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Aradhana Distribution and Its Applications

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Abstract A new one parameter continuous distribution named "Aradhana distribution" for modeling lifetime data from biomedical science and engineering has been proposed. Its mathematical and statistical properties including its shape, moments, hazard rate function, mean residual life function, stochastic ordering, mean deviations, order statistics, Bonferroni and Lorenz curves, Renyi entropy measure, stress-strength reliability have been presented. The conditions under which Aradhana distribution is over-dispersed, equi-dispersed, under-dispersed are discussed along with the conditions under which Akash, Shanker, Lindley, and exponential distributions are over-dispersed, equi-dispersed and under-dispersed. The maximum likelihood estimation and the method of moments have been discussed for estimating its parameter. The applicability and the goodness of fit of the proposed distribution over Akash, Shanker, Lindley and exponential distributions have been discussed and illustrated with two real lifetime data - sets.

Keywords Lindley distribution, Akash distribution, Shanker distribution, Mathematical and statistical properties, estimation of parameter, Goodness of fit

1. Introduction

The statistical analysis and modeling of lifetime data are essential in almost all applied sciences including, biomedical science, engineering, finance, and insurance, amongst others. A number of one parameter continuous distributions for modeling lifetime data has been introduced in statistical literature including Akash, Shanker, exponential, Lindley, gamma, lognormal, and Weibull. The Akash, Shanker, exponential, Lindley and the Weibull distributions are more popular than the gamma and the lognormal distributions because the survival functions of the gamma and the lognormal distributions cannot be expressed in closed forms and both require numerical integration. Though each of Akash, Shanker, Lindley and exponential distributions have one parameter, the Akash, Shanker, and Lindley distribution has one advantage over the exponential distribution that the exponential distribution has constant hazard rate whereas the Akash, Shanker, and Lindley distributions has monotonically increasing hazard rate. Further, it has been shown by Shanker (2015a, 2015 b) that Akash and Shanker distributions gives much closer fit in modeling lifetime data than Lindley and exponential distributions.

The probability density function (p.d.f.) and the cumulative distribution function (c.d.f.) of Lindley (1958) distribution are given by

$$f_1(x;\theta)=\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\quad ;x>0,\ \theta>0 \qquad (1.1)$$

$$F_1(x;\theta)=1-\left[1+\frac{\theta x}{\theta+1}\right]e^{-\theta x}\ ;x>0,\theta>0 \qquad (1.2)$$

The density (1.1) is a two-component mixture of an exponential distribution having scale parameter θ and a gamma distribution having shape parameter 2 and scale parameter θ with their mixing proportions $\frac{\theta}{\theta+1}$ and $\frac{1}{\theta+1}$ respectively. Detailed study about its various mathematical properties, estimation of parameter and application showing the superiority of Lindley distribution over exponential distribution for the waiting times before service of the bank customers has been done by Ghitany *et al* (2008). The Lindley distribution has been generalized, extended and modified along with their applications in modeling lifetime data from different fields of knowledge by different researchers including Zakerzadeh and Dolati (2009), Nadarajah *et al* (2011), Deniz and Ojeda (2011), Bakouch et al (2012), Shanker and Mishra (2013 a, 2013 b), Shanker and Amanuel (2013), Shanker *et al* (2013), Elbatal *et al* (2013), Ghitany *et al* (2013), Merovci (2013), Liyanage and Pararai (2014), Ashour and Eltehiwy (2014), Oluyede and Yang (2014), Singh *et al* (2014), Sharma *et al* (2015 a, 2015 b), Shanker *et al* (2015 a, 2015 b), Alkarni (2015), Pararai *et al* (2015), Abouammoh *et al* (2015) are some among others.

Shanker (2015 a) has introduced one parameter Akash

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distribution for modeling lifetime data defined by its p.d.f. and c.d.f.

$$f_2(x;\theta)=\frac{\theta^3}{\theta^2+2}\left(1+x^2\right)e^{-\theta x}\quad ;x>0,\ \theta>0 \qquad (1.3)$$

$$F_2(x;\theta)=1-\left[1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right]e^{-\theta x}\ ;x>0,\theta>0 \qquad (1.4)$$

Shanker (2015 a) has shown that density (1.3) is a two-component mixture of an exponential distribution with scale parameter θ and a gamma distribution having shape parameter 3 and a scale parameter θ with their mixing proportions $\frac{\theta^2}{\theta^2+2}$ and $\frac{2}{\theta^2+2}$ respectively. Shanker (2015 a) has discussed its various mathematical and statistical properties including its shape, moment generating function, moments, skewness, kurtosis, hazard rate function, mean residual life function, stochastic orderings, mean deviations, distribution of order statistics, Bonferroni and Lorenz curves, Renyi entropy measure, stress-strength reliability, amongst others. Shanker *et al* (2015 c) has detailed study about modeling of various lifetime data from different fields using Akash, Lindley and exponential distributions and concluded that Akash distribution has some advantage over Lindley and exponential distributions. Further, Shanker (2015 c) has obtained Poisson mixture of Akash distribution named Poisson-Akash distribution (PAD) and discussed its various mathematical and statistical properties, estimation of its parameter and applications for various count data-sets.

The probability density function and the cumulative distribution function of Shanker distribution introduced by Shanker (2015 b) are given by

$$f_3(x;\theta)=\frac{\theta^2}{\theta^2+1}(\theta+x)e^{-\theta x}\quad ;x>0,\ \theta>0 \qquad (1.5)$$

$$F_3(x,\theta)=1-\frac{\left(\theta^2+1\right)+\theta x}{\theta^2+1}e^{-\theta x}\quad ;x>0,\theta>0 \qquad (1.6)$$

Shanker (2015 b) has shown that density (1.5) is a two-component mixture of an exponential distribution with scale parameter θ and a gamma distribution having shape parameter 2 and a scale parameter θ with their mixing proportions $\frac{\theta^2}{\theta^2+1}$ and $\frac{1}{\theta^2+1}$ respectively. Shanker (2015 b) has discussed its various mathematical and statistical properties including its shape, moment generating function, moments, skewness, kurtosis, hazard rate function, mean residual life function, stochastic orderings, mean deviations, distribution of order statistics, Bonferroni and Lorenz curves, Renyi entropy measure, stress-strength reliability , some amongst others. Further, Shanker (2015 d) has obtained Poisson mixture of Shanker distribution named Poisson-Shanker distribution (PSD) and discussed its various mathematical and statistical properties, estimation of its parameter and applications for various count data-sets.

