

Arash Method and Uncontrollable Data in DEA

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Abstract

The uncontrollable data such as environmental variables are a kind of data which cannot be controlled by managers or users, but they may affect in assessing the performance evaluation of decision making units (DMU). In this paper, the Arash Method (AM) is improved to assess the efficiency scores of DMUs which have some uncontrollable data. Moreover, the paper illustrates how it would be possible to examine the effects of uncontrollable data to find the efficient DMUs among the technical efficient ones and also arrange both inefficient and technical efficient DMUs at the same time. A numerical example is also considered to depict the validity and capabilities of the proposed extension Arash Method.

Mathematics Subject Classification: 90

Keywords: Data envelopment analysis, Arash method, Uncontrollable data, Technical efficiency, Efficiency.

1. Introduction

Arash Method (AM) in data envelopment analysis (DEA) is a method which discriminates between technical efficient decision making units (DMUs) and it is able to arrange all inefficient and technical efficient DMUs at the same time. This study improves AM for assessing the efficiency of DMUs with some uncontrollable data. In the paper, Section 2 is the background, the proposed method is illustrated in Section 3 and the paper is concluded in Section 4.

2. Background

In data envelopment analysis the Pareto-Koopmans definition states that full (100%) efficiency is attained by any DMU if and only if none of its inputs or outputs can be improved without worsening some of its other inputs or outputs [3]. However, Khezrimotlagh et al. [5] proved that similar to economics the

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Pareto-Koopmans definition is only able to identify the technical efficient DMUs and the meaning of technical efficiency should not be wrongly interpreted as efficiency. They proposed some counter examples to illustrate their claims and proposed a new method called Arash Method (**AM**) to examine the Farrell frontier and arrange technical efficient DMUs. They also proved that a technical efficient DMU may even be less efficient than an inefficient one. In order to illustrate the Arash Method let us assume that there are n DMUs ($DMU_i, i = 1, 2, \dots, n$) with m nonnegative inputs ($x_{ij}, j = 1, 2, \dots, m$) and p nonnegative outputs ($y_{ik}, k = 1, 2, \dots, p$) for each DMU which at least one of its inputs and one of its outputs are not zero. The ε -**AM** is as following where $DMU_l (l = 1, 2, \dots, n)$ is evaluated, w_j^- and w_k^+ are the user specified weights obtained through values judgment, $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m)$, $\varepsilon_j \geq 0$, s_j^- 's and s_k^+ 's are nonnegative slacks, for $j = 1, 2, \dots, m$ and $k = 1, 2, \dots, p$.

ε -AM:

$$\max \sum_{j=1}^m w_j^- s_j^- + \sum_{k=1}^p w_k^+ s_k^+,$$

Subject to

$$\sum_{i=1}^n \lambda_i x_{ij} + s_j^- = x_{lj} + \varepsilon_j / w_j^-, \forall j,$$

$$\sum_{i=1}^n \lambda_i y_{ik} - s_k^+ = y_{lk}, \forall k,$$

$$\lambda_i \geq 0, \forall i,$$

$$s_j^- \geq 0, \forall j,$$

$$s_k^+ \geq 0, \forall k.$$

Targets:

$$\begin{cases} x_{lj}^* = x_{lj} + \varepsilon_j / w_j^- - s_j^-, \forall j, \\ y_{lk}^* = y_{lk} + s_k^+, \forall k, \end{cases}$$

Score:

$$A^* = \frac{\sum_{k=1}^p w_k^+ y_{lk} / \sum_{j=1}^m w_j^- x_{lj}}{\sum_{k=1}^p w_k^+ y_{lk}^* / \sum_{j=1}^m w_j^- x_{lj}^*}.$$

Moreover, if the weights w_j^- and w_k^+ are unknown they are defined as $1/x_j$ and $1/y_k$ where $x_j \neq 0$ and $y_k \neq 0$, respectively. Otherwise, they can be defined a nonnegative real number according to the goal of DMUs. In addition, it is generally defined that $\varepsilon = (\varepsilon, \varepsilon, \dots, \varepsilon)$, and when $\varepsilon > 0$ and $A_\varepsilon^* < 1$ for a DMU, ε -**AM** proposes the DMU to change its data to the new ε -**AM** target and otherwise i.e., when $A_\varepsilon^* \geq 1$, ε -**AM** warns that the DMU should not change its data, because it may decrease its efficiency score. Khezrimotlagh et al. [6] also proposed the nonlinear **AM**, proved that the **AM** is able to measure cost-efficiency of DMUs [7] and proposed the following definition of efficiency in DEA:

Definition: A technical efficient DMU is efficient with ε -degree of freedom in inputs if $A_0^* - A_\varepsilon^* \leq \delta$. Otherwise, it is inefficient with ε -degree of freedom in inputs. The proposed amount for δ is $10^{-1}\varepsilon$ or ε/m .

