Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 28 (1978), No. 1, 140–154

Persistent URL: http://dml.cz/dmlcz/101520

Czechoslovak Mathematical Journal, 28 (103) 1978, Praha

ARCHIMEDEAN KERNEL OF A LATTICE ORDERED GROUP

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(Received February 18, 1976)

For any archimedean lattice ordered group H we denote by $D(H)$ the Dedekind closure of H (cf. e.g. [1], Chap. XIII, § 13). Under the natural embedding, H is an l-subgroup of $D(H)$ such that for each element $x_0 \in D(H)$ there exists a subset $X \subseteq H$ that is upper bounded in H with $x_0 = \sup X$.

Let G be a lattice ordered group. We denote by $\mathscr{A}(G)$ the set of all convex l-subgroups of G that are archimedean. The set $\mathscr{A}(G)$ is partially ordered by inclusion. In § 1 of this paper it will be shown that $\mathscr{A}(G)$ possesses the greatest element $A(G)$. The convex l-subgroup $A(G)$ is said to be the archimedean kernel of G.

Let $\mathscr{G}$ be the class of all lattice ordered groups and let $\mathscr{R}$ be a nonempty subclass of $\mathscr{G}$ such that the following conditions are fulfilled:

(α) $\mathscr{R}$ is closed with respect to isomorphisms.

(β) If $K \in \mathscr{R}$ and K_1 is a convex l-subgroup of K, then $K_1 \in \mathscr{R}$.

(γ) If $K_2 \in \mathscr{G}$ and if $\{K_i\}_{i \in I}$ is a set of convex l-subgroups of K_2 belonging to $\mathscr{R}$, then $\bigvee_{i \in K} K_i \in \mathscr{R}$.

Under these assumptions $\mathscr{R}$ is called a *radical class* [5]. If, moreover, $\mathscr{R}$ is closed with respect to homomorphisms, then $\mathscr{R}$ is said to be a *torsion class* (MARTINEZ [6]). From the existence of the archimedean kernel we easily obtain that the class $\mathscr{A}$ of all archimedean lattice ordered groups is a radical class.

It is well-known that a homomorphic image of an archimedean lattice ordered group need not be archimedean; hence $\mathscr{A}$ fails to be a torsion class.

In § 2 we construct, for each $G \in \mathscr{G}$, a lattice ordered group $D_1(G)$ fulfilling the following conditions:

(i) G is an l-subgroup of $D_1(G)$.

(ii) $D(A(G))$ is an l-ideal of $D_1(G)$.

(iii) If $x \in G$ and X is a nonempty subset of $x + A(G)$ such that X is upper bounded in $x + A(G)$, then there is $x_0 \in D_1(G)$ with $\sup X = x_0$.

(iv) For each $x_0 \in D_1(G)$ there exists $x \in G$ and $X \subseteqq x + A(G)$ such that X is upper bounded in $x + A(G)$ and $x_0 = \sup X$.

Thus, in particular, $D_1(G)$ is an amalgam of the lattice ordered groups G and $D(A(G))$ with the common l-subgroup $A(G)$. If G is archimedean, then $D_1(G) = D(G)$. Hence $D_1(G)$ is a generalization of the notion of the Dedekind closure which can be employed also for non-archimedean lattice ordered groups. $D_1(G)$ will be called the generalized Dedekind closure of G. The lattice ordered group $D_1(G)$ is determined by the conditions (i)–(iv) up to isomorphisms.

Further, it is shown that $A(G)$ is a closed l-ideal in G and that $D(A(G))$ is a closed l-ideal in $D_1(G)$. If $X \subseteqq G$ and if g is the least upper bound of X in G, then g is also the least upper bound of X in $D_1(G)$ (and dually). A problem is proposed concerning the relations between $D_1(G)$ and the extension of G that was defined by L. FUCHS in [3] (Chap. V, § 10).

In § 3, some relations between G and $D_1(G)$ are established; e.g., it is shown that if G is abelian and divisible, then so is $D_1(G)$. There exists a one-to-one correspondence between the polars of G and the polars of $D_1(G)$. If G is representable, then $D_1(G)$ is representable as well.

For the basic notions and notation cf. BIRKHOFF [1], CONRAD [2], FUCHS [3]. In what follows all lattice ordered groups are written additively though they are not assumed to be abelian.

1. THE ARCHIMEDEAN KERNEL

Let G be a lattice ordered group. Let $\mathscr{A}(G)$ be as above and let $\mathscr{A}_1(G)$ be the set (partially ordered by inclusion) of all convex l-subgroups of G that are abelian.

1.1. Lemma. *$\mathscr{A}_1(G)$ possesses the greatest element.*

Proof. Each variety of representable l-groups being a torsion class [6], the assertion follows from (γ).

The greatest element of $\mathscr{A}_1(G)$ will be denoted by $A_1(G)$. Since each archimedean lattice ordered group is abelian, we have $A \subseteqq A_1(G)$ for each archimedean l-subgroup A of G.

An element $0 < g$ of a lattice ordered group K will be called *archimedean in K* if for each $0 < x \in K$ there exists a positive integer n such that nx non $\leqq g$. If g is archimedean in K and $0 < g_1 \in K$, $g_1 < g$, then g_1 is archimedean in K.

1.2. Lemma. *Let a, b be archimedean elements of an abelian lattice ordered group K. Then $a \vee b$ is archimedean in K.*

Proof. Denote $a - a \wedge b = a_1$, $b - a \wedge b = b_1$. Then

$$a \vee b = a \wedge b + a_1 + b_1 . \tag{1}$$

Assume that $a \vee b$ fails to be archimedean. Then there is $0 < z \in K$ such that $nz < a \vee b$ for each positive integer n. We have either $a \wedge b = 0$ or $a \wedge b$ is archimedean. Hence there is a positive integer n_1 such that $n_1 z$ non $\leqq a \wedge b$. Put

$$x = n_1 z - (n_1 z \wedge a \wedge b).$$

Thus $x > 0$. At the same time we have

$$x = n_1 z \vee (a \wedge b) - a \wedge b \leqq a \vee b - a \wedge b = a_1 + b_1 = a_1 \vee b_1,$$

since $a_1 \wedge b_1 = 0$. This implies

$$x = (x \wedge a_1) \vee (x \wedge b_1)$$

and either $x \wedge a_1$ or $x \wedge b_1$ is strictly positive. Without loss of generality we may assume that $x_1 = x \wedge a_1 > 0$. Since $x_1 \leqq x \leqq n_1 z$, we have $nx_1 \leqq a \vee b$ for each positive integer n. There is a positive integer n_2 with $n_2 x_1$ non $\leqq a$. From (1) and from $n_2 x_1 \leqq a \vee b$ we obtain that there are elements $y_1, y_2, y_3 \in K$ with $0 \leqq y_1 \leqq a \wedge b$, $0 \leqq y_2 \leqq a_1$, $0 \leqq y_3 \leqq b_1$ such that

$$n_2 x_1 = y_1 + y_2 + y_3.$$

In view of $x_1 \leqq a_1$ we have $x_1 \wedge b_1 = 0$ and hence $n_2 x_1 \wedge b_1 = 0$. Thus $y_3 = 0$ and therefore $n_2 x_1 = y_1 + y_2 \leqq a \wedge b + a_1 = a$, which is a contradiction.