Although Akash, Shanker, Lindley and exponential distributions have been used to model various lifetime data from biomedical science and engineering, there are many situations where these distributions may not be suitable from theoretical or applied point of view.

In search for a new lifetime distribution, we have proposed a new lifetime distribution which is better than Akash, Shanker, Lindley and exponential distributions for modeling lifetime data by considering a three-component mixture of an exponential distribution having scale parameter θ, a gamma distribution having shape parameter 2 and scale parameter θ, and a gamma distribution with shape parameter 3 and scale parameter θ with their mixing proportions $\frac{\theta^2}{\theta^2+2\theta+2}$, $\frac{2\theta}{\theta^2+2\theta+2}$ and $\frac{2}{\theta^2+2\theta+2}$, respectively. The probability density function (p.d.f.) of a new one parameter lifetime distribution can be introduced as

$$f_4(x;\theta)=\frac{\theta^3}{\theta^2+2\theta+2}(1+x)^2e^{-\theta x}\quad ;x>0,\ \theta>0 \qquad (1.7)$$

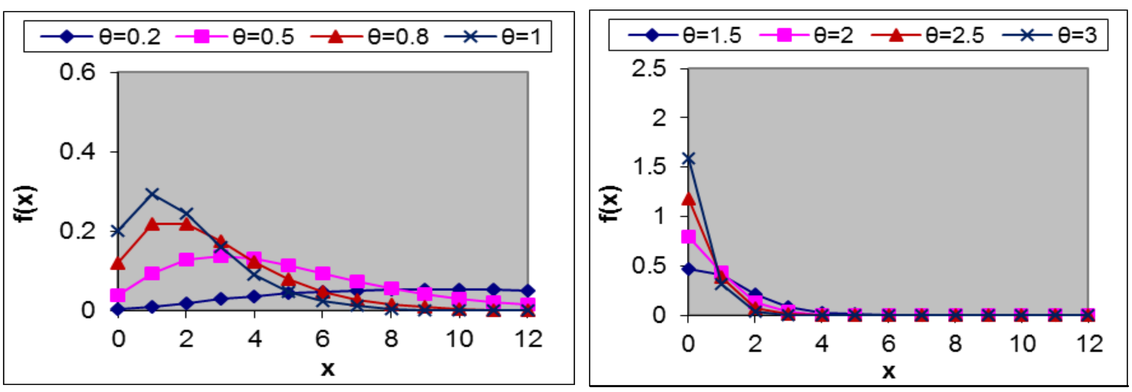

Figure 1. Graph of the pdf of Aradhana distribution for different values of parameter θ

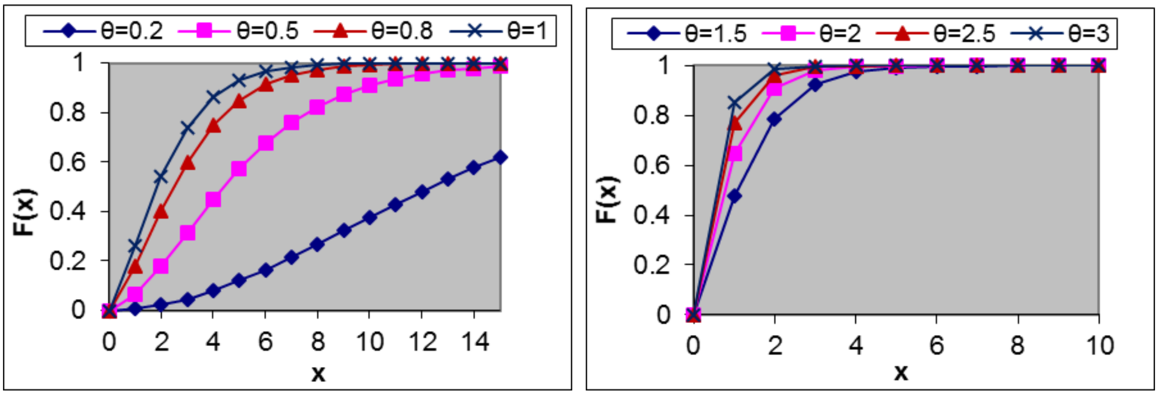

Figure 2. Graph of the cdf of Aradhana distribution for different values of parameter θ

We would call this distribution, "Aradhana distribution". The corresponding cumulative distribution function (c.d.f.) of (1.7) can be obtained as

$$F_4(x;\theta)=1-\left[1+\frac{\theta x(\theta x+2\theta+2)}{\theta^2+2\theta+2}\right]e^{-\theta x}\quad ;x>0,\theta>0 \tag{1.8}$$

The graphs of the p.d.f. and the c.d.f. of Aradhana distributions for different values of θ are shown in **figures 1 and 2.**

2. Moment Generating Function, Moments and Related Measures

The moment generating function of Aradhana distribution (1.7) can be obtained as

$$M_X(t)=\frac{\theta^3}{\theta^2+2\theta+2}\int_0^\infty e^{-(\theta-t)}(1+x)^2\,dx$$

$$=\frac{\theta^3}{\theta^2+2\theta+2}\left[\frac{1}{(\theta-t)}+\frac{2}{(\theta-t)^2}+\frac{2}{(\theta-t)^3}\right]$$

$$=\frac{\theta^3}{\theta^2+2\theta+2}\left[\frac{1}{\theta}\sum_{k=0}^{\infty}\left(\frac{t}{\theta}\right)^k+\frac{2}{\theta^2}\sum_{k=0}^{\infty}\binom{k+1}{k}\left(\frac{t}{\theta}\right)^k+\frac{2}{\theta^3}\sum_{k=0}^{\infty}\binom{k+2}{k}\left(\frac{t}{\theta}\right)^k\right]$$

$$=\sum_{k=0}^{\infty}\frac{\theta^2+2(k+1)\theta+(k+1)(k+2)}{\theta^2+2\theta+2}\left(\frac{t}{\theta}\right)^k$$