On the other hand, DMUs may sometimes have some uncontrollable inputs such as population or runway in airport. In order to consider those data, Cooper et al. [4] introduced the following radial model for uncontrollable data based on the Banker and Morey [1] model and called it NCN. Moreover, Charnes et al. [2] extended the additive model as below where the sets C and NC refer to the indexes of controllable and non-controllable data, respectively.

NCN:
 min θ ,
 Subject to
 $\sum_{i=1}^n \lambda_i x_{ij} \leq \theta x_{lj}$, for j 's in C,
 $\sum_{i=1}^n \lambda_i x_{ij} = x_{lj}$, for j 's in NC,
 $\sum_{i=1}^n \lambda_i y_{ik} \geq y_{lk}$, for k 's in C,
 $\sum_{i=1}^n \lambda_i y_{ik} = y_{lk}$, for k 's in NC,
 $\lambda_i \geq 0, \forall i$.

Extended ADD:
 max $\sum_{j=1}^m s_j^- + \sum_{k=1}^p s_k^+$,
 Subject to
 $\sum_{i=1}^n \lambda_i x_{ij} + s_j^- = x_{lj}, \forall j$,
 $\sum_{i=1}^n \lambda_i y_{ik} - s_k^+ = y_{lk}, \forall k$,
 $\lambda_i \geq 0, \forall i$,
 $s_j^- \geq 0$, for j 's in C,
 $s_k^+ \geq 0$, for k 's in C,
 $s_j^- = 0$, for j 's in NC,
 $s_k^+ = 0$, for k 's in NC.

3. Uncontrollable Arash Model

From the above models it is very easy to improve Arash Model to consider uncontrollable data, however, the corresponding constraints to the uncontrollable input values of DMU_l, i.e., $\sum_{i=1}^n \lambda_i x_{ij} = x_{lj}$, for j 's in NC, can be changed in Arash Method. In other words, without decreasing the uncontrollable inputs Arash Method is able to examine the instabilities of the combinations of data and arrange the technical efficient DMUs among all DMUs by assuming when a little error happens in the inputs data. As it is clearly illustrated in the following numerical example, it would be much significant where these two constraints (1) ' $\sum_{i=1}^n \lambda_i x_{ij} + s_j^- = x_{lj} + \varepsilon_j/w_j^-$ ' and (2) ' $s_j^- \leq \varepsilon_j/w_j^-$ ', for j 's in NC, are replaced by $\sum_{i=1}^n \lambda_i x_{ij} = x_{lj}$, in the Arash Model for uncontrollable inputs. Therefore, the Improved AM and the proposed extension of AM are as following:

Improved ε -AM:
 max $\sum_{j=1}^m w_j^- s_j^- + \sum_{k=1}^p w_k^+ s_k^+$,
 Subject to
 $\sum_{i=1}^n \lambda_i x_{ij} + s_j^- = x_{lj} + \varepsilon_j/w_j^-, \forall j$,
 $\sum_{i=1}^n \lambda_i y_{ik} - s_k^+ = y_{lk}, \forall k$,
 $\lambda_i \geq 0, \forall i$,
 $s_j^- \geq 0$, for j 's in C,
 $s_k^+ \geq 0$, for k 's in C,
 $s_j^-, \varepsilon_j = 0$, for j 's in NC,
 $s_k^+ = 0$, for k 's in NC.

Proposed Extension ε -AM:
 max $\sum_{j=1}^m w_j^- s_j^- + \sum_{k=1}^p w_k^+ s_k^+$,
 Subject to
 $\sum_{i=1}^n \lambda_i x_{ij} + s_j^- = x_{lj} + \varepsilon_j/w_j^-, \forall j$,
 $\sum_{i=1}^n \lambda_i y_{ik} - s_k^+ = y_{lk}, \forall k$,
 $\lambda_i \geq 0, \forall i$,
 $s_j^- \geq 0$, for j 's in C,
 $s_k^+ \geq 0$, for k 's in C,
 $s_j^- \leq \varepsilon_j/w_j^-$, for j 's in NC,
 $s_k^+ = 0$, for k 's in NC.

