Arbitrage-free bilateral counterparty risk valuation under collateralization and re-hypothecation with application to CDS*

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Abstract

We develop an arbitrage-free valuation framework for bilateral counterparty risk, where collateral is included with possible re-hypothecation. We show that the adjustment is given by the sum of two option payoff terms, where each term depends on the netted exposure, i.e. the difference between the on-default exposure and the pre-default collateral account. We then specialize our analysis to Credit Default Swaps (CDS) as underlying portfolios, and construct a numerical scheme to evaluate the adjustment under a doubly stochastic default framework. In particular, we show that for CDS contracts a perfect collateralization cannot be achieved, even under continuous collateralization, if the reference entity's and counterparty's default times are dependent. The impact of re-hypothecation, collateral margining frequency, and default correlation induced contagion is illustrated with numerical examples.

Keywords: Counterparty Risk, CVA, Bilateral CVA, Arbitrage-Free Credit Valuation Adjustment, Credit Default Swaps, Credit Spread Volatility, Default Correlation, Contagion, Stochastic Intensity, Collateral Margining, Netting, Re-hypotecation, Wrong Way Risk.

1 Introduction

Counterparty credit risk has proven to be one of the major drivers of the credit crisis, due the high number of credit quality deteriorations and default events experienced by several financial entities. We recall as a

*This paper is an updated and improved version of the report in Brigo and Capponi (2008), see also Brigo and Capponi (2010).

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fundamental example the seven credit events occurred in one month of 2008, involving Fannie Mae, Freddie Mac, Lehman Brothers, Washington Mutual, Landsbanki, Glitnir and Kaupthing. Furthermore, the report from BIS (2011) notices that

"Under Basel II, the risk of counterparty default and credit migration risk were addressed but mark-to-market losses due to credit valuation adjustments (CVA) were not. During the financial crisis, however, roughly two-thirds of losses attributed to counterparty credit risk were due to CVA losses and only about one-third were due to actual defaults."

Therefore, even more than the actual defaults, CVA volatility has been a major risk factor behind the losses experienced during the crisis. This highlights the importance of proper CVA valuation.

In this paper we introduce a general arbitrage-free valuation framework for bilateral counterparty default risk, inclusive of collateralization (Bilateral Collateralized Credit Valuation Adjustment, BCCVA). By "bilateral" we intend to point out that the default of both parties in the transaction is considered into the framework. The bilateral nature of counterparty risk was first considered by Duffie and Huang (1996), who present a model for valuing claims subject to default by both contracting parties, such as swap and forward contracts. In their approach, when counterparties have different default risk, the promised cash flows of the swap are discounted using a switching discount rate that, at any given state and time, is equal to the discount rate of the counterparty for whom the swap is currently out of the money. A general formula for bilateral counterparty risk valuation was given in Bielecki and Rutkowski (2001).

The ongoing financial crisis has led the Basel Committee to revisit the guidelines for OTC derivatives transactions, moving towards a new set of rules commonly called "Basel III", and reviewed in the Basel III Proposal (2010). Beside stressing the need to correctly capture the dependence between market and credit risks, which was not adequately incorporated into the Basel II framework, several other amendments have been proposed. Those include extending the margin period of risk for OTC derivatives, increasing the incentives to use central counterparties to clear trades, and enhancing the controls regarding the re-hypothecation and re-investment of collaterals. Duffie and Zhu (2010) addresses the mitigation of counterparty risk exposure through the use of central clearing counterparties, showing that resorting to a central clearing counterparty for credit default swaps can reduce the netting efficiency and lead to an increase in average exposures to counterparty defaults.

We study how counterparty risk exposure can be reduced through the use of collateralization. We provide model independent formulas for bilateral collateralized credit valuation adjustment of portfolios exchanged between a default risky investor and a default risky counterparty. Such formulas are given by the sum of option payoff terms, where each term depends on the netted exposure, i.e. the difference between the close-out amount and the on-default collateral account. We consider both the case when collateral is a risk-free asset kept into a segregate account and used upon default occurrence to reduce net exposure, and the case when collateral may be lent or re-hypothecated before default occurrence, thus making the party who posted collateral an unsecured creditor. This generalizes the framework given in Assefa et al. (2009), who build on the work of Brigo and Capponi (2008) and provide a representation formula for bilateral counterparty risk valuation adjustment for a fully netted and collateralized portfolio of contracts. Assefa et al. (2009) consider a highly stylized model for the collateral process, assuming that the collateral account is risk-free and cannot be re-hypothecated. Crépey (2011) proposes a hedging framework, where the value of a transaction is decomposed into three separate components, the collateral, the underlying portfolio, and the hedging portfolio, and he associates a funding account to each incoming cash flow, different for positive and negative cash flows. Bielecki and Crépey (2010) have developed an analytical framework for dynamic hedging of unilateral CVA, including collateralization, under a default framework where simultaneous defaults may occur with positive probability.

We then specialize our analysis to Credit Default Swaps (CDS) as underlying portfolio. Featuring a CDS as underlying, a third default time enters the picture, namely the default time for the reference credit of the CDS. We assume a doubly stochastic framework to model the default times of all three entities involved in the contract, namely the investor, counterparty and reference credit. We show that even a continuous collateralization scheme, where the two parties agree on posting collateral on a continuous mark-to-market basis, does not fully eliminate counterparty risk. This is because the computation of the pre-default exposure, which decides on the collateral to post, is computed by conditioning on the filtration excluding the sigma algebra generated by default time, whereas the computation of the on-default close out-amount is done on the enlarged filtration including the default time. We present a numerical study to evaluate the resulting BCCVA formula, and show the impact of different collateralization strategies, re-hypothecation and default correlation on the resulting credit valuation adjustment.

Earlier works on CDS with counterparty risk include Crépey et al. (2010), who employ a Markov chain copula model for computing counterparty risk, allowing for simultaneous defaults to occur. Bielecki et al. (2010) propose a hedging framework for credit default swaptions using CDS. A structural model with jumps for computing CVA on credit default swaps has been introduced by Lipton and Sepp (2009).

The rest of the paper is organized as follows. Section 2 develops a model independent formula for computing BCCVA, i.e. including counterparty risk and collateralization. Section 3 specializes the framework to credit default swaps, and illustrates how to compute on-default and pre-default survival probabilities. Section 4 specifies the credit default model, and presents numerical simulations illustrating the impact of default correlation and collateralization frequency on the resulting counterparty adjustment. Section 5 concludes the paper.

2 Arbitrage-free valuation of bilateral counterparty risk

We develop valuation formulas for bilateral credit valuation adjustment in presence of collateralization. Along the way, we highlight the relevant market standards and agreements which we follow to derive such formulas. We refer the reader to Brigo et al. (2011) for an extensive discussion of market considerations and of collateral mechanics, which also includes an analysis of credit valuation adjustments on interest rate swaps in presence of different collateralization strategies.

We refer to the two names involved in the financial contract and subject to default risk as *investor* (also called name "0") and *counterparty* (also called name "2"). In cases where the portfolio exchanged by the two parties is also a default sensitive instrument, we introduce a third name referring to the underlying reference credit of that portfolio (also called name "1").

We denote by τ_0, (τ_1) and τ_2 respectively the default times of the investor, (underlying reference credit) and counterparty. We fix the portfolio time horizon $T \in \mathbb{R}^+$, and fix the risk neutral pricing model $(\Omega, \mathcal{G}, \mathbb{Q})$, with a filtration $(\mathcal{G}_t)_{t\in[0,T]}$ such that τ_0, (τ_1) and τ_2 are $\mathcal{G}$-stopping times. This space is endowed also with a right-continuous and complete sub-filtration $\mathcal{F}_t$ representing all the observable market quantities but the default event, thus $\mathcal{F}_t \subseteq \mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t$, where $\mathcal{H}_t = \mathcal{H}^0_t(\vee\mathcal{H}^1_t)\vee\mathcal{H}^2_t$ is the right-continuous filtration generated by the default events, either of the investor (or of the underlying reference credit) or of his counterparty. Here $\mathcal{H}^i_t = \sigma(\{\tau_i \leq u\} : u \leq t)$, after completion and regularization on the right, see Belanger et al. (2004) for details. We denote by $\mathbb{E}_t$ the conditional expectation under $\mathbb{Q}$ given $\mathcal{G}_t$, and by $\mathbb{E}_{\tau_i}$ the conditional expectation under $\mathbb{Q}$ given the stopped filtration $\mathcal{G}_{\tau_i}$. We exclude the possibility of simultaneous defaults, and define the first default event between the two parties as the stopping time

$$\tau := \tau_0 \wedge \tau_2.$$

2.1 Collateral Modeling

We start by describing the mechanics of collateral posting. We assume that the collateral account is held by the *collateral taker*, with both investor and counterparty posting or withdrawing collateral during the life of the deal, to or from the collateral account. The other party is the *collateral provider*. We see all payoffs from the point of view of the investor. Therefore, when $C_t > 0$, this means that by time t the overall collateral account is in favor of the investor and the net posting has been done by the counterparty, meaning that what is in the account at t is the excess of posting done by the counterparty with respect to the investor posting. In this case the collateral account $C_t > 0$ can be used by the investor to reduce his close-out amount. On the contrary, when $C_t < 0$, this means that the overall collateral account by time t is in favor of the counterparty, and has been net-posted by the investor. In this case collateral can be used by the counterparty to reduce her close-out amount.