Thus the r th moment about origin, as given by the coefficient of $\frac{t^r}{r!}$ in $M_X(t)$, of Aradhana distributon (1.7) has been obtained as

$$\mu_r'=\frac{r!\left[\theta^2+2(r+1)\theta+(r+1)(r+2)\right]}{\theta^r(\theta^2+2\theta+2)}\quad ;r=1,2,3,...$$

and so the first four moments about origin as

$$\mu_1'=\frac{\theta^2+4\theta+6}{\theta(\theta^2+2\theta+2)},\quad \mu_2'=\frac{2(\theta^2+6\theta+12)}{\theta^2(\theta^2+2\theta+2)},\quad \mu_3'=\frac{6(\theta^2+8\theta+20)}{\theta^3(\theta^2+2\theta+2)},\quad \mu_4'=\frac{24(\theta^2+10\theta+30)}{\theta^4(\theta^2+2\theta+2)}$$

Thus the moments about mean of the Aradhana distribution (1.7) are obtained as

$$\mu_2=\frac{\theta^4+8\theta^3+24\theta^2+24\theta+12}{\theta^2(\theta^2+2\theta+2)^2}$$

$$\mu_3 = \frac{2\left(\theta^6+12\theta^5+54\theta^4+100\theta^3+108\theta^2+72\theta+24\right)}{\theta^3\left(\theta^2+2\theta+2\right)^3}$$

$$\mu_4 = \frac{3\left(3\theta^8+48\theta^7+304\theta^6+944\theta^5+1816\theta^4+2304\theta^3+1920\theta^2+960\theta+240\right)}{\theta^4\left(\theta^2+2\theta+2\right)^4}$$

The coefficient of variation $(C.V)$, coefficient of skewness $\left(\sqrt{\beta_1}\right)$, coefficient of kurtosis (β_2) and Index of dispersion (γ) of Aradhana distribution (1.7) are thus obtained as

$$C.V = \frac{\sigma}{\mu_1'} = \frac{\sqrt{\theta^4+8\theta^3+24\theta^2+24\theta+12}}{\theta^2+4\theta+6}$$

$$\sqrt{\beta_1} = \frac{\mu_3}{\mu_2^{3/2}} = \frac{2\left(\theta^6+12\theta^5+54\theta^4+100\theta^3+108\theta^2+72\theta+24\right)}{\left(\theta^4+8\theta^3+24\theta^2+24\theta+12\right)^{3/2}}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3\left(3\theta^8+48\theta^7+304\theta^6+944\theta^5+1816\theta^4+2304\theta^3+1920\theta^2+960\theta+240\right)}{\left(\theta^4+8\theta^3+24\theta^2+24\theta+12\right)^2}$$

$$\gamma = \frac{\sigma^2}{\mu_1'} = \frac{\theta^4+8\theta^3+24\theta^2+24\theta+12}{\theta\left(\theta^2+2\theta+2\right)\left(\theta^2+4\theta+6\right)}$$

Table 1. Over-dispersion, equi-dispersion and under-dispersion of Aradhana, Akash, Shanker, Lindley, and exponential distributions for varying values of their parameter θ

Distribution	**Over-dispersion** $\left(\mu<\sigma^2\right)$	**Equi-dispersion** $\left(\mu=\sigma^2\right)$	**Under-dispersion** $\left(\mu>\sigma^2\right)$
Aradhana	$\theta<1.283826505$	$\theta=1.283826505$	$\theta>1.283826505$
Akash	$\theta<1.515400063$	$\theta=1.515400063$	$\theta>1.515400063$
Shanker	$\theta<1.171535555$	$\theta=1.171535555$	$\theta>1.171535555$
Lindley	$\theta<1.170086487$	$\theta=1.170086487$	$\theta>1.170086487$
Exponential	$\theta<1$	$\theta=1$	$\theta>1$

3. Hazard Rate Function and Mean Residual Life Function

Let X be a continuous random variable with p.d.f. $f(x)$ and c.d.f. $F(x)$. The hazard rate function (also known as the failure rate function) and the mean residual life function of X are respectively defined as

$$h(x) = \lim_{\Delta x\to 0}\frac{P\left(X< x+\Delta x \mid X> x\right)}{\Delta x} = \frac{f(x)}{1-F(x)} \tag{3.1}$$

and $$m(x) = E\left[X-x \mid X>x\right] = \frac{1}{1-F(x)}\int_x^\infty \left[1-F(t)\right]\,dt \tag{3.2}$$

The corresponding hazard rate function, $h(x)$ and the mean residual life function, $m(x)$ of the Aradhana distribution (1.7) are obtained as

$$h(x)=\frac{\theta^3(1+x)^2}{\theta x(\theta x+2\theta+2)+(\theta^2+2\theta+2)} \tag{3.3}$$

and

$$m(x)=\frac{1}{\left[\theta x(\theta x+2\theta+2)+(\theta^2+2\theta+2)\right]e^{-\theta x}}\int_x^\infty\left[\theta t(\theta t+2\theta+2)+(\theta^2+2\theta+2)\right]e^{-\theta t}dt$$

$$=\frac{\theta^2x^2+2\theta x(\theta+2)+(\theta^2+4\theta+6)}{\theta\left[\theta x(\theta x+2\theta+2)+(\theta^2+2\theta+2)\right]} \tag{3.4}$$

It can be easily seen that $h(0)=\frac{\theta^3}{\theta^2+2\theta+2}=f_4(0)$ and $m(0)=\frac{\theta^2+4\theta+6}{\theta(\theta^2+2\theta+2)}=\mu_1'$. It is also obvious from the graphs of $h(x)$ and $m(x)$ that $h(x)$ is an increasing function of x, and θ, whereas $m(x)$ is a decreasing function of x, and θ.

The graphs of the hazard rate function and mean residual life function of Aradhana distribution (1.7) are shown in figures 3 and 4.