1.3. Lemma. *Let a be an archimedean element of an abelian lattice ordered group K. Then $2a$ is archimedean in K.*

Proof. Suppose that $2a$ fails to be archimedean. Then there is $0 < x \in K$ such that $2nx < 2a$ for each positive integer n, and hence $nx < a$ for each positive integer n, which is a contradiction.

1.4. Lemma. *Let K be an abelian lattice ordered group and let K_1 be the set of all elements $a \in K$ such that either $a = 0$ or $|a|$ is archimedean. Then K_1 is a convex l-subgroup of K.*

Proof. If $a \in K_1$, then $-a \in K_1$. Let $a, b \in K_1$. Then $|a|, |b| \in K_1$ and thus by Lemma 1.2, $|a| \vee |b| \in K_1$. According to Lemma 1.3 we have $2(|a| \vee |b|) \in K_1$. If $c \in K$, $0 < c \leqq |a|$, then clearly $c \in K_1$. Since

$$|a| + |b| \leqq 2(|a| \vee |b|),$$

we infer that $|a| + |b| \in K_1$. From this and from $|a + b| \leqq |a| + |b|$ we obtain $a + b \in K_1$. Hence K_1 is a subgroup of K. Since $a \in K_1$ implies $|a| \in K_1$, we infer that K_1 is directed. Being convex in K, it follows that K_1 is an l-subgroup of K.

1.5. Theorem. *Let G be a lattice ordered group. There exists a convex l-subgroup $A(G)$ of G such that* (a) *$A(G)$ is archimedean, and* (b) *if G_1 is a convex l-subgroup of G and if G_1 is archimdean, then $G_1 \leqq A(G)$.*

Proof. Put $A_1(G) = K$ and let K_1 be as in Lemma 1.4. Then K_1 is a convex l-subgroup of G and is archimedean. Let G_1 be a convex l-subgroup of G and suppose that G_1 is archimedean. Then G_1 is abelian, thus $G_1 \subseteqq K$. Moreover, each strictly positive element of G_1 must be archimedean in K, hence $G_1^+ \subseteqq K_1$. This implies $G_1 \subseteqq K_1$. Now we may put $K_1 = A(G)$.

1.6. Corollary. *The class $\mathscr{A}$ of all archimedean lattice ordered groups is a radical class.*

Proof. Obviously $\mathscr{A}$ fulfils (α) and (β). Let $G \in \mathscr{G}$ and let $\{G_i\}_{i\in I}$ be a set of convex archimedean l-subgroups of G. Then $G_i \subseteqq A(G)$ and hence $\bigvee G_i \subseteqq A(G)$. According to ($\beta$) we obtain $\bigvee G_i \in \mathscr{A}$.

1.7. Lemma. *For each $G \in \mathscr{G}$, $A(G)$ is an l-ideal of G.*

Proof. $A(G)$ being a convex l-subgroup of G it suffices to verify that $A(G)$ is normal in G. Let $g \in G$. Then $-g + A(G) + g$ is a convex l-subgroup of G isomorphic with $A(G)$. In particular, $-g + A(g) + g$ is archimedean. Hence according to Theorem 1 we have $-g + A(g) + g \subseteqq A(G)$.

When no ambiguity can occur, we shall write often A instead of $A(G)$.

2. CONSTRUCTION OF $D_1(G)$

Let L be a lattice. For $X \subseteqq L$ we denote by X^u and X^l the set of all upper bounds or the set of all lower bounds of the set X in L, respectively. Let L_1 be the system of all sets of the form $(X^u)^l$, where X is any nonempty upper bounded subset of L. Then L_1 (partially ordered by inclusion) is a conditionally complete lattice; the set L_2 of all principal ideals of L is a sublattice of L_1 isomorphic with L and each element of L_1 is a join of some elements of L_2. Hence there is a conditionally complete lattice $d(L)$ such that L is a sublattice of $d(L)$ and each element x_0 of $d(L)$ is a join of a subset X of L such that X is upper bounded in L; also, there is a subset Y of L such that Y is lower bounded in L and x_0 is the meet of the set Y in $d(L)$. The lattice $d(L)$ is determined uniquely up to isomorphism.

Let G be a lattice ordered group. Denote $A(G) = A$. For each class $x + A$ $(x \in G)$ we construt the lattice $d(x + A)$. We may assume that $d(x + A) \cap d(y + A) = \emptyset$ whenever $x + A \neq y + A$ and that $d(x + A) = D(A)$ for $x = 0$. Put

$$S = \bigcup_{x\in G} d(x + A)\,.$$

We define a binary operation $+$ on the set S as follows. Let $x_0, y_0 \in S$. There are elements $x, y \in G$ with $x_0 \in d(x + A)$, $y_0 \in d(y + A)$. Let X_0 be the set of all elements $x_i \in x + A$ with $x_i \leqq x_0$, and let Y_0 have the analogous meaning. Then X_0 and Y_0

are upper bounded in $x + A$ or $y + A$, respectively. Hence the set $Z_0 = \{x_i + y_i : : x_i \in X_0, y_i \in Y_0\}$ is an upper bounded subset of $x + y + A$ (cf. Lemma 1.7). Thus there exists $z_0 = \sup Z_0$ in $d(x + y + A)$. We put $x_0 + y_0 = z_0$.

If $x_0, y_0 \in G$, then clearly $x_0 + y_0$ in S coincides with the original operation $x_0 + y_0$ in G. Analogously, for $x_0, y_0 \in D(A)$ the operation $x_0 + y_0$ in S gives the same result as the operation $x_0 + y_0$ in $D(A)$.

