## Are coverings of graphs.

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To Enrico Bompiani on his scientific Jubilee.

1. Definitions. - In the following we shall examine certain properties of finite graphs. Such a graph $G$ is defined as usual by means of a finite vertex set $V$ and a number of associations or edges

$$
E=(a, b), a, b \in V
$$

connecting some of them. The edges are simple, that is, at most a single edge connecting any vertex pair; furthermore, there shall be no loops, that is, edges of the special form $(a, a)$. The local degree $\rho(\eta)$ of a vertex $v$ is the number of edges having $v$ as an endpoint. The total number of edges is $G$ is then

$$
v_{e}=v_{e}(G)=\frac{1}{2} \Sigma_{v} \rho(v), v \in V
$$

The complete graph $U(V)$ defined on $V$ has all possible $\frac{1}{2} n(n-1)$ edges $(a, b)$ where $a$ and $b$ run through the $n$ vertices in $V$.

A family of edges of the type

$$
\begin{equation*}
A=\left(a_{0}, a_{1}\right)\left(a_{1}, a_{2}\right) \ldots\left(a_{n-1}, a_{n}\right) \tag{1.1}
\end{equation*}
$$

is an arc of length $n$ when no vertex $\alpha_{i}$ appears more than once in it. It is a circuit when $\alpha_{0}=a_{n}$ and this is the only repeated vertex. An arc (1.1) is a Hamilton arc when it includes all vertices of $G$ and similarly for a Hamilton circuit.
2. Are coverings.

A family of $k \operatorname{arcs}$

$$
\begin{equation*}
A_{i}=\left(a_{0 i}, a_{1 i}\right)\left(a_{1 i}, a_{2_{i}}\right) \ldots\left(a_{n_{i-1}, i}, a_{n_{i}, i}\right) \quad i=1,2, \ldots k \tag{2.1}
\end{equation*}
$$

shall be given. The degenerate case where $A_{i}$ is a single vertex is permitted. The ares in (2.1) are disjoint when they have no common vertices. The vertices

$$
a_{0 i}, a_{r_{i} i}
$$

are the terminal vertices of $A_{i}$. The arcs $\left\{A_{i}\right\}$ form an arc covering of $G$ when they are disjoint and each vertex in $G$ lies on one of them.

An are covering (2.1) is maximal when it contains the greatest possible number of edges. A Hamilton arc. when it exists is a maximal covering. From now on we suppose that (2.1) is a maximal arc covering. Then there can be no edges in $G$ connecting the terminal vertices of two different ares for it could be used to produce a covering with a larger number of edges.

We select two terminal vertices $t$ and $t^{\prime}$ on different arcs $A$ and $A^{\prime}$. Suppose that for some arc $A_{i}$ there is an edge

$$
E=\left(t, a_{j t}\right), a_{j i} \text { on } A_{i}
$$

Then there cannot exist any edge

$$
E^{\prime}=\left(t^{\prime}, a_{j+1, i}\right), a_{j+1, i} \text { on } A_{i}
$$

to the following vertex on $A_{i}$. To verify this suppose first that $A_{i}$ is different from $A$ and $A^{\prime}$. (Fig. 1)


Fig. 1
One could then replace the arcs

$$
A, A^{\prime}, A_{i}
$$

in the arc covering with the two arcs

$$
\begin{aligned}
& A^{\prime}+E^{\prime}+A_{i}\left(a_{j+1, i}, \quad a_{n_{i}, i}\right) \\
& A+E+A_{i}\left(a_{j 1}, a_{0 i}\right)
\end{aligned}
$$

giving a new covering with one fewer arcs and one more edge. When $A_{i}=A^{\prime}$ (Fig. 2)


Fig. 2
one can replace the $\operatorname{arcs} A$ and $A^{\prime}$ by the single are

$$
A+E+A_{i}\left(a_{j i}, a_{0 i}\right)+E^{\prime}+A_{i}\left(a_{j+1, i}, \quad a_{n_{2}, i}\right)
$$