Thus when $C_t > 0$ this means that, at time t, the collateral taker is the investor and the collateral provider is the counterparty, whereas in the other case the collateral taker is the counterparty and the collateral provider is the investor.

We assume the collateral account to be a risk-free cash account, although in general it can be any other (defaultable) asset, which can be liquidated at the default time. Further, we assume that the collateral account is opened anew for each new deal and it is closed upon a default event or when maturity is reached. If the account is closed, then any collateral held by the collateral taker would be required to be returned to the originating party. We assume $C_t = 0$ for all $t \leq 0$, and $C_t = 0$, if $t \geq \tau \wedge T$. This suggests defining the *collateral account* C_t as the process

$$C_t := 1_{\{t<\tau\}} M_t \tag{2.1}$$

where M_t is a $\mathcal{G}_t$-adapted càdlàg process such that $M_0 = M_T = 0$. In the following, we should evaluate the collateral account at the default event to calculate close-out netting rules. Since we close the collateral account upon a default event, we need to value the collateral account just before the default event happens, referring to $C_{\tau-}$.

2.2 Bilateral Risk with Collateralization

We call $\Pi(u,s)$ the net cash flows of the claim under consideration (not including the collateral account) without investor or counterparty default risk between time u and time s, discounted back at u, as seen from the point of view of the investor. We denote by $\Pi^D(u,s,C)$ the analogous net cash flows of the claim under counterparty and investor default risk, and inclusive of collateral netting. The *counterparty valuation adjustment in presence of collateralization* is given by

$$\mathrm{BCCVA}(t,T,C) := \mathbb{E}_t\big[\,\Pi^D(t,T,C)\,\big] - \mathbb{E}_t[\,\Pi(t,T)\,]$$

This definition is based on the convention that the adjustment is to be subtracted from the risk free value in order to obtain the adjusted valuation. In order to evaluate the CVA inclusive of collateralization, we need to express $\Pi^D(t,T,C)$ in terms of risk-free quantities, default indicators and collateral. We do this in accordance with the ISDA standards, which require that upon the occurrence of a default event, the parties should terminate all transactions and do a netting of due cash-flows. The surviving party should evaluate the transactions just terminated, due to the default event occurrence, and claim for a reimbursement only after the application of netting rules, inclusive of collateral accounts.

The ISDA Master Agreement defines the term *close-out amount* to be the amount of the losses or costs the surviving party would incur in replacing or in providing for an economic equivalent of the remaining part of the deal at the time when the counterparty defaults. The time needed to complete the close-out procedure is known as the *margin period of risk.* The Basel III proposal considers, for netting sets containing one or more trades involving either illiquid collateral, or derivatives that cannot be easily replaced, a supervisory floor of twenty business days for such period.

We denote the close-out amount by ε_t, and we assume it is a $\mathcal{G}_t$-adapted process. A positive value for ε_t means that the investor is a creditor of the counterparty, while a negative value for ε_t means that the counterparty is a creditor to the investor. Notice that the close-out amount is not a symmetric quantity w.r.t. the exchange of the role of two parties, since it is valued by one party after the default of the other one, and may take into account the credit quality of the surviving party. After deriving the general formula for BCCVA, we specify how to calculate the close-out amount.

2.3 Collateral Re-Hypothecation

In case of no-default happening, at final maturity the collateral provider expects to get back from the collateral taker the remaining collateral. Similarly, in case of default happening earlier (and assuming the collateral taker, before the default event, to be the surviving party), after netting the collateral with the cash flows of the transaction, the collateral provider expects to get back the remaining collateral on the account. However, it is often considered to be important, commercially, for the collateral taker to have relatively unrestricted use of the collateral until it must be returned to the collateral provider. This unrestricted use includes the ability to sell collateral to a third party in the market, free and clear of any interest of the collateral provider. Other uses would include lending the collateral or selling it under a "repo" agreement or *re-hypothecating* it. When the collateral taker re-hypothecates the collateral, then he leaves the collateral provider as an unsecured creditor with respect to collateral reimbursement.

In case of re-hypothecation, the collateral provider must therefore consider the possibility to recover only a fraction of his collateral. If the investor is the collateral taker, we denote the recovery fraction on collateral re-hypothecated by the defaulted investor by Rec'_0, while if the counterparty is the collateral taker, then we denote the recovery fraction on collateral re-hypothecated by the counterparty by Rec'_2. Accordingly, we define the collateral loss incurred by the counterparty upon investor default by $\mathrm{Lgd}'_0 = 1 - \mathrm{Rec}'_0$ and the collateral loss incurred by the investor upon counterparty default by $\mathrm{Lgd}'_2 = 1 - \mathrm{Rec}'_2$. Typically, the surviving party has precedence on other creditors to get back his collateral, thus $\mathrm{Rec}_0 \leq \mathrm{Rec}'_0 \leq 1$, and $\mathrm{Rec}_2 \leq \mathrm{Rec}'_2 \leq 1$. Here, Rec_0 (Rec_2) denotes the recovery fraction of the market value of the transaction that the counterparty (investor) gets when the investor (counterparty) defaults.

2.4 Bilateral CVA Formula under Collateralization

We start by listing all the situations that may arise on counterparty default and investor default events. Our goal is to calculate the present value of all cash flows involved by the contract by taking into account

(i) collateral margining operations, and (ii) close-out netting rules in case of default. Notice that we can safely aggregate the cash flows of the contract with the ones of the collateral account, since on contract termination all the posted collateral is returned to the originating party. We start considering all possible situations which may arise at the default time of the counterparty, which is assumed to default before the investor. In our notation

$$X^+ := \max(X, 0)\ , \quad X^- := \min(X, 0)\,.$$

We have:

1. The investor measures a positive (on-default) exposure on counterparty default ($\varepsilon_{\tau_2} > 0$), and some collateral posted by the counterparty is available ($C_{\tau_2-} > 0$). Then, the investor exposure is reduced by netting, and the remaining collateral (if any) is returned to the counterparty. If the collateral is not enough, the investor suffers a loss for the remaining exposure. Thus, we have

$$1_{\{\tau=\tau_2<T\}} 1_{\{\varepsilon_\tau>0\}} 1_{\{C_{\tau-}>0\}} (\mathrm{Rec}_2 (\varepsilon_\tau - C_{\tau-})^+ + (\varepsilon_\tau - C_{\tau-})^-)$$

2. The investor measures a positive (on-default) exposure on counterparty default ($\varepsilon_{\tau_2} > 0$), and some collateral posted by the investor is available ($C_{\tau_2-} < 0$). Then, the investor suffers a loss for the whole exposure. All the collateral (if any) is returned to the investor if it is not re-hypothecated, otherwise only a recovery fraction of it is returned. Thus, we have

$$1_{\{\tau=\tau_2<T\}} 1_{\{\varepsilon_\tau>0\}} 1_{\{C_{\tau-}<0\}} (\mathrm{Rec}_2 \varepsilon_\tau - \mathrm{Rec}'_2 C_{\tau-})$$

3. The investor measures a negative (on-default) exposure on counterparty default ($\varepsilon_{\tau_2} < 0$), and some collateral posted by the counterparty is available ($C_{\tau_2-} > 0$). Then, the exposure is paid to the counterparty, and the counterparty gets back its collateral in full.

$$1_{\{\tau=\tau_2<T\}} 1_{\{\varepsilon_\tau<0\}} 1_{\{C_{\tau-}>0\}} (\varepsilon_\tau - C_{\tau-})$$

