Open access • Journal Article • DOI:10.1007/BF01197552

Arbitrary N-vortex solutions to the first order Ginzburg-Landau equations

— Source link

Clifford Henry Taubes

Institutions: Harvard University

Published on: 01 Oct 1980 - Communications in Mathematical Physics (Springer-Verlag)

Topics: Simultaneous equations, Higgs field, Nonlinear system and Vortex

Commun. Math. Phys. 72, 277–292 (1980)

Communications in
Mathematical
Physics

Arbitrary N-Vortex Solutions to the First Order Ginzburg-Landau Equations*

Clifford Henry Taubes

Lyman Laboratory of Physics, Harvard University, Cambridge, MA 02138, USA

Abstract. We prove that a set of N not necessarily distinct points in the plane determine a unique, real analytic solution to the first order Ginzburg-Landau equations with vortex number N. This solution has the property that the Higgs field vanishes only at the points in the set and the order of vanishing at a given point is determined by the multiplicity of that point in the set. We prove further that these are the only C^∞ solutions to the first order Ginzburg-Landau equations.

1. Introduction

A mathematical model of superconductors is given by the Ginzburg-Landau equations [1]. These equations exhibit vortex solutions which may be viewed as finite energy solutions to the equations describing the two dimensional Abelian Higgs model [2]. After suitable rescaling, the equations have one coupling constant, λ, whose value determines whether the equations describe a type I or type II superconductor. The value $\lambda = 1$ is the critical value. In this case, the energy of a configuration is bounded below by a topological invariant, a multiple of the vortex number. Any configuration which achieves this minimum energy will be a solution of the Ginzburg-Landau equations. To find solutions with this minimum bound, one need only solve a set of first order coupled equations for the vector potential and the Higgs field rather than the more general second order equations. DeVega and Schnaposnik [3] in an analysis of the equations, gave a numerical argument for the existence of a cylindrically symmetric solution to the first order equations with vortex number N which has been interpreted as N-vortices superimposed at a point. Weinberg [4] recently proved that if a solution of the Ginzburg-Landau equations with vortex number N exists then the dimension of the space of moduli of this solution must be $2N$. This led him to conjecture that there exists a $2N$ parameter family of solutions with vortex number N and that the parameters of a solution may be interpreted as being the positions in $\mathbb{R}^2$ of the N vortices. Recent

* This work is supported in part through funds provided under Contract PHY 77-18762

numerical work by Jacobs and Rebbi [5] concerning the interaction energy of a pair of vortices lends support to this conjecture.

In this paper we prove the validity of Weinberg's conjecture. To be precise, we prove that the solutions to the first order Ginzburg-Landau equations with coupling constant $\lambda=1$ are uniquely determined by a set of N not necessarily distinct points in the plane corresponding to the zeros of the Higgs field. Every set of N points determines one such solution. The vortex number of this solution is N. The solution manifold of the first order Ginzburg-Landau equations with vortex number N is isomorphic to $\mathbb{R}^{2N}$.

II. The Equations

After a suitable rescaling, the action for the two-dimensional Abelian Higgs model is:

$$\mathscr{A}=\int_{\mathbb{R}^2} d^2x\left\{\tfrac{1}{2}|(\partial_j-iA_j)\phi|^2+\tfrac{1}{4}F_{jk}F_{jk}+\frac{\lambda}{8}(|\phi|^2-1)^2\right\}. \tag{2.1}$$

The Higgs field ϕ is a complex scalar field, or alternatively it may be interpreted as a cross-section of a complex line bundle over $\mathbb{R}^2$. The A_j are the components of the connection on the line bundle; $F_{jk}=\partial_jA_k-\partial_kA_j$ is the curvature. The critical value of the coupling constant λ is 1. Henceforth we shall only discuss this case. Finite action requires that a configuration (A_j,ϕ) have the following behavior as $|x|$ approaches infinity:

$$\begin{aligned}&|\phi|\to 1\\ &D_A\phi=d\phi-iA\phi\to 0, \quad \text{as} \quad |x|\to\infty .\end{aligned} \tag{2.2}$$

The usual topological arguments imply that the vortex number,

$$n=\frac{1}{2\pi}\int_{\mathbb{R}^2} d^2xF_{12} \tag{2.3}$$

is an integer and is unchanged by finite action perturbations of the fields. The integer n is a topological invariant; the first Chern number of the complex line bundle in which A is a connection.

As Bogomol'nyi [6] pointed out, a lower bound on the action results from rewriting the action via an integration by parts as

$$\begin{aligned}\mathscr{A}=\int_{\mathbb{R}^2} d^2x\{&\tfrac{1}{2}[(\partial_1\phi_1+A_1\phi_2)\mp(\partial_2\phi_2-A_2\phi_1)]^2\\ &+\tfrac{1}{2}[(\partial_2\phi_1+A_2\phi_2)\pm(\partial_1\phi_2-A_1\phi_1)]^2\\ &+\tfrac{1}{2}[F_{12}\pm\tfrac{1}{2}(\phi_1^2+\phi_2^2-1)]^2\}\pm\tfrac{1}{2}\int_{\mathbb{R}^2} d^2xF_{12}\,.\end{aligned} \tag{2.4}$$

The upper sign corresponds to positive vortex number and the lower sign to negative vortex number. In Eq. (2.4) $\phi_1=\operatorname{Re}\phi$ and $\phi_2=\operatorname{Im}\phi$. We shall treat the case of positive vortex number only, the case of $n<0$ being completely analogous.