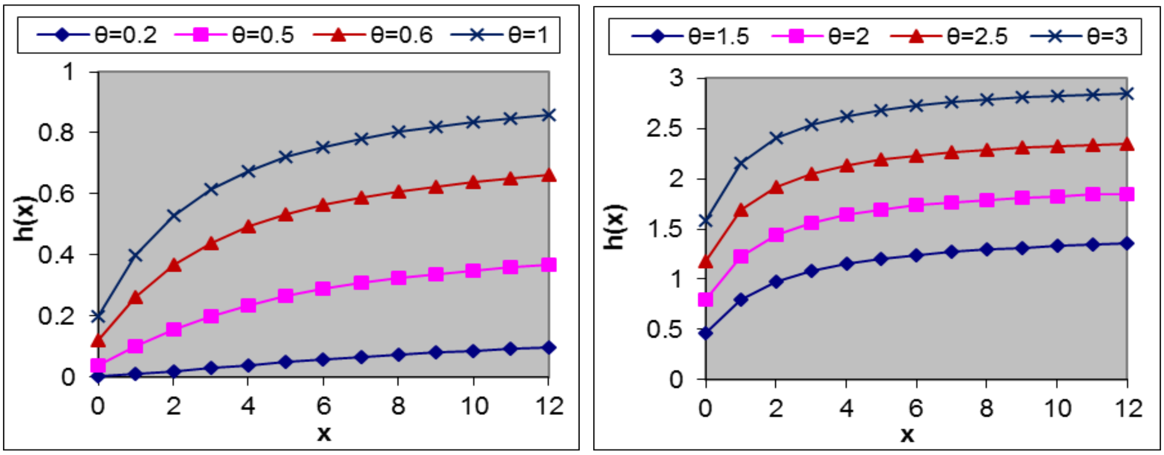

Figure 3. Graph of hazard rate function of Aradhana distribution for different values of parameter θ

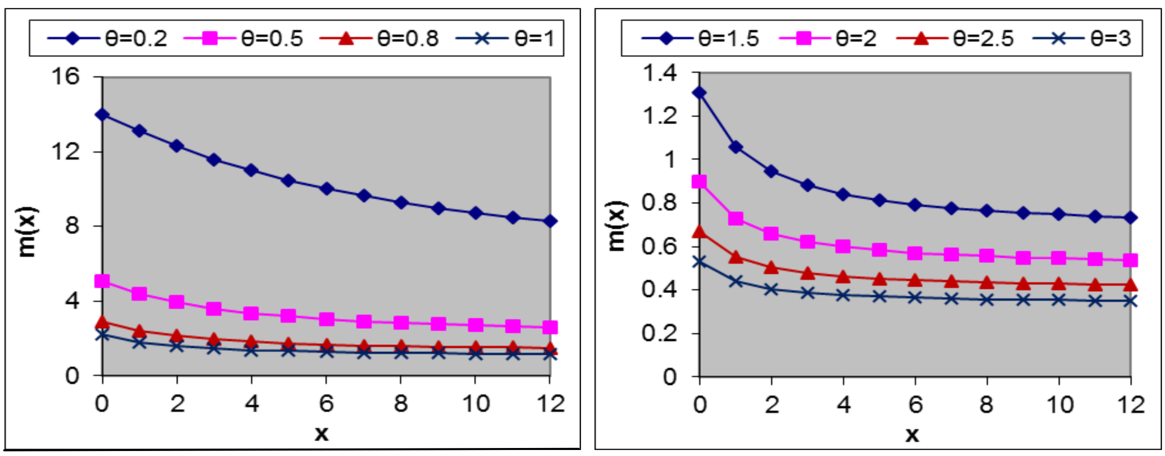

Figure 4. Graph of mean residual life function of Aradhana distribution for different values of parameter θ

4. Stochastic Orderings

Stochastic ordering of positive continuous random variables is an important tool for judging their comparative behavior. A random variable X is said to be smaller than a random variable Y in the

(i) stochastic order $\left(X \le_{st} Y\right)$ if $F_X(x) \ge F_Y(x)$ for all x

(ii) hazard rate order $\left(X \le_{hr} Y\right)$ if $h_X(x) \ge h_Y(x)$ for all x

(iii) mean residual life order $\left(X \le_{mrl} Y\right)$ if $m_X(x) \le m_Y(x)$ for all x

(iv) likelihood ratio order $\left(X \le_{lr} Y\right)$ if $\dfrac{f_X(x)}{f_Y(x)}$ decreases in x.

The following results due to Shaked and Shanthikumar (1994) are well known for establishing stochastic ordering of distributions

$$\begin{array}{c} X \le_{lr} Y \Rightarrow X \le_{hr} Y \Rightarrow X \le_{mrl} Y \\ \Downarrow \\ X \le_{st} Y \end{array}$$

The Aradhana distribution is ordered with respect to the strongest 'likelihood ratio' ordering as shown in the following theorem:

Theorem: Let X ~ Aradhana distributon $\left(\theta_1\right)$ and Y ~ Aradhana distribution $\left(\theta_2\right)$. If $\theta_1 \ge \theta_2$, then $X \le_{lr} Y$ and hence $X \le_{hr} Y$, $X \le_{mrl} Y$ and $X \le_{st} Y$.

Proof: We have

$$\frac{f_X(x)}{f_Y(x)} = \frac{\theta_1^3\left(\theta_2^2+2\theta_2+2\right)}{\theta_2^3\left(\theta_1^2+2\theta_1+2\right)} e^{-\left(\theta_1-\theta_2\right)x} \quad ; \; x>0$$

Now

$$\log\frac{f_X(x)}{f_Y(x)} = \log\left[\frac{\theta_1^3\left(\theta_2^2+2\theta_2+2\right)}{\theta_2^3\left(\theta_1^2+2\theta_1+2\right)}\right] - \left(\theta_1-\theta_2\right)x$$

This gives $\dfrac{d}{dx}\log\dfrac{f_X(x)}{f_Y(x)} = -\left(\theta_1-\theta_2\right)$

Thus for $\theta_1 \ge \theta_2$, $\dfrac{d}{dx}\log\dfrac{f_X(x)}{f_Y(x)} \le 0$. This means that $X \le_{lr} Y$ and hence $X \le_{hr} Y$, $X \le_{mrl} Y$ and $X \le_{st} Y$.