4. The investor measures a negative (on-default) exposure on counterparty default ($\varepsilon_{\tau_2} < 0$), and some collateral posted by the investor is available ($C_{\tau_2-} < 0$). Then, the exposure is reduced by netting and paid to the counterparty. The investor gets back its remaining collateral (if any) in full if it is not re-hypothecated, otherwise he only gets the recovery fraction of the part of collateral exceeding the exposure.

$$1_{\{\tau=\tau_2<T\}} 1_{\{\varepsilon_\tau<0\}} 1_{\{C_{\tau-}<0\}} ((\varepsilon_\tau - C_{\tau-})^- + \mathrm{Rec}'_2 (\varepsilon_\tau - C_{\tau-})^+)$$

Similarly, if we consider all possible situations which can arise at the default time of the investor, and then aggregate all these cash flows, along with the ones due in case of non-default, inclusive of the collateral account, we obtain, after straightforward manipulations

$$
\begin{aligned}
\Pi^D(t,T;C) = \Pi(t,T) \\
& - 1_{\{\tau<T\}} D(t,\tau)\left(\Pi(\tau,T) - \varepsilon_\tau\right) \\
& - 1_{\{\tau=\tau_2<T\}} D(t,\tau)(1-\mathrm{R{\scriptsize EC}}_2)(\varepsilon_\tau^+ - C_{\tau-}^+)^+ \\
& - 1_{\{\tau=\tau_2<T\}} D(t,\tau)(1-\mathrm{R{\scriptsize EC}}_2')(\varepsilon_\tau^- - C_{\tau-}^-)^+ \\
& - 1_{\{\tau=\tau_0<T\}} D(t,\tau)(1-\mathrm{R{\scriptsize EC}}_0)(\varepsilon_\tau^- - C_{\tau-}^-)^- \\
& - 1_{\{\tau=\tau_0<T\}} D(t,\tau)(1-\mathrm{R{\scriptsize EC}}_0')(\varepsilon_\tau^+ - C_{\tau-}^+)^-
\end{aligned}
$$

where $D(t,T)$ denotes the risk-free discount factor. Notice that the collateral account enters only as a term reducing the exposure of each party upon default of the other one, taking into account which party posted the collateral.

BCCVA General Formula

By taking risk-neutral expectation of both sides of the equation expressing BCCVA, we obtain the general expression for collateralized bilateral CVA.

$$
\begin{aligned}
\mathrm{BCCVA}(t,T;C) &= -\mathbb{E}_t\left[\, 1_{\{\tau<T\}} D(t,\tau)\left(\mathbb{E}_\tau[\,\Pi(\tau,T)\,] - \varepsilon_\tau\right)\right] \\
&\quad - \mathbb{E}_t\left[\, 1_{\{\tau=\tau_2<T\}} D(t,\tau)\left(\mathrm{L{\scriptsize GD}}_2(\varepsilon_\tau^+ - C_{\tau-}^+)^+ + \mathrm{L{\scriptsize GD}}_2'(\varepsilon_\tau^- - C_{\tau-}^-)^+\right)\right] \qquad (2.2) \\
&\quad - \mathbb{E}_t\left[\, 1_{\{\tau=\tau_0<T\}} D(t,\tau)\left(\mathrm{L{\scriptsize GD}}_0(\varepsilon_\tau^- - C_{\tau-}^-)^- + \mathrm{L{\scriptsize GD}}_0'(\varepsilon_\tau^+ - C_{\tau-}^+)^-\right)\right]
\end{aligned}
$$

The first term on right-hand side of the above equation represents the mismatch in calculating the mark-to-market price of remaining cash flows and the close-out amount. The second term is the counterparty risk due to counterparty's default (also known as counterparty valuation adjustment or CVA), and comes with a negative sign (always from the point of view of the investor). The third term represents the counterparty risk due to investor's default (also known as debit valuation adjustment or DVA) and comes with a positive sign (again from the point of view of the investor). For future references, we formally define the CVA and DVA as

$$
\mathrm{CVA}(t,T;C) = -\mathbb{E}_t\left[\, 1_{\{\tau=\tau_2<T\}} D(t,\tau)\left(\mathrm{L{\scriptsize GD}}_2(\varepsilon_\tau^+ - C_{\tau-}^+)^+ + \mathrm{L{\scriptsize GD}}_2'(\varepsilon_\tau^- - C_{\tau-}^-)^+\right)\right] \qquad (2.3)
$$

and

$$
\mathrm{DVA}(t,T;C) = -\mathbb{E}_t\left[\, 1_{\{\tau=\tau_0<T\}} D(t,\tau)\left(\mathrm{L{\scriptsize GD}}_0(\varepsilon_\tau^- - C_{\tau-}^-)^- + \mathrm{L{\scriptsize GD}}_0'(\varepsilon_\tau^+ - C_{\tau-}^+)^-\right)\right] \qquad (2.4)
$$

Notice that such definitions are different from the analogous definitions in the unilateral case, since they include a first to default indicator that avoids inconsistencies and double counting. Some market players do not include the indicator checking the first to default, and thus develop a formula that is inconsistent on closeout rules. The error induced by such approximation is discussed in the report in Brigo, Buescu and Morini (2011).

Specialization of BCCVA Formula

Starting from this section we assume that the close-out amount of investor to counterparty and of counterparty to investor at any time t is symmetric, and equal to the default-free mark-to-market price of the cash flows that would be due from the default event up to contract's maturity date, so that for the investor we have

$$\varepsilon_\tau := \mathbb{E}_\tau[\,\Pi(\tau,T)\,]\;, \tag{2.5}$$

(for the counterparty we have the opposite quantity $-\varepsilon_\tau$) and the mismatching term in the BCCVA formula drops out. Notice that we can complicate the above approximation and calculate the close-out amount by taking into account the credit worthiness of the surviving party, the cost of replacing the contract, and the funding costs. A more articolated approach can be found in Brigo et al. (2011) and in references cited therein.

If collateral re-hypothecation is not allowed ($\mathrm{L{\scriptstyle GD}}'_2 = \mathrm{L{\scriptstyle GD}}'_0 = 0$), then the BCCVA formula simplifies to

$$\begin{aligned}\mathrm{BCCVA}(t,T;C) = &- \mathbb{E}_t\big[\,1_{\{\tau=\tau_2<T\}} D(t,\tau)\mathrm{L{\scriptstyle GD}}_2(\varepsilon^+_\tau - C^+_{\tau-})^+\,\big] \\ &- \mathbb{E}_t\big[\,1_{\{\tau=\tau_0<T\}} D(t,\tau)\mathrm{L{\scriptstyle GD}}_0(\varepsilon^-_\tau - C^-_{\tau-})^-\,\big]\end{aligned} \tag{2.6}$$

On the other hand, if re-hypothecation is allowed and the surviving party always faces the worst case ($\mathrm{L{\scriptstyle GD}}'_2 = \mathrm{L{\scriptstyle GD}}_2$ and $\mathrm{L{\scriptstyle GD}}'_0 = \mathrm{L{\scriptstyle GD}}_0$), then we obtain

$$\begin{aligned}\mathrm{BCCVA}(t,T;C) = &- \mathbb{E}_t\big[\,1_{\{\tau=\tau_2<T\}} D(t,\tau)\mathrm{L{\scriptstyle GD}}_2(\varepsilon_\tau - C_{\tau-})^+\,\big] \\ &- \mathbb{E}_t\big[\,1_{\{\tau=\tau_0<T\}} D(t,\tau)\mathrm{L{\scriptstyle GD}}_0(\varepsilon_\tau - C_{\tau-})^-\,\big]\end{aligned} \tag{2.7}$$

Finally, if we remove collateralization, i.e. $C_t = 0$ for any t, then we recover the result given for example in Brigo and Capponi (2008)

$$\begin{aligned}\mathrm{BCVA}(t,T) &:= \mathrm{BCCVA}(t,T;0) \\ &= - \mathbb{E}_t\big[\,1_{\{\tau=\tau_2<T\}} D(t,\tau)\mathrm{L{\scriptstyle GD}}_2\varepsilon^+_\tau\,\big] \\ &\quad - \mathbb{E}_t\big[\,1_{\{\tau=\tau_0<T\}} D(t,\tau)\mathrm{L{\scriptstyle GD}}_0\varepsilon^-_\tau\,\big]\end{aligned} \tag{2.8}$$

If we remove collateralization ($C_t = 0$ for any t) and we consider a risk-free investor ($\tau_0 \to \infty$), we recover the classice result of Canabarro and Duffie (2004), see also Brigo and Masetti (2005) for interest rate swaps with netting and Brigo and Pallavicini (2007) for interest rate swaps with wrong way risk.

$$\mathrm{BCVA}(t,T)|_{\tau_0\to\infty} = - \mathbb{E}_t\big[\,1_{\{\tau_2<T\}} D(t,\tau_2)\mathrm{L{\scriptstyle GD}}_2\varepsilon^+_{\tau_2}\,\big] \tag{2.9}$$

2.5 Collateralization Strategies

We describe two collateralization mechanisms which we will later use in our numerical simulations. These are the collateralization through margining, and the limit case of continuous collateralization,

Collateralization through Margining

In the first case, we assume that collateral posting only occurs at discrete times on a fixed grid ($t_0 = t, \ldots, t_N = T$). Notice that a realistic margining practice also includes minimum transfer amounts, thresholds, independent amounts. Here we consider only a simple but relevant case. See Brigo et al. (2011) and references therein for more detailed examples.