Let $X_1 \subseteqq X_0$, $Y_1 \subseteqq Y_0$, $\sup X_1 = x_0$ and $\sup Y_1 = y_0$. Denote $Z_1 = \{x'_i + y'_i : x'_i \in X_1, y'_i \in Y_1\}$.

2.1. Lemma. $\sup Z_1 = x_0 + y_0$.

Proof. The set Z_1 is upper bounded in $x + y + A$, hence $\sup Z_1 = u$ exists in $d(x + y + A)$. Let $u_1 \in x + y + A$, $u_1 \geqq u$. For each $x'_i \in X_1$ and each $y'_j \in Y_1$ we have $u_1 \geqq x'_i + y'_i$, $u_1 - y'_j \geqq x'_i$, hence $u_1 - y'_j \geqq x_i$ for each $x_i \in X_0$. From $-x_i + u_1 \geqq y'_j$ we infer that $-x_i + u_1 \geqq y_j$ for each $y_j \in Y_0$. Therefore $u_1 \geqq \geqq x_i + y_j$. This implies $u_1 \geqq x_0 + y_0$. Hence $u \geqq x_0 + y_0$. Since $X_1 \subseteqq X_0$, $Y_1 \subseteqq Y_0$, we have $u \leqq x_0 + y_0$. Thus $u = x_0 + y_0$.

2.2. Lemma. *The operation + on S is associative.*

Proof. Let $x_0, y_0, t_0 \in S$ and let x, y, X_1, Y_1 be as above. There is $t \in G$ and $T_1 \subseteqq t + A$ such that $\sup T_1 = t_0$ holds in $d(t + A)$. Lemma 2.1 implies

$$(x_0 + y_0) + t_0 = \sup\{(x_1 + y_1) + t_1 : x_1 \in X_1,\ y_1 \in Y_1,\ t_1 \in T_1\} = \\ = x_0 + (y_0 + t_0)\,.$$

2.3. Lemma. $0 + x_0 = x_0 + 0 = x_0$ *for each* $x_0 \in S$.

This follows immediately from Lemma 2.1.

2.4. Lemma. *For each* $x_0 \in S$ *there are elements* $x \in G$ *and* $a \in D(A)$ *such that* $x_0 = x + a$.

Proof. There is $x \in G$ with $x_0 \in d(x + A)$ and a set $X_1 \subseteqq x + A$ such that $x_0 = \sup X_1$ is valid in $d(x + A)$ and X_1 is upper bounded in $x + A$. Put $X_2 = \{-x + x_i : x_i \in X_1\}$. Then X_2 is an upper bounded subset of A. Thus there is $a = \sup X_2$ in $D(A)$. From Lemma 2.1 we obtain $x_0 = x + a$.

2.5. Lemma. $(S; +)$ *is a group.*

Proof. From Lemma 2.2 and Lemma 2.3 it follows that it suffices to verify that for each element $x_0 \in S$ there is $y_0 \in S$ with $x_0 + y_0 = 0$. Let $x_0 \in S$ and let x, a be as in Lemma 2.4. Put $y_0 = -a + (-x)$. Then $x_0 + y_0 = 0$ by Lemma 2.2.

Let x_0, x and X_0 be as above. We denote

$$(x_0) = \{y \in G : y \geqq x_i \text{ for each } x_i \in X_0\}\,, \quad (x_0)^v = (x_0) \cap (x + A)\,.$$

Let $X_1 \subseteqq X_0$ with $\sup X_1 = x_0$ in $d(x + A)$. Clearly

$$(x_0) = \{z \in G : z \geqq x_i' \text{ for each } x_i' \in X_1\} .$$

We define a binary relation $\leqq$ on S as follows. For $x_0, y_0 \in S$ we put $x_0 \leqq y_0$ if $(y_0) \subseteqq (x_0)$. For $x_0, y_0 \in G$ the relation $x_0 \leqq y_0$ coincides with the relation $x_0 \leqq y_0$ in G, and analogously for $x_0, y_0 \in D(A)$. The relation $\leqq$ on S is obviously reflexive and transitive.

2.6. Lemma. *Let $x_0, y_0 \in S$, $x_0 \leqq y_0$ and $y_0 \leqq x_0$. Let $x, y \in G$, $x_0 \in d(x + A)$, $y_0 \in d(y + A)$. Then $x + A = y + A$.*

Proof. There are elements $x_1, t_1 \in x + A$, $y_1, t_2 \in y + A$ with $t_1 \geqq x_0 \geqq x_1$, $t_2 \geqq y_0 \geqq y_1$. From $x_0 \leqq y_0$, $y_0 \leqq x_0$ we infer that $x_1 \leqq t_2$, $y_1 \leqq t_1$. Then in the factor l-group G/A we have

$$(x_1 + A) \vee (y_1 + A) = (x_1 \vee y_1) + A \leqq (t_1 \wedge t_2) + A =$$
$$= (t_1 + A) \wedge (t_2 + A) = (x_1 + A) \wedge (y_1 + A),$$

hence $x_1 + A = y_1 + A$. Thus $x + A = y + A$.

2.7. Lemma. *Let $x_0, y_0 \in S$, $x_0 \leqq y_0$ and $y_0 \leqq x_0$, Then $x_0 = y_0$.*

Proof. According to Lemma 2.6 there is $x \in G$ such that x_0 and y_0 belong to $d(x + A)$. Moreover, we have $(x_0) = (y_0)$ and hence $(x_0)^v = (y_0)^v$. Therefore $x_0 = y_0$.

We have verified that the relation $\leqq$ is a partial order on S.

2.8. Lemma. *Let $x_0, y_0, z_0 \in S$, $x_0 \leqq y_0$. Then $x_0 + z_0 \leqq y_0 + z_0$.*

Proof. Let $x \in G$ with $x_0 \in d(x + A)$ and let $\{x_i\}$ be the set of all elements of $x + A$ that are less or equal to x_0. Let y, y_j and z, z_k have the analogous meaning with respect to y_0 and z_0. We have

$$x_0 + z_0 = \sup \{x_i + z_k\} \quad (\text{in } d(x + z + A)),$$
$$y_0 + z_0 = \sup \{y_j + z_k\} \quad (\text{in } d(y + z + A)).$$

Let $t \in G$, $t \in (y_0 + z_0)$. Then $y_j + z_k \leqq t$ for each y_j and each z_k. Hence $y_j \leqq t - z_k$ and so $y_0 \leqq t - z_k$ for each z_k. Thus $x_0 \leqq t - z_k$, hence $x_i \leqq t - z_k$, $x_i + z_k \leqq t$ for each x_i and each z_k. Thus $t \in (x_0 + z_0)$. Therefore $x_0 + z_0 \leqq y_0 + z_0$.