The constraints $s_j^- \leq \varepsilon_j/w_j^-$, in proposed extension ε -AM, guarantee the corresponding uncontrollable inputs and do not allow the model to decrease their values, because the s_j^- 's at most can be the same as ε_j/w_j^- which were added to x_{lj} 's. In order to examine the models and illustrate their capabilities, let us consider the example of public libraries with four inputs (floor area, number of books, staffs and population of wards) and two outputs (number of registered residents and borrow books) in the 12 Wards of the Tokyo Metropolitan Area in 1986 [4], which are labelled as A01 to A12 in Table 1. Assume that the

population of wards, i.e., Input4, is uncontrollable and $w_j^- = 1/x_j$, and $w_k^+ = 1/y_k$, for $j = 1,2,3,4$ and $k = 1,2$.

Table 1: The example of 12 DMUs with one uncontrollable input.

DMU	x_1	x_2	x_3	x_4	y_1	y_2
A01	11381	363116	69	192235	57279	758704
A02	10086	541658	114	194091	66137	1438746
A03	5077	511467	84	267385	65391	1562274
A04	10866	566708	118	503914	102967	1707645
A05	11469	768484	103	537746	84510	2299694
A06	10888	1148863	202	808369	191166	4096300
A07	6235	394158	77	389894	57727	1100779
A08	6500	467617	74	517318	47236	1223026
A09	10717	844949	120	622550	89401	1909698
A10	19716	1258981	242	660164	97941	3055193
A11	7781	528799	96	365844	37467	1348588
A12	7072	527457	92	332609	56064	1345185

Table 2 depicts the scores of CCR and ϵ -AM for those DMUs where the data are controllable and when only Input4 is uncontrollable.

Table 2: The outcomes of the models without uncontrollable data and where Input4 is uncontrollable.

DMU	Inputs are controllable.				Only Input4 is uncontrollable.			
	CCR	A_0^*	$A_{0.00001}^*$	$A_{0.00002}^*$	NCN	A_0^*	$A_{0.00001}^*$	$A_{0.00002}^*$
A01	1.00000	1.00000	0.99996	0.99993	1.0000	1.00000	0.99999	0.99998
A02	1.00000	1.00000	1.00000	0.99999	1.0000	1.00000	1.00000	1.00000
A03	1.00000	1.00000	0.99999	0.99998	1.0000	1.00000	1.00000	1.00000
A04	1.00000	1.00000	0.99997	0.99994	1.0000	1.00000	0.99999	0.99997
A05	1.00000	1.00000	0.99997	0.99994	1.0000	1.00000	0.99999	0.99997
A06	1.00000	1.00000	1.00000	1.00000	1.0000	1.00000	1.00000	1.00000
A07	0.84407	0.66529	0.66529	0.66529	1.0000	1.00000	0.99998	0.99997
A08	0.78671	0.56949	0.56949	0.56949	1.0000	1.00000	0.99999	0.99997
A09	0.78723	0.62539	0.62539	0.62539	0.94177	0.78658	0.78657	0.78656
A10	0.78494	0.53513	0.53513	0.53513	0.70114	0.53513	0.53513	0.53513
A11	0.72154	0.48439	0.48439	0.48439	0.80040	0.48439	0.48439	0.48439
A12	0.75835	0.64631	0.64631	0.64631	0.77275	0.64631	0.64631	0.64631

From the fifth column of Table 2, **0.00002-AM** identifies that A02, A03 and A06 are efficient with 0.00002-degree of freedom in input and arranges all DMUs together. In addition, when Input4 is considered as uncontrollable, four technical efficient DMUs A01, A02, A03 and A06 can be efficient with 0.00002-degree of freedom in input and they have the most significant combination of their data where 0.00002 errors happen in those inputs by improved **AM** as the last column of Table 2 depicts it. However, it can arrange DMUs A02, A03 and A06.

Now, let us consider the proposed extension ϵ -AM. Table 3 represents the scores of **0.00001-AM** and its suggestion targets for those DMUs in Table 1. In

order to illustrate the outcomes clearly, the amounts of slacks and $\epsilon x_j = 0.00001 \times x_j$ are represented in Table 4.

Table 3: The targets and scores of proposed model (Extended **0.00001-AM**) and $A_{0.00002}^*$.