At each collateral posting date t_i, the collateral account is updated according to changes in the mark-to-market price of the remaining cash flows. We denote by C_{t_i-} the collateral account right before the collateral update for time t_i takes place, and denote by C_{t_i} the collateral account right after the collateral update for time t_i takes place. We define the following dynamics for the collateral account $C_t =: 1_{\{t<\tau\}} M_t$ with

$$(2.10) \qquad M_{t_0} := 0 \ , \quad M_{t_n} := 0 \ , \quad M_u := \frac{M_{\beta(u)}}{D(\beta(u), u)} \ , \quad M_{t_i} := \mathbb{E}_{t_i}[\,\Pi(t_i, T)\,]$$

where $t_0 < u < t_N$ and $\beta(u)$ is the last update time before u, $\beta(u) = \sup\{t_i : t_i \leq u, i = 0, \ldots, N\}$.

Continuous Collateralization

In the second case, we consider the limit case of updating the collateral account continuously, thereby obtaining the following collateralization rule $C_t := 1_{\{t<\tau\}} M_t$ with

$$(2.11) \qquad M_t := \mathbb{E}_t[\,\Pi(t, T)\,]$$

Notice that such rule can be obtained from the previous one by setting the function β to be the identity.

The CVA and DVA terms of the BCCVA formula depend on the collateral account via the following two terms

$$\varepsilon_\tau^+ - C_{\tau-}^+ \ , \quad \varepsilon_\tau^- - C_{\tau-}^-$$

where as before τ is the first default event between the two parties, and $\varepsilon_\tau = \mathbb{E}_\tau[\,\Pi(\tau, T)\,]$.

If the mark-to-market of the remaining cash flows does not jump at the first default event, then both terms drop out when the collateral is updated continuously. Thus, in such case we can call the collateral scheme a *perfect* collateralization scheme. This will not be the case for Credit Default Swaps.

3 BCCVA for Credit Default Swaps

The price process Γ_t for a CDS selling protection $\text{L}_{\text{GD}1}$ at time t for default of the reference entity between times T_a and T_b, with $t \leq T_a < T_b$, in exchange of a periodic premium rate S_1 is given by

$$\Gamma_t := \mathbb{E}_t\left[S_1 \int_{T_a}^{T_b} D(t,u) 1_{\{\tau_1 > u\}}\, du + \text{L}_{\text{GD}1} \int_{T_a}^{T_b} D(t,u)\, d1_{\{\tau_1 > u\}} \right]$$

where, for simplicity, we assume that the premium leg pays continually. If we assume deterministic interest rates, and deterministic recovery rates, then the price process may be written as

$$\begin{aligned} \Gamma_t(T_a, T_b; S_1, \mathrm{L{\scriptstyle GD}}1) = 1_{\{\tau_1 > t\}}\, S_1 \int_{T_a}^{T_b} D(t,u)\mathbb{Q}(\tau_1 > u|\mathcal{G}_t)\, du \, + \\ 1_{\{\tau_1 > t\}}\, \mathrm{L{\scriptstyle GD}}1 \int_{T_a}^{T_b} D(t,u)\, d\mathbb{Q}(\tau_1 > u|\mathcal{G}_t) \end{aligned} \tag{3.1}$$

We next proceed to the valuation of the BCCVA adjustment for the case of a CDS contract. We use the collateralization rules detailed in Section 2.5, and we recall that $\beta(\tau)$ is the last time when collateral was updated before the default event of one of the two parties (in case of continuous collateralization the β function is simply the identity function).

If we apply the formulas in Eq. (2.6), (2.7) and (2.8) to the case of CDS contracts, then it is easily seen that when collateral re-hypothecation is not allowed, the adjustment for a receiver CDS contract (protection seller) running from time $t = T_a$ to time $T_b = T$ is given by

$$\begin{aligned} \mathrm{BCCVA}(t,T;C) = &- \mathbb{E}_t\Big[1_{\{\tau=\tau_2<T\}} D(t,\beta(\tau))\mathrm{L{\scriptstyle GD}}2 \Big(D(\beta(\tau),\tau)\Gamma_\tau^+ - \Gamma^+_{\beta(\tau)-} \Big)^+ \Big] \\ &- \mathbb{E}_t\Big[1_{\{\tau=\tau_0<T\}} D(t,\beta(\tau))\mathrm{L{\scriptstyle GD}}0 \Big(D(\beta(\tau),\tau)\Gamma_\tau^- - \Gamma^-_{\beta(\tau)-} \Big)^- \Big] \end{aligned} \tag{3.2}$$

In case when re-hypothecation is allowed, then we obtain

$$\begin{aligned} \mathrm{BCCVA}(t,T;C) = &- \mathbb{E}_t\big[1_{\{\tau=\tau_2<T\}} D(t,\beta(\tau))\mathrm{L{\scriptstyle GD}}2 \big(D(\beta(\tau),\tau)\Gamma_\tau - \Gamma_{\beta(\tau)-} \big)^+ \big] \\ &- \mathbb{E}_t\big[1_{\{\tau=\tau_0<T\}} D(t,\beta(\tau))\mathrm{L{\scriptstyle GD}}0 \big(D(\beta(\tau),\tau)\Gamma_\tau - \Gamma_{\beta(\tau)-} \big)^- \big] \end{aligned} \tag{3.3}$$

Finally, if we remove collateralization, i.e. $M_t = 0$ for any t, then we obtain

$$\begin{aligned} \mathrm{BCVA}(t,T) = &- \mathbb{E}_t\big[1_{\{\tau=\tau_2<T\}} D(t,\tau)\mathrm{L{\scriptstyle GD}}2\Gamma_\tau^+ \big] \\ &- \mathbb{E}_t\big[1_{\{\tau=\tau_0<T\}} D(t,\tau)\mathrm{L{\scriptstyle GD}}0\Gamma_\tau^- \big] \end{aligned} \tag{3.4}$$

From (3.4), we can see that the only terms we need to know in order to be able to numerically compute the credit valuation adjustments are the *on-default survival* probabilities

$$\mathbf{1}_{\tau=\tau_2\le T}\mathbf{1}_{\tau_1>\tau_2}\mathbb{Q}(\tau_1 > t|\mathcal{G}_{\tau_2}) \,, \quad \mathbf{1}_{\tau=\tau_0\le T}\mathbf{1}_{\tau_1>\tau_0}\mathbb{Q}(\tau_1 > t|\mathcal{G}_{\tau_0}) \tag{3.5}$$

Let us assume that $t > \tau_2 > u$. Then, in presence of collateralization, we can see from Eq. (3.2) and (3.3) that we need to evaluate the *pre-default survival* probabilities

$$\mathbf{1}_{\tau=\tau_2\le T}\mathbf{1}_{\tau_1>u}\mathbb{Q}(\tau_1 > t|\mathcal{G}_u) \,, \quad \mathbf{1}_{\tau=\tau_0\le T}\mathbf{1}_{\tau_1>u}\mathbb{Q}(\tau_1 > t|\mathcal{G}_u) \tag{3.6}$$

Let us define

$$\varphi_u(w,x,v) \quad = \quad \mathbb{Q}\left(\tau_0 > w, \tau_1 > x, \tau_2 > v|\mathcal{F}_u\right) \tag{3.7}$$

Then, the pre-default survival probabilities can be evaluated using the key lemma in Bielecki and Rutkowski (2001) (Lemma 5.1.2). More specifically, for $t > \tau > u$, we have

$$1_{\{\tau_1 > u\}} \mathbb{Q}(\tau_1 > t | \mathcal{G}_u) = 1_{\{\tau_1 > u\}} \frac{\varphi_u(u,t,u)}{\varphi_u(u,u,u)} \tag{3.8}$$

The on-default survival probabilities, instead, require some further analysis. For a given random variable X, when referring to the martingale process $u \to \mathbb{E}^{\mathbb{Q}}_u [X]$, we mean that we are choosing its unique càdlàg modification.