The lower bound of Bogomol'nyi is

$$\mathscr{A} \geqq n\pi \,. \tag{2.5}$$

This lower bound is realized if and only if

$$(\partial_1\phi_1 + A_1\phi_2) - (\partial_2\phi_2 - A_2\phi_1) = 0\,, \tag{2.6a}$$

$$(\partial_2\phi_1 + A_2\phi_2) + (\partial_1\phi_2 - A_1\phi_1) = 0\,, \tag{2.6b}$$

$$F_{12} + \tfrac{1}{2}(\phi_1^2 + \phi_2^2 - 1) = 0\,. \tag{2.6c}$$

Equations similar to these arise in the study of cylindrically symmetric self-dual solutions to four dimensional SU(2) Yang-Mills equations [7]. As in that case, it is possible to reduce Eqs. (2.6) to one nonlinear second order differential equation for one unknown function. To do this set

$$\hat{A} = A_1 + iA_2$$

$$\partial = \tfrac{1}{2}(\partial_1 - i\partial_2)\,; \qquad \bar{\partial} = \tfrac{1}{2}(\partial_1 + i\partial_2)\,. \tag{2.7a}$$

Equations (2.6a) and (2.6b) are the real and imaginary parts of the complex valued equation

$$2\bar{\partial}\phi - i\hat{A}\phi = 0 \tag{2.7b}$$

This equation may be solved for $\hat{A}$

$$\hat{A} = -2i\bar{\partial}\ln\phi\,. \tag{2.8}$$

Define the complex valued function $f = f_1 + if_2$ with f_1 and f_2 real valued on $\mathbb{R}^2$ by

$$\phi = e^f\,. \tag{2.9}$$

In terms of the real functions f_1 and f_2 the components of A and the curvature are

$$\begin{aligned} A_1 &= \partial_2 f_1 + \partial_1 f_2 \\ A_2 &= -\partial_1 f_1 + \partial_2 f_2 \\ F_{12} &= -\Delta f_1\,. \end{aligned} \tag{2.10}$$

The asymptotic condition that $|\phi| = 1$ as $|x|$ approaches infinity demands that

$$\lim_{|x|\to\infty} f_1(x) = 0\,. \tag{2.11}$$

Equation (2.6c) is a second order nonlinear differential equation for f_1 [5]:

$$-\Delta f_1 + \tfrac{1}{2}(e^{2f_1} - 1) = 0\,. \tag{2.12}$$

We reinterpret the topological invariant n as defined in equation (2.3) in terms of the functions f_1 and f_2. For large values of $|x|$,

$$\phi \to e^{if_2}\,, \quad \text{as} \quad |x| \to \infty \tag{2.13a}$$

and

$$f_2(\theta, |x|) = f_2(\theta + 2\pi, |x|) + 2\pi n\,, \tag{2.13b}$$

where n is the vortex number defined in Eq. (2.3). Homotopy considerations imply that no continuous function on $\mathbb{R}^2$ can have this asymptotic behavior. The Ginzburg-Landau equations leave f_2 undetermined. This is the manifestation of the gauge invariance of the original lagrangian. Any f_2 with the above asymptotic behavior must be singular on some set in the plane. If we are interested in at least continuous solutions to the first order Ginzburg-Landau equations, then ϕ must vanish on a nontrivial set in the plane and there must exist a function f_2 satisfying Eq. (2.13b) which has its singularities only on this set. In the Appendix we shall prove that a necessary condition for ϕ to solve the first order Ginzburg-Landau equations is that the zero set of ϕ be discrete. For now we shall assume that this is the case. If we demand that our solutions be C^1 on $\mathbb{R}^2$, we are further restricted to have ϕ vanish as an integer power of $(x-a_i)$ as x approaches a_i for a_i a zero of ϕ. Any solution to the first order equations determines a set of N points $(a_1, \ldots, a_N)$ in $\mathbb{R}^2$. These points need not be distinct, the number of times a given point occurs corresponds to the order of vanishing of ϕ at that point. As $x \to a_k$ the function f_1 must have the behavior:

$$\lim_{|x| \to a_k} f_1 \to \frac{n_k}{2} \ln (x-a_k)^2, \tag{2.14}$$

where n_k is the order of vanishing of ϕ at a_k. Given f_1 with this behavior, we can find a function f_2 with the asymptotic behavior expressed in (2.13b) which is in fact C^∞ in $\mathbb{R}^2 \setminus \bigcup_{k=1}^{N} \{a_k\}$:

$$f_2 = \sum_{k=1}^{N} \theta_k, \tag{2.15}$$

where

$$\theta_k = \tan^{-1} \frac{(x-a_k)_1}{(x-a_k)_2}.$$

The topological charge, n, is equal to the size, N, of the set $(a_1, a_2, \ldots, a_N)$.

A simple calculation shows that with f_2 given above and f_1 as in Eq. (2.14), vanishing as $|x| \to \infty$ and C^∞ in $\mathbb{R}^2 \setminus \bigcup_{k=1}^{n} \{a_k\}$ the components of the connection [Eq. (2.10)] are in fact infinitely differentiable on $\mathbb{R}^2$.

III. The Results

Up to now, solutions to the first order Ginzburg-Landau equations with vortex number n are thought to exist only when the n points $(a_1, \ldots, a_n)$ are the same [3]. The main result of this paper is the following theorem:

Theorem 1. *To each point* $(a_1, \ldots, a_n)$ *in* $\mathbb{R}^2 \times \ldots \times \mathbb{R}^2 = \mathbb{R}^{2n}$, *there exists a unique globally* C^∞ *solution to Eqs.* (2.6a)–(2.6c) *which satisfies the conditions of Eqs.* (2.2) *and* (2.3) *and satisfies the further conditions that*

$$\{x \in \mathbb{R}^2 | \phi(x) = 0\} = \bigcup_{k=1}^{n} \{a_k\} \tag{3.1}$$

with the order of vanishing of ϕ at a point a_0 being precisely the number of times a_0 appears in the set $\{a_1, \ldots, a_n\}$. □

From the discussion of the last section, we will have proved Theorem 1 if we can prove that Eq. (2.12) has a unique solution, f_1, such that f_1 is C^∞ in $\mathbb{R}^2 \setminus \bigcup_{k=1}^{n} \{a_k\}$; Eq. (2.11) is satisfied as $|x| \to \infty$ and as x approaches each point a_k, f_1 has the behavior described by Eq. (2.14). To do this, let

$$u_0 = -\sum_{k=1}^{n} \ln\left(1 + \frac{\lambda}{(x-a_k)^2}\right) \tag{3.2}$$

with $\lambda > 4n$ a real number. On $\mathbb{R}^2 \setminus \bigcup_{k=1}^{n} \{a_k\}$, u_0 is infinitely differentiable and formally

$$-\Delta u_0 = 4 \sum_{k=1}^{n} \frac{\lambda}{((x-a_k)^2 + \lambda)^2} + 4\pi \sum_{k=1}^{n} \delta(x - a_k). \tag{3.3}$$