We conclude that when (2.1) is a maximal arc covering there exists to each edge ( $t, a_{j i}$ ) a unique vertex $a_{j+1, i}$ to which there can be no edge from $t^{\prime}$. Thus if $r_{i}$ and $r_{i}^{\prime}$ denote the number of edges from $t$ and $t^{\prime}$ to $A_{i}$ then

$$
\begin{equation*}
r_{i}+r_{i}^{\prime} \leqq n_{i} \tag{2.2}
\end{equation*}
$$

In a maximal are covering (2.1) with $k \geqq 2$ ares the condition (2.2) must be satisfied for each are $A_{\imath}$ and all pairs of terminal vertices $t$ and $t^{\prime}$. Let us add all these inequalities for a fixed pair of vertices $t$ and $t^{\prime}$. Since

$$
n=\Sigma_{i}\left(n_{i}+1\right)=\Sigma_{i} n_{i}+k
$$

it follows that the local degrees of $G$ at $t$ and $t^{\prime}$ must satisfy the condition

$$
\rho(t)+\rho\left(t^{\prime}\right) \leqq n-k .
$$

This yields the result:
Theorem 2.1. - When a maximal arc covering (2.1) contains $k \geqq 2$ arcs then

$$
\begin{equation*}
k \leqq n-\rho(t)-\rho\left(t^{\prime}\right) \tag{2.3}
\end{equation*}
$$

where $n$ is the number of vertices in $G$ and $t$ and $t^{\prime}$ two vertices not connected by an edge.

In particular one has

$$
\begin{equation*}
k \leqq n-\rho_{1}-\rho_{2} \tag{2.4}
\end{equation*}
$$

Where $\rho_{1}$ and $\rho_{2}$ are the two smallest local degrees.
3. Hamilton arcs. - From the condition (2.4) follows as a special case:

Theorem 3.1. - When the local degrees of the graph $G$ satisfy the conditions

$$
\begin{equation*}
\rho(a)+\rho(b) \geqq n-1 \tag{3.1}
\end{equation*}
$$

for all vertices $a$ and $b$ not connected by an edge then it has a Hamilton arc.
This is a companion result to a theorem obtained previously for Hamilron circuits (O. Ore, Note on Hamilton circuits, «Am. Math. Monthly », v. 67 (1960) p. 55 ):

Theorem 3.2. - When the local degrees satisfy

$$
\begin{equation*}
p(a)+p(b) \geqq n \tag{3.2}
\end{equation*}
$$

for all vertices $a$ and $b$ not connected by an edge then $G$ has a Hamilton circuit.
4. Maximal graphs without Hamilton circuits. The complete graph on $n$ vertices has a Hamilon are and when $n \geq 3$ also a Hamilion circuit. Thus it is to be expected that a graph with $n$ vertices will have the same properties when its number of edges $\nu_{e}(G)$ is sufficiently large. The preceding results yield the specific conditions:

Theorem 4.1. - When the number of edges in a graph satisfies

$$
\begin{equation*}
\nu_{e}(G) \geqq \frac{1}{2}(n-1)(n-2)+1 \tag{4.1}
\end{equation*}
$$

then $G$ has a Hamilton arc. The graphs without Hamilton arcs and

$$
\begin{equation*}
v_{c}(G)=\frac{1}{2}(n-1)(n-2) \tag{4.2}
\end{equation*}
$$

consist of an isolated vertex and a complete graph on $n-1$ vertices; in addition when $n=4$ there is the star graph consisting of three edges from the same vertex.

Proof. - When the condition (4.1) is fulfilled the graph may be considered to have been obtained from the complete graph $D_{n}$ through the elimination of at most

$$
\frac{1}{2} n(n-1)-\frac{1}{2}(n-1)(n-2)-1=n-2
$$

edges. But then no relation

$$
\rho(a)+\rho(b) \leqq n-2
$$

can hold for any pair of vertices not connected by an edge since this would imply that at least

$$
(n-1-p(a))+(n-1-\rho(b))-1 \geqq n-1
$$

edges would have been eliminated. According to Theorem 3.1 the graph has a Hamilton are.