Proposition 3.1. *Assume $\varphi_u(u,x,y)$ and $\varphi_u(y,x,u)$ are differentiable with respect to y. If $t > \tau_2$, we have*

$$\mathbf{1}_{\tau=\tau_2 \le T} \mathbf{1}_{\tau_1 > \tau_2} \mathbb{Q}(\tau_1 > t | \mathcal{G}_{\tau_2}) = \lim_{u \downarrow \tau_2} \mathbf{1}_{u \le T} \mathbf{1}_{\tau_1 > u} \frac{\left.\frac{\partial}{\partial y}\varphi_u(u,t,y)\right|_{y=\tau_2}}{\left.\frac{\partial}{\partial y}\varphi_u(u,u,y)\right|_{y=\tau_2}} \tag{3.9}$$

Similarly, if $t > \tau_0$, we have

$$\mathbf{1}_{\tau=\tau_0 \le T} \mathbf{1}_{\tau_1 > \tau_0} \mathbb{Q}(\tau_1 > t | \mathcal{G}_{\tau_0}) = \lim_{u \downarrow \tau_0} \mathbf{1}_{u \le T} \mathbf{1}_{\tau_1 > u} \frac{\left.\frac{\partial}{\partial y}\varphi_u(y,t,u)\right|_{y=\tau_0}}{\left.\frac{\partial}{\partial y}\varphi_u(y,u,u)\right|_{y=\tau_0}} \tag{3.10}$$

Proof. We only prove Eq. (3.9), as the proof of Eq. (3.10) is analogous and omitted here. Since $\mathcal{G}_t$ is a right-continuous filtration and ς a finite $\mathcal{G}$-stopping time, then for any integrable random variable X, it follows from the right continuity of $\mathbb{E}_u[X]$ and from the martingale convergence theorem, see also Protter (2004), that

$$\mathbb{E}_\varsigma [X] = \lim_{u \downarrow \varsigma} \mathbb{E}_u [X] \tag{3.11}$$

Using Eq. (3.11), we obtain

$$\mathbf{1}_{\tau=\tau_2 \le T} \mathbf{1}_{\tau_1 > \tau_2} \mathbb{Q}(\tau_1 > t | \mathcal{G}_{\tau_2}) \quad = \quad \lim_{u \downarrow \tau_2} f(u)$$

where

$$f(u) = \mathbf{1}_{u \le T} \mathbf{1}_{\tau_1 > u} \mathbb{Q}(\tau_1 > t | \mathcal{G}_u).$$

Notice that on the set $u > \tau_2$, we have $\mathcal{G}_u \supseteq \mathcal{G}_{\tau_2} \supseteq \sigma(\tau_2)$. Therefore, using the key lemma, we have

$$\begin{aligned} f(u) &= 1_{\{u \le T\}} 1_{\{\tau_1 > u\}} \frac{\mathbb{E}^{\mathbb{Q}}\left[1_{\{\tau_1 > t\}} 1_{\{\tau_1 > u\}} | \mathcal{F}_u \vee \mathcal{H}^0_u \vee \sigma(\tau_2)\right]}{\mathbb{E}^{\mathbb{Q}}\left[1_{\{\tau_1 > u\}} | \mathcal{F}_u \vee \mathcal{H}^0_u \vee \sigma(\tau_2)\right]} \\ &= 1_{\{u \le T\}} 1_{\{\tau_1 > u\}} \frac{\mathbb{E}^{\mathbb{Q}}\left[1_{\{\tau_1 > t\}} | \mathcal{F}_u \vee \mathcal{H}^0_u \vee \sigma(\tau_2)\right]}{\mathbb{E}^{\mathbb{Q}}\left[1_{\{\tau_1 > u\}} | \mathcal{F}_u \vee \mathcal{H}^0_u \vee \sigma(\tau_2)\right]} \end{aligned} \tag{3.12}$$

Further application of the key lemma yields

$$\mathbb{E}^{\mathbb{Q}}\left[1_{\{\tau_1 > t\}} | \mathcal{F}_u \vee \mathcal{H}^0_u \vee \sigma(\tau_2)\right] \quad = \quad 1_{\{\tau_0 > u\}} \frac{\left.\frac{\partial}{\partial y}\varphi_u(u,t,y)\right|_{y=\tau_2}}{\left.\frac{\partial}{\partial y}\varphi_u(u,0,y)\right|_{y=\tau_2}} \tag{3.13}$$

and

$$\mathbb{E}^{\mathbb{Q}}\left[1_{\{\tau_1>u\}}|\mathcal{F}_u \vee \mathcal{H}_u^0 \vee \sigma(\tau_2)\right] = 1_{\{\tau_0>u\}} \frac{\frac{\partial}{\partial y}\varphi_u(u,u,y)\Big|_{y=\tau_2}}{\frac{\partial}{\partial y}\varphi_u(u,0,y)\Big|_{y=\tau_2}} \tag{3.14}$$

Plugging the expressions in Eq. (3.13) and in Eq. (3.14) into Eq. (3.12), we obtain

$$f(u) = 1_{\{u\leq T\}}1_{\{\tau_1>u\}} \frac{\frac{\partial}{\partial y}\varphi_u(u,t,y)\Big|_{y=\tau_2}}{\frac{\partial}{\partial y}\varphi_u(u,u,y)\Big|_{y=\tau_2}}$$

and the desired result follows. □

Notice that, in general, the on-default and pre-default survival probabilities have a different term structure. However, in some special case, they turn out to be the same, as stated the next corollary.

Corollary 3.1. *Assume*

$$\varphi(w,x,v) = \mathbb{Q}\left(\tau_0 > w|\mathcal{F}_u\right)\mathbb{Q}\left(\tau_1 > x|\mathcal{F}_u\right)\mathbb{Q}\left(\tau_2 > v|\mathcal{F}_u\right).$$

Then

$$1_{\{\tau_1>\tau\}}\mathbb{Q}(\tau_1 > t|\mathcal{G}_\tau) = 1_{\{\tau_1>\tau\}}\mathbb{Q}(\tau_1 > t|\mathcal{F}_\tau) \tag{3.15}$$

The proof follows immediately using the independence assumption and the fact that simultaneous defaults are excluded.

Remark I: Notice that, if the default events are not conditionally independent given the reference filtration, it is no longer true that on-default and pre-default survival probabilities are the same when $u = \tau_2^-$. In financial terms, this means that in the case of a credit default swap contract, continuous collateralization ($\beta(\tau) = \tau$) does not fully eliminate counterparty risk. It does so only if the default times of the counterparties and of the reference entity are conditionally independent. Moreover, observe that this is a feature of products such as credit default swaps, where the mark-to-market price of the residual transaction, if taken as defining the exposure ε_τ, may experience a jump at τ. The same would not occur if the product were, for instance, an interest rate swap, as we discuss instead in Brigo et al. (2011). In that case the adjustments, which depend on the collateral account via the following terms $\varepsilon_\tau^+ - C_{\tau-}^+$ and $\varepsilon_\tau^- - C_{\tau-}^-$, would drop to zero, thus eliminating completely the counterparty risk.

4 Numerical Simulations

In this section, we specify credit and default correlation models, and then present numerical simulations to evaluate the bilateral CVA of credit default swap contracts. In order to keep the computation tractable, we consider a square root diffusion model driving the intensity of investor, counterparty and CDS reference credit, and correlate the default events through a copula structure. We measure the impact of default correlation on the resulting adjustment.