5. Mean Deviations

The amount of scatter in a population is measured to some extent by the totality of deviations usually from their mean and median. These are known as the mean deviation about the mean and the mean deviation about the median and are defined as

$$\delta_1(X) = \int_0^\infty |x-\mu| f(x)\,dx \quad \text{and} \quad \delta_2(X) = \int_0^\infty |x-M| f(x)\,dx,$$

respectively, where $\mu = E(X)$ and $M = \text{Median}(X)$. The measures $\delta_1(X)$ and $\delta_2(X)$ can be computed using the following simplified relationships

$$\begin{aligned} \delta_1(X) &= \int_0^\mu (\mu-x) f(x)\,dx + \int_\mu^\infty (x-\mu) f(x)\,dx \\ &= \mu F(\mu) - \int_0^\mu x f(x)\,dx - \mu\left[1-F(\mu)\right] + \int_\mu^\infty x f(x)\,dx \\ &= 2\mu F(\mu) - 2\mu + 2\int_\mu^\infty x f(x)\,dx \\ &= 2\mu F(\mu) - 2\int_0^\mu x f(x)\,dx \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} \delta_2(X) &= \int_0^M (M-x) f(x)\,dx + \int_M^\infty (x-M) f(x)\,dx \\ &= M\,F(M) - \int_0^M x f(x)\,dx \\ &\quad -M\left[1-F(M)\right] + \int_M^\infty x f(x)\,dx \\ &= -\mu + 2\int_M^\infty x f(x)\,dx \\ &= \mu - 2\int_0^M x f(x)\,dx \end{aligned} \tag{5.2}$$

Using p.d.f. (1.7) and expression for the mean of Aradhana distribution (1.7), we get

$$\int_0^{\mu} x\, f_4(x)\,dx = \mu - \frac{\left\{\theta^3\left(\mu^3+2\mu^2+\mu\right)+\theta^2\left(3\mu^2+4\mu+1\right)+2\theta\left(3\mu+2\right)+6\right\}e^{-\theta\mu}}{\theta\left(\theta^2+2\theta+2\right)} \tag{5.3}$$

$$\int_0^{M} x\, f_4(x)\,dx = \mu - \frac{\left\{\theta^3\left(M^3+2M^2+M\right)+\theta^2\left(3M^2+4M+1\right)+2\theta\left(3M+2\right)+6\right\}e^{-\theta M}}{\theta\left(\theta^2+2\theta+2\right)} \tag{5.4}$$

Using expressions (5.1), (5.2), (5.3) and (5.4), the mean deviation about mean, $\delta_1(X)$ and the mean deviation about median, $\delta_2(X)$ of Aradhana distribution (1.7), after a little algebraic simplification, are obtained as

$$\delta_1(X) = \frac{2\left\{\theta^2\left(\mu^2+2\mu+1\right)+4\theta\left(\mu+1\right)+6\right\}e^{-\theta\mu}}{\theta\left(\theta^2+2\theta+2\right)} \tag{5.5}$$

$$\delta_2(X) = \frac{2\left\{\theta^3\left(M^3+2M^2+M\right)+\theta^2\left(3M^2+4M+1\right)+2\theta\left(3M+2\right)+6\right\}e^{-\theta M}}{\theta\left(\theta^2+2\theta+2\right)} - \mu \tag{5.6}$$

6. Order Statistics

Let $X_1, X_2, ..., X_n$ be a random sample of size n from Aradhana distribution (1.7). Let $X_{(1)} < X_{(2)} < ... < X_{(n)}$ denote the corresponding order statistics. The p.d.f. and the c.d.f. of the k th order statistic, say $Y = X_{(k)}$ are given by

$$f_Y(y) = \frac{n!}{(k-1)!(n-k)!} F^{k-1}(y)\left\{1-F(y)\right\}^{n-k} f(y)$$

$$= \frac{n!}{(k-1)!(n-k)!}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l F^{k+l-1}(y) f(y)$$

and

$$F_Y(y) = \sum_{j=k}^{n}\binom{n}{j} F^j(y)\left\{1-F(y)\right\}^{n-j}$$

$$= \sum_{j=k}^{n}\sum_{l=0}^{n-j}\binom{n}{j}\binom{n-j}{l}(-1)^l F^{j+l}(y),$$

respectively, for $k = 1, 2, 3, ..., n$.

Thus, the p.d.f. and the c.d.f of k th order statistics of Aradhana distribution (1.7) are obtained as

$$f_Y(y) = \frac{n!\theta^3(1+x)^2 e^{-\theta x}}{\left(\theta^2+2\theta+2\right)(k-1)!(n-k)!}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l\left[1-\frac{\theta x\left(\theta x+2\theta+2\right)+\left(\theta^2+2\theta+2\right)}{\theta^2+2\theta+2}e^{-\theta x}\right]^{k+l-1}$$

and

$$F_Y(y) = \sum_{j=k}^{n}\sum_{l=0}^{n-j}\binom{n}{j}\binom{n-j}{l}(-1)^l\left[1-\frac{\theta x\left(\theta x+2\theta+2\right)+\left(\theta^2+2\theta+2\right)}{\theta^2+2\theta+2}e^{-\theta x}\right]^{j+l}$$

7. Bonferroni and Lorenz Curves

The Bonferroni and Lorenz curves (Bonferroni, 1930) and Bonferroni and Gini indices have applications not only in economics to study income and poverty, but also in other fields like reliability, demography, insurance and medicine. The Bonferroni and Lorenz curves are defined as

$$B(p)=\frac{1}{p\mu}\int_0^q x\,f(x)\,dx=\frac{1}{p\mu}\left[\int_0^\infty x\,f(x)\,dx-\int_q^\infty x\,f(x)\,dx\right]=\frac{1}{p\mu}\left[\mu-\int_q^\infty x\,f(x)\,dx\right] \tag{7.1}$$

$$\text{and}\quad L(p)=\frac{1}{\mu}\int_0^q x\,f(x)\,dx=\frac{1}{\mu}\left[\int_0^\infty x\,f(x)\,dx-\int_q^\infty x\,f(x)\,dx\right]=\frac{1}{\mu}\left[\mu-\int_q^\infty x\,f(x)\,dx\right] \tag{7.2}$$

respectively or equivalently

$$B(p)=\frac{1}{p\mu}\int_0^p F^{-1}(x)\,dx \tag{7.3}$$

$$\text{and}\quad L(p)=\frac{1}{\mu}\int_0^p F^{-1}(x)\,dx \tag{7.4}$$

respectively, where $\mu=E(X)$ and $q=F^{-1}(p)$.