Here, $\delta(x)$ is the Dirac delta function. Define a function g_0 by

$$g_0 = 4 \sum_{k=1}^{n} \frac{\lambda}{((x-a_k)^2 + \lambda)^2} \tag{3.4}$$

so that on $\mathbb{R}^2 \setminus \bigcup_{k=1}^{n} \{a_k\}$, g_0 and $-\Delta u_0$ agree. Next we define a new unknown function v by:

$$2f_1 = u_0 + v. \tag{3.5}$$

If f_1 satisfies Eq. (2.12) and has the behavior at infinity and at the points $\{a_k\}$ discussed above, then v is a solution to the equation

$$-\Delta v + g_0 - 1 + e^{u_0} e^{v} = 0, \tag{3.6a}$$

$$\lim_{|x| \to \infty} v = 0. \tag{3.6b}$$

Theorem I then follows from

Theorem I′. *For every point $(a_1, \ldots, a_n)$ in $\mathbb{R}^2 \times \mathbb{R}^2 \times \ldots \times \mathbb{R}^2 = \mathbb{R}^{2n}$ and u_0 and g_0 defined by Eqs.* (3.2) *and* (3.4), *respectively, there exists a unique function v, real analytic in $\mathbb{R}^2$, satisfying* (3.6a) *and* (3.6b). □

We observe that (3.6a) is formally the variational equations of the functional

$$a(v) = \int_{\mathbb{R}^2} d^2x \left(\tfrac{1}{2}\partial_\mu v \partial_\mu v - v(1 - g_0) + e^{u_0}(e^{v} - 1)\right). \tag{3.7}$$

The strategy for proving Theorem I′ is to define the functional $a(v)$ on an appropriate Banach space and then appeal to the following results of functional analysis. For a detailed exposition of the techniques used here see e.g., Vainberg [8].

Proposition 3.1. *If a strictly convex functional $G(x)$ defined in a linear space E has a minimum at a point x_0, then x_0 is an absolute minimum point, and there are no other minimum points.* □

For a proof, see [8], p. 96.

In Sect. IV we show that there exists an appropriate Banach space on which $a(v)$ is strictly convex. This proves the uniqueness assertion of Theorem I′.

The existence of a solution will follow if we can show that $a(v)$ satisfies the hypothesis of

Proposition 3.2. *Let $G(x)$ be a real Gâteaux differentiable functional defined in a real reflexive Banach space E, which is weakly lower semi-continuous and satisfies the condition*

$$\langle G'(x), x\rangle > 0 \tag{3.8}$$

for any vector $x \in E$, $\|x\| = R > 0$, and $G'(x) = \operatorname{grad} G(x)$. Then there exists an interior point x_0 of the ball $\|x\| \leqq R$ at which $f(x)$ has a local minimum so that $G'(x) = 0$. □

For a proof, see [8], p. 100.

We prove in Sect. V that $a(v)$ satisfies the hypothesis of Proposition 3.2. This proves the existence of weak solution to Eq.(3.6a).

In Sect. VI we prove that the solution must in fact be C^∞ on $\mathbb{R}^2$.

IV. Properties of $a(v)$

We begin by defining a Banach space H to be the completion of $C_0^\infty(\mathbb{R}^2)$ (the space of infinitely differentiable functions with compact support) in the norm

$$\|v\|_{1,2} = \left[\int_{\mathbb{R}^2} d^2x(\partial_\mu v \partial_\mu v + v^2)\right]^{1/2}. \tag{4.1}$$

The space H is the first Sobolev space on $\mathbb{R}^2$. If Ω is any Lebesgue measurable set in $\mathbb{R}^2$, define for $1 \leqq p, q < \infty$

$$\|v\|_{1,p;\Omega} = \left[\int_\Omega d^2x\left(\sum_{\mu=1}^{2} |\partial_\mu v|^p + |v|^p\right)\right]^{1/p}$$

$$\|v\|_{0,q;\Omega} = \left[\int_\Omega d^2x |v|^q\right]^{1/q}. \tag{4.2}$$

The functional $a(v)$ as defined in Eq. (3.7) is defined on $C_0^\infty(\mathbb{R}^2)$ which is dense in H.

We state the fundamental properties of the functional $a(v)$ in the following set of propositions and their corollaries:

Proposition 4.1. *Define the functional $a(v)$ on C_0^∞ by*

$$a(v) = \int_{\mathbb{R}^2} \{\tfrac{1}{2}\partial_\mu v \partial_\mu v - v(1-g_0) + e^{u_0}(e^v - 1)\} \tag{4.3}$$

with u_0 and g_0 defined by Eqs. (3.2) and (3.4); then $a(v)$ extends to a nonlinear functional on H with domain H. □

Proposition 4.2. *The Gâteaux derivative of $a(v)$, $a'(v;h)$, exists for all $v, h \in H$ and*

$$a'(v,h) \equiv \lim_{t\to 0} \frac{1}{t}(a(v+th) - a(v)) = \int_{\mathbb{R}^2} \{\partial_\mu v \partial_\mu h - h(1-g_0-e^{u_0}) + he^{u_0}(e^v - 1)\}. \tag{4.4}$$

Furthermore, for fixed v, $a'(v,\cdot)$ is a bounded linear functional on H, denoted $\langle \operatorname{grad} a(v), h\rangle$. For fixed $h\in H$, $a'(\cdot, h)$ is a nonlinear functional on H with domain H. □

Proposition 4.3. *The functional $a(v)$ is strictly convex on H.* □

Before proving these propositions we state and prove some immediate corollaries:

Corollary 4.4. *The functional $a(v)$ is weakly lower semi-continuous on H.* □

Proof of Corollary 4.4. From Proposition 4.2 the Gâteaux differential $a'(v,\cdot)$ for fixed $v\in H$ is bounded and hence continuous on H. The result follows from Theorem 8.6 of Vainberg [8], p. 82.

Corollary 4.5. *v_0 is a minimum of the functional $a(v)$ if and only if* $\operatorname{grad} a(v_0)=0$. □

Proof of Corollary 4.5. From Proposition 4.2 Gâteaux differential, $a'(v,\cdot)$ for fixed v is linear. From Proposition 4.3, $a(v)$ is convex on H. The result follows from Theorem 9.1 of Vainberg [8], p. 91.