4.1 Credit Spread Model

For the stochastic intensity models we set

$$\lambda_t^i = y_t^i + \psi^i(t;\beta^i) \ , \quad i \in \{0,1,2\} \tag{4.1}$$

where whenever we omit the upper index we refer to quantities for all indices. Each ψ^i is a deterministic function, depending on the model's parameter vector β^i, that is integrable on closed intervals. The initial condition y_0^i is one more parameter at our disposal. We are free to select its value as long as

$$\psi^i(0;\beta^i) = \lambda_0^i - y_0^i \ .$$

We take each y^i to be a Cox Ingersoll Ross process:

$$dy_t^i = \kappa^i(\mu^i - y_t^i)dt + \nu^i\sqrt{y_t^i}dZ^i(t) \ , \quad i \in \{0,1,2\}$$

where the parameter vector is $\beta^i := (\kappa^i, \mu^i, \nu^i, y_0^i)$ and each parameter is a positive deterministic constant, thus implying the condition $\kappa\mu > 0$. As usual, Z^i is a standard Brownian motion process under the risk neutral measure. We relax the Feller condition $2\kappa^j\mu^j > (\nu^j)^2$ so that we do not limit the CDS implied volatility generated by the model. The three processes y^i are assumed to be independent, i.e. the driving brownian motion processes Z^i are independent. This is assumed in order to simplify the parametrization of the model and to focus on default correlation rather than spread correlation, but the assumption can be removed if one is willing to complicate the parametrization of the model. In general, when stochastic intensities follow diffusion processes, spread correlation is of second order with respect to default correlation in most cases, as it has been recognized in Jouanin et al. (2001), for example. We also define the integrated quantities

$$\Lambda^i(t) := \int_0^t \lambda_s^i ds \ , \quad Y^i(t) := \int_0^t y_s^i ds \ , \quad \Psi^i(t,\beta^i) := \int_0^t \psi^i(s,\beta^i) ds \ .$$

Default Dependence

We place ourselves in a Cox process setting, where

$$\tau_i = \left(\Lambda^i\right)^{-1}(\xi_i) \ , \ i = 0,1,2 \tag{4.2}$$

with ξ_0, ξ_1 and ξ_2 being standard exponential random variables whose associated uniforms $U_i := 1 - \exp\{-\xi_i\}$ are connected through a Gaussian trivariate copula function $C_{\boldsymbol{R}}(u_0,u_1,u_2) := \mathbb{Q}(U_0 < u_0, U_1 < u_1, U_2 < u_2)$, where $\boldsymbol{R} = [r_{i,j}]_{i,j=0,1,2}$ is the correlation matrix parameterizing the Gaussian copula.

Calibration to CDS Market Quotes

Since the survival probabilities in our model are given by

$$(4.3)\quad \mathbb{Q}(\tau_i > t)_{model} = \mathbb{E}_0\Big[e^{-\Lambda^i(t)}\Big] = \mathbb{E}_0\big[\exp\big(-\Psi^i(t,\beta^i) - Y^i(t)\big)\big] = \exp\big(-\Psi^i(t,\beta^i)\big)\,\mathbb{E}_0\big[\exp\big(-Y^i(t)\big)\big]$$

we need to guarantee that

$$\exp\big(-\Psi^i(t,\beta^i)\big)\,\mathbb{E}_0\big[\exp\big(-Y^i(t)\big)\big] = \mathbb{Q}(\tau_i > t)_{\text{CDSmarket}} \tag{4.4}$$

where the parameters β^i are chosen so that we have a positive function ψ^i (i.e. a non-decreasing Ψ^i). Notice that the expectation on the left hand side of Eq. (4.4) represents the bond price in the time-homogeneous CIR model with parameters β^i. Thus, if ψ^i is selected according to this last formula, as we will assume from now on, the model is easily and automatically calibrated to the market survival probabilities (possibly stripped from CDS data).

Realistic Market Data-Set for CDS Options

Once we have calibrated CDS data through $\psi^i(\cdot,\beta^i)$, we are left with the parameters β^i, which can be used to fit the price of further products, such as single-name option data. However, single-name options on the credit derivatives market are not liquid. Indeed, typically the bid-ask spreads for single name CDS options are large and suggest to consider these quotes with caution, see Brigo (2005). At the moment we content ourselves of calibrating only CDS's for the credit part. To help specifying β without further data we set some values of the parameters implying possibly reasonable values for the implied volatility of hypothetical at the money CDS options on the three entities. Such CDS options are options to enter t years from now into a CDS selling protection up to a future time T. They are at the money, in that if the option is exercised the future CDS is entered at time t at a spread given by the initial CDS spread at time 0 for maturity $t+T$. Also, the CDS to be entered is a receiver CDS and we do not consider front end protection for defaults up to t. The implied volatilities are calculated by matching Black's formula for CDS options to the model price. The model option price is obtained through Jamshidian's decomposition as described in Brigo and Mercurio (2006) and more rigorosuly in Brigo and El-Bachir (2010), and the values are reported in Brigo et al. (2011).

We focus on three different sets of CDS quotes reported in Table 2, that we name hereafter *Low*, *Mid* and *High* risk settings. We then calibrate the parameters of the CIR model to these quotes and recover the parameters given in Table 1, assuming that recovery rates are at 40% level.

4.2 Numerical Evaluation of the Credit Spread Model

In order to simulate the three $y^i(t)$, we use the well known fact that the distribution of $y^i(t)$ given $y^i(u)$, for some $u < t$ is, up to a scale factor, a noncentral chi-square distribution, see Cox et al. (1985). More precisely, the transition law of $y^i(t)$ given $y^i(u)$ can be expressed as

$$y^i(t) = \frac{(\nu^i)^2(1-e^{-\kappa^i(t-u)})}{4\kappa^i}\chi'_d\left(\frac{4\kappa^i e^{-\kappa^i(t-u)}}{(\nu^i)^2(1-e^{-\kappa^i(t-u)})}y^i(u)\right) \tag{4.5}$$

where

$$d = \frac{4\kappa^i\mu^i}{(\nu^i)^2} \tag{4.6}$$

and $\chi'_u(v)$ denotes a non-central chi-square random variable with u degrees of freedom and non centrality parameter v. In this way, if we know $y^i(0)$, we can simulate the process $y^i(t)$ exactly on a discrete time grid by sampling from the non-central chi-square distribution. Let us denote by $\Phi^{t,u}_{CIR,i}(x)$ the cumulative distribution function of the integrated shifted CIR process $\Lambda^i(t)$ conditional on $\mathcal{F}_u$ evaluated at x. Such distribution may be obtained through inversion of the characteristic function of the integrated CIR process, which is well known from the work of Cox et al. (1985), and from the literature on Brownian motion, since it is closely associated with the Lévy's stochastic area formula, see also Yor (1992). Moreover, let $\Upsilon(z) := -\log(1-\Phi(z))$, where $\Phi(x)$ denotes the cumulative distribution function of the univariate Gaussian. Under the copula model, we have that

$$\varphi_u(v,s,w) = \mathbb{E}^{\phi_R}\left[\Phi^{v,u}_{CIR,0}(\Upsilon(Z_0))\Phi^{s,u}_{CIR,1}(\Upsilon(Z_1))\Phi^{w,u}_{CIR,2}(\Upsilon(Z_2))\right] \tag{4.7}$$

where (Z_0, Z_1, Z_2) is a standard Gaussian vector with density ϕ_R, and R denotes the correlation matrix.

4.3 Numerical Study

We consider an investor trading a five-years CDS contract on a reference name with a counterparty. Both the investor and the counterparty are subject to default risk. We consider three different levels of credit risk (Low, Mid and High), specified by the parameters of the CIR processes in Table 1. We measure the counterparty adjustments under the two different collateralization strategies described in Section 2.5, i.e. continuous collateralization and collateralization with three-months margining frequency, and under no collateralization.

We consider two sets of simulations. In both cases, investor and counterparty have middle credit risk profile, while the reference entity has high credit risk profile. Moreover, all three names are equally correlated to each other. We consider 5y CDS. In the first setup, we set the CDS spread S_1 in the premium leg to 100 basis points, while in the second scenario we set it to 500 basis points. The break-even or fair spread value for S_1 that would make the total value of the CDS equal to zero at time zero is 251 basis points. In this numerical investigation we implement a proper quarterly spaced premium leg, rather than the idealized continually-paying premium leg. Results are displayed in Figures 1 and 2, respectively. Let us begin by analyzing our results in the case where the protection payment is 100 basis points. Given that the fair spread is 251, in this setup the payer CDS has a markedly positive initial value, whereas the receiver CDS has a markedly negative one. We discuss the results for the payer CDS contract, as the results for the receiver exhibit a specular pattern. If the investor holds a payer CDS, he is buying protection from the counterparty, i.e. he is a protection buyer. Given that the payer CDS will be positive in most scenarios, when the investor defaults it is quite unlikely that the net present value be in favor of the counterparty. Hence, one expects the DVA to be small or null in most cases due to out-moneyness of the related option. This is what we see from the middle panel of Figure 1, except in the case with zero correlation and under re-hypothecated collateral, where the DVA for zero correlation is about 3.5 basis points rather than zero. This can be explained as follows. Under collateralization with re-hypothecation, since the net present value (NPV) is in most cases in favor of the investor, the counterparty will post collateral to the investor. However, if the investor is allowed to re-hypothecate and then defaults, the counterparty will get back only a recovery fraction of the collateral, and the investor will have a discount on the collateral she needs to give back to the counterparty. This discount generates a non-zero, albeit small DVA. However, when default correlation goes up, it becomes more unlikely that the investor defaults alone and first, without the counterparty and the underlying CDS defaulting as well, and therefore there will be less scenarios where the DVA payoff term will be activated by the first default of the investor.