The Bonferroni and Gini indices are thus defined as

$$B=1-\int_0^1 B(p)\,dp \tag{7.5}$$

$$\text{and}\quad G=1-2\int_0^1 L(p)\,dp \tag{7.6}$$

respectively.

Using p.d.f. (1.7), we get

$$\int_q^\infty x\,f_4(x)\,dx=\frac{\left\{\theta^3\left(q^3+2q^2+q\right)+\theta^2\left(3q^2+4q+1\right)+2\theta(3q+2)+6\right\}e^{-\theta q}}{\theta\left(\theta^2+2\theta+2\right)} \tag{7.7}$$

Now using equation (7.7) in (7.1) and (7.2), we get

$$B(p)=\frac{1}{p}\left[1-\frac{\left\{\theta^3\left(q^3+2q^2+q\right)+\theta^2\left(3q^2+4q+1\right)+2\theta(3q+2)+6\right\}e^{-\theta q}}{\theta^2+4\theta+6}\right] \tag{7.8}$$

$$L(p)=1-\frac{\left\{\theta^3\left(q^3+2q^2+q\right)+\theta^2\left(3q^2+4q+1\right)+2\theta(3q+2)+6\right\}e^{-\theta q}}{\theta^2+4\theta+6} \tag{7.9}$$

Now using equations (7.8) and (7.9) in (7.5) and (7.6), the Bonferroni and Gini indices of Aradhana distribution (1.7) are obtained as

$$B=1-\frac{\left\{\theta^3\left(q^3+2q^2+q\right)+\theta^2\left(3q^2+4q+1\right)+2\theta(3q+2)+6\right\}e^{-\theta q}}{\theta^2+4\theta+6} \tag{7.10}$$

$$G=-1+\frac{2\left\{\theta^3\left(q^3+2q^2+q\right)+\theta^2\left(3q^2+4q+1\right)+2\theta(3q+2)+6\right\}e^{-\theta q}}{\theta^2+4\theta+6} \tag{7.11}$$

8. Renyi Entropy

An entropy of a random variable X is a measure of variation of uncertainty. A popular entropy measure is Renyi entropy (1961). If X is a continuous random variable having probability density function $f(.)$, then Renyi entropy is defined as

$$T_R(\gamma)=\frac{1}{1-\gamma}\log\left\{\int f^{\gamma}(x)\,dx\right\}$$

where $\gamma>0$ and $\gamma\neq 1$.

Thus, the Renyi entropy for the Aradhana distribution (1.7) can be obtained as

$$T_R(\gamma)=\frac{1}{1-\gamma}\log\left[\int_0^{\infty}\frac{\theta^{3\gamma}}{\left(\theta^2+2\theta+2\right)^{\gamma}}(1+x)^{2\gamma}e^{-\theta\gamma x}dx\right]$$

$$=\frac{1}{1-\gamma}\log\left[\int_0^{\infty}\frac{\theta^{3\gamma}}{\left(\theta^2+2\theta+2\right)^{\gamma}}\sum_{j=0}^{\infty}\binom{\gamma}{j}(x)^{j}e^{-\theta\gamma x}dx\right]$$

$$=\frac{1}{1-\gamma}\log\left[\sum_{j=0}^{\infty}\binom{\gamma}{j}\frac{\theta^{3\gamma}}{\left(\theta^2+2\theta+2\right)^{\gamma}}\int_0^{\infty}e^{-\theta\gamma x}x^{j+1-1}dx\right]$$

$$=\frac{1}{1-\gamma}\log\left[\sum_{j=0}^{\infty}\binom{\gamma}{j}\frac{\theta^{3\gamma-j-1}}{\left(\theta^2+2\theta+2\right)^{\gamma}}\frac{\Gamma(j+1)}{(\gamma)^{j+1}}\right]$$

9. Stress-strength Reliability

The stress - strength reliability describes the life of a component which has random strength X that is subjected to a random stress Y. When the stress Y applied to it exceeds the strength X, the component fails instantly and the component will function satisfactorily till $X>Y$. Therefore, $R=P(Y<X)$ is a measure of component reliability and is known as stress-strength reliability in statistical literature. It has wide applications in almost all areas of knowledge especially in engineering such as structures, deterioration of rocket motors, static fatigue of ceramic components, aging of concrete pressure vessels etc.

Let X and Y be independent strength and stress random variables having Aradhana distribution (1.7) with parameter θ_1 and θ_2 respectively. Then the stress-strength reliability R can be obtained as

$$R=P(Y<X)=\int_0^{\infty}P(Y<X\mid X=x)f_X(x)\,dx$$

$$=\int_0^{\infty}f_4(x;\theta_1)\,F_4(x;\theta_2)\,dx$$

$$=1-\frac{\theta_1^{3}\left[\begin{array}{l}\theta_2^{6}+2(2\theta_1+3)\theta_2^{5}+2\left(3\theta_1^{2}+10\theta_1+10\right)\theta_2^{4}+2\left(2\theta_1^{3}+12\theta_1^{2}+25\theta_1+20\right)\theta_2^{3}\\+\left(\theta_1^{4}+12\theta_1^{3}+42\theta_1^{2}+60\theta_1+40\right)\theta_2^{2}+2\left(\theta_1^{3}+7\theta_1^{2}+12\theta_1+10\right)\theta_1\theta_2\\+2\left(\theta_1^{2}+2\theta_1+2\right)\theta_1^{2}\end{array}\right]}{\left(\theta_1^{2}+2\theta_1+2\right)\left(\theta_2^{2}+2\theta_2+2\right)\left(\theta_1+\theta_2\right)^{5}}.$$