Analogously we obtain: if $x_0, y_0, z_0 \in S$, $x_0 \leqq y_0$, then $z_0 + x_0 \leqq z_0 + y_0$. Thus $(S, +, \leqq)$ is a partially ordered group.

2.9. Lemma. *S is lattice ordered.*

Proof. Let $x_0, y_0 \in S$ and let x, y, x_i, y_k have the same meaning as in the proof of Lemma 2.8. Let Z be the set consisting of all elements $x_i \vee y_k$. Then Z is an upper-bounded subset of $(x \vee y) + A$. Hence there is $z_0 = \sup Z$ in $(x \vee y) + A$. If $t \in (z_0)$, then $x_i \leqq t$ and $y_j \leqq t$ for each x_i and each y_j, hence $x_0 \leqq z_0$ and $y_0 \leqq z_0$. Let $z_1 \in S$, $x_0 \leqq z_1$, $y_0 \leqq z_1$ and let $t_1 \in (z_1)$. Then $x_i \leqq t_1$ and $y_j \leqq t_1$, hence $x_i \vee y_j \leqq t_1$ and thus $z_0 \leqq z_1$. Therefore $z_0 = x_0 \vee y_0$. This implies that S is a lattice ordered group.

2.10. Lemma. *G is an l-subgroup of S and D(A) is an l-ideal in S.*

Proof. Let $x_0, y_0 \in G$. From the method of constructing $x_0 \vee y_0$ in S (cf. the proof of Lemma 2.9) it follows that $x_0 \vee y_0$ in S coincides with $x_0 \vee y_0$ in G. Since $x_0 \wedge y_0 = -(-x_0 \vee -y_0)$ holds in G and since G is a subgroup of S we infer that G is an l-subgroup in S. Analogously we verify that $D(A)$ is an l-subgroup in S.

Let $0 < x_0 \in D(A)$, $0 < y_0 \in S$, $y_0 < x_0$. There is $y \in G$ with $y_0 \in d(y + A)$. Further, there are elements $x_1 \in A$, $y_1 \in y + A$ with $0 < y_1 \leqq y_0$, $x_0 < x_1$. Thus $0 < y_1 < x_1$ and hence according to Theorem 1.5 we have $y_1 \in A$. Hence $y \in A$ and so $d(y + A) = D(A)$. Thus $y_0 \in D(A)$. Therefore $D(A)$ is a convex l-subgroup of S.

Let $d \in D(A)$. There is a subset $\{a_i\}$ in A that is upper bounded in A and such that $d = \bigvee a_i$ holds in $D(A)$. This together with the convexity of $D(A)$ in S shows that $d = \bigvee a_i$ is valid in S. Let $g \in G$. Then

$$-g + d + g = -g + \bigvee a_i + g = \bigvee(-g + a_i + g)$$

holds in S and according to Lemma 1.7, $-g + a_i + g \in A$. Moreover, the set $\{-g + a_i + g\}$ is upper bounded in A. Hence $-g + d + g$ belongs to $D(A)$ for each $g \in G$; thus $-g + D(A) + g = D(A)$.

Let $x_0 \in S$ and let x, a be as in 2.4. Then $x_0 = x + a$ and

$$-x_0 + D(A) + x_0 = -a - x + D(A) + x + a = -a + D(A) + a = D(A).$$

Hence $D(A)$ is a normal subgroup of S. Thus $D(A)$ is an l-ideal in S.

2.11. Lemma. *For each $x \in G$ we have $d(x + A) = x + D(A)$.*

Proof. Let $x_0 \in d(x + A)$. By Lemma 2.4 we have $x_0 = x + a$ for some $a \in D(A)$. Hence $d(x + A) \subseteqq x + D(A)$. Conversely, let $x_0 \in x + D(A)$, thus $x_0 = x + a_1$ for some $a_1 \in D(A)$. There exists an upper bounded subset $\{a_i\}$ of A such that $\bigvee a_i = a_1$. Then $\{x + a_i\}$ is an upper bounded subset of $x + A$ and $x + a_1 = \sup \{x + a_i\}$ according to the definition of the operation $+$ in S (the operation sup being taken with respect to $d(x + A)$). Hence $x + D(A) \subseteqq d(x + A)$.

2.12. Corollary. *Each set $d(x + A)$ is convex in S. Thus if $\{x_i\}$ is an upper bounded subset in $d(x + A)$ and if $x_0 = \bigvee x_i$ holds in $d(x + A)$, then $x_0 = \bigvee x_i$ is valid in S.*

Denote $S = D_1(G)$.

2.13. Theorem. *$D_1(G)$ is a lattice ordered group fulfilling the conditions (i)–(iv).*

Proof. By Lemma 2.9, $D_1(G)$ is a lattice ordered group. According to Lemma 2.10, the conditions (i) and (ii) are fulfilled. The conditions (iii) and (iv) follow from 2.11, 2.12 and from the construction of the set S.

2.14. Proposition. *Let $G \in \mathcal{G}$. Then* (a) *$A(D_1(G)) = D(A)$, and* (b) *$D_1(G) = G$ if and only if $A(G)$ is conditionally complete.*

Proof. $D(A)$ being conditionally complete, it is archimedean and hence $D(A) \subseteqq A(D_1(G))$. Let $0 < x_0 \in D_1(G)$, x_0 non $\in D(A)$. Then there is $x \in G$ such that $x \notin A$ and $x_0 \in d(x + A)$. Further, there is $x_1 \in x + A$ with $0 < x_1 \leqq x_0$. Thus x_1 non $\in A$ and hence there is $0 < y \in G$ such that $ny < x_1 \leqq x_0$ holds for each positive integer n. This shows that x_0 fails to be archimedean. Hence $A(D_1(G))^+ \subseteqq D(A)$ and so $A(D_1(G)) \subseteqq D(A)$. Therefore (a) is valid.

Let $A(G)$ be conditionally complete. Then $D(A) = A(G)$ and hence according to Lemma 2.4 we have $D_1(G) = G$. Conversely, assume that $D_1(G) = G$. Then in view of (a) we have

$$A(G) = A(D_1(G)) = D(A),$$

hence $A(G)$ is conditionally complete.