We now analyze the CVA term. Again, given that the payer CDS will be positive in most scenarios, we expect the CVA term to be relevant, given that the related option will be mostly in the money. This is confirmed by our outputs. We see in the figure a relevant CVA term starting at about 10 and ending up at 60 basis points when under high correlation. We also see that, for zero correlation, collateralization succeeds in completely removing CVA, which goes from 10 to 0 basis points. However, collateralization seems to become less effective as default dependence grows, in that collateralized and uncollateralized CVA become closer and closer, and for high correlations we still get 60 basis points of CVA, even under collateralization. The reason for this is the instantaneous default contagion that, under positive dependency, pushes up the intensity of the survived entities, as soon as there is a default of the counterparty. Indeed, we can clearly see from Figure 3 that the term structure of the on-default survival probabilities lies below the one of the pre-default survival probabilities conditioned on $\mathcal{G}_{\tau-}$. Moreover, we can see that for larger values of default correlation (see the case when the default correlation is 0.9), the on-default survival curve lies significantly below the pre-default curve. The result is that the default leg of the CDS will increase in value due to contagion, and instantaneously the Payer CDS will be worth more. This will instantly increase the loss to the investor, and most of the CVA value will come from this jump. Given the instantaneous nature of the jump, it is clear that the value after the jump will be quite different from the value at the last date of collateral posting, before the jump, and this explains the limited effectiveness of collateral under significantly positive default dependence. We refer also to Fujii and Takahashi (2011) for a similar example of contagion based on the Clayton copula, even if the effects seem to be more limited in that context.

We now turn to the case where the premium leg running spread is at 500 bps, thus roughly twice as much as the equilibrium CDS spread. Hence, in this setup, the payer CDS has a markedly negative initial value, whereas the receiver CDS has a markedly positive one. Let us focus first on the Payer CDS. If the investor holds a payer CDS, he is a protection buyer as we remarked earlier. Given that the payer CDS will be negative in most scenarios, when the investor defaults it is quite likely that the net present value be in favor of the counterparty, and therefore we should expect a relevant DVA term. On the other hand, for the same reason, we should expect a small or even zero CVA term. This is what can be seen from Figure 2, for the uncollateralized case of DVA under small values of default correlation. However, as we increase correlation, we can see that, even uncollateralized DVA decreases whereas the CVA becomes relevant. This can be again explained in terms of contagion. When the investor defaults, under positive dependence the default leg of the underlying CDS jumps up, increasing the value of the Payer CDS. This will increase the option moneyness embedded in the CVA term and decrease that for the DVA term, leading to the observed effects. Moreover, notice that the higher the dependence, the higher the effect. We can also notice the impact of collateral and of re-hypotecation. More specifically, collateral makes the DVA term very small even for zero default correlation, where there is no contagion. Indeed, under low or zero correlation the underlying CDS spread will not move much upon default of the investor, so that the last posted collateral will be close to the on-default value of the underlying CDS, bringing the loss due to sudden default of the investor near zero. As for the CVA term, we see that for all values of the correlation parameter there is difference mostly between the re-hypotecated case and all other cases. Indeed, with a largely negative value for the underlying CDS, it will be the investor who will have to post collateral as a guarantee towards the counterparty in most scenarios. If the counterparty defaults first, as in the CVA term, she will only give back a fraction of the collateral received by the investor, increasing the loss for the investor and, consequently, the related CVA term. With no rehypotecation allowed, this does not happen as the counterparty will give all the collateral back to the investor. We also notice that in the CVA term, collateralization does almost nothing to reduce CVA. This is because the moneyness of the contract is always in favor of the DVA term, thus there will be almost never collateral posted from the counterparty as a guarantee to alleviate the CVA term. On the contrary, in most scenarios the moneyness will cause the investor to post collateral in favor of the counterparty, thus reducing the DVA term. The bilateral counterparty risk adjustments in Figures 1 and 2 can be explained exactly in terms of the embedded CVA and DVA.

One may argue from the above analysis that, in some instances, the effect of contagion is so dramatic to change CVA and DVA patterns even in presence of a strong and adverse moneyness in the underlying option terms. This is a feature of copula models that is worth keeping in mind when modeling bilateral counterparty risk. On the other hand, it is easy to simulate and allows for decomposing block dependence into pairwise dependence easily, and is largely used and understood in limitations by practitioners, even if it is used in an extremely stylized and simplistic way when dealing with synthetic CDO's, see for example the analysis in Brigo et al. (2010), leading to a number of problems. The situation is however less dramatic when the number of entities who can default is small. Even then, care must be taken in assessing the size of contagion effects, in order to make sure that the model gives realistic contributions.

5 Conclusions

In this paper we have provided a complete framework for bilateral CVA risk-neutral pricing, inclusive of collateralization, and considering also the case when collateral can be re-hypothecated. We derived model independent formulas showing that the adjustment is given by the sum of two option terms, with each term depending on the netted exposure, i.e. the difference between the on-default exposure and the pre-default collateral account. We have specialized our analysis to the case where the underlying portfolio is sensitive to a third credit event, and in particular a credit default swap written on a third reference entity. Through a numerical study, we have analyzed the impact of collateralization frequency, collateral re-hypothecation, and default correlation on the resulting counterparty adjustments. The results obtained confirm that the adjustments are monotonic with respect to the level of default correlation. Moreover, higher frequency of collateralization reduce counterparty exposure, while re-hypothecation enhance the absolute size of the adjustment due to the possibility that the collateral provider can only recover a fraction of his posted collateral. Finally, contagion effects play a key role in limiting the effectiveness of collateral in the CDS case.

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Table 1: Credit spread parameters. From top to bottom: low, mid and high risk settings.

	$\boldsymbol{y}_0$	$\boldsymbol{\kappa}$	$\boldsymbol{\mu}$	$\boldsymbol{\nu}$
Low	0.0003	0.1000	0.0005	0.0100
Mid	0.0100	0.8000	0.0200	0.2000
High	0.0300	0.5000	0.0500	0.5000

Maturity	Low Risk	Middle Risk	High risk
1y	0	92	234
2y	0	104	244
3y	0	112	248
4y	1	117	250
5y	1	120	251
6y	1	122	252
7y	1	124	253
8y	1	125	253
9y	1	126	254
10y	1	127	254

Table 2: Break-even spreads in basis points generated using the parameters of the CIR processes in Table 1. The first column is generated using low credit risk and credit risk volatility. The second column is generated using middle credit risk and credit risk volatility. The third column is generated using high credit risk and credit risk volatility.