10. Estimation of Parameter

10.1. Maximum Likelihood Estimation

Let $(x_1, x_2, x_3, \ldots, x_n)$ be a random sample from Aradhana distribution (1.7). The likelihood function, L of Aradhana distribution (1.7) is given by

$$L = \left(\frac{\theta^3}{\theta^2 + 2\theta + 2} \right)^n \prod_{i=1}^{n} (1 + x_i)^2 e^{-n\theta \bar{x}}$$

The natural log likelihood function is thus obtained as

$$\ln L = n \ln \left(\frac{\theta^3}{\theta^2 + 2\theta + 2} \right) + 2\sum_{i=1}^{n} \ln(1 + x_i) - n\theta \bar{x}$$

Now $\dfrac{d \ln L}{d\theta} = \dfrac{3n}{\theta} - \dfrac{2n(\theta + 1)}{\theta^2 + 2\theta + 2} - n\bar{x}$

where $\bar{x}$ is the sample mean.

The maximum likelihood estimate, $\hat{\theta}$ of θ is the solution of the equation $\dfrac{d \ln L}{d\theta} = 0$ and so it can be obtained by solving the following non-linear equation

$$\bar{x}\theta^3 + (2\bar{x} - 1)\theta^2 + 2(\bar{x} - 2)\theta - 6 = 0 \qquad (10.1.1)$$

10.2. Method of Moment Estimation

Equating the population mean of the Aradhana distribution to the corresponding sample mean, the method of moment (MOM) estimate, $\tilde{\theta}$, of θ is same as given by equation (10.1.1).

11. Applications and Goodness of Fit

The Aradhana distribution has been fitted to a number of lifetime data-sets relating to biomedical science and engineering. In this section, we present the fitting of Aradhana distribution to two real lifetime data- sets and compare its goodness of fit with the one parameter Akash, Shanker, Lindley and exponential distributions.

Data set 1: The first data - set represents the lifetime's data relating to relief times (in minutes) of 20 patients receiving an analgesic and reported by Gross and Clark (1975, P. 105). The data are as follows:

1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6, 2.0

Data set 2: The second data - set is the strength data of glass of the aircraft window reported by Fuller *et al* (1994)

18.83, 20.80, 21.657, 23.03, 23.23, 24.05, 24.321, 25.50, 25.52, 25.80, 26.69, 26.77, 26.78, 27.05, 27.67, 29.90, 31.11, 33.20, 33.73, 33.76, 33.89, 34.76, 35.75, 35.91, 36.98, 37.08, 37.09, 39.58, 44.045, 45.29, 45.381

In order to compare the goodness of fit of Aradhana, Akash, Shanker, Lindley and exponential distributions, $-2\ln L$, AIC (Akaike Information Criterion), AICC (Akaike Information Criterion Corrected), BIC (Bayesian Information Criterion),K-S Statistics (Kolmogorov-Smirnov Statistics) for two real lifetime data - sets have been computed and presented in table 2. The formulae for computing AIC, AICC, BIC, and K-S Statistics are as follows:

$$AIC = -2\ln L + 2k \quad , \quad AICC = AIC + \frac{2k(k+1)}{(n-k-1)} \quad ,$$

$$BIC = -2\ln L + k \ln n \quad \text{and} \quad D = \sup_x \left| F_n(x) - F_0(x) \right| \quad ,$$

where k = the number of parameters, n = the sample size and $F_n(x)$ is the empirical distribution function.

The best distribution for modeling lifetime data is the distribution corresponding to lower values of $-2\ln L$, AIC, AICC, BIC, and K-S statistics

Table 2. MLE's, $-2\ln L$, AIC, AICC, BIC, and K-S Statistics of the fitted distributions of data -sets 1 and 2

	Model	Parameter estimate	$-2\ln L$	AIC	AICC	BIC	K-S statistic
Data 1	Aradhana	1.1232	56.4	58.4	58.6	59.4	0.3019
	Akash	1.1569	59.5	61.7	61.7	62.5	0.3205
	Shanker	0.8039	59.7	61.8	62.0	62.8	0.3151
	Lindley	0.8161	60.5	62.5	62.7	63.5	0.3410
	Exponential	0.5263	65.7	67.7	67.9	68.7	0.3895
Data 2	Aradhana	0.09432	242.2	244.2	244.3	245.6	0.2739
	Akash	0.09706	240.7	242.7	242.8	244.1	0.2664
	Shanker	0.06470	252.3	254.3	254.5	255.8	0.3263
	Lindley	0.06299	254.0	256.0	256.1	257.4	0.3332
	Exponential	0.03246	274.5	276.7	276.7	277.9	0.4264

It can be easily seen from above table that in data-set 1, Aradhana distribution gives better fitting than Akash, Shanker, Lindley, and exponential distributions and in data-set 2, Akash distribution gives better fitting than Aradhana, Shanker, Lindley and exponential distributions.

12. Conclusions

A new one parameter lifetime distribution named, “Aradhana distribution” has been suggested for modeling lifetime data-sets from engineering and medical science. Its important mathematical and statistical properties including shape, moments, skewness, kurtosis, hazard rate function, mean residual life function, stochastic ordering, mean deviations, order statistics have been discussed. Further, expressions for Bonferroni and Lorenz curves, Renyi entropy measure and stress-strength reliability of the suggested distribution have been derived. The method of moments and the method of maximum likelihood estimation have also been discussed for estimating its parameter. Finally, the goodness of fit test using K-S Statistics (Kolmogorov-Smirnov Statistics) for two real lifetime data-sets have been presented to demonstrate the applicability and comparability of Aradhana, Akash, Shanker, Lindley and exponential distributions for modeling lifetime data - sets.