2.15. Proposition. *Let D' be a lattice ordered group. Assume that D' fulfils the conditions* (i)–(iv) *with D' instead of $D_1(G)$. Then there exists an isomorphism φ of $D_1(G)$ onto D' such that $\varphi(x) = x$ and $\varphi(a) = a$ for each $x \in G$ and each $a \in D(A(G))$.*

Proof. Let $x_0 \in D_1(G)$. There is $x \in G$ with $x_0 \in d(x + A)$. Let $\{x_i\} = X$ be the set of all elements of the set $x + A$ that are less or equal to x_0. The set $\{x_i\}$ is bounded in $x + A$ and hence there exists $x'_0 = \sup \{x_i\}$ in D' by (iii). Put $\varphi(x_0) = x'_0$. If $x_0 \in G$ or $x_0 \in D(A(G))$, then clearly $\varphi(x_0) = x_0$.

(a) Let $\{x'_j\} = X_1 \subseteqq X$ such that $\sup X_1 = x_0$ holds in $D_1(G)$. Then the set X_1 is upper bounded in $x + A$, hence there exists $\sup X_1 = x''_0$ in D'. Both sets $\{x_i - x\}$, $\{x'_j - x\}$ are upper bounded subsets in A, hence $\bigvee(x_i - x)$ and $\bigvee(x'_j - x)$ belong to $D(A)$. Moreover, since $D(A)$ is an l-ideal in both $D_1(G)$ and D' (cf. (ii)), $\bigvee(x_i - x)$ calculated in $D_1(G)$ gives the same result as $\bigvee(x_i - x)$ with respect to D', and analogously for $\bigvee(x'_j - x)$. By calculating in $D_1(G)$ we obtain $\bigvee(x_i - x) = x_0 - x = \bigvee(x'_j - x)$; in D' it holds $\bigvee(x_i - x) = x'_0 - x$, $\bigvee(x'_j - x) = x''_0 - x$. Hence $x'_0 = x''_0$.

(b) Let $y'_0 \in D'$. There is $x \in G$ and a subset $Y \subseteqq x + A$ such that Y is upper bounded in $x + A$ and $y'_0 = \sup Y$ in D'. There exists $y_0 \in D_1(G)$ with $\sup Y = y_0$ in $D_1(G)$. According to (a) we have $\varphi(y_0) = y'_0$. Hence φ is surjective.

(c) Let $x_0, y_0 \in D_1(G)$ and suppose that $\varphi(x_0) = \varphi(y_0)$. There are $x, y \in G$ and $X_1, Y_1 \subset G$ such that X_1 is an upper bounded subset in $x + A$, Y_1 is an upper bounded subset in $y + A$ and $\sup X_1 = x_0$, $\sup Y_1 = y_0$ holds in $D_1(G)$. Then according to (a) we have $\sup X_1 = \varphi(x_0) = \varphi(y_0) = \sup Y_1$ in D'. Hence $x - y = (x - \varphi(x_0)) + (\varphi(y_0) - y) \in D(A)$, since both $x - \varphi(x_0)$ and $\varphi(y_0) - y$ belong to $D(A)$ (to verify this, we can use an analogous method as in (a)). Thus without loss of generality we can suppose that $x = y$. By calculating in D' we obtain that both elements $\sup(X_1 - x)$, $\sup(Y_1 - x)$ belong to $D(A)$ and that $\sup(X_1 - x) = \sup(Y_1 - x)$ holds in $D(A)$; this implies $\sup X_1 = \sup Y_1$ in $D_1(G)$. Hence φ is a monomorphism.

(d) Let x_0, y_0, x, y, X_1, Y_1 be as in (c) with the distinction that we do not assume $\varphi(x_0) = \varphi(y_0)$. Put $X_1 = \{x_i\}$, $Y_1 = \{y_i\}$.

In $D_1(G)$ we have $x_0 + y_0 = \sup\{x_i + y_j\}$ and the set $\{x_i + y_j\}$ is an upper bounded subset of $x + y + A$. Hence in D' we get

$$\varphi(x_0 + y_0) = \sup\{x_i + y_j\} = \bigvee x_i + \bigvee y_j = \varphi(x_0) + \varphi(y_0)\,.$$

Thus φ is an isomorphism with respect to the group operation. Further, in $D_1(G)$ we have $x_0 \vee y_0 = \sup\{x_i \vee y_j\}$ and $\{x_i \vee y_j\}$ is an upper bounded subset of $x \vee y + A$. Thus in D' it holds

$$\varphi(x_0 \vee y_0) = \sup\{x_i \vee y_j\} = \bigvee x_i \vee \bigvee y_j = \varphi(x_0) \vee \varphi(y_0)\,.$$

Hence φ is an isomorphism with respect to $\vee$. Since $x_0 \wedge y_0 = -((-x_0) \vee (-y_0))$, φ is also an isomorphism with respect to the operation $\wedge$.

2.16. Theorem. *For each lattice ordered group G, $D(A(G))$ is a closed l-subgroup of $D_1(G)$.*

Proof. It suffices to verify that if $\emptyset \neq \{a'_i\}_{i\in I} \subseteqq D(A(G))^+$ and if $\bigvee a'_i = b$ holds in $D_1(G)$, then $b \in D(A(G))$. Assume that b does not belong to $D(A(G))$. Then there is $0 < x \in G$ with $b \in x + D(A(G))$, $x < b$, x non $\in A(G)$. Put $a'_i \wedge x = a_i$. From the infinite distributivity of $D_1(G)$ we obtain $\bigvee a_i = x$. Clearly $\{a_i\}_{i\in I} \subseteqq D(A(G))$.

Since x does not belong to $A(G)$, it fails to be archimedean and hence there is $0 < c_1 \in G$ such that $nc_1 < x$ for each positive integer n. If $c_1 \wedge a_i = 0$ for each $i \in I$, then $c_1 \wedge x = 0$, which is a contradiction. Hence there is $j \in I$ such that $a_j \wedge c_1 = c > 0$. Then $c \in D(A(G))$ and $nc < x$ for each positive integer n.