We remark that Proposition 4.3 implies that if $a(v)$ has a minimum on H, that minimum is unique. Corollary 4.5 states that if the minimum exists, then it is a weak solution to Eqs. (3.6a) and (3.6b).

Proof of Proposition 4.1. We must show that if $v\in H$ then $a(v)$ is finite. Write $a(v)$ as a sum of three terms:

$$a(v)=\int_{\mathbb{R}^2}\tfrac{1}{2}\partial_\mu v\partial_\mu v-\int_{\mathbb{R}^2} v(1-g_0-e^{u_0})+\int_{\mathbb{R}^2} e^{u_0}(e^v-1-v)\,. \tag{4.5}$$

We note that the first term on the right in Eq. (4.5) is bounded by $1/2\|v\|_{1,2}^2$. Using Holder's inequality we have for the second term

$$\begin{aligned}\left|\int_{\mathbb{R}^2} v(1-g_0-e^{u_0})\right| &\leqq \left[\int_{\mathbb{R}^2} v^2\right]^{1/2}\left[\int_{\mathbb{R}^2}(1-g_0-e^{u_0})^2\right]^{1/2}\\ &\leqq \|v\|_{1,2}\|(1-g_0-e^{u_0})\|_{0,2;\mathbb{R}^2}\,.\end{aligned} \tag{4.6}$$

Since $1-g_0-e^{u_0}$ is in $L_2(\mathbb{R}^2, d^2x)$ the second term is bounded on H. For the third term in Eq. (4.5) we have

$$\left|\int_{\mathbb{R}^2} e^{u_0}(e^v-1-v)\right| \leqq \int_{\mathbb{R}^2}|e^v-1-v|\,. \tag{4.7}$$

It remains to be shown that the right hand side of Eq. (4.7) is finite for all $v\in H$. For $\varrho>0$, define $I_1(\varrho v)$ by

$$I_1(\varrho v)=\int_{\mathbb{R}^2}\{\exp(\varrho^2 v^2)-1\}\,dx\,. \tag{4.8}$$

Given $v\in H$, there exists $\varrho>0$ such that $I_1(\varrho v)<\infty$ (cf. [9], pp. 242–246). Fix such a ϱ. To estimate $\int_{\mathbb{R}^2}|e^v-1-v|$ define sets Ω_1, $\Omega_2\subset\mathbb{R}^2$ by

$$\begin{aligned}\Omega_1&=\{x\in\mathbb{R}^2|v(x)\geqq\varrho^{-2}\}\,,\\ \Omega_2&=\{x\in\mathbb{R}^2|v(x)<\varrho^{-2}\}\,.\end{aligned} \tag{4.9}$$

In Ω_1 we have $\varrho^2 v^2 \geqq v$, hence

$$I_1 \geqq \int_{\Omega_1} \{\exp(\varrho^2 v^2) - 1\} \geqq \int_{\Omega_1} |e^v - 1| \geqq \int_{\Omega_1} |e^v - 1 - v| \,. \tag{4.10}$$

In Ω_2, v is bounded. Hence there exists a constant $c > 0$ such that

$$|e^v - 1 - v| \leqq c v^2 \quad \text{in} \quad \Omega_2 \,. \tag{4.11}$$

Therefore:

$$\int_{\mathbb{R}^2} |e^v - 1 - v| \leqq I_2 + c \int_{\mathbb{R}^2} v^2 < \infty \,, \tag{4.12}$$

and we have proved

Lemma 4.6. *For all $v \in H$, the functional*

$$R(v) = \int_{\mathbb{R}^2} |e^v - 1 - v|$$

is finite.

Proof of Proposition 4.2. Let v, $h \in H$. The Gateaux derivative of $a(v)$ is by definition

$$\begin{aligned} a'(v; h) &\equiv \lim_{t \to 0} \frac{a(v + th) - a(v)}{t} \\ &= \lim_{t \to 0} \left\{ \int_{\mathbb{R}^2} \partial_\mu v \partial_\mu h + \frac{t}{2} \partial_\mu h \partial_\mu h - h(1 - g_0) + e^{u_0} e^v \frac{(e^{th} - 1)}{t} \right\} \\ &= \int_{\mathbb{R}^2} \{\partial_\mu v \partial_\mu h - h(1 - g_0) + h e^{u_0 + v}\} \\ &\quad + \lim_{t \to 0} \left\{ \int_{\mathbb{R}^2} \frac{t}{2} \partial_\mu h \partial_\mu h + e^{u_0} e^v \frac{(e^{th} - 1 - th)}{t} \right\} . \end{aligned} \tag{4.13}$$

To prove Proposition 4.2 we must first show that Eq. (4.4) is correct. Once we have established Eq. (4.4) it remains for us to demonstrate that for fixed $v \in H$, $a'(v, \cdot)$ is a bounded functional on H and for fixed $h \in H$, the nonlinear functional $a'(\cdot, h)$ has domain H.

Lemma 4.7. *For any $h \in H$*

$$\lim_{t \to 0} \left\{ \int_{\mathbb{R}^2} \frac{t}{2} \partial_\mu h \partial_\mu h + e^{u_0} e^v \frac{(e^{th} - 1 - th)}{t} \right\} = 0 \,. \tag{4.14}$$

Proof of Lemma 4.7. We first note that

$$\lim_{t \to 0} t \int \partial_\mu h \partial_\mu h \leqq \lim_{t \to 0} t \|h\|_{1,2}^2 = 0 \,. \tag{4.15}$$

For the remaining term in Eq. (4.14) we have the bound

$$\begin{aligned} \lim_{t \to 0} \left| \int_{\mathbb{R}^2} e^{u_0} e^v \frac{(e^{th} - 1 - th)}{t} \right| &\leqq \lim_{t \to 0} \int_{\mathbb{R}^2} \left| e^{u_0} \frac{(e^{th} - 1 - th)}{t} \right| \\ &+ \lim_{t \to 0} \int_{\mathbb{R}^2} \left| e^{u_0} \frac{(e^{th} - 1 - th)}{t} (e^v - 1) \right| . \end{aligned} \tag{4.16}$$

In order to show that the right hand side of (4.16) is zero in the limit that t goes to zero we need the estimate

Lemma 4.8. *For any $v \in H$ and $k \geqq 2$*

$$\|v\|^k_{0,k;\mathbb{R}^2} \leqq 2^{k/2+2} k^k \|v\|^k_{1,2;\mathbb{R}^2} . \tag{4.17}$$

Proof of Lemma 4.8. This estimate is proved in Adams [9], pp. 103–106.