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REFERENCES

[1] Abouammoh, A.M., Alshangiti, A.M. and Ragab, I.E. (2015): A new generalized Lindley distribution, *Journal of Statistical Computation and Simulation,* preprint http://dx.doi.org/10.1080/ 00949655.2014.995101.

[2] Alkarni, S. (2015): Extended Power Lindley distribution-A new Statistical model for non-monotone survival data, *European journal of statistics and probability*, 3(3), 19 – 34

[3] Ashour, S. and Eltehiwy, M. (2014): Exponentiated Power Lindley distribution, *Journal of Advanced Research,* preprint http://dx.doi.org/10.1016/ j.jare. 2014.08.005.

[4] Bakouch, H.S., Al-Zaharani, B. Al-Shomrani, A., Marchi, V. and Louzad, F. (2012): An extended Lindley distribution, *Journal of the Korean Statistical Society*, 41, 75 – 85

[5] Bonferroni, C.E. (1930): Elementi di Statistca generale, Seeber, Firenze.

[6] Deniz, E. and Ojeda, E. (2011): The discrete Lindley distribution-Properties and Applications, *Journal of Statistical Computation and Simulation*, 81, 1405 – 1416.

[7] Elbatal, I., Merovi, F. and Elgarhy, M. (2013): A new generalized Lindley distribution, *Mathematical theory and Modeling*, 3 (13), 30-47.

[8] Fuller, E.J., Frieman, S., Quinn, J., Quinn, G., and Carter, W. (1994): Fracture mechanics approach to the design of glass aircraft windows: A case study, SPIE Proc 2286, 419-430.

[9] Ghitany, M.E., Atieh, B. and Nadarajah, S. (2008): Lindley distribution and its Application, *Mathematics Computing and Simulation*, 78, 493 – 506.

[10] Ghitany, M., Al-Mutairi, D., Balakrishnan, N. and Al-Enezi, I. (2013): Power Lindley distribution and associated inference, *Computational Statistics and Data Analysis*, 64, 20 –33.

[11] Gross, A.J. and Clark, V.A. (1975): Survival Distributions: Reliability Applications in the Biometrical Sciences, John Wiley, New York.

[12] Lindley, D.V. (1958): Fiducial distributions and Bayes’ theorem, *Journal of the Royal Statistical Society, Series B*, 20, 102- 107.

[13] Liyanage, G.W. and Pararai. M. (2014): A generalized Power Lindley distribution with applications, *Asian journal of Mathematics and Applications*, 1 – 23.

[14] Merovci, F. (2013): Transmuted Lindley distribution, *International Journal of Open Problems in Computer Science and Mathematics*, 6, 63 – 72.

[15] Nadarajah, S., Bakouch, H.S. and Tahmasbi, R. (2011): A generalized Lindley distribution, *Sankhya Series B*, 73, 331 – 359.

[16] Oluyede, B.O. and Yang, T. (2014): A new class of generalized Lindley distribution with applications, *Journal of Statistical Computation and Simulation*, 85 (10), 2072 – 2100.

[17] Pararai, M., Liyanage, G.W. and Oluyede, B.O. (2015): A new class of generalized Power Lindley distribution with applications to lifetime data, *Theoretical Mathematics & Applications,* 5(1), 53 – 96.

[18] Renyi, A. (1961): On measures of entropy and information, in proceedings of the 4th berkeley symposium on Mathematical Statistics and Probability, 1, 547 – 561, Berkeley, university of California press.

[19] Shaked, M. and Shanthikumar, J.G. (1994): Stochastic Orders and Their Applications, Academic Press, New York.

[20] Shanker, R. (2015 a): Akash distribution and Its Applications, *International Journal of Probability and Statistics*, 4(3), 65 – 75.

[21] Shanker, R. (2015 b): Shanker distribution and Its Applications, *International Journal of Statistics and Applications*, 5(6), 338 – 348.

[22] Shanker, R. (2015 c): The discrete Poisson-Akash distribution, communicated.

[23] Shanker, R. (2015 d): The discrete Poisson-Shanker distribution, communicated.

[24] Shanker, R., Hagos, F, and Sujatha, S. (2015a): On modeling of Lifetimes data using exponential and Lindley distributions, *Biometrics & Biostatistics International Journal*, 2(5), 1-9.

[25] Shanker, R., Hagos, F. and Sharma, S. (2015b): On two parameter Lindley distribution and Its Applications to model Lifetime data, *Biometrics & Biostatistics International Journal,* 3 (2), 1 – 8.

[26] Shanker, R., Hagos, F, and Sujatha, S. (2015 c): On modeling of Lifetimes data using one parameter Akash, Lindley and exponential distributions, submitted to *Biometrics & Biostatistics International Journal.*

[27] Shanker, R. and Mishra, A. (2013 a): A two-parameter Lindley distribution, *Statistics in Transition-new series*, 14 (1), 45-56.

[28] Shanker, R. and Mishra, A. (2013 b): A quasi Lindley distribution, *African Journal of Mathematics and Computer Science Research*, 6(4), 64 – 71.

[29] Shanker, R. and Amanuel, A.G. (2013): A new quasi Lindley distribution, *International Journal of Statistics and Systems*, 8 (2), 143 – 156.

[30] Shanker, R., Sharma, S. and Shanker, R. (2013): A two-parameter Lindley distribution for modeling waiting and survival times data, *Applied Mathematics*, 4, 363 – 368.

[31] Sharma, V., Singh, S., Singh, U. and Agiwal, V. (2015): The inverse Lindley distribution- A stress-strength reliability model with applications to head and neck cancer data, *Journal of Industrial & Production Engineering*, 32 (3), 162 – 173.

[32] Singh, S.K., Singh, U. and Sharma, V.K. (2014): The Truncated Lindley distribution- inference and Application, *Journal of Statistics Applications& Probability*, 3(2), 219 – 228.

[33] Smith, R.L, and Naylor, J.C. (1987): A comparison of Maximum likelihood and Bayesian estimators for the three parameter Weibull distribution, *Applied Statistics*, 36, 358 - 369.

[34] Zakerzadeh, H. and Dolati, A. (2009): Generalized Lindley distribution, *Journal of Mathematical extension*, 3 (2), 13 – 25.