Since $D(A(G))$ is conditionally complete, the element

$$c_i = \bigvee(a_i \wedge nc) \quad (n = 1, 2, \ldots)$$

exists for each $i \in I$. Let

$$(c)^\gamma = \{g \in D_1(G) : |g| \wedge c = 0\},$$

$$K = \{h \in D_1(G) : |h| \wedge |g| = 0 \text{ for each } g \in (c)^\gamma\}.$$

Because $D_1(G)$ is a complete lattice ordered group, both K and $(c)^\gamma$ are direct factors of $D_1(G)$. We shall show that c_i is the component of a_i in K. It suffices to verify that c_i is the greatest element of the set

$$K_i = \{k \in K : 0 \leqq k \leqq a_i\}.$$

Clearly $c_i \in K_i$. Suppose that c_i fails to be the greatest element of K_i. Then there is $0 < t_1 \in D_1(G)$ with $t_1 + c_i \in K$, $t_1 + c_i \leqq a_i$. Hence $t_1 \wedge c = t > 0$. For each positive integer n we have

$$t + (a_i \wedge nc) \leqq t_1 + c_i \leqq a_i,$$

$$t + (a_i \wedge nc) \leqq c + nc = (n + 1)c,$$

thus $t + (a_i \wedge nc) \leqq a_i \wedge (n + 1)c$ and therefore

$$c_i < t + c_i = t + \bigvee_{n=1}^{\infty}(a_i \wedge nc) = \bigvee_{n=1}^{\infty}(t + (a_i \wedge nc)) \leqq \bigvee_{n=2}^{\infty}(a_i \wedge nc) = c_i,$$

which is a contradiction. Hence c_i is the component of a_i in K and therefore

$$d_i = a_i - c_i$$

is the component of a_i in $(c)^\gamma$. This implies immediately that $d_i \wedge c_i = 0$, hence $a_i = c_i \vee d_i$. Further, we have $d_i \wedge nc = 0$ for each positive integer n, since $nc \in K$ and $d_i \in (c)^\gamma$.

Let N be the set of all positive integers. Then

$$x = \bigvee_{i\in I} a_i = \bigvee_{i\in I}(c_i \vee d_i) = \bigvee_{i\in I}\bigvee_{n\in N}(a_i \wedge nc) \vee d_i.$$

Since $nc < x$, we get

$$x = \bigvee_{i\in I}\bigvee_{n\in N}(nc \vee d_i).$$

At the same time we have obviously

$$x = \bigvee_{i\in I}\bigvee_{n\in N}((n + 1)c \vee d_i).$$

Then

$$c + x = c + \bigvee_{i\in I}\bigvee_{n\in N}(nc \vee d_i) = \bigvee_{i\in I}\bigvee_{n\in N}((n + 1)c \vee (c + d_i)) =$$

$$= \bigvee_{i\in I}\bigvee_{n\in N}((n + 1)c \vee (c \vee d_i)) = \bigvee_{i\in I}\bigvee_{n\in N}((n + 1)c \vee d_i) = x,$$

which is a contradiction, since $c > 0$. Thus $b \in D(A(G))$.

2.17. Lemma. *Let $x \in G$, $b_1 \in D_1(G)$, $b_1 \notin G$, $b_1 < x$. Then there is $x_1 \in G$ with $b_1 < x_1 < x$.*

Proof. Put $b_2 = b_1 - x$, $b_3 = -b_2$. Then $0 < b_3$ and $b_3 \notin G$. Hence there is $Y \subseteqq G^+$ with $\sup Y = b_3$. Choose $0 < y \in Y$. We have $-y + x \in G$ and $b_1 < < -y + x < x$.

2.18. Theorem. *For each lattice ordered group G, $A(G)$ is a closed l-subgroup of G.*

Proof. Again, it suffices to verify that if $\emptyset \neq \{a_i\}_{i \in I} \subseteqq A(G)$ and $b = \bigvee a_i$ holds in G, then $b \in A(G)$. If $b = \sup \{a_i\}$ is valid in $D_1(G)$, then according to Theorem 2.16 we have $b \in D(A(G))$ and thus, since $b \in G$, we obtain $b \in A(G)$.

Assume that $b \neq \sup \{a_i\}$ in $D_1(G)$. Hence there is $b_1 \in D_1(G)$ with $b_1 \notin G$ such that $a_i < b_1$ for each $i \in I$ and $b_1 < b$. According to Lemma 2.17 there is $x_1 \in G$ with $b_1 < x_1 < b$. Hence $a_i < x_1$ for each $i \in I$, thus $x_1 \geqq b$, which is a contradiction.

2.19. Corollary. *Let $\emptyset \neq \{a_i\}$ be a set of archimedean elements in a lattice ordered group G and let $\bigvee a_i = b$ be valid in G. Then b is archimedean in G.*

2.20. Proposition. *Let $\{x_i\} \subset G$ and let x be the least upper bound of the set $\{x_i\}$ in G. Then x is the least upper bound of the set $\{x_i\}$ in $D_1(G)$.*

Proof. Since G is an l-subgroup of $D_1(G)$, we have $x_i \leqq x$ for each x_i. Assume that x fails to be the least upper bound of the set $\{x_i\}$ in $D_1(G)$. Then there is $y \in D_1(G)$ such that $y < x$ and $x_i \leqq y$ for each x_i. Thus y non $\in G$. Hence $0 < x - y$ and $x - y$ does not belong to G. Hence there is $z \in G$ such that $0 < z < x - y$. This yields $y < -z + x < x$ and clearly $-z + x \in G$, $x_i < -z + x < x$ for each x_i. This is a contradiction.

Analogously we can verify the assertion dual to 2.20.

Let G be a partially ordered group. In [3], Chap. V, § 10, L. Fuchs has defined an extension of G such that if G is an archimedean lattice ordered group then this extension coincides with $D(G)$; we denote this extension by $F(G)$. Let us recall the definition of $F(G)$.

Let $F_1(G)$ be the system consisting of all sets $(X^u)^l$, where X is any nonempty subset of G that is upper bounded in G. The system $F_1(G)$ is partially ordered by the inclusion. For $X_1, Y_1 \in F_1(G)$ we put $X_1 +_1 Y_1 = (\{x_1 + y_1 : x_1 \in X_1,\ y_1 \in Y_1\}^u)^l$. Then $(F_1(G); \leqq, +_1)$ is a partially ordered semigroup with a neutral element $(\{0\}^u)^l$. We denote by $F(G)$ the set of all elements of $F_1(G)$ that have an inverse in $F_1(G)$. Then $F(G)$ is a partially ordered group. If we identify the element $g \in G$ with $(\{g\}^u)^l$, then $F(G)$ turns out to be an extension of G.

Problem 1. Let G be a lattice ordered group. What relations exist between $F(G)$ and $D_1(G)$? In particular, when do $F(G)$ and $D_1(G)$ coincide? (If this is the case, then the above results give a rather constructive description of the structure of $F(G)$.)

Problem 2. Let G be a partially ordered group. Let $A(G)$ be the system of all convex subgroups G_1 of G having the property that G_1 is an archimedean lattice ordered group under the induced partial order. When has $A(G)$ the greatest element?