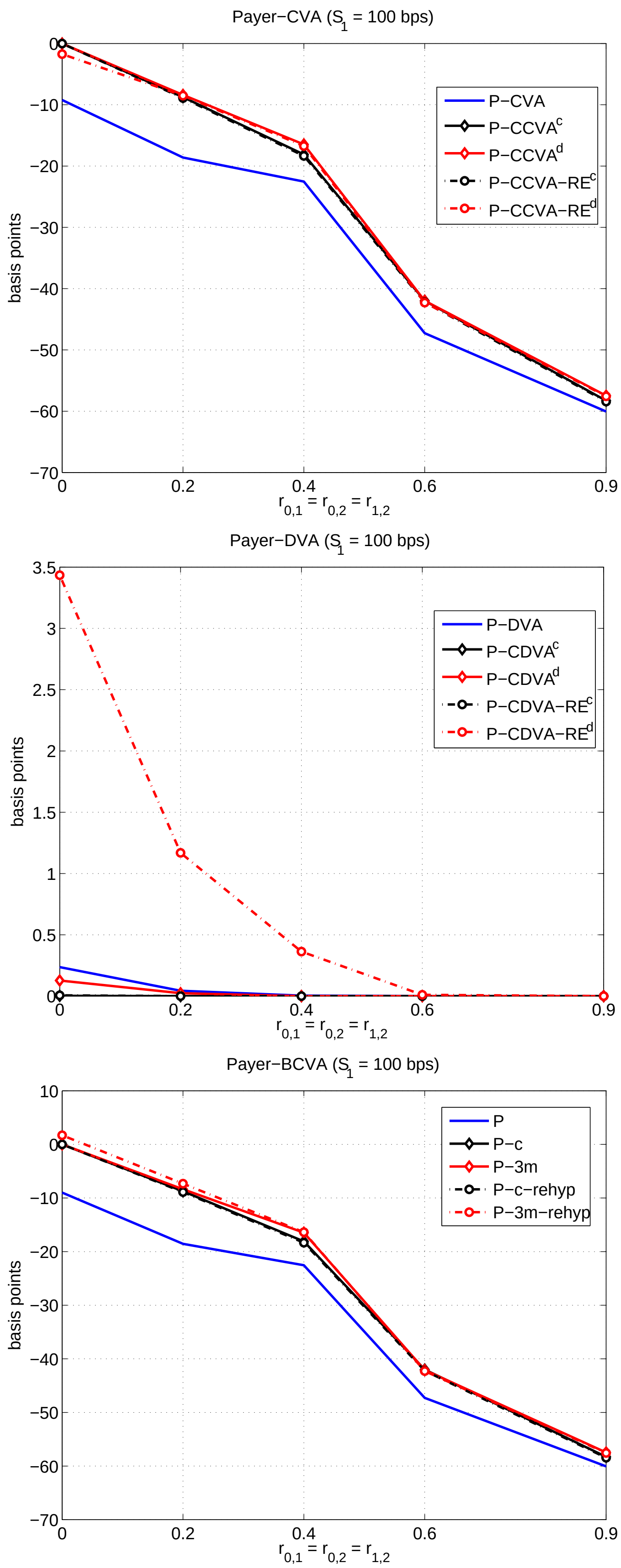

Figure 1: Counterparty Value Adjustments versus default correlation under the different collateralization strategies for the five-year Payer CDS contract. The 5-year CDS spread is set to 100 basis points

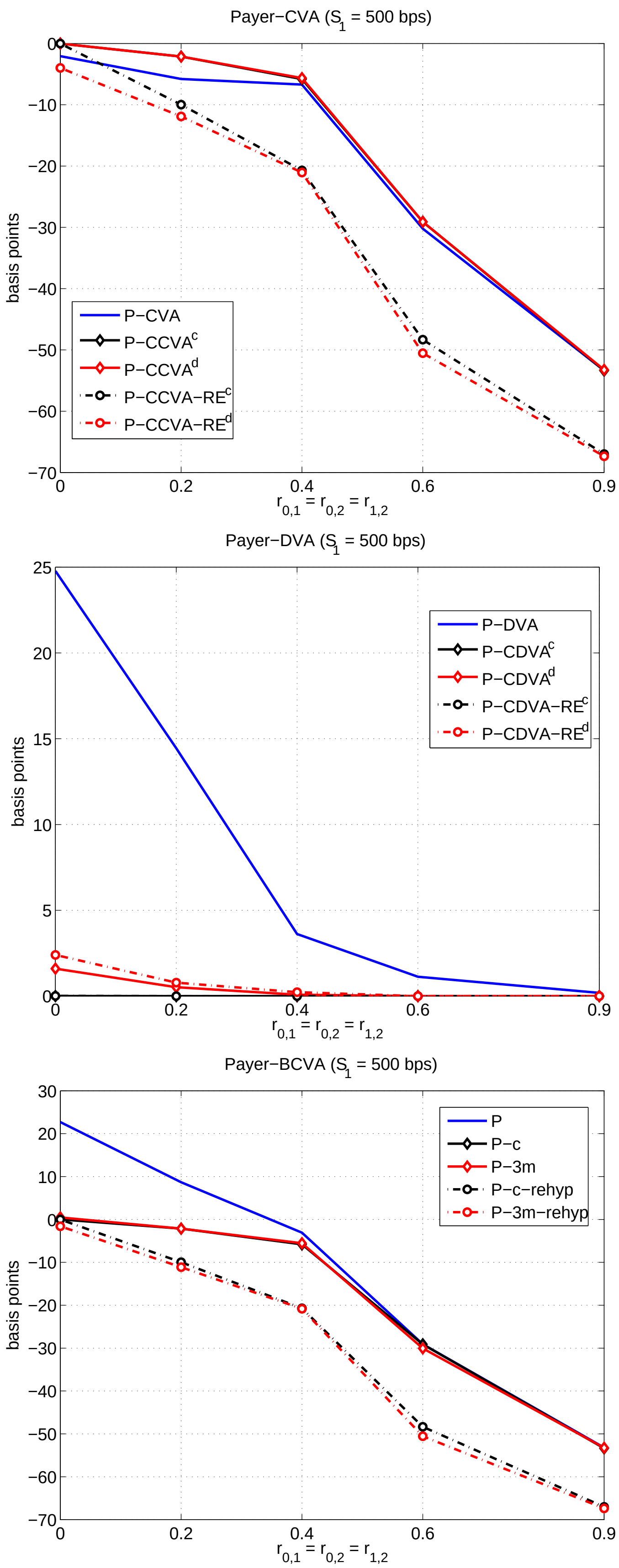

Figure 2: Counterparty Value Adjustments versus default correlation under the different collateralization strategies for the five-year Payer CDS contract. The 5-year CDS spread is set to 500 basis points

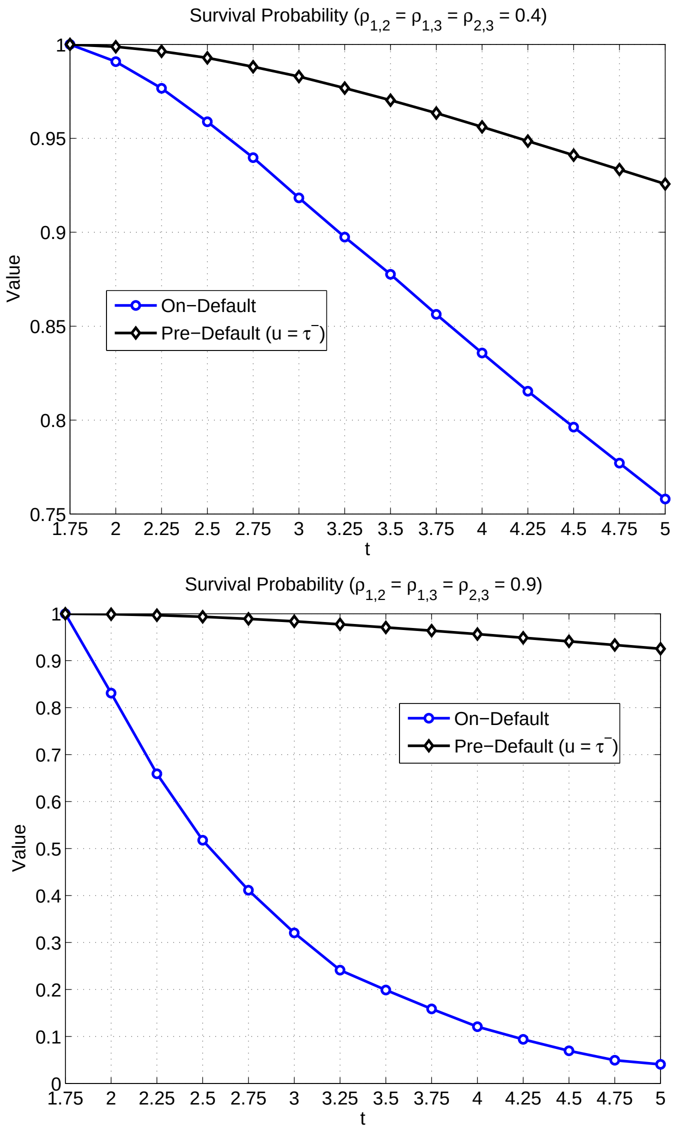

Figure 3: On-default survival probability and Pre-Default survival probability. The default time is $\tau = \tau_0 = 1.75$.