When the number of edges is given by (4.2) there might be a pair of vertices not connected by an edge such that

$$
\rho(a)+\rho(b)=n-2 .
$$

Then there remains

$$
\frac{1}{2}(n-2)(n-3)
$$

edges so that these must form a complete graph $U_{n-2}$ on the other $n-2$ vertices. One readily verifies that a complete graph has a Hammon are connecting any pair of its vertices. Consequently also $G$ has a Hamilton are if there are edges from $a$ and $b$ to two different vertices in $U_{n-2}$. Thus only in the two following cases can there be no Hamilton are:

1. Either $a$ or $b$ is an isolated vertex, for instance

$$
\rho(a)=0, \quad \rho(b)=n-2
$$

giving the first type of graphs.
2. There is a single edge from $a$ and $b$ to the same vertex in $U_{n-2}$. Then

$$
\rho(a)=\rho(b)=1, \quad n=4, \quad v_{e}(G)=3
$$

giving the second type.
An immediate consequence of Theorem 4.1 is:
Theorem 4.2 - A graph with

$$
\nu_{e}(G) \geqq \frac{1}{2}(n-1)(n-2)+1
$$

edges is connected. A graph with

$$
\nu_{e}(G)=\frac{1}{2}(n-1)(n-2)
$$

edges can only be disconnected when it consists of an isolated vertex and a complete graph on $n-1$ vertices.

This result could also have been obtained directly by a simple argument.
Theorem 4.3. - A graph with

$$
\begin{equation*}
v_{e}(G) \geq \frac{1}{2}(n-1)(n-2)+2 \tag{4.3}
\end{equation*}
$$

edges has a Hamilton circuit. When

$$
\begin{equation*}
v_{e}(G)=\frac{1}{2}(n-1)(n-2)+1 \tag{4.4}
\end{equation*}
$$

the only graph without a Hamilton circuit consists of a complete graph, $U_{n-1}$ and a single edge connecting it with an outside vertex; in addition, for $n=5$ there is the exceptional graph depicted in Fig. 3.


Fig. 3
Proof. - It follows by the same reasoning as before that when (4.3) holds there can be no vertices $a$ and $b$ not connected by an edge such that

$$
\rho(a)+\rho(b) \leqq n-1
$$

so that $G$ has a Hamiton circuit according to Theorem 3.2.
To prove the second part of the theorem we notice that when (4.4) holds there may be a pair of vertices $a$ and $b$ not connected by an edge such that

$$
\begin{equation*}
\rho(a)+\rho(b)=n-1 . \tag{4.5}
\end{equation*}
$$

The remaining

$$
\frac{1}{2}(n-2)(n-3)
$$

edges must define a complete graph $U_{n-2}$ on the other vertices. From this observation the result is readily verified for the small values $n \leqq 5$. It may be assumed therefore that $n \geqq 6$. According to Theorem 4.2 the graph is connected so that $\rho(a) \geqq 1$. The relation (4.5) shows that when $\rho(a)=1$ then $\rho(b)=n-2$ and we have a graph of the type indicated. Clearly it has no Hamllton circuit.

There remains the case where

$$
\rho(a) \geq 2, \quad \rho(b) \geqq 3 .
$$

As we shall show there exists a Hamilton circuit under these conditions. There must then exist four edges

$$
\left(a, a_{1}\right)\left(a, a_{2}\right)\left(b, a_{3}\right)\left(b, a_{4}\right)
$$

from $a$ and $b$ to $U_{n-2}$ such that at least three of the vertices $a_{i}$ are distinct. If all are distinct we form the are

$$
Q=\left(a_{1}, a\right)\left(a, a_{2}\right)\left(a_{2}, a_{3}\right)\left(a_{3}, b\right)\left(b, a_{4}\right) .
$$

The graph obtained from $G$ by omitting $a, b, a_{2}, a_{3}$ and all edges from these vertices is a complete graph $U_{n-4}$, hence it contains a Hamiliton arc $P\left(a_{1}, a_{4}\right)$ which when combined with $Q$ gives a Hamilion circuit for $G$. When $a_{2}=a_{3}$ one obtains a Hamilion circuit by an analogous reasoning.

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