precession five-halves as large as Russell's. This it should do; for ignoring the variation of the tidal distortion with time, during the orbital motion, amounts to setting R, in equation (9), equal to A. When the operator $(1/e)(\partial/\partial e)$ is later applied to $2(A/R)^3$ instead of to $(A/R)^6$, a tidal precession must be obtained exactly two-fifths as large in the former case as in the latter, since the coefficient of e^2 in the former expansion is 3, and in the latter 15/2. The present results * are therefore consistent with Russell's.

For any given model, the present treatment leads to an expected rate of apsidal precession that is, roughly, about twice as rapid as Russell's would predict. Hence, when applied to the same observational data, the present treatment implies a considerably higher degree of central condensation in the components than does Russell's.

6. *Comparison with Cowling's Results.*—A treatment very different from the present one has been given recently by Cowling †. Cowling's treatment does not take into account deformations of higher order than those corresponding to $n=2$, and so his equations (40) and (41) should be, respectively,

* That the consideration of varying tides should lead to such different results from the assumption of constant tides is due to the circumstance that terms in e^2 in the disturbing function (although they approach zero as e approaches zero), since they yield mere constants after the operator $(1/e)$ $(\partial/\partial e)$ is applied to them, are for small eccentricities the most important terms, in producing secular apsidal motions, in the disturbing function.

† Cowling, *ibid.*

the same as equations (14) and (14 *a*) here. Equations (14) and (14 *a*) then carry the results to Cowling's order of accuracy and are seen to be the same as his.

The present analysis leads to considerably higher degrees of central condensation than Cowling's, when applied to eclipsing binaries for which (a_i/A) is large.

7. *Summary*.—The secular motion of the apse, in the orbits of binary stars whose components are assumed to be composed of compressible fluids, is found in the limiting case where the orbital period is long compared with the free harmonic periods of the components. Account is taken of the variation of the tidal deformations with time. Terms of an order up to but not including the tenth power of (a_i/A), where the mean radii are a_1 and a_2 and the semi-major axis of the relative orbit is A, are retained in the final results. The orbital eccentricity, e, enters the analysis exactly through functions which are represented by closed expressions valid for all values of e less than unity. The motion is found to be a precession, and the results are the same as Cowling's (*M.N.*, **98**, 734, 1938) to his order of accuracy, which includes terms of the fifth order in (a_i/A). The present more complete results, which are given in a form applicable to all models, lead to still higher degrees of central condensation than Cowling's when (a_i/A) is not very small. The superficial tidal and rotational distortions are found incidentally, in forms applicable to all stellar models. For given stellar models, masses, a_i's, rotations, and A, the predicted apsidal precession increases rapidly with e and approaches infinity as e approaches unity.

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