We return to the proof of Lemma 4.7. For the first term on the right hand side of (4.16) we have

$$\frac{1}{t} \int_{\mathbb{R}^2} |e^{u_0}(e^{th} - 1 - th)| \leqq \frac{1}{t} \int_{\mathbb{R}^2} |e^{th} - 1 - th| \leqq \int_{\mathbb{R}^2} \sum_{k=2}^{\infty} \frac{1}{k!} t^{k-1} |h|^k . \tag{4.18}$$

In (4.18) we have defined e^{th} by its power series. By the Monotone Convergence Theorem we can interchange the order of integration and summation. Using (4.17) we have

$$\frac{1}{t} \int_{\mathbb{R}^2} |e^{u_0}(e^{th} - 1 - th)| \leqq 4 \sum_{k=2}^{\infty} t^{k-1} \frac{k^k}{k!} (\sqrt{2} \|v\|_{1,2;\mathbb{R}^2})^k . \tag{4.19}$$

Next we remark that by Stirling's formula, n^n is asymptotic to $n!e^n(2\Pi n)^{1/2}$ and that $\exists N > 0$ such that $\forall n > N$

$$n^n \leqq n! e^n . \tag{4.20}$$

Choose N such that (4.20) holds. Then

$$\begin{aligned} \frac{1}{t} \int_{\mathbb{R}^2} |e^{u_0}(e^{th} - 1 - th)| \leqq & 4 \sum_{k=2}^{N} t^{k-1} \frac{k^k}{k!} (\sqrt{2} \|v\|_{1,2;\mathbb{R}^2})^k \\ & + 4 \sum_{k=N+1}^{\infty} t^{k-1} (\sqrt{2} e \|v\|_{1,2;\mathbb{R}^2})^k . \end{aligned} \tag{4.21}$$

The first term on the right hand side of (4.21) is finite and vanishes as $t \to 0$. The second term is an infinite sum which converges for $t < (\sqrt{2} e \|v\|_{1,2;\mathbb{R}^2})^{-1}$ and vanishes as t^N as $t \to 0$.

For the second term in (4.16) we use Holder's inequality to write

$$\begin{aligned} \lim_{t \to 0} \int_{\mathbb{R}^2} \left| e^{u_0} \frac{(e^{th} - 1 - th)}{t} (e^v - 1) \right| \leqq & \left[\int_{\mathbb{R}^2} (e^v - 1)^2 \right]^{1/2} \lim_{t \to 0} \\ & \cdot \frac{1}{t} \left[\int_{\mathbb{R}^2} |e^{th} - 1 - th|^2 \right]^{1/2} . \end{aligned} \tag{4.22}$$

The function $(e^v - 1)$ for $v \in H$ is square integrable on $L^2(\mathbb{R}^2, dx)$. This follows from the bound

$$\int_{\mathbb{R}^2} (e^v - 1)^2 \leqq \int_{\mathbb{R}^2} |e^{2v} - 2v - 1| + 2 \int_{\mathbb{R}^2} |e^v - v - 1| \tag{4.23}$$

which is finite due to Lemma 4.6. We remark that $\int_{\mathbb{R}^2} |e^{th} - 1 - th|^2$ is order t^4.

For, given any $u\in H$ we have

$$\int_{\mathbb{R}^2} |e^u - 1 - u|^2 \leqq \int_{\mathbb{R}^2} \sum_{j,k=2}^{\infty} \frac{1}{j!k!} |u|^{j+k} \leqq \sum_{n=4}^{\infty} \sum_{k=0}^{n} \frac{1}{(n-k)!k!} \|u\|_{0,n;\mathbb{R}^2}^n . \tag{4.24}$$

The last step follows from the Monotone Convergence Theorem and the fact that the sum has strictly positive terms. Using Lemma 4.8 to bound $\|u\|_{0,n;\mathbb{R}^2}^n$ and choosing an integer N so that inequality (4.20) holds we have

$$\int_{\mathbb{R}^2} |e^u - 1 - u|^2 \leqq 2 \sum_{n=4}^{N} \sum_{k=0}^{n} \frac{n^n}{(n-k)!k!} 2^{n/2} \|u\|_{1,2;\mathbb{R}^2}^n + 2 \sum_{n=N+1}^{\infty} (2\sqrt{2e}\|u\|_{1,2;\mathbb{R}^2})^n . \tag{4.25}$$

Setting $u = th$, we see that for sufficiently small t, the right hand side of (4.25) is finite and order t^4. This implies that as $t\to 0$ the right hand side of (4.22) vanishes, proving Lemma 4.7.

To prove the remaining assertions of Proposition 4.2 notice that

$$|a'(v;h)| \leqq \int_{\mathbb{R}^2} |\partial_\mu v \partial_\mu h| + \int_{\mathbb{R}^2} |h|\,|1 - g_0 - e^{u_0}| + \int_{\mathbb{R}^2} |h|\,|e^v - 1| . \tag{4.26}$$

We use Holder's inequality to bound $|a'(v,h)|$ by

$$|a'(v,h)| \leqq \|h\|_{1,2} \left\{ \|v\|_{1,2} + \left(\int_{\mathbb{R}^2} (1 - g_0 - e^{u_0})^2 \right)^{1/2} + \left(\int_{\mathbb{R}^2} |e^v - 1|^2 \right)^{1/2} \right\} . \tag{4.27}$$

The right hand side of (4.27) is finite for any $v\in H$. This completes the proof of Proposition 4.2.