3. SOME FURTHER PROPERTIES OF THE GENERALIZED DEDEKIND COMPLETION

In what follows G denotes a lattice ordered group.

3.1. Lemma. *$D_1(G)$ is abelian if and only if G is abelian.*

Proof. Since G is an l-subgroup of $D_1(G)$, the assertion 'only if' is obvious. Let G be abelian and let $x_0, y_0 \in D_1(G)$. Let x, y, X_0, Y_0 be as in the definition of $x_0 + y_0$ (cf. § 2). Then

$$x_0 + y_0 = \sup \{x_i + y_j : x_i \in X_0,\ y_j \in Y_0\} = \\ = \sup \{y_j + x_i : x_i \in X_0,\ y_j \in Y_0\} = y_0 + x_0 .$$

3.2. Proposition. *Let G be abelian and divisible. Then $D_1(G)$ is abelian and divisible.*

Proof. According to 3.1, $D_1(G)$ is abelian. Let $x_0 \in D_1(G)$. There is $x \in G$ such that $x_0 \in x + D(A)$. Let n be a positive integer. Since G is divisible, there is $y \in G$ with $ny = x$. Put $y_0 = y + x_0 - x$. We have $y_0 - y \in D(A)$. Since A is a convex l-subgroup of G, it must be divisible. In [4] it was shown that if H is an archimedean divisible lattice ordered group, then $D(H)$ is a vector lattice. Thus $D(A)$ is a vector lattice. In particular, $D(A)$ is divisible and hence there is $t \in D(A)$ with $y_0 - y = nt$. Therefore $x_0 = x + y_0 - y = ny + nt = n(y + t)$. Hence $D_1(G)$ is divisible.

Let us remark that if G is abelian and divisible, then $D_1(G)$ need not be a vector lattice (cf. Example 1 below).

Problem 3. Is $D_1(G)$ divisible for each divisible lattice ordered group G?

3.3. Proposition. *Let G be a vector lattice. Then $D_1(G)$ is a vector lattice as well.*

Proof. Each convex l-subgroup of a vector lattice is again a vector lattice; hence A is a vector lattice. Thus $D(A)$ is a vector lattice as well. Let us choose in each class $x + A$ of the factor l-group G/A a fixed element $x_1 = f(x + A)$. Let $x_0 \in D_1(G)$. There is $x \in G$ such that $x_0 \in x + D(A)$. Let $x_1 = f(x + A)$ and let α be a real. Then $x_0 - x_1 \in D(A)$, hence $\alpha(x_0 - x_1)$ is defined. We put

$$\alpha x_0 = \alpha x + \alpha(x_0 - x_1) .$$

If $x_0 \in G$ or $x_0 \in D(A)$, then this definition of αx_0 coincides with the product αx_0 defined in G or $D(A)$, respectively. It is a routine to verify that under this definition of multiplication of elements of $D_1(G)$ by reals the lattice ordered group $D_1(G)$ turns out to be a vector lattice.

Let $\emptyset \neq X \subseteqq G$, $\emptyset \neq X_0 \subseteqq D_1(G)$. Denote

$$X^\delta = \{g \in G : |g| \wedge |x| = 0 \text{ for each } x \in X\},$$
$$X_0^\beta = \{g_0 \in D_1(G) : |g_0| \wedge |x_0| = 0 \text{ for each } x_0 \in X_0\}.$$

X^δ and X_0^β are said to be *polars in* G and in $D_1(G)$, respectively (cf. Šik [7]). For each polar X^δ of G we denote by $f(X^\delta)$ the set of all elements $y_0 \in D_1(G)$ such that $|y_0|$ is a join of a certain subset of X^δ.

3.4. Proposition. *For each polar X^δ of G, $f(X^\delta)$ is a polar of $D_1(G)$. Moreover, f is a one-to-one mapping of the set of all polars of G onto the set of all polars of $D_1(G)$.*

Proof. Let $y_0 \in f(X^\delta)$. There is a subset $X_1 = \{x_j\}$ of X^δ with $|y_0| = \bigvee x_j$. Without loss of generality we may suppose that $x_j \geqq 0$ is valid for each x_j. If $x \in X$, then $|x| \wedge x_j = 0$ for each x_j and hence by the infinite distributivity of $D_1(G)$ we obtain $|x| \wedge |y_0| = 0$. Thus $f(X^\delta) \subseteqq X^\beta$. Let $y_1 \in X^\beta$. There exists a system $\{y_k\} \subset G^+$ with $\bigvee y_k = |y_1|$. For each $x \in X$ we have $|x| \wedge |y_1| = 0$ and hence $|x| \wedge y_k = 0$ for each y_k. Thus $\{y_k\} \subset X^\delta$ and hence $y_1 \in f(X^\delta)$. Therefore $f(X^\delta) = X^\beta$ and so $f(X^\delta)$ is a polar in $D_1(G)$.

Let X_0^β be a polar of $D_1(G)$. We denote by X the set of all elements $x \in G$ such that $0 \leqq x \leqq |x_0|$ for some $x_0 \in X_0$. Let $y_1 \in f(X^\delta)$ and $x_0 \in X_0$. Then there is a subset $\{x_i\} \subseteqq X$ and a subset $\{y_j\} \subseteqq X^\delta$ such that $\{x_i\} \subseteqq G^+$, $\{y_j\} \subseteqq G^+$ and $\bigvee x_i = |x_0|$, $\bigvee y_j = |y_1|$. Using the infinite distributivity of $D_1(G)$ we obtain $|y_1| \wedge |x_0| = 0$, hence $f(X^\delta) \subseteqq X_0^\beta$. Conversely, let $y_1 \in X_0^\beta$. There is a subset $\{y_j\} \subseteqq G^+$ such that $\bigwedge y_j = |y_1|$. Let $x \in X$. There is $x_0 \in X_0$ with $x \leqq |x_0|$. Hence $0 \leqq y_j \wedge x \leqq y_1 \wedge x_0 = 0$. Thus $\{y_j\} \subseteqq X^\delta$ and therefore $y_1 \in f(X^\delta)$. Summarizing, we conclude $X_0^\beta = f(X^\delta)$. Hence f is onto.

Let X, Y be nonempty subsets of G and suppose that $X^\delta \neq Y^\delta$, $f(X^\delta) = f(Y^\delta)$. Without loss of generality we may suppose that X^δ is not a subset of Y^δ. Thus there are elements $0 < x_1 \in X^\delta$, $y = Y$ such that $x_1 \wedge |y| > 0$. Further, from $f(X^\delta) = f(Y^\delta)$ we get $x_1 \in f(Y^\delta)$ and hence by the infinite distributivity $x_1 \wedge |y| = 0$, which is a contradiction. Therefore f is one-to-one.