DMU	$A_{0.00001}^*$	x_1^*	x_2^*	x_3^*	x_4^*	y_1^*	y_2^*	$A_{0.00002}^*$
A01	0.99996	11380.43	363119.11	69.00	192236.92	57279.00	758750.31	0.99993
A02	1.00000	10085.86	541655.05	114.00	194092.94	66137.00	1438746.00	0.99999
A03	0.99999	5076.94	511462.21	84.00	267387.67	65391.49	1562274.00	0.99998
A04	0.99999	10865.72	566713.67	118.00	503914.00	102967.00	1707678.16	0.99997
A05	0.99999	11468.69	768477.95	103.00	537746.00	84511.40	2299694.00	0.99997
A06	1.00000	10888.00	1148863.00	202.00	808369.00	191166.00	4096300.00	1.00000
A07	0.99998	6234.87	394161.94	77.00	389894.00	57727.00	1100803.74	0.99997
A08	0.99999	6500.01	467620.58	74.00	517318.00	47237.01	1223044.72	0.99997
A09	0.78657	8084.16	712579.91	120.00	622550.00	98896.97	2254841.55	0.78656
A10	0.53513	8891.81	938232.41	164.97	660164.00	156117.95	3345291.31	0.53513
A11	0.48439	4927.64	519946.76	91.42	365847.66	86516.97	1853883.27	0.48439
A12	0.64631	4479.94	472707.60	83.11	332609.00	78656.57	1685450.89	0.64631

Table 4: The slacks and the amount of one hundred thousandth errors in input values by **0.00001-AM**.

DMU	s_1^{-*}	ϵx_1	s_2^{-*}	ϵx_2	s_3^{-*}	ϵx_3	s_4^{-*}	ϵx_4	s_1^{+*}	s_2^{+*}
A01	0.69	0.11	0.52	3.63	0.00	0.00	0.00	1.92	0.00	46.31
A02	0.24	0.10	8.37	5.42	0.00	0.00	0.00	1.94	0.00	0.00
A03	0.11	0.05	9.91	5.11	0.00	0.00	0.00	2.67	0.49	0.00
A04	0.38	0.11	0.00	5.67	0.00	0.00	5.04	5.04	0.00	33.16
A05	0.42	0.11	13.74	7.68	0.00	0.00	5.38	5.38	1.40	0.00
A06	0.11	0.11	11.49	11.49	0.00	0.00	8.08	8.08	0.00	0.00
A07	0.19	0.06	0.00	3.94	0.00	0.00	3.90	3.90	0.00	24.74
A08	0.06	0.07	1.09	4.68	0.00	0.00	5.17	5.17	1.01	18.72
A09	2632.94	0.11	132377.54	8.45	0.00	0.00	6.23	6.23	9495.97	345143.55
A10	10824.38	0.20	320761.18	12.59	77.04	0.00	6.60	6.60	58176.95	290098.31
A11	2853.44	0.08	8857.53	5.29	4.58	0.00	0.00	3.66	49049.97	505295.27
A12	2592.13	0.07	54754.67	5.27	8.89	0.00	3.33	3.33	22592.57	340265.89

As it can be seen, the optimum values of s_4^{-*} 's in Table 4 cannot be more than ϵx_4 . However, the values of s_4^{-*} for DMUs A01, A02, A03 and A11 are 0. This outcome means that the combinations of their controllable data can be improved even if those DMUs increase their uncontrollable Input4. For example, the proposed extension **0.00001-AM** suggests A01 (technical efficient DMU) to decrease its Input1 to 11380.43 and increase its output2 to 758750.31, even if it increases 0.001% of its input2 and input4. Although, A01 is able to reject increasing the input2 and input4 values, it is recommended to improve its input1 and output2 in order to have a good combination of its data. In other words, **AM** suggests A01 to concentrate to decrease its input1 and increase output2. This capability of **AM** clearly represents its worth in comparison with other current DEA models.

Moreover, the proposed extension **0.00001-AM** declares that the technical efficient DMU A02 should focus to decrease its input1 and input2 even if it

increases its uncontrollable input⁴. Therefore, it cannot be more efficient than A06. In fact, **0.00002-AM** characterizes that A02, A03 and A06 are efficient with 0.00002-degree of freedom in input as the last column of Table 3 illustrates it. Indeed, just 0.002% errors in each input is enough that **AM** discriminates the differences between those technical efficient DMUs in Table 1 and arrange all DMUs at the same time. As a result, the proposed extension **ϵ -AM** has more advantages than the first improved **AM**.

4. Conclusion

This paper proposes an extension of **AM** to consider uncontrollable data. The results significantly demonstrate the validity of the proposed model. In addition, although, the paper focuses on uncontrollable data, **AM** is able to improve for non-discretionary and bounded variables, too.

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Received: June, 2012