Proof of Proposition 4.3. To prove that the functional $a(v)$ is strictly convex we must show that

$$a(\gamma v_1 + (1-\gamma) v_2) \leqq \gamma a(v_1) + (1-\gamma) a(v_2) . \tag{4.28}$$

For all $v_1, v_2 \in H$; $\gamma \in (0,1)$ with equality only when $v_1 = v_2$. This result follows from the strict convexity of the functional $\int \partial_\mu v \partial_\mu v$ on H and the strict convexity of the exponential function. We leave the details to the reader.

V. The Existence of Weak Solutions

To prove the existence assertion of Theorem I′ it is sufficient to show that the functional $a(v)$ satisfies the conditions set forth in Proposition 3.2. Gâteaux differentiability of $a(v)$ was proven in Proposition 4.2. Weak lower semi-continuity of the functional is a result of Corollary 4.4. It remains for us to show that there exists a positive number R such that Eq. (3.8) is obeyed by $\langle \operatorname{grad} a(v), v\rangle$ for all v with norm equal to R. The existence of such an R follows from the following estimate:

Proposition 5.1. *There exists constants* $\alpha>0$, b *and* $k>0$ *such that for all* $v\in H$

$$a'(v;v)\geqq\frac{\alpha\|v\|_{1,2}^2}{(1+k\|v\|_{1,2})}-b\,. \quad \square \tag{5.1}$$

The proof of Proposition 5.1 rests on the following properties of the functions u_0 and g_0:

Lemma 5.2. *Let* u_0 *and* g_0 *be defined by Eq.* (3.2) *and* (3.4) *respectively. Let* $\lambda>4n$. *Then*

a) *There exists a constant* $c_1>0$ *such that for all* $x\in\mathbb{R}^2$

$$1-g_0(x)\geqq c_1\,. \tag{5.2}$$

b) *For all* $x\in\mathbb{R}^2$

$$1-g_0(x)-e^{u_0(x)}>0\,. \tag{5.3}$$

Proof of Lemma 5.2. For part (a) we note that for $\lambda>4n$

$$g_0\leqq\frac{4n}{\lambda}<1\,. \tag{5.4}$$

For part (b) we have

$$g_0(x)+e^{u_0(x)}=4\sum_{k=1}^{n}\frac{\lambda}{((x-a_k)^2+\lambda)^2}+\prod_{k=1}^{n}\frac{(x-a_k)^2}{((x-a_k)^2+\lambda)}\,. \tag{5.5}$$

Define $\gamma=4n/\lambda<1$ and

$$z_k=((x-a_k)^2/\lambda+1)^{-1}\,. \tag{5.6}$$

With this notation we rewrite (5.5) as

$$g_0+e^{u_0}=\prod_{k=1}^{n}(1-z_k)+\frac{\gamma}{n}\sum_{k=1}^{n}z_k^2\,. \tag{5.7}$$

Each z_k is less than 1 so that

$$\prod_{k=1}^{n}(1-z_k)<1-\frac{1}{n}\sum_{k}z_k\,. \tag{5.8}$$

Using this inequality in (5.7) gives finally

$$g_0+e^{u_0}<1-\frac{1}{n}\sum_{k=1}^{n}z_k+\frac{\gamma}{n}\sum_{k=1}^{n}z_k^2<1-\frac{(1-\gamma)}{n}\sum_{k=1}^{n}z_k<1\,. \tag{5.9}$$

Inequality (5.9) proves part (b) of the lemma.

Proof of Proposition 5.1. Recall that the functional $a'(v;v)$ is given by

$$a'(v;v)=\int_{\mathbb{R}^2}[\partial_\mu v\partial_\mu v+v(e^{\mu_0+v}-1+g_0)]\,. \tag{5.10}$$

We consider the expression $v(e^{u_0+v}-1+g_0)$ pointwise.

Case 1. If $v(x) \geqq 0$ then

$$v(e^{u_0+v}-1+g_0)=v^2+v(u_0+g_0)+v(e^{u_0+v}-1-(u_0+v)) . \tag{5.11}$$

It is simple to show that for any $x \in \mathbb{R}^2$,

$$e^x-1-x>0 \tag{5.12}$$

with equality only at $x=0$, so the third term in (5.11) is positive. Completing the square in (5.11) to eliminate the linear term we have for $v(x) \geqq 0$ and any $\beta \in (0,1)$ the inequality

$$v(e^{u_0+v}-1+g_0) \geqq \beta v^2 - \frac{1}{(1-\beta)}(u_0+g_0)^2 . \tag{5.13}$$

Case 2. If $v(x) \leqq 0$ we have

$$v(e^{u_0+v}-1+g_0)=|v|(1-g_0-e^{u_0})+|v|e^{u_0}(1-e^{-|v|}) . \tag{5.14}$$

To continue further, notice that for $x \geqq 0$

$$1-e^{-x} \geqq \frac{x}{1+x} \tag{5.15}$$

with equality only when $x=0$. Inequality (5.15) implies that

$$v(e^{u_0+v}-1+g_0) \geqq |v|(1-g_0-e^{u_0})+\frac{|v|^2 e^{u_0}}{1+|v|} \geqq \frac{|v|^2 c_1}{1+|v|} . \tag{5.16}$$

The last inequality in (5.16) results from the inequalities (a) and (b) of Lemma 5.2.

Because β, c_1 and $(1+|v|)^{-1}$ are smaller than 1, we have proved that for any $\beta \in (0,1)$ and any $v \in H$

$$a'(v,v) \geqq \int_{\mathbb{R}^2} \left(\partial_\mu v \partial_\mu v + \frac{\beta c_1 |v|^2}{1+|v|}\right) - \frac{1}{(1-\beta)} \int_{\mathbb{R}^2} (u_0+g_0)^2 . \tag{5.17}$$

The function $(u_0+g_0)^2$ is integrable. We choose $\beta=1/2$ and define

$$b=2\|u_0+g_0\|^2_{0,2;\mathbb{R}^2} . \tag{5.18}$$

In order to continue we prove

Lemma 5.3. *For any* $v \in H$

$$\int_{\mathbb{R}^2} \frac{|v|^2}{(1+|v|)} \geqq \frac{\|v\|^4_{0,2;\mathbb{R}^2}}{(\|v\|^2_{0,2;\mathbb{R}^2}+\|v\|^3_{0,3;\mathbb{R}^2})} . \tag{5.19}$$

Proof of Lemma 5.3. We use Holder's inequality to write

$$\begin{aligned}\int_{\mathbb{R}^2} |v|^2 &= \int_{\mathbb{R}^2} \frac{v}{(1+|v|)^{1/2}}(1+|v|)^{1/2} v \\ &\leqq \left[\int_{\mathbb{R}^2} \frac{|v|^2}{1+|v|}\right]^{1/2} \left[\int_{\mathbb{R}^2} (|v|^2+|v|^3)\right]^{1/2} .\end{aligned} \tag{5.20}$$

Inequality (5.19) is true if $v=0$ so we may assume in (5.20) that $v \neq 0$. Squaring both sides of (5.20) and dividing by $(\|v\|_{0,2;\mathbb{R}^2}^2 + \|v\|_{0,3;\mathbb{R}^2}^3)$ yields the result.