Each polar of a lattice ordered group is a convex l-subgroup [7]. A lattice ordered group is said to be representable if it is a subdirect product of linearly ordered groups. It is well-known that a lattice ordered group is representable if and only if each its polar is a normal subgroup (cf. e.g. [2]).

3.5. Theorem. *Let G be a representable lattice ordered group. Then $D_1(G)$ is also representable.*

To prove this we need the following lemmas.

3.6. Lemma. *Let G be a representable lattice ordered group. Let B be a polar in $D_1(G)$ and let $g \in G$. Then $-g + B + g = B$.*

Proof. As we have already proved there exists a polar B_1 of G such that for each $0 < b \in B$ there is a subset $S \subset B_1$ with $\sup S = b$. The mapping $\psi(t) = -g + t + g$ $(t \in D_1(G))$ is an automorphism on $D_1(G)$, thus $-g + B + g$ is a polar of $D_1(G)$. Since G is representable, we have $-g + B_1 + g = B_1$ and thus $B_1 \subseteqq -g + B + g$. Each polar being a closed sublattice (cf. [7]) we obtain $B^+ \subseteqq -g + B + g$ and hence $B \subseteqq -g + B + g$. By putting $-g$ instead of g we get $B \subseteqq g + B - g$, thus $B = -g + B + g$.

3.7. Lemma. *Let G be a representable lattice ordered group. Let B be a polar in $D_1(G)$ and let $a \in D(A)$. Then $-a + B + a = B$.*

Proof. Because each element of $D(A)$ can be written as a difference of two elements belonging to $D(A)^+$, it suffices to prove the assertion for $a > 0$. Then there exists a subset $\{a_i\} \subset A^+$ such that $\{a_i\}$ is upper bounded in A and $\bigvee a_i = a$. Let a_1 be an upper bound of $\{a_i\}$ in A. Without loss of generality we may suppose that $\{a_i\}$ possesses the least element a_0. Let $b \in B$. According to 3.6 there are elements b_i, b' and b'' in B such that

$$a_i + b = b_i + a_i\,, \quad a_1 + b = b' + a_1\,. \tag{2}$$

For $a_i = a_0$ we denote $b_i = b''$. All elements b_i, b', b'' belong to $b + D(A)$. We have $a_i + b \leqq a_1 + b$, thus $b_i + a_i \leqq b' + a_1$ and hence $b_i \leqq b' + a_1$. Since $b' + a_1 \in b + D(A)$, the set $\{b_i\}$ is upper bounded in $b + D(A)$ and hence there exists a least upper bound b_1 of the set $\{b_i\}$ in $b + D(A)$. Clearly $b_1 = \bigvee b_i$ is valid in $D_1(G)$. Since each polar is a closed sublattice, we get $b_1 \in B$.

From $a_0 + b \leqq a_i + b$ we obtain $b'' + a_0 \leqq b_i + a_i$ and thus

$$b'' + a_0 - a \leqq b'' + a_0 - a_i \leqq b_i\,.$$

Since $b'' + a_0 - a \in b + D(A)$, the set $\{b_i\}$ is lower bounded in $b + D(A)$ and hence there exists the greatest lower bound b_2 of $\{b_i\}$ in $b + D(A)$. Then $\bigwedge b_i = b_2$ is valid in $D_1(G)$ and $b_2 \in B$.

From (2) we get

$$b_2 + a_i \leqq a_i + b \leqq b_1 + a_i\,,$$

hence

$$b_2 + a \leqq a + b \leqq b_1 + a\,.$$

Because $b_1 + a$, $b_2 + a \in B + a$ and $B + a$ is a convex subset of $D_1(G)$ we infer that $a + b \in B + a$. Thus $a + B \subseteqq B + a$. Analogously we can verify that $B + a \subseteqq a + B$.

Proof of Theorem 3.5. Let B be a polar of $D_1(G)$ and $x_0 \in D_1(G)$. There are $g \in G$ and $a \in D(A)$ such that $x_0 = g + a$. Now from 3.6 and 3.7 we obtain $-x_0 + B + x_0 = B$. Thus $D_1(G)$ is representable.

3.8. Proposition. *Let $G_1 = (G; \leqq_1, +_1)$, $G_2 = (G; \leqq_2, +_2)$ be lattice ordered groups defined on the same underlying set G such that*

(i) $(G; \leqq_1) = (G; \leqq_2)$,

(ii) *the partition of G corresponding to the l-ideal $A(G_1)$ (consisting of classes $x +_1 A(G_1)$, $x \in G$) coincides with the partition of G corresponding to the l-ideal $A(G_2)$.*

Then there exists an isomorphism ψ of the lattice $(D_1(G_1); \leqq_1)$ onto the lattice $(D_1(G_2), \leqq_2)$ such that $\psi(g) = g$ for each $g \in G$.

Proof. The assertion follows immediately from the definition of the partial order in $D_1(G_1)$ or $D_1(G_2)$, respectively (cf. § 2).

Let us remark that the condition (ii) is not a consequence of (i) (cf. Example 3.10 below).

3.9. Example. Let R_0 and R be the additive group of all reals or all rationals, respectively, with the natural linear order. Let $G = R_0 \circ R$ be the lexicographic product of R_0 and R (cf. [3]). Then $A(G) = D(A(G))$ is the set of all $(x, y) \in R_0 \circ R$ with $x = 0$, hence $D_1(G) = G$, G is divisible and $D_1(G)$ fails to be a vector lattice.

3.10. Example. Let R_0 be as in 3.9. Put $G_1 = R_0$, $G'_2 = R_0 \circ R_0$. The lattice $(G'_2, \leqq)$ is isomorphic with the lattice $(R_0, \leqq)$, hence there is a lattice ordered group $G_2 = (R_0; \leqq, +_1)$ defined on the set R_0 such that G_2 is isomorphic with G'_2. Thus the condition (i) from 3.8 is fulfilled. We have $A(G_1) = G_1$, hence $G_1/A(G_1)$ is a one-element set. On the other hand, $G_2/A(G_2)$ is isomorphic with R_0, hence the condition (ii) from 3.8 fails to be valid.

Added in proof. In a recent paper by R. H. Redfield (Archimedean and basic elements in completely distributive lattice ordered groups, Pacif. J. Math. *63* (1976), 247—254) there is given a different proof of Theorem 1.5. (Redfield's paper appeared in March 1976.)

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