Given a $v \in H$, we define the number $\sigma \in (0,1)$ by

$$\begin{aligned} \int_{\mathbb{R}^2} |v|^2 &= (1-\sigma)\|v\|_{1,2;\mathbb{R}^2}^2 \\ \int_{\mathbb{R}^2} \partial_\mu v \partial_\mu v &= \sigma \|v\|_{1,2;\mathbb{R}^2}^2 . \end{aligned} \tag{5.21}$$

From Lemma 4.9 we have the inequality

$$\|v\|_{0,3;\mathbb{R}^2}^3 \leqq k\|v\|_{1,2;\mathbb{R}^2}^3 , \tag{5.22}$$

where $k = 8\sqrt{2}\cdot 27$. These definitions and Lemma 5.3 give

$$a'(v,v) \geqq \sigma\|v\|_{1,2;\mathbb{R}^2}^2 + \frac{\beta c_1(1-\sigma)^2\|v\|_{1,2;\mathbb{R}^2}^2}{(1+k\|v\|_{1,2;\mathbb{R}^2})} - b . \tag{5.23}$$

Finally, if we set $\alpha = 3/4\beta c_1$ we have for any $v \in H$ the result:

$$a'(v,v) \geqq \frac{\alpha\|v\|_{1,2;\mathbb{R}^2}^2}{(1+k\|v\|_{1,2;\mathbb{R}^2})} - b . \tag{5.24}$$

This completes the proof of the existence and uniqueness of a weak solution to Eqs. (3.6a) and (3.6b).

The functions u_0 and g_0 as defined in Eqs. (3.2) and (3.4) depend not only on the points $(a_1, \ldots, a_n) \in \mathbb{R}^2 \times \ldots \times \mathbb{R}^2$ but also on the parameter λ. Our estimates required $\lambda > 4n$. Any two values, $\lambda_1, \lambda_2 > 0$ of λ give the same solution. For let v_i satisfy (3.6a) and (3.6b) with

$$\begin{aligned} u_0(i) &= -\sum_{k=1}^{n} \ln\left(1 + \frac{\lambda_i}{(x-a_k)^2}\right) \qquad i = 1,2 . \\ g_0(i) &= 4\sum_{k=1}^{n} \frac{\lambda_i}{((x-a_k)^2+\lambda_i)^2} \end{aligned} \tag{5.25}$$

We note that $u_0(1) - u_0(2)$ is in H. By uniqueness then we have

$$v_2 = u_0(1) - u_0(2) + v_1 . \tag{5.26}$$

VI. Properties of the Solution

We now prove the regularity of the weak solution obtained in Sect. V. The Sobolev space $W^{m,p}(\mathbb{R}^2)$ is defined as the completion of $C_0^\infty(\mathbb{R}^2)$ in the norm

$$\|v\|_{m,p} = \sum_{\alpha_1+\alpha_2 \leqq m} \left\| \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} v \right\|_{0,p;\mathbb{R}^2} \tag{6.1}$$

with α_1, α_2 nonnegative integers.

We restate the final assertion of Theorem I′ in a proposition:

Proposition 6.1. *The unique weak solution, v_0 to Eqs.* (3.6a) *and* (3.6b) *in H is real analytic in $\mathbb{R}^2$.* □

Proof of Proposition 6.1. Since v_0 is a weak solution of (3.6a) and (3.6b), $\Delta v_0 \in L^2(\mathbb{R}^2)$.

Thus v_0 is in the Sobolev space $W^{2,2}(\mathbb{R}^2)$. By the Sobolev Imbedding Theorem ([9], p. 97), $v_0 \in C^0(\mathbb{R}^2)$. Repeating this argument for derivatives of v_0 we obtain $v \in C^k(\mathbb{R}^2)$ for $k=0,1,\ldots$. By a standard theorem ([10], pp. 170–180) we have v_0 is real analytic in $\mathbb{R}^2$.

Appendix

The purpose of this appendix is to prove the following proposition.

Proposition A1.1 *Let A and ϕ be respectively a C^∞ connection and Higgs field on $\mathbb{R}^2$ which satisfy the first order Ginzburg-Landau equations. Then $\{x \in \mathbb{R}^2 | \phi(x)=0\}$ is discrete.* □

Proof of Proposition A1.1. Our plan is to assume the converse and show that a contradiction results. Therefore assume that there exists A *and* ϕ *satisfying the conditions of Proposition* A1.1 *with* $\{x \in \mathbb{R}^2 | \phi(x)=0\}$ not discrete. Denote by Z the zero set of ϕ. If Z is not discrete then there exists a Jordan arc, γ (for definitions, see e.g. Ahlfors [11], p. 69), in Z. Given any open set U intersecting γ there exists an open set $V \subset U$ such that γ divides V into two nonempty sets V_+ and V_- such that $V_+ \cap V_- = \gamma$ and $\gamma \cap V_+$ is open in γ. Because Z is the zero set of the C^∞ function $|\phi|^2$ on $\mathbb{R}^2$, it is possible to choose a Jordan arc $\gamma \subset Z$ and an open set V as above such that

$$\begin{aligned} &|\phi|^2|_\gamma = 0 \\ &|\phi|^2|_{V_+} > 0 . \end{aligned} \tag{A1.1}$$

By taking a smaller open set if necessary we may assume that V is simply connected.

Select an open set $W \subset V$ such that γ divides W into two nonempty sets W_+ and W_- with W_+ simply connected; its closure $\bar{W}_+$ compact; its boundary $\partial \bar{W}_+$ a Jordan curve and $\gamma \cap W_+$ open in γ.

In the interior of V_+, $\ln|\phi|^2 = u_+$ is a C^∞ function which satisfies

$$-\Delta u_+ + |\phi|^2 - 1 = 0 . \tag{A1.2}$$

By assumption, $|\phi|^2$ is C^∞ in V. Standard arguments (see e.g., Lions and Magenes [12]) imply that there exists a unique C^∞ function h on $\bar{W}_+$ satisfying

$$\begin{aligned} &-\Delta h + |\phi|^2 - 1 = 0 \quad \text{in} \quad \operatorname{int} \bar{W}_+ \\ &\text{with} \\ &\qquad h = 0 \quad \text{on} \quad \partial \bar{W}_+ . \end{aligned} \tag{A1.3}$$

In the interior of $\bar{W}_+$ we have

$$-\Delta(h - u_+) = 0 . \tag{A1.4}$$

Hence, in the interior of $\bar{W}_+$

$$u_+ = h + u \tag{A1.5}$$

with u a harmonic function in int $\bar{W}_+$. Since $|\phi|^2 = 0$ on γ and is continuous in V and $h|_\gamma = 0$ it follows that for fixed $y \in \gamma$

$$\lim_{\substack{|x-y|\to 0 \\ x\in \mathrm{int}\,\bar{W}_+}} u(x) \to -\infty . \tag{A1.6}$$

Thus we are led to consider the possible behavior on the boundary, $\partial\bar{W}_+$, of a function u harmonic in int $\bar{W}_+$. We note that it is sufficient to study this question for $\bar{W}_+$ the unit disc in $\mathbb{R}^2$ and γ a subset of the unit circle. This follows from the following two fundamental theorems on conformal mappings.

Theorem A1.2 (Riemann Mapping Theorem). *The interior of any simply-connected domain R with more than one frontier-point can be represented on the interior of the unit circle by means of a one to one conformal transformation.* □

Theorem A1.3. *If one Jordan domain is transformed conformally into another, then the transformation is one-one and continuous in the closed domain, and the two frontiers are described in the same sense by moving a point on one and the corresponding point on the other.* □

For proofs and discussions of these two theorems, see, e.g., Carathéodory [13], pp. 70–86.

Without loss of generality we take for $\theta_0 > 0$

$$\gamma = \{e^{i\theta} \in S^1 | 0 \leqq \theta \leqq \theta_0\} .$$

Here we are representing the unit disc, D, in $\mathbb{R}^2$ as the set of complex numbers with modulus less than or equal to one.

Lemma A1.4. *Given* $n > 0$, $u(x)$ *as above and* $\bar{W}_+ = D$, *there exists* $0 < \varrho_n < 1$ *such that*

L) *The set* $\{\theta \in [0, 2\pi] | u(\varrho_n e^{i\theta}) < -n\}$ *has Lebesgue measure* $\theta_n > \theta_0$.

Proof of Lemma A1.4. Statement L) follows from (A1.6) and the continuity of u in int D.

Let $\{\varrho_n\}_{n=1}^{\infty}$ be a sequence of positive numbers which satisfy the conditions and statement L) of Lemma A1.4, with the added proviso that $\varrho_n < \varrho_{n+1}$ for $n = 1, \ldots \infty$. We have $\lim_{n\to\infty} \varrho_n = 1$.

Since u is harmonic in int D,

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(\varrho_n e^{i\theta}) d\theta . \tag{A1.7}$$

(For a proof of this statement, see e.g., Ahlfors [11], p. 164.) $u(0)$ is a finite constant independent of n. Equation (A1.7) implies that

$$\sup_{\theta \in [0, 2\pi]} u(\varrho_n e^{i\theta}) \geqq \frac{n\theta_0}{2\pi} + u(0) . \tag{A1.8}$$

Let $z_n \in \{|z| = \varrho_n\}$ be such that

$$\sup_{\theta \in [0, 2\pi]} u(\varrho_n e^{i\theta}) = u(z_n) . \tag{A1.9}$$

The compactness of the interval $[0, 2\pi]$ insures that z_n exists. The set $\{z_n\}_{n=1}^{\infty}$ form a sequence in D and hence have a limit point $z_\infty \in D$. Since u is harmonic, $z_\infty \in \partial D$ and $u(z_\infty) = +\infty$. Applying the inverse mapping of D into $\bar{W}_+$ we see that this contradicts that $|\phi|^2 = \exp\{u+h\}$ is continuous in the set V. Q.E.D.

Acknowledgements. We wish to thank Professor A. Jaffe and Thordur Jonsson for their suggestions and critical comments.

References

1. Ginzburg, V.L., Landau, L.D.: Zh. Eksp. Teor. Fiz. **20**, 1064 (1950)
2. Nielsen, H.B., Olesen, P.: Nucl. Phys. B **61**, 45 (1973)
3. deVega, H.J., Schnaposnik, F.A.: Phys. Rev. D **14**, 1100 (1976)
4. Weinberg, E.: Phys. Rev. D **19**, 3008 (1979)
5. Jacobs. L., Rebbi, C.: Phys. Rev. B **19**, 4486 (1979)
6. Bogomol'nyi, E.B.: Sov. J. of Nucl. Phys. **24**, 449 (1976)
7. Witten, E.: Phys. Rev. Lett. **38**, 121 (1977)
8. Vainberg, M.M.: Variational method and method of monotone ɜ theory of nonlinear equations. New York: John Wiley 1973
9. Adams, R.: Sobolev spaces. New York: Academic Press 1975
10. Morrey, C.: Multiple integrals in the calculus of variations. Berlin, Heidelberg, New York: Springer 1966
11. Ahlfors, L.V.: Complex analysis. New York: McGraw-Hill 1966
12. Lions, J., Magenes, E.: Nonhomogeneous boundary value problems and applications. Berlin, Heidelberg, New York: Springer 1972
13. Carathéodory, C.: Conformal representation. Cambridge: Cambridge University Press 1963

Communicated by A. Jaffe

Received June 21, 1979; in revised form November 6, 1979